q-CALCULUS AND THE DISCRETE INVERSE SCATTERING

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ABSTRACT

The discrete inverse scattering in one-dimension has been reidentified with lattice calculus. By transforming the deformation parameter, the coordinate and the partial derivatives from lattice space to q -space, the Schrödinger equation with a potential is systematically analysed. The potential calculated for given phase shifts confirms with the classical result in form. The possibility of the deformation parameter q identified with curvature of space is briefly commented.

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I. Introduction

The one-dimensional inverse scattering problem in quantum mechanics was investigated by Case and Kac¹ using the difference method. The problem was addressed to determine the potential for all $E > 0$ given the phase shifts $\delta(E)$. Here we examine the inverse scattering problem using q -calculus after establishing the transformation of the deformation parameter, the coordinate and the partial differential operators from lattice space to q -space.² Following Gelfand-Levitan procedure³ and making necessary approximation, we find that the potential agrees with the classical result in form.¹ The q -dependence of the potential is reflected in the structure of the wave function $\psi(E, n)$. The possibility of identifying the deformation parameter q with the curvature of the space-time is examined.⁴ This comes handy as a deformation of the Pythagoras theorem, i.e., $\cos^2\Theta + \sin^2\Theta = q$. Such a relation between q and the curvature was long speculated but never proved to our knowledge. We feel such an identification of the parameter q will have far reaching bearing on the physical applicability of quantum groups.

Our material is arranged as follows.

In Section II, we briefly survey the non-commutative geometry and lattice calculus and the passage from lattice framework to q -calculus. Section III is devoted to a systematic analysis of the q -deformed one-dimensional inverse scattering. Using Gelfand-Levitan procedure, the Schrödinger equation for $\psi(E, x)$, $0 \leq x < \infty$, with the boundary conditions

$$\psi(E, x) = 0, \psi'(E, 0) = 1$$

is recast in the discrete version and then the passage to q -calculus is facilitated. In Section IV, we discuss the solution of the inverse scattering with an example. The connection between the deformation parameter q and the curvature of the space is examined in

the light of the deformation of the Pythagoras theorem in Section V. In an appendix, we discuss some of the properties of the q -deformed trigonometric functions relevant to our analysis.⁵

II. Survey of the Non-commutative Differential Calculus and Lattice Calculus

a. Lattice Calculus

Non-commutative geometry as a tool to investigate the ultraviolet divergence problem in quantum field theory is very well-known.⁶ Instead of postulating the non-commuting property of the coordinates, one can instead achieve the similar results by assuming the coordinates commutative but functions not commuting with the differential.^{2,7} We thus consider the deformation of the classical form

$$[x^i, dx^j] = 0, \quad i, j = 1, 2, \dots, n, \quad (2.1)$$

$$\text{as } [x^i, x^j] = 0$$

$$\text{and } [x^i, dx^j] = \sum_k dx^k C_k^{ij}, \quad (2.2)$$

where the constrained constants C_k^{ij} satisfy

$$C_k^{ij} = C_k^{ji},$$

$$C_l^{ik} C_m^{jl} = C_l^{jk} C_m^{il},$$

$$\text{and } C_l^{ki} C_m^{jl} - C_l^{kj} C_m^{il} = 0. \quad (2.3)$$

In one dimension, we have from (2.2)

$$[x, dx] = dx \cdot a, \quad (2.4)$$

where $a > 0$ and real. Further, we have

$$x^n dx = dx (x+a)^n$$

$$\text{and } f(x) dx = dx f(x+a) \text{ for any } f. \quad (2.5)$$

(2.4) and (2.5) reveal the existence of a discrete translational symmetry. We introduce the right partial derivative

$$df = dx \cdot (\partial_a f)(x). \quad (2.6)$$

Using the modified Leibnitz rule

$$(\partial_a fh)(x) = \partial_a f(x) h(x) + f(x+a) (\partial_a h)(x)$$

we obtain

$$(\partial_a f)(x) = \frac{1}{a} [f(x+a) - f(x)]. \quad (2.7)$$

We note that (2.7) yields the ordinary derivative in the $\lim a \rightarrow 0$. We observe that $df = 0$ defines f to be a periodic function of x with periodicity 'a'. To define the left partial derivative, we make use of

$$[f(x), dx] = a df$$

and the modified Leibnitz rule

$$(\partial_a fh)(x) = (\partial_a f)(x) h(x) + f(x-a) (\partial_a h)(x)$$

yielding

$$(\partial_a f)(x) = \frac{1}{a} [f(x) - f(x-a)]. \quad (2.8)$$

If we replace the constant 'a' by an arbitrary function $a(x)$, then the above formulas still hold good, e.g., let $y = y(x)$. Then, it can be shown that

$$[y, dy] = dy a (\partial_a y)(x).$$

Thus, by substituting $(\partial_a y)(x) = a(y)/a$ we find

$$[y, dy] = dy a(y).$$

b. q-Calculus

Let q , y , ∂_q and $\bar{\partial}_q$ be the deformation parameter, coordinate, the right and left partial derivatives respectively in q -space. Then the lattice space is related to the q -space by the following mappings :

$$a \rightarrow q = 1-a, \quad 0 < a < 1, \quad 0 < q < 1;$$

$$x \rightarrow y = q^{x/1-q},$$

$$\partial_a \rightarrow \partial_q \quad \text{and} \quad \bar{\partial}_a \rightarrow \bar{\partial}_q, \quad (2.9)$$

$$\text{where } (\partial_q f)(y) = \frac{f(qy) - f(y)}{(1-q)y}.$$

$$\text{and } (\bar{\partial}_q f)(y) = \frac{f(y) - f(q^{-1}y)}{(1-q)y}. \quad (2.10)$$

In the present investigation, we consider the period 'a' to be very close to 1 so that the deformation parameter $q < 1$. We introduce the second derivative from (2.10) as⁵

$$\frac{f(qy) - (1+q)f(y) + qf(q^{-1}y)}{(1-q)^2 y^2} \quad (2.11)$$

III. The Inverse Scattering Problem on R^1

Let us consider the one-dimensional Schrödinger equation

$$\frac{1}{2} \frac{d^2 \psi(E, x)}{dx^2} + [E - V(x)] \psi(E, x) = 0 \quad (3.1)$$

with the boundary conditions

$$\psi(E, 0) = 0, \psi'(E, 0) = 1 \quad (3.2)$$

and $\psi(E, x) : x \in [0, \infty)$.

Further, for +ve E and suitable well-behaved $V(x)$, $\psi(E, x)$ is asymptotically proportional to $\sin(\sqrt{2E}x + \delta(E))$, $x \rightarrow \infty$. The inverse scattering problem is concerned with the determination of $V(x)$ given the phase shifts $\delta(E)$ for all $E > 0$.

If there are no bound states, then it is possible to estimate $V(x)$. But if $E_i < 0$, i.e., if there exist bound states, then in addition to $\delta(E_i)$, we need have to know the normalisation constants C_i :

$$C_i = \int_0^{\infty} \psi^2(E, x) dx. \quad (3.3)$$

a. Lattice Quantum Mechanics

$$\text{Let } na = x, \quad (3.4)$$

where n is an integer and 'a' is the linear spacing of a lattice. Then, the Schrodinger equation (3.1) can be written as

$$\begin{aligned} \frac{1}{2a^2} \{ \psi(E, (n+1)a) - 2\psi(E, na) + \psi(E, (n-1)a) \} \\ + [E - V(na)] \psi(E, na) = 0. \end{aligned} \quad (3.5)$$

(3.5) can be written in a more general form by making the ansatz

$$\begin{aligned} \frac{1}{2} \{ \psi(E, (n+1)a) + \psi(E, (n-1)a) \} \\ = (1 - Ea^2) e^{a^2 V(na)/2} \psi(E, na). \end{aligned} \quad (3.6)$$

We see that as $a \rightarrow 0$, (3.6) reduces to (3.5).

b. q-Deformed Quantum Mechanics

Using the transformations (2.9) and (2.10) in (3.5), we have

$$\frac{1}{2} \{ \psi(E, qy) - (1+q) \psi(E, y) + q \psi(E, q^{-1}y) \} / (1-q)^2 y^2 + [E - V(y)] \psi(E, y) = 0. \quad (3.7)$$

Since $q = 1-a$ and $x = na$ and $y = q^{n-1}$, we have

$$y = q^n, qy = q^{n+1}, q^{-1}y = q^{n-1}. \quad (3.8)$$

$$\text{Let } \Delta^2 = (1-q)^2 y^2 = (1-q)^2 q^{2n}. \quad (3.9)$$

Then, (3.7) reduces to

$$\frac{1}{2\Delta^2} \{ \psi(E, q^{n+1}) - (1+q) \psi(E, q^n) + q \psi(E, q^{n-1}) \} + [E - V(q^n)] \psi(E, q^n) = 0. \quad (3.10)$$

$$\begin{aligned} \text{Or, } & \frac{1}{2} \{ \psi(E, q^{n+1}) + q \psi(E, q^{n-1}) \} \\ & = \left(\frac{1+q}{2} - E\Delta^2 \right) e^{2\Delta^2 V(q^n)} \psi(E, q^n). \end{aligned} \quad (3.11)$$

(3.11) represents the generalised form of (3.10) where we have neglected $o(q\Delta^2)$ terms since $q \ll 1$.

Substituting

$$\lambda = \frac{1+q}{2} - E\Delta^2,$$

$$v(n) = 2\Delta^2 V(q^n),$$

$$\text{and } \phi(\lambda, n) = e^{v(n)/2} \psi(E, q^n) \quad (3.12)$$

in (3.11), we have

$$\begin{aligned} & \frac{1}{2} \{ e^{-[v(n)+v(n+1)]/2} \phi(\lambda, n+1) + q e^{-[v(n-1)+v(n)]/2} \phi(\lambda, n-1) \} \\ & = \lambda \phi(\lambda, n). \end{aligned} \quad (3.13)$$

(3.13) can be written in the form

$$\tilde{A} \phi = Q A \phi = \lambda \phi, \quad (3.14)$$

where the (m, n) element a_{mn} of A is given by

$$a_{mn} = \frac{1}{2} e^{-[v(n)+v(m)]/2} \delta_{m, n-1} = a_{nm}$$

$$\text{and } Q_{n, n+1} = 1, Q_{n, n-1} = q. \quad (3.15)$$

Setting $v(0) = 0$, we find $\phi(\lambda, 0) = \psi(E, 0) = 0$.

Let us choose $\phi(\lambda, 0) = 1$ as the other boundary condition. Let A^0 correspond to $v(n) = 0$,

$$\text{i.e., } a_{mn}^0 = \frac{1}{2} (\delta_{m, n+1} + \delta_{m, n-1}). \quad (3.16)$$

Then, (3.13) reduces to

$$\frac{1}{2} \{ \phi^0(\lambda, n+1) + q \phi^0(\lambda, n-1) \} = \lambda \phi^0(\lambda, n). \quad (3.17)$$

The solution of (3.17) can be written as

$$\phi^0(\lambda, n) = \frac{[\lambda + (\lambda^2 - q)^{1/2}]^n - [\lambda - (\lambda^2 - q)^{1/2}]^n}{2(\lambda^2 - q)^{1/2}} \quad (3.18)$$

with the spectral property as

$$\sigma(\lambda) = \begin{cases} 0, & \lambda < -\sqrt{q}, \\ \frac{2}{\sqrt{q}\pi} \int_{-\sqrt{q}}^{\lambda} (q - \mu^2)^{1/2} d\mu, & -\sqrt{q} < \lambda < \sqrt{q}, \\ \sqrt{q}, & \lambda > \sqrt{q}. \end{cases} \quad (3.19)$$

Now, let $\rho(\lambda)$ be the spectral distribution of A such that

$$\int \phi(\lambda, m) \phi(\lambda, n) d\rho(\lambda) = \delta_{mn}, \quad (3.20)$$

$$\int \lambda \phi(\lambda, m) Q_{mn}^{-1} \phi(\lambda, n) d\rho(\lambda) = a_{m,n}. \quad (3.21)$$

$\phi(\lambda, n)$ are the orthogonal polynomials (properly normalised) with respect to the weight $\rho(\lambda)$. Given the spectral distributions $\rho(\lambda)$, we have to determine $v(n)$. Now, from (3.15)

$$a_{n,n-1} = \frac{1}{2} e^{-[v(n)+v(n-1)]/2} = a_{n-1,n}. \quad (3.22)$$

Positivity of $a_{n,n-1}$ implies that

$$\int \lambda \phi(\lambda, n) \phi(\lambda, n-1) d\rho(\lambda) > 0. \quad (3.23)$$

Further, we must have

$$\int \lambda \phi^2(\lambda, n) d\rho(\lambda) = a_{n,n} = 0. \quad (3.24)$$

Thus, a priori, we must have suitable conditions on the spectral function $\rho(\lambda)$ so that (3.23) and (3.24) are satisfied. Let us choose, e.g.,

$$\rho(-\lambda) = 1 - \rho(\lambda). \quad (3.25)$$

(3.25) generates the even (odd) properties of the polynomials $\phi(\lambda, n)$.

$$\text{Let } \phi(\lambda, n) = K(n, n) + \phi^0(\lambda, n) \sum_{m=1}^{n-1} K(n, m) \phi^0(\lambda, m). \quad (3.26)$$

Thus, the orthogonality condition (3.20) can be rewritten as

$$\int \phi(\lambda, n) \phi^0(\lambda, m) d\rho(\lambda) = 0, \quad m < n. \quad (3.27)$$

$$\text{Let } g(m, r) = \int \phi^0(\lambda, m) \phi^0(\lambda, r) d(\rho(\lambda) - \sigma(\lambda)). \quad (3.28)$$

Then, (3.27) yields

$$0 = K(n, m) g(m, n) + K(n, m) + \sum_{r=1}^{n-1} K(n, r) g(r, m), \quad n > m. \quad (3.29)$$

Thus, the normalisation condition

$$\int \phi^2(\lambda, n) d\rho(\lambda) = 1$$

yields on substituting (3.26)

$$K(n, n) \int \phi(\lambda, n) \phi^0(\lambda, n) d\rho(\lambda) = 1. \quad (3.30)$$

$$\text{Let } \chi(n, m) = K(n, m)/K(n, n), \quad m < n. \quad (3.31)$$

Then, (3.29) can be rewritten as

$$0 = g(n, m) + \chi(n, m) + \sum_{r=1}^{n-1} \chi(n, r) g(r, m). \quad (3.32)$$

After a little manouvering, we find from (3.26), (3.29) and (3.30)

$$K(n, n) = [1 + g(n, n) + \sum_{r=1}^{n-1} \chi(n, r) g(r, n)]^{-1/2}. \quad (3.33)$$

To estimate $v(n)$, we have from (3.21)

$$\begin{aligned}
a_{n,n-1} &= q^{-1} \int \lambda \phi(\lambda, n) \phi(\lambda, n-1) d\rho(\lambda) \\
&= q^{-1} K(n, n) \int \lambda \phi^0(\lambda, n) \phi(\lambda, n-1) d\rho(\lambda).
\end{aligned} \tag{3.34}$$

Using $\lambda \phi^o(\lambda, n) = \frac{q}{2} \phi^o(\lambda, n-1) + \frac{1}{2} \phi^o(\lambda, n+1)$ from (3.17) in (3.22), we have finally

$$\begin{aligned}
a_{n,n-1} &= K(n, n) \int \phi^0(\lambda, n-1) \phi(\lambda, n-1) d\rho(\lambda), \\
&= \frac{K(n, n)}{K(n-1, n-1)} \quad (\text{using eq. (3.30)}).
\end{aligned} \tag{3.35}$$

Or, equivalently we obtain from (3.22)

$$\frac{1}{2} [v(n-1) + v(n)] = \log K(n-1, n-1) - \log K(n, n). \tag{3.36}$$

For a unique determination of $v(n)$, we need have to know further that asymptotically $v(n)$ vanishes.

IV. An Example

$$\text{a. Let } \lambda = \sqrt{q} (\cos \theta)_q, \quad -\sqrt{q} < \lambda < \sqrt{q}. \tag{4.1}$$

$$\begin{aligned}
\text{Then, } \phi^o(\lambda, n) &= \frac{[\lambda + (\lambda^2 - q)^{1/2}]^n - [\lambda - (\lambda^2 - q)^{1/2}]^n}{2(\lambda^2 - q)^{1/2}} \\
&= q^{(n-1)/2} \frac{(\sin n\theta)_q}{(\sin \theta)_q}.
\end{aligned}$$

For q -deformed properties of the trigonometric functions, see Appendix I.

$$\text{Thus, } \phi^o(\lambda, 0) = 0,$$

$$\phi^o(\lambda, 1) = 1,$$

$$\phi^o(\lambda, 2) = e^{i\theta} + qe^{-i\theta} = 2\sqrt{q} (\cos \theta)_q = 2\lambda.$$

From (4.1), we have further

$$\lambda \approx \frac{1+q}{2} - \frac{1+q}{2} \theta^2/2. \quad (4.3a)$$

$$\text{But } \lambda = \frac{1+q}{2} - E\Delta^2 \quad (\text{from eq. (3.12)}). \quad (4.3b)$$

Thus, from (4.3) we have

$$\theta \approx 2\Delta \sqrt{E/(1+q)}. \quad (4.4)$$

Substituting (4.4) in (4.2), we find

$$\begin{aligned} \Delta \phi(E, n) &= q^{\frac{n-1}{2}} K(n, n) \Delta \frac{(\sin(2m\Delta \sqrt{E/(1+q)}))_q}{(\sin(2\Delta \sqrt{E/(1+q)}))_q} \\ &+ \Delta \sum_{m=1}^{n-1} q^{\frac{m-1}{2}} K(n, m) \frac{(\sin(2m\Delta \sqrt{E/(1+q)}))_q}{(\sin(2\Delta \sqrt{E/(1+q)}))_q}. \end{aligned} \quad (4.5)$$

$$\text{As } q \rightarrow 1, q^n \rightarrow e^{na}.$$

However, from (3.9), we have

$$n\Delta \approx n(1-q)q^n. \quad (4.6)$$

Thus, neglecting terms $o(a^2)$ in the above, we finally have

$$n\Delta \approx na = x. \quad (4.7)$$

Using (4.7) in (4.5), we are led to the generalised Gelfand-Levitan equation.

$$\text{Thus, } \psi(E, x) = \alpha \frac{\sin x \sqrt{2E}}{\sqrt{2E}} + \int_0^x K(x, \xi) \frac{\sin \xi \sqrt{2E}}{\sqrt{2E}} d\xi \quad (4.8)$$

$$\text{where } \alpha = \lim_{q \rightarrow 1} q^{(n-1)/2} K(n, n).$$

We note further that whenever $\psi(E, x) = 1$ is satisfied $\alpha = 1$.

b. Let us take $\lambda = \sqrt{q} (\cos \theta)_q$ as usual.

$$\text{Then } \psi_{\pm}(n, \theta) \sim e^{\pm i n \theta} \quad (4.9)$$

are solutions for the recursion relation

$$\frac{q}{2} \psi(\lambda, n-1) + \frac{1}{2} \psi(\lambda, n+1) = \lambda e^{v(n)} \psi(\lambda, n) \quad (4.10)$$

for sufficiently large n , i.e., $\lim_{n \rightarrow \infty} v(n) \rightarrow 0$.

Following Case and Kac¹, we substitute

$$\psi_{\pm}(\theta, 0) = P(e^{\pm i \theta}), \quad (4.11)$$

where $P(e^{\pm i \theta})$ is a polynomial in $e^{\pm i \theta}$.

We also have

$$\psi(\theta, n) = e^{-v(n)} \phi(\theta, n) \quad (4.12)$$

which is a solution of (4.10) satisfying the boundary conditions

$$\psi(\theta, 0) = 0, \quad \psi(\theta, 1) = e^{-v(1)/2}. \quad (4.13)$$

$$\text{Let } \psi(\theta, n) = A(\theta) \psi_+(\theta, n) + B(\theta) \psi_-(\theta, n). \quad (4.14)$$

From (4.11) and (4.14), we have

$$0 = A(\theta) P(e^{i\theta}) + B(\theta) P(e^{-i\theta}). \quad (4.15)$$

After a little simplification we get

$$\psi_{-}(\theta, 0) \psi_{+}(\theta, 1) - \psi_{+}(\theta, 1) \psi_{-}(\theta, 1) \approx \frac{2i \sin \theta}{q}. \quad (4.16)$$

We further have

$$e^{-v(1)/2} = A(\theta) \psi_{+}(\theta, 1) + B(\theta) \psi_{-}(\theta, 1). \quad (4.17 a)$$

$$\text{and } 0 = A(0) \psi_{+}(\theta, 0) + B(\theta) \psi_{-}(\theta, 0) \quad (4.17 b)$$

which combined with eq. (4.16) yield

$$A(\theta) = q e^{-v(1)/2} [\psi_{-}(\theta, 0) / 2 i \sin \theta] \quad (4.18 a)$$

$$\text{and } B(\theta) = q e^{-v(1)/2} [\psi_{+}(\theta, 0) / 2 i \sin \theta]. \quad (4.18 b)$$

It can be shown following ref. (1) that the measure

$$d\rho(\lambda) = \frac{1}{2\sqrt{q}\pi} \frac{d\lambda}{A(\cos^{-1} \lambda/\sqrt{q})_q (q-\lambda^2)^{\frac{1}{2}}} \quad (4.19)$$

$$\text{and } |A(\theta)|^2 = q^2 e^{-v(1)} P(e^{-i\theta}) P(e^{i\theta}) / 4 \sin^2 \theta \quad (4.20)$$

where $|A(\theta)|^2 = A(\theta) B(\theta) = A(\theta) \cdot A^*(\theta)$.

We observe here that $|A(\theta)|^2$ can be determined from the phase shift analysis.

V. Deformation of the Pythagoras Theorem

We make use of the deformation properties of the circular functions as outlined in Appedix I.

$$\text{Let } \cos \Theta = \frac{e^{i\theta} + qe^{-i\theta}}{2} = \sqrt{q} (\cos \theta)_q$$

$$\text{and } \sin \Theta = \frac{e^{i\theta} - qe^{-i\theta}}{2} = \sqrt{q} (\sin \theta)_q. \quad (5.1)$$

$$\text{Theorem : } \cos^2 \Theta + \sin^2 \Theta = q. \quad (5.2)$$

Proof : obvious.

Clearly, (5.2) is a deformation of the Pythagoras theorem. For $q \rightarrow 1$, we recover the usual result of Euclidean geometry. In other words, (5.2) may be designated as the Pythagoras theorem in a q -space which is a curved space with $q < 1$, i.e., the space is a hyperbolic space.

Since Pythagoras theorem is so basic in mathematics, our analysis will have far reaching implications in mathematics and theoretical physics. We have thus established the connection between the deformation parameter q and the curvature of the space and hence non-linearity. We think that this connection will set to rest many questions in the field of quantum groups and open up new vistas in describing the physical world.

V. Conclusions

Although, our analysis is pedagogical in nature, we have successfully studied the discrete inverse scattering problem on R^1 establishing the kinship existing between lattice and q -calculus. The new expressions for the circular functions have yielded the 'modified' Pythagoras theorem in a curved space with $q < 1$.

Equivalently, our analysis ventilated the passage from Euclidean geometry to hyperbolic geometry providing negative curvature of the space.

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APPENDIX -- I

q-Deformed Trigonometric Functions

Define

$$\begin{aligned} (\cos n\theta)_q &= \frac{e^{in\theta} + q^n e^{-in\theta}}{2 q^{n/2}}, \\ (\sin n\theta)_q &= \frac{e^{in\theta} - q^n e^{-in\theta}}{2 i q^{n/2}}, \quad 0 < q < 1. \end{aligned} \quad (\text{A.1})$$

We give blow some of the properties of q-deformed trigonometric functions defined in (A.1).

$$1. \quad (\cos \theta)_q^2 + (\sin \theta)_q^2 = 1.$$

$$\text{Proof : } \left(\frac{e^{i\theta} + q e^{-i\theta}}{2\sqrt{q}} \right)^2 + \left(\frac{e^{i\theta} - q e^{-i\theta}}{2 i \sqrt{q}} \right)^2 = 1. \quad (\text{A.2})$$

$$2. \quad (\cos \theta)_q^2 - (\sin \theta)_q^2 = (\cos 2\theta)_q.$$

$$\begin{aligned} \text{Proof : } & \left(\frac{e^{i\theta} + q e^{-i\theta}}{2\sqrt{q}} \right)^2 - \left(\frac{e^{i\theta} - q e^{-i\theta}}{2 i \sqrt{q}} \right)^2 \\ &= \frac{e^{2i\theta} + q^2 e^{-2i\theta}}{2q} = (\cos 2\theta)_q. \end{aligned} \quad (\text{A.3})$$

$$3. \quad 2(\sin \theta)_q (\cos \theta)_q = (\sin 2\theta)_q. \quad (\text{A.4})$$

Proof : Follows directly from (A.1).

$$4. \quad (i) \quad (\cos n\theta)_q + i (\sin n\theta)_q = \frac{e^{in\theta}}{q^{n/2}},$$

$$(ii) \quad (\cos n\theta)_q - i (\sin n\theta)_q = q^{n/2} e^{-in\theta}. \quad (\text{A.5})$$

$$5. \quad \partial_q (\cos \theta)_q = - (\sin \theta)_q. \quad (\text{A.6})$$

Proof :
$$\partial_q (\cos \theta)_q = \partial_q \left(\frac{e^{i\theta} + q e^{-i\theta}}{2\sqrt{q}} \right)$$

$$= \frac{1}{2\sqrt{q}} \partial_q e^{i\theta} + \frac{q}{2\sqrt{q}} \partial_q e^{-i\theta}.$$

$$\partial_q e^{i\theta} = \frac{e^{iq\theta} - e^{i\theta}}{(1-q)\theta} = \frac{e^{i\theta} [e^{i(q-1)\theta} - 1]}{(1-q)\theta}$$

$$\partial_q e^{-i\theta} = \frac{e^{-iq\theta} - e^{-i\theta}}{-(1-q)\theta} = -\frac{e^{-i\theta} [e^{-i(q-1)\theta} - 1]}{(1-q)\theta}.$$

$$\therefore \partial_q (\cos \theta)_q = \frac{1}{2\sqrt{q}(1-q)\theta} [C e^{i\theta} + D q e^{-i\theta}] \quad (\text{A.7})$$

where $C = e^{i(q-1)\theta} - 1$ and $D = e^{-i(q-1)\theta} - 1$.

For $q < 1$, $C \approx i(q-1)\theta$ and $D \approx -i(q-1)\theta$.

Thus,
$$\partial_q (\cos \theta)_q \approx \frac{i(1-q)\theta}{2\sqrt{q}(1-q)\theta} [e^{i\theta} - q e^{-i\theta}]$$

$$= - (\sin \theta)_q.$$

Similarly, it can be shown that

$$\partial_q (\sin \theta)_q \approx (\cos \theta)_q. \quad (\text{A.8})$$

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Here it has been shown that on deformation of the super Poincaré algebra, one gets the super-conformal algebra. The deformation parameter is identified with the curvature of space-time.
5. See e.g., H.Exton, q-hypergeometric functions and applications, Ellis Harwood (Publishers) (1993).
6. S. Majid, Class. Quantum Grav. 5, 1987, (1993).
7. A. Connes, Geometric Non-commutative (Paris : Inter Editors), Publ. IHES 62, 257 (1990).