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ABSTRACT

We apply Nahm's normal ordered product to the Z_k parafermion and prove (up to the level $k = 4$) that different orders of the product of generating parafermion with its conjugate, except a normalization constant, can be identified with the generators in a $W_k \equiv W(2, 3, \dots, k)$ algebra. The scheme we describe is also an illustration of how to enlarge an algebra W_k to W_{k+1} .

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Recently, much interest has been devoted to the W -algebras by theoretical physicists¹⁾, because they may concern the classification of conformal field theory and the theory of gravity. For a comprehensive review we refer the reader to the nice papers in Ref.2.

In a recent paper, we have constructed a W_k -algebra from Z_k parafermion (PF) with $k = 3$ ³⁾. Before us, Fateev and Lykyanov and Bais *et al.* realized that the central charges of $Z_k^{(k+1)}$ model and $V_r[A_{N-1}^{(1)} \oplus A_{N-1}^{(1)}(1, 1)]$ coset model coincide with the Z_k PFs⁴⁾. Narganes-Quijano found the relations between the bosonic expressions of W_3 algebra and the free field realization of $SU(2)$ PF⁵⁾. But our method is to construct W -algebra directly from PF without using free field realization. Two further questions follow immediately: (1) How to find the set of generators for general level k ? (2) Can we generalize the underlying Z_k PF to the PF of higher rank Lie algebra? The second question was solved in a published paper in which we constructed a W_3 algebra from $SU(3)$ PF successively⁶⁾. To solve the first question we devote this paper to Z_k ($SU(2)$) PF.

Three years ago, Nahm proposed a quasiprimeralized normal ordered product of field operator⁷⁾. He used it with his cooperators for establishing a Lie bracket method of W algebra^{8),9)}. We now apply it to the Z_k PF and prove (up to $k = 4$) that Nahm's products of generating PF with its conjugate are primary, not only quasiprimary, except a normalization constant which can be identified with the generators in the $W_k = W(2, 3, \dots, k)$ algebra for special level k . Although we did not give the general proof for any k , we guess that it is true. The scheme we supply may also show the way how to enlarge an algebra W_k to W_{k+1} .

To construct a W_k -algebra (generators $T, W^{(3)}, \dots, W^{(k)}$) from Z_k PF let us write down the stress-energy tensor first,

$$T(z) = \frac{k}{k+2} [\psi_\alpha \psi_{-\alpha}]_0(z), \quad (1)$$

which is the $SU(2)$ form of the general case

$$T(z) = \sum_{\alpha} \frac{k}{2(k+g)} [\psi_\alpha \psi_{-\alpha}]_0(z), \quad (2)$$

where k and g are level and dual-Coxeter number respectively; ψ_α is a generating PF field operator with charge (root) α , and the sum is over all of the roots, $[\psi_\alpha \psi_{-\alpha}]_{-n}$ is the coefficients of the n -th order of $(z-w)$ in the operator product expansion (OPE)

$$\psi_\alpha(z) \psi_\beta(w) (z-w)^{\alpha \cdot \beta / k} = \frac{\delta_{\alpha, -\beta}}{(z-w)^2} + \frac{\epsilon_{\alpha \beta} / \sqrt{k}}{z-w} \psi_{\alpha+\beta}(w) + \sum_{n=0}^{\infty} (z-w)^n [\psi_\alpha \psi_\beta]_{-n}(w) \quad (3)$$

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in which $\epsilon_{\alpha\beta}$ is the structure constant of Lie algebra, $\epsilon_{\alpha\beta} = 0$ if $\alpha + \beta$ is not a root. The stress-energy tensor is equivalent to the Sugawara construction if the relation between current operator and PF is considered ¹⁰⁾.

For the generator $W^{(3)}$ we use the following expression ³⁾

$$W^{(3)}(z) = B_3 \{ [\psi_\alpha \psi_{-\alpha}]_{-1} - [\psi_{-\alpha} \psi_\alpha]_{-1} \} (z), \quad (B_3)^2 = \frac{k^3}{2(k+2)(k-2)(3k+4)}. \quad (4)$$

By using the technique to be stated later we find the OPEs

$$T(z) W^{(3)}(w) = \frac{3}{(z-w)^2} W^{(3)}(w) + \frac{1}{z-w} \partial W^{(3)}(w) + \text{regular terms}, \quad (5)$$

and

$$\begin{aligned} W^{(3)}(z) W^{(3)}(w) &= \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} + \frac{1}{(z-w)^2} \left\{ 2b^2 \Lambda(w) + \frac{3}{10} \partial^2 T(w) \right. \\ &\quad \left. + 2C_{33}^4 \sqrt{\frac{k-3}{k-2}} W^{(4)}(w) \right\} + \frac{1}{z-w} \left\{ b^2 \partial \Lambda(w) + \frac{1}{15} \partial^3 T(w) \right. \\ &\quad \left. + C_{33}^4 \sqrt{\frac{k-3}{k-2}} \partial W^{(4)}(w) \right\} + \text{regular terms}, \end{aligned} \quad (6)$$

in which central charge $c = \frac{2(k-1)}{k+2}$, $b^2 = \frac{8(k+2)}{16k+17}$, $C_{33}^4 = \frac{2k(2k+3)}{k+2} \sqrt{\frac{2k+1}{3k+4}}$, $\Lambda(w) = [T(w) T(w)]_0 - \frac{3}{10} \partial^2 T(w)$, and

$$\begin{aligned} W^{(4)}(z) &= B_4 \left\{ [\psi_\alpha \psi_{-\alpha}]_{-2} + [\psi_{-\alpha} \psi_\alpha]_{-2} - \frac{3(k+2)}{10k} \partial^2 T + \frac{(k+2)^2(6k-5)}{k^2(16k+17)} \Lambda \right\} (z), \\ (B_4)^2 &= \frac{k^4(16k+17)}{8(k+2)(2k+1)(3k+4)(k-2)(k-3)}, \end{aligned} \quad (7)$$

and the two-point function $\langle W^{(4)} W^{(4)} \rangle = \frac{c/4}{(z-w)^2}$. Due to the factor $\sqrt{k-3}$, the term related to $W^{(4)}$ in (6) is a null operator when $k=3$, and the algebra is closed, exactly the Zamolodchikov's W_3 algebra. However, why we have the expression of $W^{(3)}$ and $W^{(4)}$? Can we find a general construction for generators in a W_k algebra? Thanks to Nahm's quasiprimerized normal ordered product of field operators ⁷⁾, and applying it to the PF we find the following

$$\begin{aligned} N(\psi_\alpha^{(\nu)} \psi_{-\alpha})(w) &= N(\psi_\alpha^{(\nu)} \psi_{-\alpha})(w) \\ &\quad + \sum_{r=1}^{\nu} (-)^r \binom{\nu+1}{r} \binom{2\nu+2}{r}^{-1} \binom{\nu+2/k}{r} \partial^r N(\psi_\alpha^{(\nu-r)} \psi_{-\alpha})(w) \\ &\quad - \sum_k C^k \nu! \binom{\nu+2/k}{\nu} \Phi_k(w), \end{aligned} \quad (8)$$

in which $\psi_\alpha^{(\nu)} \equiv \partial^\nu \psi_\alpha$, $N(\psi_\alpha^{(\nu)} \psi_{-\alpha}) \equiv [\psi_\alpha^{(\nu)} \psi_{-\alpha}]_0 = \nu! \binom{\nu-2/k}{\nu} [\psi_\alpha \psi_{-\alpha}]_{-\nu}$ is the ordinary normal ordered product, and $\Phi_k(w)$ with dimension $\nu+2$ are quasiprimerized field operators.

Because of the semilocality of PF, $\Phi_k(w)$ appears in every order of the OPE, not just in the singular part of OPE like the Bosonic (or Fermionic) case ⁸⁾. The structure constants C^k are determined in terms of 3-point and 2-point functions.

Comparing with T , $W^{(3)}$ (and $W^{(4)}$) mentioned above we make the following conjecture for generators in higher level.

$$W^{(l)}(z) \propto \mathcal{N}(\psi_\alpha^{(l-2)} \psi_{-\alpha} + (-1)^l \psi_{-\alpha}^{(l-2)} \psi_\alpha)(z). \quad (9)$$

The first five are T , $W^{(3)}$, (we use the same symbols for quasiprimerized fields like Ref.8).

$$W^{(4)}(z) = B_4 \left\{ [\psi_\alpha \psi_{-\alpha}]_{-2} + [\psi_{-\alpha} \psi_\alpha]_{-2} - \frac{3}{10} \partial^2 [\psi_\alpha \psi_{-\alpha}]_0 - 2C^4 \Lambda \right\} (z) \quad (10)$$

$$\begin{aligned} W^{(5)}(z) &= B_5 \left\{ [\psi_\alpha \psi_{-\alpha}]_{-3} - [\psi_{-\alpha} \psi_\alpha]_{-3} - \frac{1}{7} \partial^2 ([\psi_\alpha \psi_{-\alpha}]_{-1} - [\psi_{-\alpha} \psi_\alpha]_{-1}) \right. \\ &\quad \left. - 2C^4 \Lambda \right\} (z), \end{aligned} \quad (11)$$

$$A(z) \equiv [T(z) W^{(3)}(z)]_0 - \frac{3}{14} \partial^2 W^{(3)}(z), \quad (12)$$

$$\begin{aligned} W^{(6)}(z) &= B_6 \left\{ [\psi_\alpha \psi_{-\alpha}]_{-4} + [\psi_{-\alpha} \psi_\alpha]_{-4} - \frac{5}{36} \partial^2 ([\psi_\alpha \psi_{-\alpha}]_{-2} + [\psi_{-\alpha} \psi_\alpha]_{-2}) \right. \\ &\quad \left. + \frac{5}{168} \partial^4 [\psi_\alpha \psi_{-\alpha}]_0 - 2(C^4 T + C^{\Delta} \Delta + C^{\Omega} \Omega + C^{\rho} \rho + C^T T) \right\} (z) \end{aligned} \quad (13)$$

$$\Gamma(z) \equiv [\partial^2 T(z) T(z)]_0 - \frac{5}{18} \partial^2 \Lambda(z) - \frac{1}{42} \partial^4 T(z), \quad (14)$$

$$\Delta(z) \equiv [T(z) \Lambda(z)]_0 - \frac{1}{6} \partial^2 \Lambda(z), \quad (15)$$

$$\Omega(z) \equiv [T(z) W^{(4)}(z)]_0 - \frac{1}{6} \partial^2 W^{(4)}(z), \quad (16)$$

$$\rho(z) \equiv [W^{(3)}(z) W^{(3)}(z)]_0 - \frac{1}{84} \partial^4 T(z) - \frac{20(k+2)}{9(16k+17)} \partial^2 \Lambda(z) - \frac{5}{9} C_{33}^3 \partial^2 W^{(4)}(z),$$

$$C_{33}^3 = (2k+3) \left[\frac{8(2k+1)(k-3)}{(k+2)(3k+4)(16k+17)(k-2)} \right]^{1/2}, \quad (17)$$

$$\Upsilon(z) \equiv [\partial T(z) W^{(3)}(z)]_0 - \frac{2}{5} \partial \Lambda(z) - \frac{1}{28} \partial^3 W^{(3)}(z). \quad (18)$$

To prove this conjecture (up to $k=4$ of course) we have to calculate OPEs of $W^{(3)} W^{(4)}$ and $W^{(4)} W^{(4)}$ and all of structure constants. Before doing this we would like to describe the technique we use in this paper. Let $A_\alpha(z)$ and $B_\beta(w)$ be two field operators with PF charges α and β , local ($\alpha, \beta = 0$) or semilocal ($\alpha, \beta = \text{roots of underlying algebra}$). The OPE of them can be written as follows

$$R(A_\alpha(z) B_\beta(w)) (z-w)^{\alpha\beta/k} = \sum_{n=-[h_\alpha+h_\beta]}^{\infty} [AB]_{-n}(w) (z-w)^n \quad (19)$$

in which $[h_A]$ means the integral part of dimension A . Hence we have

$$[AB]_n(w) = \oint_w dz A_\alpha B_\beta(w) (z-w)^{n-1+\alpha\beta/k}, \quad (20)$$

and two relations

$$[\partial A B]_{-n}(w) = (n+1-\alpha \cdot \beta/k)[AB]_{-(n+1)}(w), \quad (21)$$

$$[A \partial B]_n(w) = (n-1+\alpha \cdot \beta/k)[AB]_{n-1}(w) + \partial[A B]_n(w). \quad (22)$$

It is easy to find a relation between three fold radial ordering products

$$\begin{aligned} & \left\{ \oint_w du \oint_w dz R(A(u)) R(B(z)C(w)) - \oint_w dz \oint_w du (-)^{\alpha\beta/k} R(B(z)) R(A(u)C(w)) \right. \\ & \quad \left. - \oint_w dz \oint_z du R(R(A(u)B(z))C(w)) \right\} (z-w)^{p-1+\beta\gamma/k} (u-w)^{q-1+\gamma\alpha/k} (u-z)^{r-1+\alpha\beta/k} \\ & = 0, \end{aligned} \quad (23)$$

where the integers p, q, r are in the region $-\infty < p \leq [h_B + h_C]$, $-\infty < q \leq [h_C + h_A]$, $-\infty < r \leq [h_A + h_B]$. The contours are selfevident. From Eq.(23) we obtain the following Jacobi identity-like relations 5),11),

$$\begin{aligned} & \sum_{i=p}^{[h_B+h_C]} C_{r-1+\alpha\beta/k}^{(i-p)} [A[B C]_i]_{Q-1}(w) + (-)^r \sum_{j=q}^{[h_C+h_A]} C_{r-1+\alpha\beta/k}^{(j-q)} [B[A C]_j]_{Q-j}(w) \\ & = \sum_{k=r}^{[h_A+h_B]} (-)^{k-r} C_{q-1+\gamma\alpha/k}^{(k-r)} [[AB]_k C]_{Q-k}(w), \end{aligned} \quad (24)$$

in which $Q = p + q + r - 1$, $C_x^y = \frac{(-)^x \Gamma(x-1)}{\Gamma!} \frac{(x-\ell+1)}{\Gamma!}$.

By using analytic continuation one more equation is useful,

$$[B A]_r(w) = \sum_{t=r}^{[h_A+h_B]} \frac{(-)^t}{(t-r)!} \partial^{t-r} [A B]_t(w). \quad (25)$$

And two special cases should be noted ($n \geq 0$),

$$[A \text{ const.}]_{-n}(w) = \text{const.} \frac{1}{n!} \partial^n A(w), \quad (26)$$

$$[\text{const.} B]_{-n}(w) = \text{const.} \delta_{n0} B(w). \quad (27)$$

In all of the above equations A, B, C can be a compound operator such as: $[\psi_\alpha \psi_\beta]_n, [\partial^2 T]_n, [[\psi_\alpha \psi_\beta]_n \psi_\gamma]_m$ etc. We can calculate any coefficient in OPE from fundamental ones and Eq.(3).

Now, let us look for the constant C^A in $W^{(4)}$. Similar to Ref.8, we calculate the 3-point function $[[\psi_\alpha \psi_{-a}]_{-2} \Lambda]_8 = \frac{(k-1)(6k-5)}{5k^2}$, and the 2-point function $[\Lambda \Lambda]_8 = \frac{2(k-1)(16k+17)}{5(k+2)^2}$, so that

$$C^A = [[\psi_\alpha \psi_{-a}]_{-2} \Lambda]_8 / [\Lambda \Lambda]_8 = \frac{(k+2)^2(6k-5)}{2k^2(16k+17)}. \quad (28)$$

This makes the formula (10) equivalent to Eq.(7), and $W^{(4)}$ is made to be a primary field.

Next, we obtain the OPE

$$\begin{aligned} W^{(3)}(z)W^{(4)}(w) &= C_{34}^3 \left\{ \frac{3}{(z-w)^4} W^{(3)}(w) + \frac{1}{(z-w)^3} \partial W^{(3)}(w) \right. \\ & \quad + \frac{3}{(z-w)^2} \left(\frac{1}{14} \partial^2 W^{(3)}(w) + \frac{26(k+2)}{64k+107} A(w) \right. \\ & \quad + \frac{5k^3(3k+4)(16k+17)}{24(2k+3)} \left(\frac{k(64k+107)(k-2)(k-4)^{1/2}}{(2k+1)^3(5k+8)(k-3)} \right) W^{(5)}(w) \\ & \quad + \frac{1}{z-w} \left(\frac{1}{28} \partial^3 W^{(3)}(w) + \frac{156(k+2)}{5(64k+107)} \partial A(w) + \frac{3(k+2)}{2(2k+1)} T(w) \right. \\ & \quad \left. \left. + \frac{k^3(3k+4)(16k+17)}{12(2k+3)} \left(\frac{k(64k+107)(k-2)(k-4)^{1/2}}{(2k+1)^3(5k+8)(k-3)} \right) \partial W^{(5)}(w) \right) \right\} \end{aligned} \quad (29)$$

in which there is also a null operator when $k=4$. And $W^{(3)}(z)$ is normalized so that

$$\langle W^{(3)}(z)W^{(3)}(w) \rangle = \frac{c/5}{(z-w)^{10}}, \quad B_2^2 = \frac{k^5(64k+107)}{4(k+2)(2k+1)(5k+8)(k-2)(k-3)(k-4)}.$$

It is almost the same to calculate the structure constant C^A as to get C^A . We have

$$[[\psi_\alpha \psi_{-a}]_{-3} A]_{10} = \frac{2(k-1)(k-2)(3k+4)(10k-7)}{21k^4}, \quad (30)$$

$$[A A]_{10} = \frac{4(k-1)(k-2)(3k+4)(64k+107)}{21k^3(k+2)}, \quad (31)$$

$$C^A = [[\psi_\alpha \psi_{-a}]_{-3} A]_{10} / [A A]_{10} = \frac{(k+2)(10k-7)}{2k(64k+107)}, \quad (32)$$

which in turn guarantees that $W^{(3)}$ is also a primary field.

We are most interested in calculating the OPE of $W^{(4)}W^{(4)}$, because it is the first example in which there are more than one quasiprimary field operators having the same conformal dimension. The result is

$$\begin{aligned} W^{(4)}(z)W^{(4)}(w) &= \frac{c/4}{(z-w)^8} + \frac{2T(w)}{(z-w)^6} + \frac{\partial T(w)}{(z-w)^5} + \frac{1}{(z-w)^4} \left\{ \frac{3}{10} \partial^2 T(w) \right. \\ & \quad + \frac{21(k+2)}{16k+17} \Lambda(w) \\ & \quad + 2C_{44}^4 W^{(4)} \left\{ + \frac{1}{(z-w)^3} \left\{ \frac{1}{15} \partial^3 T(w) + \frac{21(k+2)}{2(16k+17)} \partial A(w) \right. \right. \\ & \quad \left. \left. + C_{44}^4 \partial W^{(4)}(w) \right\} \right\} \\ & \quad + \frac{1}{(z-w)^2} \left\{ \frac{1}{84} J^4 T(w) + \frac{35(k+2)}{12(16k+17)} \partial^2 \Lambda(w) \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{5}{18} C_{44}^4 \partial^2 W^{(4)}(w) + K^\Gamma \Gamma(w) + K^\Delta \Delta(w). \\
& + K^\Omega \Omega(w) + K^\rho \rho(w) + C_{44}^6 B_6^{-1} W^{(6)}(w) \left\{ \right. \\
& + \frac{1}{z-w} \left\{ \frac{1}{560} \partial^5 T(w) + \frac{7(k+2)}{12(16k+17)} \partial^3 \lambda(w) \right. \\
& + \frac{1}{18} C_{44}^4 \partial^2 W^{(4)}(w) + \frac{1}{2} K^\Gamma \partial \Gamma(w) + \frac{1}{2} K^\Delta \partial \Delta(w) \\
& + \frac{1}{2} K^\Omega \partial \Omega(w) + \frac{1}{2} K^\rho \partial \rho(w) \\
& \left. \left. + \frac{1}{2} C_{44}^6 B_6^{-1} \partial W^{(6)}(w) \right\} \right\}, \tag{33}
\end{aligned}$$

in which $C_{44}^4 = \frac{9(4k^3 - 15k^2 - 33k - 4)}{2(k+2)(2k+1)(3k+4)(16k+17)(k-2)(k-3)^{1/2}}$, $C_{44}^6 = \frac{k^3(9k+10)(16k+17)}{(k+2)(2k+1)(3k+4)(k-2)(k-3)}$, and $(F \equiv \frac{16(9k+10)}{k})$

$$\begin{aligned}
K^\Gamma &= F B_4^2 C^\Gamma - \frac{2(k+2)^2}{5k^4(16k+17)} (36k^3 + 12k^2 + 65k + 220) B_4^2, \\
K^\Delta &= F B_4^2 C^\Delta - \frac{8(k+2)^4(6k-5)^2}{k^4(16k+17)^2} B_4^2, \\
K^\Omega &= F B_4^2 C^\Omega - \frac{16(k+2)^2(6k-5)}{k^2(16k+17)} B_4, \\
K^\rho &= F B_4^2 C^\rho - \frac{4(k+2)(k-2)(3k+4)}{k^4} B_4^2. \tag{34}
\end{aligned}$$

Let us determine the structure constants now. It is easy to find that the following 2-point functions are zero,

$$[\Gamma\lambda]_{12} = [\sigma\Gamma]_{12} = [\Delta\Omega]_{12} = [\Omega\Delta]_{12} = 0. \tag{35}$$

From 3-point functions

$$[[\psi_\alpha \psi_{-\alpha}]_{-4}\Gamma]_{12} = 0, \tag{36}$$

$$\begin{aligned}
[[\psi_\alpha \psi_{-\alpha}]_{-4}\Gamma]_{12} &= C^\Gamma [\Gamma\Gamma]_{12} + C^\Delta [\Delta\Gamma]_{12} + C^\rho [\rho\Gamma]_{12}, \\
[[\psi_\alpha \psi_{-\alpha}]_{-4}\Delta]_{12} &= C^\Gamma [\Gamma\Delta]_{12} + C^\Delta [\Delta\Delta]_{12} + C^\rho [\rho\Delta]_{12}, \\
[[\psi_\alpha \psi_{-\alpha}]_{-4}\Omega]_{12} &= C^\Omega [\Omega\Omega]_{12} + C^\rho [\rho\Omega]_{12}, \\
[[\psi_\alpha \psi_{-\alpha}]_{-4}\rho]_{12} &= C^\Gamma [\Gamma\rho]_{12} + C^\Delta [\Delta\rho]_{12} + C^\Omega [\Omega\rho]_{12} + C^\rho [\rho\rho]_{12} \tag{37}
\end{aligned}$$

we have

$$C^\Gamma = 0, \tag{38}$$

and four equations. It can be found that only three of them are independent, which are

$$C^\Gamma = \alpha C - \beta, \quad C^\Delta = \delta - \gamma C, \quad B_4 C^\Omega = \zeta - \eta C, \quad C^\rho \equiv C; \tag{39}$$

in which

$$\begin{aligned}
\alpha &= \left(\frac{1408(k+2)}{9(97k+149)} - \frac{43}{63} \right) / \left(\frac{2(169k-82)}{63(k+2)} - \frac{128(16k+17)}{9(97k+149)} \right), \\
\beta &= \left(\frac{8(k+2)^2(6k-5)(5k-3)}{9k^3(97k+149)} - \frac{(k+2)(15k-14)}{63k^2} \right) / \left(\frac{4(169k-82)}{63(k+2)} - \frac{256(16k+17)}{9(97k+149)} \right), \\
\gamma &= \left(\frac{344(k+2)}{3(169k-82)} - \frac{176(k+2)}{3(16k+17)} \right) / \left(\frac{448(16k+17)}{5(169k-82)} - \frac{97k+149}{5(k+2)} \right), \\
\delta &= \left(\frac{4(k+2)^2(15k-14)}{3k^2(169k-82)} - \frac{(k+2)^2(6k-5)(5k-3)}{6k^3(16k+17)} \right) / \left(\frac{448(16k+17)}{5(169k-82)} - \frac{97k+149}{5(k+2)} \right), \\
\eta &= \frac{44k^2(2k+3)}{3(k-2)(3k+4)(13k+23)}, \\
\zeta &= \frac{(k+2)(5k-3)}{6k(13k+23)}. \tag{40}
\end{aligned}$$

We can also prove that $W^{(6)}$ is a primary field under these three conditions. Besides, the 2-point function of $B_6^{-2} W^{(6)} W^{(6)}$ has zero points at $k=4$ and $k=5$ (zero points $k=1, 2, 3$ can be found trivially) if these same three conditions are tenable. Therefore, we can choose normalization constant B_6 so that

$$\langle W^{(6)}(z) W^{(6)}(w) \rangle = \frac{c/6}{(z-w)^2}, \tag{41}$$

and there must exist a factor $[(k-1)(k-2)(k-3)(k-4)(k-5)]^{1/2}$ in B_6^{-1} . Therefore the W_4 -algebra is closed when $k=4$. Since the operator $W^{(4)}(z)$ appears in a position which is a null current in the OPE of $W^{(3)} W^{(3)}$ when $k=3$. One may expect that $W^{(5)}(z)$ and $W^{(6)}(z)$ are also generators in a higher k W -algebra.

If we choose a suitable C , the structure constants $K^\Gamma, K^\Delta, K^\Omega, K^\rho$ will coincide with structure constants in Ref.8 if $k=4$ ($c=1$). But in our case the constant C is definite and satisfies a quadratic equation:

$$\begin{aligned}
& (\alpha^2 B_{\Gamma\Gamma} + \gamma^2 B_{\Delta\Delta} + \eta^2 B_{\Omega\Omega} + B_{\rho\rho} - \alpha\gamma B_{\Gamma\Delta} + \alpha B_{\Gamma\rho} - \gamma B_{\Delta\rho} - \eta B_{\Omega\rho}) C^2 \\
& - \{\alpha A_\Gamma - \gamma A_\Delta - \eta A_\Omega + A_\rho + 2\alpha\beta B_{\Gamma\Gamma} + 2\gamma\delta B_{\Delta\Delta} + 2\eta\zeta B_{\Omega\Omega} \\
& - (\alpha\delta + \beta\gamma) B_{\Gamma\Delta} + \beta B_{\Gamma\rho} - \delta B_{\Delta\rho} - \zeta B_{\Omega\rho}\} C \\
& + (\beta A_\Gamma - \delta A_\Delta - \zeta A_\Omega + \beta^2 B_{\Gamma\Gamma} + \delta^2 B_{\Delta\Delta} + \zeta^2 B_{\Omega\Omega} - \beta\delta B_{\Gamma\Delta}) = 0, \tag{42}
\end{aligned}$$

in which

$$\begin{aligned}
A_\Gamma &= \frac{2(15k-14)}{63k^2}, \quad A_\Delta = \frac{(6k-5)(5k-3)}{15k^3}, \\
A_\Omega &= \frac{2(k-2)(k-3)(2k+1)(3k+4)(5k-3)}{3k^5(16k+17)},
\end{aligned}$$

$$\begin{aligned}
A_p &= \frac{1360k^4 - 91k^3 - 4768k^2 - 1540k + 1904}{63k^3(k+2)(16k+17)}, & B_{rr} &= \frac{8(169k - 82)}{63(k+2)^2}, \\
B_{\Delta\Delta} &= \frac{2(16k+17)(97k+149)}{25(k+2)^3}, & B_{nn} &= \frac{4(k-2)(k-3)(2k+1)(3k+4)(13k+23)}{k^4(k+2)(16k+17)}, \\
B_{pp} &= \frac{2(25136k^4 + 51411k^3 - 78776k^2 - 237284k - 133552)}{63(k+2)^2(k-2)(3k+4)(16k+17)}, \\
B_{r\Delta} &= \frac{64(16k+17)}{15(k+2)^2}, & B_{rp} &= \frac{344}{63(k+2)}, \\
B_{\Delta p} &= \frac{704}{15(k+2)}, & B_{np} &= \frac{352(k-3)(2k+1)(2k+3)}{3k^2(k+2)(16k+17)}.
\end{aligned} \tag{43}$$

This is from a relation between 2-point functions and 3-point functions:

$$\begin{aligned}
& [|\psi_\alpha \psi_{-\alpha}\rangle_{-1}, (C^r \Gamma + C^e \Delta + C^n \Omega + C^o \rho)]_{12} \\
&= [\Gamma \Gamma]_{12} (C^r)^2 + [\Delta \Delta]_{12} (C^e)^2 + [\Omega \Omega]_{12} (C^n)^2 + [\rho \rho]_{12} (C^o)^2 \\
&+ ([\Gamma \Delta]_{12} + [\Delta \Gamma]_{12}) C^r C^e + ([\Gamma \rho]_{12} + [\rho \Gamma]_{12}) C^r C^o \\
&+ ([\Delta \rho]_{12} + [\rho \Delta]_{12}) C^e C^o + ([\Omega \rho]_{12} + [\rho \Omega]_{12}) C^n C^o.
\end{aligned} \tag{44}$$

Due to the complicity of coefficients we prefer to give the numerical values of C (double valued) instead of the analytic formula

k	4	5	6	7	8
C_1	$-0.489 \cdot 10^{-2}$	$0.635 \cdot 10^{-2}$	$-0.145 \cdot 10^{-12}$	$0.515 \cdot 10^{-13}$	$-0.439 \cdot 10^{-13}$
C_2	$-0.427 \cdot 10^6$	$-0.131 \cdot 10^6$	$-0.851 \cdot 10^4$	$-0.310 \cdot 10^4$	$-0.172 \cdot 10^4$
k	9	10			
C_1	$-0.101 \cdot 10^{-13}$	$0.315 \cdot 10^{-14}$			
C_2	$-0.116 \cdot 10^4$	$-0.865 \cdot 10^3$			

As mentioned above we have proven our conjecture (9), the construction for general k , up to $k=4$. It seems to us, however, one might establish a programme so that it can be proven up to any k in terms of a computer.

Since we have found a construction for W_3 -algebra from $SU(3)$ PF ⁶⁾

$$W^{(3)}(z) = B_3 \sum_{\alpha_i} (|\psi_{\alpha_i} \psi_{-\alpha_i}\rangle_{-1} - |\psi_{-\alpha_i} \psi_{\alpha_i}\rangle_{-1})(z), \tag{45}$$

and also a primary current

$$\begin{aligned}
T^{(4)}(z) &= B_4 \left\{ \sum_{i=1}^3 \left(|\psi_{\alpha_i} \psi_{-\alpha_i}\rangle_{-2} + |\psi_{-\alpha_i} \psi_{\alpha_i}\rangle_{-2} - \frac{3(k+2)^2}{k^2(4k+3)} \Lambda^{(\alpha_i)} \right) (z) \right. \\
&\quad \left. - \frac{(k-3)(k+3)^2(12k+11)}{k^2(4k+3)(13k+9)} \Lambda(z) - \frac{3(k+3)}{10k} \partial^2 T(z) \right\},
\end{aligned} \tag{46}$$

it is natural to ask if there exists also construction for general k . However, it is well known that the central charges of Z_k PF coincide with the minimal W -algebra series. When $k=3$, the central charge of $SU(3)$ PF agrees with a limit value of the minimal series. Nevertheless the central charges will be different from the minimal theory if we could construct a W_k ($k \geq 4$) algebra from $SU(3)$ PF. Does there exist construction for $SU(3)$ PF for any k ? It must be a very interesting question to be investigated.

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