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Nonperturbative Methods in Quantum Chromodynamics

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ABSTRACT

Several methods and models to treat nonperturbative effects in QCD are presented.

Foreword and Content

It is hard to start a review of this kind without stating that Quantum-Chromodynamics (QCD), i.e. the field theory of quarks and gluons, is generally assumed to be the fundamental theory of strong interactions, but that - alas - we have no analytic tool to handle its nonperturbative aspects. Indeed many of the great successes of QCD are rather based on model assumptions inspired from QCD than on a rigorous calculus. In this review I shall expound on some of these nonperturbative treatments. Since an exhaustive treatment of these methods would not only fill a book, but a small library I must make a (subjective) selection. Nevertheless I shall not concentrate on a single approach, but present a whole variety of them and try to point out their connections, their successes and their shortcomings. After a nontechnical introduction in section 1 I shall shortly review in section 2 the QCD-Lagrangian, its quantization by functional integration and some important aspects of general field theory, as composite operators, their renormalization, and their expansion. In section 3 I present the lattice regularization of QCD and discuss shortly the (non-)convergence of the perturbative series expansion of QCD. Section 4 is devoted to models. After a rather short discussion of non-relativistic quark models and an even shorter one of instantons I treat in more detail some practical aspects of the important method of QCD sum rules. A short presentation of the model of the stochastic vacuum closes this section. As an practical application I report in the last section on formfactors and compare there the results obtained by different approaches. In Appendix 1 I make some remarks on dimensional considerations. In Appendix 2 the numerical values of QCD parameters are collected and discussed.

1 Nontechnical Introduction

In the late sixties hadron physics presented an intriguing puzzle: There was ample evidence that the strongly interacting "elementary particles" where not more elementary than the atoms are indivisible and that they consisted of constituents, the so-called quarks. These were introduced by Gell-Mann (1964) and Zweig (1964). A comparison of the "hadronic spectrum" of a nucleon (i.e. all states with baryon number 1 and isospin 1/2) and the familiar spectrum of the hydrogen atom gives a graphic hint to that composite structure of hadrons (see Figs. 1a,b).

Figure 1: Lowest lying states of the spectra of the nucleon, hydrogen and bottomonium

The intensive search for quarks was not crowned by success, in spite of the easy signature for their experimental detection, namely the fractional charge. So one had to assume that the quarks are tied very strongly together, perhaps even with infinite strength. An even stronger indication for the composite nature of hadrons came in the late sixties from the so-called deep inelastic scattering experiments, in which reactions of electrons and hadrons with a large momentum transfer from the electrons to the hadrons were investigated. One can view these scattering experiments as microscopic investigations of the hadrons where the resolution of the microscope is determined by the wave length corresponding to the exchanged momentum. In these experiments a novel feature appeared. Viewed at small scales (a small fraction of a fermi), the constituents behaved practically like free particles, called partons. The fact that at small spacetime scales partons behave (nearly) as free particles has been called asymptotic freedom. The fact that they cannot be separated, i.e. that at large scales the interaction becomes very (infinitely) strong, is sometimes called infrared slavery.

Such a behaviour is typical for quantum mechanical systems bound together by a potential which is not too singular at zero distance and which increases infinitely with increasing distance. The Greens function for such a system behaves for short times nearly like a free one. This can be seen e.g. directly for a harmonic oscillator where the Greens function $G(\vec{x}, \vec{x}'; t)$ is given by

$$G(\vec{x}, \vec{x}'; t) = \left(\frac{m\omega}{2\pi\hbar \sin \omega t} \right)^{3/2} \cdot \exp \left[\frac{im\omega}{2\hbar \sin \omega t} \left((\vec{x}^2 + \vec{x}'^2) \cos \omega t - 2\vec{x}\vec{x}' \right) \right]. \quad (1)$$

For small times t , i.e. ($\omega t \ll 1$) this Greens function is indeed well approximated by that of a free particle:

$$G_0(\vec{x}, \vec{x}'; t) = \left(\frac{m}{2\pi\hbar t} \right)^{3/2} \exp \left[\frac{im}{2\hbar t} (\vec{x} - \vec{x}')^2 \right]. \quad (2)$$

It is thus not astonishing that nonrelativistic potential models of quarks were able to describe characteristic features of these systems with infrared slavery and asymptotic freedom. A purely quantum mechanical description of the hadrons is, however, not fully satisfactory. After all, we want to incorporate Lorentz invariance and know that relativistic quantum mechanics is, due to particle-antiparticle production, not a fully consistent theory.

In relativistic quantum field theory which to our present knowledge is the most adequate description of subatomic systems, the situation is more complex. The only well studied realistic and really successful relativistic quantum theory was at that time quantum electrodynamics, and this theory showed just the opposite behaviour: Viewed from a large distance, the electric charge of an electron is small ($e^2/\hbar c = 1/137$) but the nearer one approaches the particle, the larger the charge becomes. In an intuitive picture one can say that the bare charge of an electron is infinity, but that this bare charge is screened to a

finite value due to the virtual pairs of charged particles and antiparticles created in the vacuum. The nearer one comes to the electron, the more one feels the large charge. The formal device to investigate these effects is the so-called renormalization group evolution, going back to Gell-Mann and Low (1954). The increase of the electric charge is rather moderate. At distances of 1/500 fm (corresponding to a momentum of 100 GeV), the fine structure "constant" $e^2/\hbar c$ has increased from 1/137 to 1/129. In a naive extrapolation (which is perhaps not so naive after all) the charge becomes infinite only at an energy which is much higher than the mass of the universe. This infinity is called the Landau pole (1955).

Quantum electrodynamics thus has just the opposite behaviour of what we need in quark physics. At this point non-Abelian gauge theories enter the game. Since I have to introduce later the Lagrangian for such a theory on a rather technical level, let me praise here a bit its beauty and introduce it on an elementary level. Since gauge invariance plays a crucial role not only in nonperturbative QCD but in all fundamental field theories, I have a good conscience in doing so.

Let me start with general quantum physics. If a state vector is multiplied by a constant phase factor $e^{i\alpha}$ (a real), this induces no observable consequences. Now imagine a wave function with two very distinct peaks. H. Weyl (1929) was disturbed by the fact that if you multiply this wave function at one peak by a phase factor, you have to choose exactly the same one for the very distant peak, too. He did not believe in such a parallelism at distance and postulated local gauge invariance, i.e. invariance under a space (space-time) dependent phase factor $\exp[i\alpha(x)]$. But evidently this local gauge invariance does not hold even for a free nonrelativistic particle. If $\psi(\vec{x}, t)$ is the solution of the free Schrödinger equation,

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(\vec{x}, t) = i\hbar \partial_t \psi(\vec{x}, t) \quad (3)$$

the product $\exp[i\alpha(\vec{x})]\psi(\vec{x}, t)$ will in general not be a solution of 3. We thus might conclude that particles are not free but must interact with other particles (fields) in such a way that the local gauge invariance is restored. Indeed if we consider the Schrödinger equation in a magnetic field with the vector potential $\vec{A}(\vec{x}, t)$, then the free equation 3 is modified to

$$\frac{\hbar^2}{2m} \left(\vec{\nabla} - \frac{e}{c} \vec{A}(\vec{x}, t) \right)^2 \psi(\vec{x}, t) = i\hbar \partial_t \psi(\vec{x}, t). \quad (4)$$

With the wave function $\psi(\vec{x}, t)$ also the product $\exp[i\alpha(\vec{x})]\psi(\vec{x}, t)$ is a solution of 4, if at the same time the vector potential $\vec{A}$ is changed:

$$\vec{A}(\vec{x}, t) \rightarrow \vec{A}(\vec{x}, t) - \frac{c}{e} \vec{\nabla} \alpha(\vec{x}) \quad (5)$$

(Note that this is a gauge transformation and does not influence the magnetic field). The moral of the story is: If we insist on local gauge invariance of the wave function, we must introduce a coupling of the particle to a vector field (see 4). In a relativistic theory the phase factors depend of course on space and time, i.e. we have the common transformation for a "matter field" $\psi(x)$ and a "gauge field" $A_\mu(x)$

$$\psi(x) \rightarrow e^{i\alpha(x)} \psi(x); \quad A_\mu(x) \rightarrow A_\mu(x) - \partial_\mu \alpha(x). \quad (6)$$

The phase factors $\exp[i\frac{e}{c}\alpha(x)]$ form an Abelian group.

If we have several matter fields $\psi_i(x); \quad i = 1 \dots N_c$ we can introduce the transformation

$$\psi_i(x) \rightarrow \sum_k U_{ik}(x) \psi_k(x);$$

with space time dependent unitary matrices $\{U_{ik}\}$ of rank N_c , (generally they are also chosen with determinant 1). Since these matrices form a non-Abelian group, one speaks of a non-Abelian gauge

theory [Yang and Mills 1954]. In that case we have to introduce a multiplet of gauge fields $A_\mu^a(x)$ in order to insure local gauge invariance. The minimal form of the interaction is completely determined by the requirement of local gauge invariance, and a special feature of a non-Abelian theory with several gauge fields is, that these fields have to couple among themselves. The explicit form of the Lagrangian is given in sect. 2.1. The self-coupling of the gauge fields has a far-reaching consequence. Now it is possible also to form pairs of gauge quanta out of the vacuum and the effect of these pairs is just opposite to that of fermion-antifermion pairs, and leads to anti-screening. The coupling increases with growing and falls off (logarithmically) with decreasing distance. We thus have for a non-Abelian gauge theory asymptotic freedom. The increase with large distances is less certain. If we extrapolate naively, we obtain an infinite coupling at a scale of a fermi or so, but this extrapolation is based on perturbation theory and this is not supposed to work for large coupling. There are even strong hints that even a resummation of all terms in a perturbative series does miss essential features of such a non-Abelian gauge theory (see e.g. sect. 3.2).

Unfortunately we dispose of no analytical treatment of the effects happening at large distances as we have in perturbation theory for short-distance effects and are therefore bound to use numerical calculations or models. Of these methods the one based on numerical evaluation of quantum field theoretical Greens functions plays a special role, since it is - in principle - free of assumptions going beyond the form of the interaction, and the choice of the fundamental parameters (sect.3.1). These parameters are the quark masses and a scale which fixes the coupling strength. The method is based on the representation of the Greens functions by functional integrals and the latter are approximated by very high-dimensional Riemannian ones. This procedure corresponds to the approximation of a Riemann integral by sums. As mentioned above the method draws its results directly from the fundamental assumptions, but to come to numbers comparable with experiment is a very hard job. The high-dimensional integrals to be performed devour a lot of computer time, even if sophisticated Monte Carlo algorithms are used. The direct result of the calculations are lattice approximations to Greens functions from which the observables like masses or decay constants have to be extracted by extrapolation. Furthermore an "exact" integration of the fermions is, given realistic constraints on computer time, not possible. Therefore results from these calculations are normally expensive (computer time) and have large errors. We shall treat the foundation of this method very shortly in sects. 3.2 and 3.3 and quote some results of it in the applications.

For these reasons and also in order to get a more intuitive understanding of the main mechanisms, analytical methods are highly desirable. As mentioned in the beginning, potential models (sect. 4.1) are a simple and efficient tool to reproduce essential features of non-Abelian gauge theories, but their relation to the underlying field theory is rather indirect. Furthermore the treatment of an important symmetry of QCD, namely chiral symmetry (see sect. 2.4) of the light quark sector is highly unsatisfactory. Whereas nature has chosen to break this symmetry by the groundstate of hadrons (the hadronic vacuum), it is in quark models explicitly broken by constituent masses.

Instanton models (sect. 4.2) are much more directly related to quantum field theory. In these models certain classical configurations (instantons), which certainly do contribute to the functional integrals are singled out and treated with various degrees of approximations.

An important tool in nonperturbative QCD are the QCD sum rules [Shifman et al. 1979]. In this method nonperturbative effects are parametrized by a few universal parameters typical for QCD, the so-called condensates. The method is discussed in some more detail in sect. 4.3.

As an example for a phenomenological model with many simplifying assumptions, but with many applications I shall also discuss the model of the stochastic vacuum, developed by Simonov and myself (see sect. 4.4). In its most restrictive form the nonperturbative contributions are approximated by a Gaussian stochastic process of the field strength tensor. Hence all Greens functions are in this model reducible to the fundamental correlator of the Gaussian process.

I am perfectly aware that there are many more models which deserve to be discussed and have their own merits, but as I stated in the foreword, this would be material for a library and not a review.

Let me just mention the dielectric model, on which a review of Pinner (1992) came out recently, and the model of dual QCD which is treated in Baker et. al. (1991). Particularly interesting seems to me also an investigation of Johnson, Lellouch and Polonyi (1991) in which specific features of the lattice regularization are incorporated in the evaluation of the functional integrals in the continuum. Finally I draw your attention on the 1992 proceedings edited by Fried and B.Müller (1993) in which more material on QCD and vacuum structure can be found.

2 General Quantum Chromodynamics

2.1 The QCD-Lagrangian and Gauge Transformations

Quantum Chromodynamic is the field theory of quarks and gluons. Although it contains many coupling terms between these fields there is only one single coupling constant and the form of the coupling terms is fixed. This is a consequence of local gauge invariance, i.e. the invariance of the Lagrangian under a space time dependent symmetry group [Weyl (1929), Yang and Mills (1954)]. For the fundamental field theory of strong interactions this symmetry group has turned out to be $SU(3)$. It is called for historical reasons $SU(3)_C$ [Fritzsch et al. (1973)]. The quarks transform under this group as triplets (i.e. under the so called fundamental representation) and the gluons as octets (the so called adjoint representation). For some applications it is advantageous to consider as symmetry group generally the group $SU(N_C)$ since some interesting rigorous results can be obtained in the limit $N_C \rightarrow \infty$ [t'Hooft 1974] which can be taken as an approximation to the realistic case with three colours. Therefore I shall mostly quote results for a general value of N_C , the realistic case can be obtained by putting $N_C = 3$. For your convenience I collect a few important features of the groups $SU(N_C)$, which at the same time serve to fix the notation in the following.

An element U of the group $SU(N_C)$, i.e. a unitary $N_C \times N_C$ -matrix with determinant one can be represented in the form:

$$U = \exp(i\omega_F \tau_F) \quad (7)$$

where ω_F are real numbers and τ_F are $N_C^2 - 1$ hermitean and traceless matrices, the so called generators of the group $SU(N_C)$. In the fundamental (defining) representation of $SU(N_C)$ they are matrices of rank N_C . For the moment we need only the normalization and the commutation relations of the generators:

$$tr(\tau_F \tau_G) = \frac{1}{2} \delta_{FG} ; \quad [\tau_F, \tau_G] = i f_{FGH} \tau_H \quad (8)$$

with the totally antisymmetric real "structure constants" f_{FGH} .

In case $N_C = 3$, the generators τ_F are in the fundamental representation the eight Gell-Mann matrices λ_F divided by 2. For $N_C = 2$ they are proportional to the Pauli matrices: $\tau_F = \sigma_F/2$ and the structure constants form the totally antisymmetric tensor in three dimensions, i.e. $f_{FGH} = \epsilon_{FGH}$.

A space time dependent element of $SU(N_C)$ has the form:

$$U(x) = \exp(i\omega_F(x) \tau_F) \quad (9)$$

with $N_C^2 - 1$ real functions $\omega_F(x)$.

Quarks occur in different families distinguished by a quantum number called flavour: at the moment we know off five different flavours, a sixth one is firmly believed to exist. The quarks carry spin 1/2 and thus in a relativistic quantum field theory are described by Dirac-fields $\psi_a^C(x)$, which apart from the spinor index a carry a colour index C , $1 \leq C \leq N_C$, and a flavour index f . The Lagrangian is diagonal in the flavour index. The gluon field-potentials are described by the vector fields $A_\mu^F(x)$ with

a colour index F which runs from 1 to $N_C^2 - 1$. The Lagrangian describing the dynamics of these fields is given by:

$$\mathcal{L}_{QCD}(x) = i\bar{\psi}^C(x) \gamma^\mu D_\mu^E \psi^E(x) - m_f \bar{\psi}^C(x) \psi^C(x) - \frac{1}{4} F_{\mu\nu}^F(x) F_{\mu\nu}^F(x). \quad (10)$$

We use the Einstein convention, i.e. sum over all doubly occurring indices. In this Lagrangian a so called "covariant derivative" D_μ^E occurs. It is defined as:

$$D_\mu^E := \partial_\mu \delta^{CE} - ig_s \tau_F^{CE} A_\mu^F \quad (11)$$

The strong coupling g_s and the quark masses m_f are the free parameters in the Lagrangian. The "gluon field-tensor" $F_{\mu\nu}^F$ is defined as:

$$F_{\mu\nu}^F := \partial_\mu A_\nu^F - \partial_\nu A_\mu^F + g_s f_{GHF} A_\mu^G A_\nu^H \quad (12)$$

I just remind you that the nonlinear term in the gluon field-tensor, which is not present in the electrodynamic Lagrangian, leads through the last term in 10 to cubic and quartic terms in the fields $A_\mu^F(x)$ in the QCD-Lagrangian and thus to non-linear equivalents of the Maxwell equations.

The Lagrangian 10 has been constructed in such a way that it is invariant if the quark and gluon fields are transformed by space-time dependent gauge transformations of the group $SU(N_C)$ (local gauge invariance). For the case that the functions $\omega_F(x)$ in 9 are small as compared to one these gauge transformations of the fields can be written in the infinitesimal form:

$$\begin{aligned} \psi^C(x) &\rightarrow \psi^{C'}(x) = \psi^{C'}(x) + i\omega^F(x) \tau_F^{CE} \psi^E(x) + O(\omega^2) \\ A_\mu^F(x) &\rightarrow A_\mu^{F'}(x) = A_\mu^F(x) - f_{FGH} \omega^G(x) A_\mu^H(x) - \partial_\mu \omega^F(x) + O(\omega^2) \end{aligned} \quad (13)$$

In order to express the gauge transformations in a non-infinitesimal form, it is convenient to introduce matrix valued potentials, covariant derivatives, and field tensors:

$$A_\mu(x) := A_\mu^F(x) \tau_F ; \quad D_\mu := \partial_\mu \mathbf{1} - ig_s A_\mu ; \quad F_{\mu\nu}(x) = D_\mu A_\nu - D_\nu A_\mu \quad (14)$$

The gauge transformation of the fields are then given by the matrix equations:

$$\begin{aligned} A_\mu(x) &\rightarrow A'_\mu(x) = U(x) A_\mu(x) U^\dagger(x) + \frac{i}{g_s} U(x) \partial_\mu U^\dagger(x) \\ F_{\mu\nu}(x) &\rightarrow F'_{\mu\nu}(x) = U(x) F_{\mu\nu}(x) U^\dagger(x) ; \quad \psi_a^C(x) \rightarrow \psi_a^{C'}(x) = U_{CD}(x) \psi_a^D(x) \end{aligned} \quad (15)$$

with the $N_C \times N_C$ -matrices $U(x)$ given by 9; $U^\dagger$ is the Hermitean adjoint of U .

2.2 Quantization

Up to now we have treated QCD like a classical theory. To promote it to a quantum theory there are two ways: Either one considers the gluon and quark fields as operators with the canonical commutation relations or one introduces the Feynman integrals in function spaces. The latter procedure seems much more adequate for gauge theories and especially their treatment by non-perturbative methods. I shall only sketch very briefly this quantization by path integration and refer to modern textbooks of quantum field theory. Analogously to the path integral formulation of quantum mechanics, we write for the quantum mechanical expectation value of a functional of quark and gluon fields $F[\psi, A]$ the Feynman integral:

$$\langle F(\psi, A) \rangle = \frac{1}{N} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A F(\psi, A) e^{iS_{QCD}[\psi, A]} \quad (16)$$

where the normalization constant $\mathcal{N}$ is given by the integral without the function F :

$$\mathcal{N} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A e^{iS(\psi, \bar{\psi}, A)} \quad (17)$$

and $S_{QCD}[\psi, \bar{\psi}, A]$ is the action, i.e. the integral of the QCD Lagrangian 10 over the space time continuum:

$$S[\psi, \bar{\psi}, A] = \int d^4x d\tau \mathcal{L}_{QCD} \quad (18)$$

The function F might be for instance an expression like:

$$\Pi_{\mu\nu} = \bar{\psi}^{CI}(x) \gamma_\mu \psi^{CI}(x) \bar{\psi}^{Dg}(0) \gamma_\nu \psi^{Dg}(0). \quad (19)$$

The Feynman integral can be viewed in two ways.

1) One defines the fields in an appropriate way on a 4-dimensional lattice, expressing differential quotients as difference quotients of adjacent points and integrals by sums over lattice points. Then F and S can be viewed as conventional functions of the fields on the lattice, $\psi_i, \bar{\psi}_i, A_i$ and the integral is a (very high dimensional) conventional one. In section 3.1 we shall do this explicitly.

2) One keeps only terms quadratic in the fields in the exponent of $\exp[iS_{QCD}]$ and expands all cubic and higher terms. Since all these terms contain at least one power of the strong coupling g_s , one obtains a power series in g_s . The functional integration with a quadratic term in the exponential is a Gaussian one which can be performed explicitly and one obtains in this way a perturbative expansion. At the end of this subsection I give some examples for a Gaussian integration. Since I assume, that you are at least superficially acquainted with perturbation theory and since I shall discuss the lattice representation in more detail in sect.3 I do not dwell on these points and leave the functional integrals in their highly symbolic form 16.

Before we proceed further, we notice that the imaginary and oscillating weight function $e^{iS_{QCD}(\psi, \bar{\psi}, A)}$ in 16 is not very pleasant, it certainly is much nicer to have a positive definite weight function. This is achieved by transition from the space time continuum with Minkowski metric $g_{\mu\nu}$ to a four dimensional continuum with Euclidean metric $-\delta_{\mu\nu}$. The transition between the two continua is simply achieved by replacing the Minkowski time x_0 by $-ix_4$, where x_4 is the "Euclidean time" or vice versa. Correspondingly the zeroth component of the gluon potential in the Minkowski continuum, A_0^F is replaced by $-iA_4^F$ in the Euclidean one and the gamma matrix γ_4 is replaced by $\gamma_4 = i\gamma_0$. The integral measure $d\tau_0 dx_k$ of the Minkowski continuum becomes $-id x_4 dx_k$ in the Euclidean one and hence $e^{iS_{QCD}(\psi, \bar{\psi}, A)}$ becomes $e^{S_{QCD}^E(\psi, \bar{\psi}, A)}$ and one can convince oneself that the Euclidean action is negative definite.

After we have evaluated expectation values in the Euclidean continuum we may obtain the (realistic) Minkowski expressions by analytic continuation:

$$G_{Minkowski}(x_0, x_1, x_2, x_3) = G_{Euclid}(x_1, x_2, x_3, x_4 = ix_0). \quad (20)$$

In the following I shall use capital letters $XY PQ$ in order to denote Euclidean coordinates and momenta and B_μ and G_μ for Euclidean potentials and field strengths.

I close this section with some formulae for Gaussian functional integration (in Euclidean space-time) which are also useful outside perturbation theory:

Be $\phi(X)$ a real scalar field. Then one can show quite easily (using e.g. translational invariance):

$$\begin{aligned} IV[J] &:= \frac{1}{\mathcal{N}} \int \mathcal{D}\phi \exp\left(-\frac{1}{2} \int dX dX' \phi(X) K(X, X') \phi(X') + \int dX J(X) \phi(X)\right) \\ &= \exp\left\{\frac{1}{2} \int dX dX' J(X) K^{-1}(X, X') J(X')\right\} \end{aligned} \quad (21)$$

where K^{-1} is the inverse of the operator K and $\mathcal{N}$ is the normalization:

$$\mathcal{N} = \int \mathcal{D}\phi \exp\left\{-\frac{1}{2} \int dX dX' \phi(X) K(X, X') \phi(X')\right\} \quad (22)$$

Functional integrals over polynomials in the fields are easily obtained by functional derivation, e.g.

$$\begin{aligned} G(Y, Y') &:= \frac{1}{\mathcal{N}} \int \mathcal{D}\phi \phi(Y) \phi(Y') \exp\left\{-\frac{1}{2} \int dX dX' \phi(X) K(X, X') \phi(X')\right\} \\ &= \frac{\delta}{\delta J(Y)} \frac{\delta}{\delta J(Y')} W[J]_{J=0} = K^{-1}(Y, Y') \end{aligned} \quad (23)$$

Though the integration over fermion fields is conceptually different (see sect. 3.1), very similar formulae hold also for that case.

2.3 Perturbation Theory and Renormalization Group Improvement

For further reference I shortly repeat the steps which lead to a form of the Lagrangian which is suitable for a perturbative expansion in the coupling g_s . A straightforward application of the perturbative procedure mentioned in the last section leads to a problem, which is ultimately due to the gauge freedom of the theory. In order to overcome this problem one has to take care that the functional integration over a whole class of potentials which are connected by the gauge transformations 16 is counted only as a single contribution. This is done by introducing a gauge fixing term which selects only one representative of the class. This in turn leads to a functional Jacobi determinant which can be evaluated with the help of so called "ghost fields" $\eta^F, F = 1 \dots N_C^2 - 1$, which behave like scalar fermions. A widely used gauge fixing term is based on the Lorentz gauge and has the form:

$$G[A] = -\frac{1}{2a} \partial^\mu A_\mu^F(x) \partial^\nu A_\nu^F(x) \quad (24)$$

If a , the gauge parameter, is chosen to be $a = 1$ we obtain the Feynman gauge with the gluon propagator in momentum space $-g_{\mu\nu}/k^2$ and in the limit $a = 0$ the Landau gauge with the propagator $\{-g_{\mu\nu} + k_\mu k_\nu/k^2\}/k^2$. We separate the resulting gauge fixed Lagrangian into quadratic terms, collected in $\mathcal{L}_{QCD}^0$, and higher terms collected in $\mathcal{L}_{QCD}^{int}$:

$$\mathcal{L}_{QCD,GF} = \mathcal{L}_{QCD}^0 + \mathcal{L}_{QCD}^{int} \quad (25)$$

with:

$$\begin{aligned} \mathcal{L}_{QCD}^0 &= i\bar{\psi}^{CI}(x) \gamma^\mu \partial_\mu \psi^{E,F}(x) - m_f \bar{\psi}^{CI}(x) \psi^{CI}(x) - \bar{\eta}^F(x) \partial^2 \eta^F(x) \\ &\quad - \frac{1}{4} (\partial_\mu A_\nu^F(x) - \partial_\nu A_\mu^F(x)) (\partial^\mu A^\nu F(x) - \partial^\nu A^\mu F(x)) - \frac{1}{2a} \partial^\mu A_\mu^F(x) \partial^\nu A_\nu^F(x) \end{aligned} \quad (26)$$

$$\begin{aligned} \mathcal{L}_{QCD}^{int} &= g_s \bar{\psi}^{CI}(x) \gamma^\mu \tau_F^{CE} \psi^{E,F}(x) A_\mu^F(x) - \frac{1}{2} g_s f_{FGH} (\partial_\mu A_\nu^F(x) - \partial_\nu A_\mu^F(x)) A^\mu G(x) A^\nu H(x) \\ &\quad - \frac{1}{4} g_s^2 f_{FGH} f_{FGH} A^\mu G(x) A^\nu H(x) A_\mu^G(x) A_\nu^H(x) + g_s f_{FGH} \partial^\mu \eta^G(x) \eta^H(x) A_\mu^F(x) \end{aligned} \quad (27)$$

The terms containing the ghost fields η^F are due to the functional Jacobi determinant, the last term of 27 is the gauge fixing term mentioned above. For this Lagrangian the Feynman rules are well defined and can be easily derived. But as is well known, even this Lagrangian poses problems as it stands, since Feynman diagrams with loops can be divergent. In order to tackle this problem one first regularizes the Lagrangian, i.e. brings it into a form for which the perturbation series is well defined term by term. This can be done by cut-offs in momentum space, by introducing a Pauli Villars regulariser or by going to a space time continuum with less than four dimensions. The last procedure seems at a first

glance rather weird, but if the dimension of the space time continuum is small enough, the integrals which occur due to the internal loops converge. This dimensional regularization procedure respects the gauge invariance of the Lagrangian. In a fictitious low dimensional world the Lagrangian is modified in such a way that the expressions one evaluates can be analytically continued to the 4 dimensional world and yield finite values. The modification of the Lagrangian, the so called counterterms can in a renormalizable theory like QCD be absorbed in renormalization constants of the fields, of the coupling g , and of the masses m_f . The renormalized Lagrangian has a form like 27,28, but the fields and constants occurring in it are formally diverging in the limit of 4 dimensions. They can be split into a product of (finite) renormalized quantities and renormalization constants Z_f , Z_m , Z_g etc. These constants are formal power series in the renormalized (and hence finite) coupling constant g . Since the coupling g is dimensionless (in natural units of $\hbar$ and c , see App. 1) only in $d = 4$ dimensions one replaces in $d = 4 - 2\epsilon$ dimensions the coupling constant g by $g\mu^{2\epsilon}$ with a dimensionless g , and the “scale” μ with the dimension of a mass (1 in our convention). If we speak of the mass of quarks and the strength of the coupling g , we mean renormalized quantities with a fixed scale μ and referring to a special regularization and renormalization scheme. The fact that observable quantities, like hadron masses or scattering amplitudes, cannot depend on the arbitrarily scale μ can be exploited and gives rise to a powerful tool, namely the renormalization group equation. The solution of this equation tells us how a observable quantity Γ behaves if all momenta occurring in it are scaled by the value σ :

$$\Gamma(\sigma g, \alpha_s(\mu), m_f(\mu), \mu) = \sigma^D \exp\left\{\int_1^\sigma \frac{\gamma(\alpha_s(s\mu))}{s} ds\right\} \Gamma(g, \alpha_s(\sigma\mu), m_f(\sigma\mu), \mu) \quad (28)$$

The value of the canonical dimension D can be found by simple dimensional considerations (see App. 1). The strong fine structure constant, α_s , is introduced in analogy to the fine structure constant of quantum electrodynamics: $\alpha_s := g_s^2/(4\pi)$. In this equation a running coupling constant $\alpha_s(\mu)$, a running mass $m_f(\mu)$, and an anomalous dimension $\gamma(\alpha_s(\mu))$ occur.

The scale dependence of the coupling constant can be calculated in perturbation theory. One finds in the so called minimal subtraction schemes:

$$\mu \frac{\partial}{\partial \mu} \alpha_s(\mu) := \beta(\alpha_s) = -\frac{\beta_0}{2\pi} \alpha_s^2 + O(\alpha_s^3) \quad \text{with } \beta_0 = \frac{11}{3} N_c - \frac{2}{3} n_f \quad (29)$$

In order to give an explicit solution of equation 29 one has to introduce an integration constant. A convenient choice is given to lowest order by:

$$\frac{1}{\alpha_s(\mu)} = \frac{\beta_0}{2\pi} \ln \frac{\mu}{\Lambda} \quad (30)$$

Note that in this approximation the coupling becomes infinite for the scale $\mu = \Lambda$. This is the so called Landau pole. The coupling at a different scale can easily be evaluated:

$$\frac{1}{\alpha_s(\mu')} = \frac{1}{\alpha_s(\mu)} + \frac{\beta_0}{2\pi} \ln \frac{\mu'}{\mu} \quad (31)$$

The “anomalous dimension” $\gamma(\alpha_s)$ can be determined in perturbation theory, it depends on the quantity. For the mass one finds to lowest order:

$$m_f(\mu) = \tilde{m}_f \left(\ln \frac{\mu}{\Lambda}\right)^{\gamma_0/\beta_0} \quad \text{with } \gamma_0 = \frac{3(N_c^2 - 1)}{2N_c} \quad (32)$$

Taking into account the anomalous dimension in 28 amounts to a summation of the leading powers of $\ln[\sigma^2/\mu^2]$ to all orders in α_s . For four flavours in the widely used $\overline{MS}$ renormalization scheme one finds:

$\Lambda \approx 0.23 \text{ GeV}$ and for the masses of the known quarks, i.e. the up, down, strange, charmed, and beautiful quark:

$$m_u \approx 0.005 \text{ GeV} \quad m_d \approx 0.009 \text{ GeV} \quad m_s \approx 0.140 \text{ GeV} \quad m_c \approx 1.4 \text{ GeV} \quad m_b \approx 4.7 \text{ GeV} \quad (33)$$

The three light quark masses are running masses in the $\overline{MS}$ scheme renormalized at the scale of 1 GeV, the heavy quark masses are pole masses i.e. defined like in quantum electrodynamics. The relation to the running mass in the $\overline{MS}$ scheme is to order α_s given by:

$$m_{pole} = \overline{m}(m_{pole}) \left\{1 + \frac{4\alpha_s}{3\pi}\right\} \quad (34)$$

2.4 Flavour and Chiral Symmetry

If all masses in the Lagrangian 10 were equal, it would possess in addition to the colour symmetry a (global) symmetry under unitary transformations between quark fields of different flavour, i.e. under the transformation

$$\psi^C \rightarrow \psi^C U^f \quad \text{with the unitary flavour matrix } U_{flavour} \quad (35)$$

From 33 we see that the three lightest quarks differ in their mass values by less than 0.15 GeV and hence we expect an approximate $SU(3)$ symmetry in these flavours which is broken by this scale. This is the famous $SU(3)$ -symmetry (now called $SU(3)$ -flavour symmetry) introduced by Gell-Mann (1961) and Neuman (1961). The splitting between the up and down quark is only a few MeV and indeed, the resulting $SU(2)$ isospin symmetry [Heisenberg (1932)] is only broken by this scale.

In the limit that the up and down quark are considered to be massless, there is still another symmetry of the Lagrangian, the so called chiral symmetry. The Lagrangian is for massless quarks invariant under the transformation $\psi \rightarrow \gamma_5 \psi$. If we introduce the left and right-handed spinors (the name right- and left-handed comes from the polarization of the spinors)

$$\psi_L = \frac{1}{2}(1 - \gamma_5)\psi \quad \psi_R = \frac{1}{2}(1 + \gamma_5)\psi, \quad (36)$$

one can easily show that in the absence of mass terms the Lagrangian is invariant under a separate $SU(2)$ transformation of the right and left handed up and down quark. This is the $SU(2)_L \otimes SU(2)_R$ symmetry of the light quarks. Although we expect this symmetry to be observed on the level of a few MeV, there is no sign of it in the hadron spectra. If it were exact one would expect a partly degeneracy of all hadrons whereas parity partners are generally split by a few hundred MeV (see Fig. 1a). From this one concludes that the symmetry is spontaneously broken: Although the Lagrangian is (approximately) invariant under a $SU(2)_L \otimes SU(2)_R$ transformation the ground state, i.e. the hadronic vacuum is not. Such a spontaneous symmetry breaking has a far-reaching consequence. It follows that there exist massless bosons, the Goldstone Bosons, which in our case must transform as a triplet under the isospin transformation and which are identified with the pions. Since the light quarks are not exactly massless the symmetry $SU(2)_L \otimes SU(2)_R$ is not only spontaneously broken, but also by the mass terms of the up and down quark in the Lagrangian. Therefore the pion is not exactly massless, but has the – as compared to other hadrons – small mass of ca. 140 MeV. The sign for the breaking of the chiral symmetry is the existence of a non-vanishing vacuum expectation value of the operator $\psi^C \psi^C + \psi^C \gamma_5 \psi^C$, which plays also an important role in other nonperturbative approaches. Although chiral symmetry and especially chiral perturbation theory is an important tool in low energy hadron physics, I shall not treat it in this review and refer e.g. to the review of Gasser and Leutwyler (1982) and the literature quoted there.

2.5 Interpolating Fields and Composite Operators

As the space of physically observable hadrons, or shortly physical space, we define the Hilbert space which contains all states of "observable" particles like protons, neutrons, π -mesons, B-mesons etc. Generally we shall also accept states which are not stable under strong interactions, like the ρ -meson, the Δ resonance, as observable particles. An interpolating ψ field of a particle P is a field operator which couples to that state, i.e. for which the matrix element

$$\langle \Omega | \psi(x) | P \rangle \neq 0. \quad (37)$$

The state $|\Omega\rangle$ denotes the "no-hadron state", the so-called physical vacuum. In spite of extensive searches, particles which couple to the fundamental fields of QCD, i.e. to the quark and gluon fields, have never been detected and there is strong evidence, although no proof, that the dynamics induced by the QCD Lagrangian does not allow the existence of such states. Since "open colour" has never been observed, one may postulate that all *observable* states are singlets under the colour gauge group (colour-neutral, gauge-invariant). Since in order to form a colour singlet state out of colour octets or triplets one needs at least two of them, interpolating fields for hadrons must be composite fields consisting of at least of two quark or gluon fields.

In genuine field theoretical investigations of QCD like lattice-gauge calculations or QCD sum rules one generally evaluates Greens functions of interpolating hadron fields and extracts hadronic parameters from them. Be $\psi(x)$ an interpolating field of the hadron P and $\bar{G}(p^2)$ the Greens function in momentum space

$$\bar{G}(p^2) = \int d^4x e^{ipx} \langle \Omega | T \psi(x) \bar{\psi}(0) | \Omega \rangle. \quad (38)$$

By inserting intermediate states one finds easily that $\bar{G}(p^2)$ has a pole at the mass squared of the hadron P

$$\bar{G}(p^2) = \frac{|\langle \Omega | \psi | P \rangle|^2}{p^2 - m_P^2} \dots \quad (39)$$

where the dots indicate contributions of other states coupling to the field ψ . From the knowledge of the Greens function we can thus deduce the mass of the hadron P and its coupling to the vacuum $\langle \Omega | \psi | P \rangle$.

In order to study hadronic properties it would be ideal to find an operator which couples just to one hadron. Such an operator would be a complicated nonlocal superposition of field operators, and we shall therefore discuss first such objects.

We first introduce the "string operator"

$$\phi(x, x', C) = P \exp \{ i g_s \int_C A_\mu(z) dz^\mu \} \quad (40)$$

where the gluon field A_μ takes its values in the Lie algebra of the gauge group (see 14). The line integral is to be taken along the path C connecting the point x with x' . Since the (matrix-valued) fields do not commute and since for non-commuting exponents $\exp\{\Sigma A_i\} \neq \prod_i \exp\{A_i\}$ we have to give a prescription of how to construct the exponential of the integral in (40). Such a prescription is indicated by the "path-ordering symbol" P , which tells us to cut the path in infinitesimal pieces and multiply the exponentials of these pieces:

$$P \exp \{ i g_s \int_C A_\mu(z) dz^\mu \} = \lim_{z_{i+1} - z_i \rightarrow 0} \prod_{i=N-1}^1 \exp \left\{ i g_s A_\mu \left(\frac{z_{i+1} + z_i}{2} \right) (z_{i+1} - z_i)^\mu \right\} \quad (41)$$

where z_i are ordered points on the path C with $z_N = x'$, $z_1 = x$. If the path is parametrized by the parameter τ with $0 \leq \tau \leq 1$, one finds in a power series expansion in g_s of (41) τ -ordering:

$$P \exp \{ i g_s \int_C A_\mu(z) dz^\mu \} = 1 + i g_s \int_0^1 d\tau A_\mu(x(\tau)) \frac{d}{d\tau} z^\mu(\tau) - g_s^2 \int_0^1 d\tau \int_0^\tau d\tau' A_\mu(z(\tau)) \frac{d}{d\tau} z^\mu(\tau) A_\nu(z(\tau')) \frac{d}{d\tau'} z^\nu(\tau') + \dots \quad (42)$$

Since A is an element of the Lie algebra of the gauge group, the "string" ϕ is an element of this gauge group and it is easy to show that it behaves under a local gauge transformation $U(x)$ (see 16) in the following way:

$$\phi(x, x', C) \rightarrow U(x') \phi(x, x', C) U^\dagger(x) \quad (43)$$

An immediate consequence of this transformation property is that the trace of a string with a closed path is gauge-invariant

$$\text{tr}(\phi(x, x, L)) \rightarrow \text{tr}(U(x) \phi(x, x, L) U^\dagger(x)) = \text{tr} \phi(x, x, L) \quad (44)$$

Of great practical importance is the fundamental representation of the string operator in which

$$A_\mu = A_\mu^F F \quad (45)$$

where the F^F are traceless hermitian $N_c \times N_c$ matrices (the sum in F running of course from 1 to $N_c^2 - 1$). In that case the string $\phi(x, x', C)$ is an element of the fundamental representation of the gauge group $SU(N_c)$, i.e. an unitary matrix of rank N_c . If I give no further specification, always a string in the fundamental representation is understood. From (43) we obtain also that the expression

$$\bar{\psi}(x') \phi(x, x', C) \psi(x) \quad (46)$$

where $\psi(x)$ is a quark field, is a colour singlet, since under a local gauge transformation the quark fields which are colour N_c -tuplets transform as (see 16)

$$\psi(x) \rightarrow U(x) \psi(x); \quad \bar{\psi}(x') \rightarrow \bar{\psi}(x') U^\dagger(x'), \quad (47)$$

In the same way the nonlocal gluon operator

$$F_{\mu\nu}^F(x') \phi_{FG}^{a\bar{b}}(x, x', C) F_{\mu\nu}^G(x) \quad (48)$$

is gauge- (and Lorentz) invariant. Here ϕ is to be taken in the adjoint representation, i.e. is a $(N_c^2 - 1) \times (N_c^2 - 1)$ matrix. A more convenient form of a gauge-invariant two-gluon operator is given by

$$\text{tr}(\phi(x, x', C) \mathbf{F}_\mu(x) \phi^\dagger(x, x', C) \mathbf{F}_\mu(x')) \quad (49)$$

where now ϕ and $\mathbf{F}$ are in the fundamental representation of the group and Lie algebra, respectively. The two operators (46) and (48) are interpolating fields for mesons, the first one coupling most strongly to ordinary mesons, the second one to glue balls, i.e. mesons consisting mainly of gluons. An interpolating composite field for a baryon for case of $N_c = 3$ is given by:

$$\psi_a^i(x_1) \psi_\beta^j(x_2) \psi_\gamma^k(x_3) (C\gamma_5)_{\alpha\beta} \phi_{AA'}(x_1, y, C_1) \phi_{BB'}(x_2, y, C_2) \phi_{CC'}(x_3, y, C_3) \epsilon_{ABC'} \quad (50)$$

where C is the charge conjugation matrix and $\epsilon_{ABC'}$ the totally antisymmetric tensor in three dimensions.

Figure 2: Graphical representation of the singular part of the scalar composite operator $\bar{\psi}^C(X)\psi^C(X)$ up to order α_s .

Quantum corrections to these non-local operators are awful, and we therefore shall concentrate next on the simpler case of local operators. Gauge invariant and Lorentz covariant local operators are generally interpolating hadron fields, i.e. they couple to hadrons, but unfortunately not only to one, but also to all with the same quantum numbers. So the vector field

$$V_\mu^+ = \bar{\psi}^C(x)\gamma_\mu\psi^C(x) \quad (51)$$

(d and u stand for the flavours up and down) does not only couple to the $\rho^+(770)$ -meson, but also to the $\rho^+(1450)$ and generally to all multi-meson states with the resulting quantum numbers $J^C(J^P) = 1^+(1^-)$. As it is well known even from free field theory, there is a problem with the multiplication of operators at the same space-time point. Indeed the canonical commutation relation for fermion fields,

$$\bar{\psi}(\vec{x}, t)\psi(\vec{y}, t) = i\delta(\vec{x} - \vec{y}), \quad (52)$$

tells us that the product of ψ and $\bar{\psi}$ at the same space-time point is singular. In the operator approach this problem is generally solved by the normal ordering prescription, i.e. in the product of two operators all annihilation operators are shuffled to the right. In the functional approach it is a classical renormalization problem which necessitates the introduction of (diverging) counter-terms. We shall study the situation in some more detail at the important example of the scalar operator in the Euclidean continuum:

$$O_s(X) = \bar{\psi}^C(X)\psi^C(X) \quad (53)$$

It should be noted that the vacuum expectation value of a product of fields in the Euclidean world becomes under the continuation 20 the vacuum expectation value of a time ordered product in the Minkowski world. The vacuum expectation value of this operator (indicated by a dot) is to lowest (i.e. zeroth) order in α_s given by the diagram Fig 2

Let us first consider the vacuum expectation value of the nonlocal operator, diagram 2b).

$$\langle \bar{\psi}^C(X)\psi^C(Y) \rangle = N_c \int \frac{d^4 P}{(2\pi)^4} \text{Tr} \frac{-\gamma P + m}{P^2 + m^2} e^{-iP(X-Y)} \quad (54)$$

Here and in the following $\langle \cdot \rangle$ means the perturbatively evaluated vacuum expectation value. In the limit $X \rightarrow Y$ the integral is clearly diverging, this indicates the fact that the local operator product is singular. It is thus reasonable to define the renormalized product

$$\{\bar{\psi}^C(X)\psi^C(X)\}_{ren} = \bar{\psi}^C(X)\psi^C(X) - \text{singular part of } \langle \bar{\psi}^C(X)\psi^C(X) \rangle \quad (55)$$

If the renormalized operator product is inserted in a Green function, i.e. if we form

$$\langle \psi^C(X)\{\bar{\psi}^C(X)\psi^C(X)\}_{ren}\bar{\psi}^C(Z) \rangle \quad (56)$$

we obtain for that expression

$$S(Y - X)S(X - Z) - S(Y - Z) \times \text{singular part of } \langle \bar{\psi}^C(X)\psi^C(X) \rangle \quad (57)$$

where $S(V)$ is the fermion propagator.

The "operator part" of the composite operator is thus perfectly well-behaved (to zeroth order) but the "c-number part" is singular and has to be renormalized. In order to isolate the singular part from $\langle \bar{\psi}^C(X)\psi^C(X) \rangle$ we can use dimensional regularization, i.e. we evaluate the vacuum expectation value 54 for $X = Y$ in $d = 4 - 2\epsilon$ dimensions, leading to

$$\begin{aligned} \langle \bar{\psi}^C(X)\psi^C(X) \rangle_d &= -N_c \int \frac{d^d P}{(2\pi)^d} \text{Tr} \frac{-\gamma P + m}{P^2 + m^2} = -N_c \int \frac{d\Omega^d P^{d-1} dP}{(2\pi)^d} \cdot \frac{4m}{P^2 + m^2} \\ &= N_c \frac{m^3}{4\pi^2} \left\{ \frac{1}{\epsilon} - \ln \frac{m^2}{4\pi} - \gamma + 1 + O(\epsilon) \right\} \end{aligned} \quad (58)$$

where γ is the Euler constant $\gamma \approx 0.577216$.

If we subtract the full expression 58 from the product $\bar{\psi}^C(X)\psi^C(X)$, we obtain the normal-ordered product (to lowest order in α_s):

$$:\bar{\psi}^C(X)\psi^C(X): = \bar{\psi}^C(X)\psi^C(X) - \langle \bar{\psi}^C(X)\psi^C(X) \rangle \quad (59)$$

(to be understood for $d < 4$ in the limit $d \rightarrow 4 - 0$) with

$$\langle : \bar{\psi}^C(X)\psi^C(X) : \rangle = 0 \quad (60)$$

The inconvenience of this procedure is that the subtraction term contains the mass explicitly. In QCD calculations generally the so-called $\overline{MS}$ -scheme (modified minimal subtraction scheme) is used, where only the term proportional $1/\epsilon + \ln 4\pi - \gamma$ is subtracted. We furthermore must also introduce a mass scale ν , in order to give the $\log m^2$ a scale-independent meaning and thus we are finally led to the $\overline{MS}$ prescription for the renormalized product:

$$\{\bar{\psi}^C(X)\psi^C(X)\}_{ren(\overline{MS})} = \bar{\psi}^C(X)\psi^C(X) - N_c \frac{m^3}{4\pi} \left\{ \frac{1}{\epsilon} + \ln \frac{4\pi^2}{\nu^2} - \gamma \right\} \quad (61)$$

leading to the "perturbative" vacuum expectation value of the renormalized product

$$\langle \{\bar{\psi}^C(X)\psi^C(X)\}_{ren(\overline{MS})} \rangle = N_c \frac{m^3}{4\pi^2} (1 + \ln \frac{\nu^2}{m^2}) \quad (62)$$

Since the scale ν^2 plays here (as mostly in dimensional regularization) a very formal role, and since it plays an important role in the theoretical foundation of QCD sum rules, I shall try to shed some light on it by a different regularization scheme, namely that by a cutoff of the internal momenta. If we define

$$\langle \bar{\psi}^C(X)\psi^C(X) \rangle_M = N_c \int_{p_1 < M^2} \frac{d^4 P}{(2\pi)^4} \text{Tr} \frac{-\gamma P + m}{P^2 + m^2} \quad (63)$$

the renormalization prescription

$$\{\bar{\psi}^C(X)\psi^C(X)\} = \lim_{M \rightarrow \infty} \left(\bar{\psi}^C(X) \cdot \psi^C(X) - \langle \bar{\psi}^C(X)\psi^C(X) \rangle_M \right) \quad (64)$$

corresponds to normal ordering. But we might also subtract only the internal momentum contributions $p^2 > \nu^2$, which of course also leads to a well-defined operator product

$$\{\bar{\psi}^C(X)\psi^C(X)\}_\nu = \lim_{M \rightarrow \infty} (\bar{\psi}^C(X) \cdot \psi^C(X) - \langle \bar{\psi}^C(X)\psi^C(X) \rangle_M - \langle \bar{\psi}^C(X)\psi^C(X) \rangle_\nu) \quad (65)$$

In that case the vacuum expectation value of $\{\bar{\psi}^C(X)\psi^C(X)\}_\nu$ is

$$\langle \{\bar{\psi}^C(X)\psi^C(X)\}_\nu \rangle = N_c \int_0^{\nu^2} d^3\Omega \frac{1}{2} P^2 dP^2 \frac{4m}{P^2 + m^2} = \frac{N_c}{4\pi^2} m \left\{ \nu^2 - m^2 \ln \frac{\nu^2 + m^2}{m^2} \right\} \quad (66)$$

Here the renormalization scale ν has a well-defined physical meaning. Only the internal momenta p larger than ν are thrown away by the renormalization procedure, the lower ones contribute (as a c-number) to the renormalized operator. For up and down quarks this "perturbative" contribution to the vacuum expectation value is numerically completely negligible, it only might be necessary for the cancellation of mass singularities [Broadhurst and Generalis 1982, Pich and Rafael 1985].

I have dwelled so long on this simple point since it is a non-trivial example for the dependence of the vacuum expectation values on the treatment of perturbation theory. This played an important role in objections against the sum rule method and will be shortly discussed in the next subsection.

As a next step we have to take into account radiative corrections to the scalar operator. They are represented (to order α_s) by the graphs 2c-e. Graph 2c-e is just the radiative correction to the term treated above, but graph 2e) is genuine for the operator product.

The divergent parts of the diagrams have to be taken into account by divergent counter-terms in the Lagrangian, leading to the renormalized scalar operator:

$$\{\bar{\psi}^C(X)\psi^C(X)\}_{ren} = Z_s^{-1} \cdot \bar{\psi}_0^C(X)\psi_0^C(X) - \text{divergent } c - \text{number} \quad (67)$$

Here $\bar{\psi}_0^C(X)\psi_0^C(X)$ denotes the unrenormalized product of unrenormalized fields, i.e. ψ_0^C occurs in a Lagrangian without counter-terms. In dimensional regularization the divergent part of Z_s can be easily evaluated to order α_s , yielding

$$Z_s = 1 + 3 - \frac{N_c^2 - 1}{2N_c} \frac{\alpha_s}{\pi} \epsilon + \text{finite terms} \quad (68)$$

Due to the regularization of the divergent diagrams 2d,e) a scale ν^2 enters also in the operator part of $\{\bar{\psi}^C(X)\psi^C(X)\}_{ren}$, i.e. the part which contributes to the connected part of the Green function $\langle \psi^D(Y)\{\bar{\psi}^C(X)\psi^C(X)\}_{ren}\psi^E(Z) \rangle$. This scale dependence comes out in dimensional regularization in a rather formal way, but as before it can be also viewed as the upper limit of internal integration.

The dependence of the renormalized operator on the scale ν is governed by its anomalous dimension γ_s :

$$\gamma_s = \nu \frac{d}{d\nu} \ln Z_s \quad (69)$$

If we denote by $\{\bar{\psi}^C(X)\psi^C(X)\}_\nu$ the scalar operator renormalized at scale ν , then we have the relation to one loop:

$$\{\bar{\psi}^C(X)\psi^C(X)\}_\nu = \left(\frac{\alpha_s(\nu_2)}{\alpha_s(\nu_1)} \right)^{\gamma_0/\beta_0} \{\bar{\psi}^C(X)\psi^C(X)\}_{\nu_2} \quad (70)$$

where β_0 is the first coefficient of the Gell-Mann-Low function β (29) and γ_1 the leading coefficient of the anomalous dimension of the scalar operator γ_s : $\gamma_0 = \frac{3}{2} \frac{N_c^2 - 1}{N_c}$. It should be emphasized that this rescaling makes sense only if both scales ν_1 and ν_2 are high enough in order to justify perturbation theory. A comparison with the scale dependence of the mass (32) shows that the product $m \cdot \bar{\psi}^C(X)\psi^C(X)$ is scale-independent, i.e. has $Z = 1$. This can be seen also to all orders by realizing that the insertion of $\bar{\psi}^C(X)\psi^C(X)$ in a diagram amounts to a differentiation with respect to m . It can be shown generally [Preparata and Weisberger 1968] that the vector and axial vector operator $\bar{\psi}^C(X)\gamma_\mu\psi^C(X)$ and $\bar{\psi}^C(X)\gamma_5\psi^C(X)$ are not renormalized, i.e. have $Z_{V,A} = 1$.

Figure 3: Radiative corrections to the two gluon operator

A gauge-invariant operator containing the gluon field is

$$tr(\mathbf{F}_{\mu\nu}\mathbf{F}_{\mu\nu}) = \frac{1}{2}F_{\mu\nu}^F(X)F_{\mu\nu}^F(X). \quad (71)$$

In dimensional regularization (but not in other schemes) its vacuum expectation value is zero, so in that scheme we have only renormalization effects to order α_s (see Fig.3). To that order the diverging diagrams 3a),b) necessitate the introduction of a divergent counterterm, which can be absorbed into the renormalization constant

$$\{F_{\mu\nu}^F(X)F_{\mu\nu}^F(X)\} = Z_\alpha(F_{\mu\nu}^F(X)F_{\mu\nu}^F(X)) \quad (72)$$

where the subscript zero again indicates bare fields and Z_α is the charge renormalization. Since $\alpha_s = Z_\alpha^{-1} \cdot \alpha_{s0}$ we obtain the important result that, to one-loop level, the product of α_s with the two-gluon operator is renormalization group-invariant, i.e. $\{\alpha_s F_{\mu\nu}^F(X)F_{\mu\nu}^F(X)\}$ does not depend on a renormalization scale.

There is another in principle very important effect: the mixing of the gluon with the scalar operator introduced above. Indeed, if we evaluate the expectation value $\langle \bar{\psi}^D(z)F_{\mu\nu}^F(X)F_{\mu\nu}^F(X)\psi^E(z) \rangle$, it does not vanish, due to the contribution of graph 3d). Putting results of diagrams 3 together, one finds that to order α_s in $\overline{MS}$ the linear combination

$$\frac{1}{4}\partial_0\left\{\frac{\alpha_s}{\pi}FF\right\} + \gamma_0\frac{\alpha_s}{\pi}\{m\bar{\psi}^C(X)\psi^C(X)\} \quad (73)$$

is renormalization group-invariant (i.e. scale-independent). It can be shown [Collins *et al.* 1977, Nielsen 1977] that the trace of the energy momentum tensor between physical states is given by

$$\theta_{\mu\mu} = (1 + \gamma_m) \sum_f m_f \bar{\psi}^f \psi^f + \frac{1}{4}\beta F_{\mu\nu}^F F_{\mu\nu}^F. \quad (74)$$

In a massless theory this trace should be zero, the deviation from it indicates the broken scale invariance. The anomalous dimensions of the operators used here are given in App. 2 table 2.

2.6 Operator Product Expansion

An important tool in field theory is the so-called operator product expansion, first conjectured by Wilson (1969,1971). It states that for short (Euclidean) distances a nonlocal product of operators $A(x) \cdot B(y)$ can be expressed as a series of local operators O_n :

$$A(x) \cdot B(y) \xrightarrow{(x-y)^2 \rightarrow 0} \sum_{n=1}^{\infty} C_n^A B(x-y) O_n \quad (75)$$

The coefficient functions $C_n(x-y)$ are in general distributions. By the relation of Fourier transformation the short distances correspond of course to large momenta.

It has been shown rigorously that 75 holds order by order in perturbation theory [Zimmermann 1971] and that the coefficient distributions can be calculated unambiguously in perturbation theory. Since we are, however, interested in nonperturbative effects, there is the legitimate question if there is a unique way in order to evaluate the coefficient functions also there. There has been a long discussion about this point where the possibility was challenged mainly in ref. [David 1982, David 1984]. We are here interested in the case of QCD, but since there we have no rigorous analytical tool, most of the discussions have to be performed in models, from which one has to draw conclusions on QCD by analogy. Novikov et al. (1984a) have argued convincingly that in most objections the important role played by the renormalization point in the definition of the local operators O_n was overlooked. Indeed these operators will be in general composite ones and therefore they have to be renormalized, as discussed shortly in the last section. Thus they depend in general on a renormalization scale ν (and on the regularization scheme). The scale now plays a crucial role in the physical interpretation of the OPE. The coefficient functions are calculated by integrating over internal fields with frequencies larger than ν , whereas the lower frequencies are absorbed in the definition of the operators. A change of the ν will therefore also lead to a change in the coefficient distributions and a mixing of them, which is governed by the renormalization group. Since in QCD the contributions of internal fields with high frequencies can be evaluated by perturbation theory (or manageable non-perturbative contributions such as small-size instantons) and take care of non-perturbative effects in the definition of the operators.

In this review the operator product expansion will be used exclusively in connection with sum rules and I shall postpone a discussion of the practical aspects of it to that section.

Suggested reading for sect. 2: There are many good textbooks on modern quantum field theory, so I quote only a few which I found particularly helpful for several reasons: Itzykson and Zuber (1980) (encyclopedia), Ryder (1985) (good mixture between operator and functional approach), Pascual and Tarach (1984) (very practical), Collins (1984) (specialized on renormalization group and related subjects). The relation to phenomenology is excellently treated in Nachtmann (1990).

3 Non-Perturbative QCD

3.1 Lattice Gauge Theory

3.1.1 The QCD Lagrangian on a Lattice

In this section we introduce the essential ingredients for the *lattice regularization* of a non-Abelian gauge theory like QCD. In contrast to the more formal expressions of sect. 2 all equations introduced and derived in this section are well defined and can - in principle - be evaluated numerically. It is this feature which has made lattice gauge theory one of the most important tools of non-perturbative QCD. The functional integral on a field $\Phi(x)$ introduced in the last section has to be understood as limiting value of a high dimensional volume integral with the fields $\Phi(i), \Phi(j), \dots$ at the lattice points $i, j, \dots$ as integration variables. We introduce in the Euclidean continuum a finite hypercubic lattice, with lattice spacing a and Volume $V = (Na)^4$. A point on the lattice is denoted by its coordinates in units of a , i.e. by a fourtuple of integers: $(j) = (j_1, j_2, j_3, j_4)$, representing the point with the coordinates $X = (a j_1, a j_2, a j_3, a j_4)$.

By $(j + \mu)$ we denote the neighbour of the point (j) in μ -direction.

A special feature of lattice gauge theories is the important role played by links between neighbouring lattice points.

The link from point j to its neighbour in μ -direction, $j + \mu$ is denoted by $(j, j + \mu)$. In a lattice gauge

theory to each link an element of the gauge group is attached, to the link in the opposite direction its inverse. This was first done by Wegner (1971) for the group Z_2 (i.e. the group consisting of the elements $+1$ and -1). It was extended to QCD by Wilson (1974) mapping each link of the lattice on elements of the group $SU(3)$ (more generally $SU(N_C)$):

$$(j, j + \mu) \rightarrow U(j, j + \mu) \in SU(N_C); \quad (j + \mu, j) \rightarrow U^*(j, j + \mu). \quad (76)$$

The group elements $U(j, j + \mu)$ can be expressed by the generators of the group $\mathcal{T}_F$ (see 8):

$$U(j, j + \mu) = \exp[-iag B_\mu^F(j) \mathcal{T}_F] \quad (77)$$

Through 76 and 77 we have a mapping of the links on the Lie-algebra valued vector potentials:

$$(j, j + \mu) \rightarrow B_\mu(j) := B_\mu^F(j) \mathcal{T}_F \quad (78)$$

For the case of the Abelian gauge group $U(1)$ we have the mapping: $(j, j + \mu) \rightarrow B_\mu(j)$

In a local gauge theory an element of the gauge group is attached to each *point* on the lattice

$$U(j) = \exp(i g \mathcal{T}_F A^F(j)) \quad (79)$$

and we define as gauge transformation for the group element $U(j, j + \mu)$

$$U(j, j + \mu) \rightarrow U'(j, j + \mu) = U(j) U(j, j + \mu) U^*(j + \mu). \quad (80)$$

Note that there is no inhomogeneous term in the lattice version of a gauge transformation. The continuum limit of the lattice version is achieved by connecting quantities attached to neighbouring lattice points through Taylor expansion and retaining only the nontrivial contributions lowest in lattice spacing a . So for the elements of the gauge group we have:

$$U^*(j + \mu) = U^*(x) + a \partial_\mu U^*(x) + O(a^2) \quad (81)$$

Using $U(j, j + \mu) = 1 + iag B_\mu(x) + O(a^2)$ (see 77) the gauge transformation 80 translates to:

$$B_\mu(x) \rightarrow U(x) B_\mu(x) U^*(x) + \frac{i}{g} U(x) \partial_\mu U^*(x) \quad (82)$$

This is the non-Abelian gauge transformation for the vector fields in the continuum (see 16)

A product of group elements $U(j, j + \mu)$ along a closed contour L on the lattice is called a Wegner-Wilson loop $W[L]$ (see Fig. 4a).

$$W[L] = U(j, j + \mu) U(j + \mu, j + \lambda) \dots U(j + \nu, j) \quad (83)$$

Its trace is a gauge invariant quantity. Under a gauge transformation 80 the loop becomes:

$$\begin{aligned} \text{Tr} W[L] &\rightarrow \text{Tr} U(j) U(j, j + \mu) U(j + \mu, j + \lambda) U(j + \lambda, j + \mu + \lambda) U(j + \mu + \lambda, j) \dots \\ &\dots U(j + \nu) U(j + \nu, j) U(j) = \text{Tr} W[L] \end{aligned} \quad (84)$$

since $U U^* = 1$ and since the trace is cyclic: $\text{Tr}\{ABC\} = \text{Tr}\{BCA\}$

A *plaquette* is the simplest non-trivial Wegner-Wilson loop, namely the product of four group elements attached to a square with sidelength a and lattice points as corners (see Fig. 4b).

$$\begin{aligned} P(j, \mu, \nu) &= U(j + \mu, j + \mu + \nu) U(j + \mu + \nu, j + \nu) U(j + \nu, j) U(j, j + \mu) \\ &= U(j + \mu, j + \mu + \nu) U^*(j + \nu, j + \mu + \nu) U^*(j, j + \nu) U(j, j + \mu) \\ &= \exp\{iga B_\nu(j + \mu)\} \exp\{-iga B_\mu(j)\} \exp\{iga B_\nu(j)\} \exp\{iga B_\mu(j)\} \end{aligned}$$

Figure 4: a) Rectangular Wegner-Wilson loop. b) Plaquette. To each link the Group element $U(k, k + \rho)$ is attached

In order to obtain the continuum limit of a plaquette we first make use of the Campbell-Hausdorff formula:

$$e^{aA} e^{bB} = e^{aA + bB + \frac{1}{2}[A, B] + O(a^3)} \quad (85)$$

Applying it to the first and second pair of exponentials we obtain:

$$P(j, \mu, \nu) = \exp\{-iag[B_\nu(j + \mu) - B_\mu(j + \nu)] + a^2 g^2 [B_\nu(j + \mu), B_\mu(j + \nu)]/2 + iO(a^3)\} \cdot \exp\{iag[B_\nu(j) - B_\mu(j)] + a^2 g^2 [B_\nu(j), B_\mu(j)]/2 + iO(a^3)\}.$$

We now make use of the Taylor expansion:

$$B_\nu(j + \nu) = B_\nu(j) + a\partial_\nu B_\nu(j) + O(a^2), \quad (86)$$

apply again 85 and obtain:

$$P(j, \mu, \nu) = \exp\{ia^2 g^2 \{G_{\mu\nu}(j) + O(a)\}\} \quad (87)$$

with the usual definition for the Lie-algebra valued field tensor:

$$G_{\mu\nu}(x) = G_{\mu\nu}^F(x) \tau_F = \partial_\mu B_\nu(x) - \partial_\nu B_\mu(x) - ig[B_\mu, B_\nu] \quad (88)$$

Using 29 and the commutation relations for the generators $[\tau_G, \tau_H] = if_{FGH}\tau_F$, we see that 88 is equivalent to the definition 12. Now we are in the position to define a real and gauge invariant action on the lattice:

$$S_L = -\frac{1}{g^2} \sum_j \sum_{\mu < \nu} \text{Tr} (P(j, \mu, \nu) + P(j, \mu, \nu)^*) \quad (89)$$

First we show that, up to a additive constant, the continuum limit of 89 is the pure gauge field part non-Abelian action 10, 18: For that we insert 87 into 89 and obtain:

$$S_L = \frac{1}{g^2} \sum_j \sum_{\mu < \nu} 2\Re \text{Tr} \exp(-ia^2 g^2 \{G_{\mu\nu}(j) + O(a)\}) \\ = \frac{1}{g^2} \sum_j \sum_{\mu < \nu} \Re \text{Tr} \left(1 - ia^2 g^2 G_{\mu\nu}(j) + iO(a^3) - \frac{1}{2} a^4 g^4 G_{\mu\nu}(j) G_{\mu\nu}(j) + O(a^6) \right) \quad (90)$$

Using $\text{Tr} \tau_F = 0$ and $\text{Tr} (\tau_F \tau_G) = \frac{1}{2} \delta_{FG}$ we obtain

$$S_L = -\frac{1}{4} \sum_j \sum_{\mu < \nu} a^4 \{G_{\mu\nu}^F(j) G_{\mu\nu}^F(j) + O(a^2) + \text{Constant}\} \quad (91)$$

i.e. in the limit $a \rightarrow 0$ and $V \rightarrow \infty$ the lattice action S_L goes, up to an irrelevant (infinite) constant to the continuum action.

The quantum expectation value of a function of the fields, $F[U(j, j + \mu)]$, is given by the integral:

$$\langle F[U(j, j + \mu)] \rangle = \int \mathcal{D}U e^{S_L} F[U(j, j + \mu)] / \int \mathcal{D}U e^{S_L} \quad (92)$$

where $\mathcal{D}U = \prod_{k, \nu} dU(k, k + \nu)$ and $dU(k, k + \nu)$ is the invariant measure on the group (Haar measure) attached to the link $(k, k + \nu)$. The product runs over all links of the lattice. For the Abelian group $U(1)$ the measure is particularly simple:

$$dU(k, k + \nu) = d(aB_\nu(k)) \quad \text{with} \quad -\pi/a \leq B_\nu(k) < \pi/a \quad (93)$$

For the group $SU(N_C)$ it is:

$$dU(k, k + \nu) = \sqrt{\det \left(\frac{1 - \cos aB_\nu(k)}{(aB_\nu(k))^2} \right)}^{N_C^2 - 1} \prod_{F=1}^{N_C^2 - 1} da B_\nu^F(k) \quad (94)$$

In order to describe the full action of QCD on a lattice, we also have to incorporate fermions. Since fermions obey the Pauli exclusion principle they are described on the classical level (i.e. before functional integration) by anticommuting variables, which form a so called Grassmann-Algebra. These among themselves anticommuting quantities commute with complex numbers. Thus we attach to each lattice point with coordinates (j) ($N_C \times 8$) anticommuting quantities (generators of a Grassmann-Algebra): $\psi_\alpha^C(j)$, $\bar{\psi}_\alpha^C(j)$, where the spinor index α runs from 1 to 4 and the color index C from 1 to N_C . Under a $SU(N_C)$ gauge such a field shall transform as:

$$\psi_\alpha^C(j) \rightarrow U_{CD}(j) \psi_\alpha^D(j) \quad \bar{\psi}_\alpha^C(j) \rightarrow \bar{U}_{CD}(j) \bar{\psi}_\alpha^D(j) \quad (95)$$

i.e. terms like $\bar{\psi}_\alpha^C(j) \psi_\alpha^C(j)$ and $\bar{\psi}_\alpha^C(j + \mu) U_{CD}(j + \mu, j) \psi_\alpha^D(j)$ are gauge invariant, as follows immediately from 95 and 80

To the pure gauge lattice action S_G , 89, we add the gauge invariant expression:

$$\sum_j M \bar{\psi}_\alpha^C(j) \psi_\alpha^C(j) \\ + \sum_{j, \mu} p [\bar{\psi}_\alpha^C(j + \mu) \{h\delta_{\alpha\beta} - i(\gamma_\mu)_{\alpha\beta}\} U_{CD}(j, j + \mu) \psi_\beta^D(j) \\ + \bar{\psi}_\alpha^C(j) \{h\delta_{\alpha\beta} + i(\gamma_\mu)_{\alpha\beta}\} U_{CD}(j, j + \mu) \psi_\beta^D(j + \mu)] \quad (96)$$

Here we have explicitly indicated the summation over spinor indices and the indices of the unitary matrices $U_{CD}(j, l + \mu)$, in the following we shall omit the explicit indication of these indices in order to make the formulae less clumsy. The quantity h is an arbitrary real number larger than zero (it must be introduced in order to avoid the so called "fermion doubling"). The classical continuum limit of 96 is easily obtained in the standard way:

$$\sum_j M \bar{\psi}(j) \psi(j) \\ + \sum_{j, \mu} p [\bar{\psi}(j) \{h + i\gamma_\mu\} (1 - iag B_\mu(j) + O(a^2)) (\psi(j) + a\partial_\mu \psi(j) + O(a^2)) \\ + (\bar{\psi}(j) + a\partial_\mu \bar{\psi}(j) + O(a^2)) \{h - i\gamma_\mu\} (1 + iag B_\mu(j) + O(a^2)) \psi(j)] \\ = m \sum_j \bar{\psi}(j) \psi(j) + \frac{1}{2} \sum_{j, \mu} a^4 \{ (i\bar{\psi}(j) \gamma_\mu \partial_\mu \psi(j) - i\partial_\mu \bar{\psi}(j) \gamma_\mu \psi(j) \\ - g \bar{\psi}(j) \tau_F \psi(j)) A_\mu^F(j) + h p \partial_\mu (\bar{\psi}(j) \psi(j)) \} \quad (97)$$

with:

$$M = ma^4 - 4a^3h, \quad p = \frac{a^3}{2} \quad (98)$$

The last term $h p \partial_\mu (\bar{\psi}(j)\psi(j))$ vanishes after summation over j (integration over X) and thus the continuum limit 98 is just the action of the fermion field and its interaction with the gauge field given in 10. In order to define Greens functions we have still to define the integration over Grassmann variables. It is very simple. Be $\eta_1 \eta_2 \dots \eta_n$ a set of n Grassmann variables, e.g. the fermion variables attached to a point, $\psi_a^C(j)$, $\bar{\psi}_a^C(j)$. Then the integral is defined as:

$$\int d\eta_1 d\eta_2 \dots d\eta_n \eta_1 \eta_2 \dots \eta_n = 1 \quad \text{all other integrals are zero} \quad (99)$$

We give a few examples:

$$\int d\eta_1 d\eta_2 \eta_1 \eta_2 = - \int d\eta_1 d\eta_2 \eta_1 \eta_2 = -1 \quad \int d\eta_1 d\eta_2 \eta_2 = 0 \quad \int d\eta_1 \eta_2 = 0$$

With this definition any analytic function of Grassmann variables can be integrated. One has just to Taylor expand it and then apply 99. Note that the Taylor expansion has only a finite number of terms, since due to the anticommutativity $\eta_i^n = 0$ for $n > 1$. As example we consider the integral over a Grassmann algebra with 4 generators $\bar{\eta}_j$, η_j , $j = 1, 2$, the A_{ij} are complex numbers :

$$\begin{aligned} & \int \prod_{i,j=1}^2 d\eta_i d\bar{\eta}_j \exp \left\{ \sum_{i,j=1}^2 \bar{\eta}_i A_{ij} \eta_j \right\} \\ &= \int \prod_{j=1}^2 d\eta_j d\bar{\eta}_j \{ \bar{\eta}_1 A_{11} \eta_1 \bar{\eta}_2 A_{22} \eta_2 + \bar{\eta}_1 A_{12} \eta_2 \bar{\eta}_2 A_{21} \eta_1 \} = (A_{11} A_{22} - A_{12} A_{21}) = \det(A) \end{aligned} \quad (100)$$

One can prove easily that 100 holds for an arbitrary number of $\bar{\eta}_i$, η_i .

Collecting all things, we obtain in the gauge invariant lattice regularization for the expectation value of a function of the vector and fermion fields $F[B_\mu^F(k), \bar{\psi}(l)\psi(m)]$

$$\langle F[B_\mu^F(k), \bar{\psi}(l)\psi(m)] \rangle = \frac{1}{N} \int \prod_{j,\mu} dU(j,\mu) \prod_{j,a,C} d\bar{\psi}_a^C(j) d\psi_a^C(j) F[B_\mu^F(k), \bar{\psi}(l)\psi(m)] e^{-S_L} \quad (101)$$

$$N = \int \prod_{j,\mu} dU(j,\mu) \prod_{j,a,C} d\bar{\psi}_a^C(j) d\psi_a^C(j) e^{-S_L}$$

with the action given by 89 and 96 :

$$\begin{aligned} S_L &= -\frac{1}{g^2} \sum_{j,\mu < \nu} \text{Tr} (P(j,\mu,\nu) + P(j,\mu,\nu)^*) \\ &+ \sum_j M \bar{\psi}(j)\psi(j) \\ &+ \sum_{j,\mu} p [\bar{\psi}(j + \mu) \{h - i\gamma_\mu\} U^*(j,j + \mu) \psi(j) \\ &+ \bar{\psi}(j) \{h + i\gamma_\mu\} U(j,j + \mu) \psi(j)] \end{aligned} \quad (102)$$

The attribution of links and points to the fields is again illustrated in Fig. 5.

3.1.2 Applications of Lattice Gauge Theory

We are now able, at least in principle, to calculate Greens functions using 102. We first consider the Greens function of a pair of a static (very heavy) anti-quark and quark at lattice points j and $j + n\mu$ respectively. A gauge invariant expression for such a state is given by:

$$\bar{\psi}(k) U(k, k + \nu) \dots U(k + (n-1)\nu, k + n\nu) \psi(k + n\nu). \quad (103)$$

Figure 5: The attribution of links to vector gauge fields $B_\mu(j) = A_\mu^F(j) \tau_F$ and points to fermion fields $\psi(j)$, $\bar{\psi}(j)$

Its propagation in Euclidean space-time is described by the Greens function which is a functional integral according to 102

$$G[(k, k + n\nu), (j, j + n\nu)] = \langle \{ \bar{\psi}(j) U(j, j + \nu) \dots U(j + (n-1)\nu, j + n\nu) \psi(j + n\nu) \}^* \bar{\psi}(k) U(k, k + \nu) \dots U(k + (n-1)\nu, k + n\nu) \psi(k + n\nu) \rangle \quad (104)$$

where the lattice point j is displaced with respect to k by r units in 4 direction. The integration over the fermion fields poses in principle no problem, since in the action the fermionic variables $\psi \bar{\psi}$ occur quadratically and hence the integration is Gaussian. One can show that for M going to infinity the leading term of the Greens function 104 is given by:

$$G[(k, k + n\nu), (j, j + n\nu)] \propto (p/M)^{2n} < \text{Tr} W[L] >_U \quad (105)$$

where $W[L]$ is the (rectangular) W -loop with the corners j , k , $k + n\mu$, $j + n\mu$ and $< \dots >_U$ denotes averaging over the pure gauge fields according to 92.

We can only give some hints, how this result appears. Consider the integration over the fermion fields at the points k . At this point we have in the integrand a factor $\bar{\psi}(k)$ from the state 103. In order that the fermion integral does not vanish, we need an additional factor $\psi(k)$, which we obtain by expanding the interaction term in e^{S_L} . This yields, among others a term $p \bar{\psi}(k + \mu) (h - i\gamma_\mu) U^*(k, k + \mu) \psi(k)$. After fermion integration at the point k the fermion field $\bar{\psi}(k)$ is no longer present, but now we have a fermion field at the position $k + \mu$ and a factor $p \bar{\psi}(k + \mu) (h - i\gamma_\mu) U^*(k, k + \mu)$. We thus hopped with the fermion field from k to $k + \mu$. So we may hop from k to j , and from $j + \nu$ to $k + \nu$. For other points on the lattice we have to expand the mass term e^{S_L} in order to have the necessary factor $\bar{\psi}$ (see 99). This yields a factor M for each of these points. After deviding through the normalization N (see 102) we obtain the factor $(p/M)^{2n}$ and the group elements U attached to the links of the loop. Those on the sides from j to $j + \nu$ and from $k + \nu$ to k are due to the state (see 104), those on the other two sides due to the fermion integration.

On the other hand it follows from quantum mechanics that the Greens-function 104 is proportional to e^{-ET} , where E is the static energy of the pair and T the Euclidean time $T = n a$. Hence we come to the conclusion: The static energy of a quark anti-quark pair with separation $R = na$ is given by

$$E(R) = - \lim_{T \rightarrow \infty} \frac{1}{T} \log < W[L] >_U \quad (106)$$

where $< W[L] >_U$ is the pure gauge expectation value of a Wegner-Wilson loop with width R and length T . The expectation value $< W[L] >_U$ can be easily evaluated as a power series in $(1/g^2)$ (strong coupling expansion) and one finds:

$$< W[L] >_U \propto \frac{1}{g^2} r^n = \frac{1}{g^2} \left(1 + O\left(\frac{1}{g^2}\right) \right) \quad (107)$$

Hence in that approximation the static energy of the quarks increases linearly with the spatial distance and in order to separate them one would need infinite energy. Unfortunately such a result can be also obtained in an Abelian gauge theory and since we do not observe confinement in electrodynamics we have to assume that there is a phase transition in that theory between a confining phase in the strong

coupling and a deconfined phase in the weak coupling regime. No formal proof exists that such a transition does not exist in non-Abelian theories like QCD. One can however evaluate the expectation value $\langle W[L] \rangle > 0$ numerically and obtains the area law 107 also for small coupling. This is the strongest indication that confinement is a consequence of the QCD-Lagrangian.

Directly observable quantities can be obtained by evaluating Greens functions of interpolating fields, e.g. the vector field $\bar{\psi}(k)\gamma_\mu\psi(k)$. For large Euclidean times T this Greens function should behave like $\exp[-m_\rho T]$ since the ρ -meson is the lightest particle interpolated by the vector field $\bar{\psi}(k)\gamma_\mu\psi(k)$. This follows from the Euclidean Fourier transform of eq. 39. The strategy consists thus in evaluating the Greens function $\langle \bar{\psi}(j)\gamma_\mu\psi(j)\bar{\psi}(k)\gamma_\mu\psi(k) \rangle > 0$ according to 102 and investigate the dependence of it on the Euclidean time difference. In that way we can in principle calculate the whole hadron spectrum. The input parameters are one scale (e.g. the ρ -mass) and the quark masses. One principal question is off course: if we approximate the functional integral by a product of Riemann integrals, when have we reached the continuum limit? Here asymptotic freedom and the renormalization group helps. The lattice spacing a corresponds to a cutoff in momentum space at momentum $p \propto 1/a$ and hence the coupling constant depends on the lattice spacing through (see 30):

$$\alpha_s(a) := \frac{g^2(a)}{4\pi} = 2\pi / \{ \beta_0 \ln \frac{1}{\Lambda_L a} \} \quad (108)$$

Thus with decreasing a also g decreases or vice versa and we may select a small lattice spacing by choosing a small coupling constant g . If g is small enough the coefficients of the renormalization group equation should take the values as calculated in perturbation theory and we can check, if the numerical results of the lattice calculation behave as predicted by the renormalization group given by 28. If that is the case one may safely assume to have reached the continuum limit. In practice there are however thousand difficulties, some of technical, some of more than technical nature. I shall just quote a few of them in order to make the restrictions mentioned in lattice calculations more understandable. If the lattice spacing is small (say a small fraction of a fermi) the lattice must be large in order to accommodate a hadron which has an extension of about one fermi. Thus lattices should have at least 10^4 lattice points, preferably of course more. Since for $SU(3)$ we have for each lattice point 8 group integrations (see 94) and 24 fermionic integrations it is clear that very sophisticated integration methods have to be applied. But even with these methods an exact evaluation of the fermionic integrals is, given the present capacity of supercomputers, impossible, and thus one has to rely on approximations. A particular simple one is the so called quenched approximation in which only these fermionic integrations are performed which are necessary to take care of the fermion fields in the integrand. Another more technical obstacle are the small masses of the light quarks. Generally one makes the numerical calculations with larger masses and extrapolates to the values near zero. Finally since one only evaluates Greens functions with finite errors the extraction of hadronic parameters is not an easy task. At the moment good lattice results for masses, decay constants, and formfactors have typical errors between 5 and 20 %, with some additional uncertainties due to the above mentioned approximations; for a recent review see e.g. [Mackenzie 1993].

Suggested reading for sects. 3.1,3.2: Abundant information can be found in the following monographs on lattice gauge theory: M. Creutz (1983), C. Rebbi (1983), H.J. Rothe (1992) Newest results can be obtained from the proceedings of the annual lattice conferences (Nucl. Phys. B (Proc. Suppl.))

3.2 Renormalons

Although nobody expects that the perturbative series of QCD is convergent, one might have the hope that it is Borel summable, i.e. that even if the power series

$$D(\alpha) = \sum_{n=0}^{\infty} D_n \alpha^n \quad (109)$$

Figure 6: Radiative correction to the vector two point function

has no finite radius of convergence, the Borel improved series

$$\bar{D}(b) = \sum_{n=1}^{\infty} D_{n+1} \frac{b^n}{n!} \quad (110)$$

is convergent and that the function $D(\alpha)$ can be obtained (up to a constant) through:

$$D(\alpha) - D(0) = \int_0^{\infty} db \bar{D}(b) \exp\left(-\frac{b}{\alpha}\right) \quad (111)$$

The relation 111 is correct order by order in perturbation theory, but there are good arguments that it cannot be true for a full Greens function: There are probably poles of $\bar{D}(b)$ on the positive real axis of the complex b -plane, the so called infrared renormalons. It is however "important to emphasize that there are no proven results concerning the asymptotic nature of QCD perturbation expansion. Thus the discussion that follows is rather a collection of folklore and prejudices" [Mueller 1993]. In the following I shortly present some arguments given by A.H. Mueller (1985).

We consider the vector two point function:

$$\begin{aligned} \Pi_{\mu\nu}(Q) &= \int d^4X e^{iQX} \langle \bar{\psi}(X) \gamma_\mu \psi(X) \bar{\psi}(0) \gamma_\nu \psi(0) \rangle \\ &= \Pi(Q) (\delta_{\mu\nu} Q^2 - Q_\mu Q_\nu) \quad \text{with } Q = \sqrt{Q^2} \end{aligned} \quad (112)$$

The lowest order perturbative correction (see Fig. 6) can be written in the form:

$$\Pi_1(Q/\mu, \alpha_s) = \int \frac{d^4K}{(2\pi)^4} \alpha_s G(K, Q) \quad (113)$$

We improve this expression by introducing the running coupling $\alpha_s(K)$, $K = \sqrt{K^2}$ defined by (see 29)

$$K \frac{\partial}{\partial K} \alpha_s(K) = \beta(\alpha_s) \quad \text{with } \beta(\alpha_s) = -\frac{\beta_0}{2\pi} \alpha_s^2 - \frac{\beta_1}{8\pi^2} \alpha_s^3 + \dots \quad (114)$$

To the one-loop level (only β_0 is taken into account) integration of this equation yields:

$$-\frac{1}{\alpha_s(K)} + \frac{1}{\alpha_s} = -\frac{\beta_0}{2\pi} \log\left(\frac{K}{\mu}\right) \quad (115)$$

where α_s is the coupling at a fixed scale μ , i.e. $\alpha_s = \alpha_s(\mu)$. Renormalization group improvement of 113 can be achieved by replacing α_s by $\alpha_s(K)$:

$$\Pi_1(Q/\mu, \alpha_s) = \int \frac{d^4k}{(2\pi)^4} G(k, Q) \left\{ \frac{1}{\alpha_s} - \frac{\beta_0}{2\pi} \log\left(\frac{K}{\mu}\right) \right\}^{-1} \quad (116)$$

The Borel improved function 110 can be expressed by $\Pi(Q/\mu, \alpha_s)$ through the integral along an imaginary line:

$$\bar{\Pi}_1(Q/\mu, b) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} d\left(\frac{1}{\alpha_s}\right) \exp\left(\frac{b}{\alpha_s}\right) \Pi(Q/\mu, \alpha_s) \quad (117)$$

where α is a real constant to the left of the singularities of $\Pi(Q/\mu, \alpha)$ in the $\frac{1}{\alpha}$ -plane. Inserting 116 into 117 the contour can be closed around the pole in the $\frac{1}{\alpha}$ -plane (Landau pole) and we obtain:

$$\tilde{\Pi}_1(Q/\mu, b) = \int \frac{K^3 dK d\Omega}{(2\pi)^4} G(K, Q) \exp\{-b \frac{\beta_0}{2\pi} \log(\frac{K}{\mu})\} \quad (118)$$

There is indeed a singularity in b due to small values of K . In order to separate it we may confine ourselves to values $K < \Delta$ in the integral. Then for $Q \gg \Delta$ one obtains easily by cutting the internal gluon lines in Fig. 6

$$G(K, Q) \approx \frac{-4\lambda C}{\pi Q^4} \quad (119)$$

and 118 yields, if we choose $\Delta = \mu$:

$$\tilde{\Pi}_1(Q/\mu, b) = -\frac{N_c}{2\pi^3} \left(\frac{\mu}{Q}\right)^4 \frac{1}{4 - \frac{30}{\pi}b} \quad (120)$$

We observe a singularity (here a pole) for positive real b at $b = \frac{8\pi}{A_0}$. Taking into account higher loop corrections to the β function does not influence the position of the singularity, but its nature. Two loop corrections lead to the singularity

$$(1 - \frac{3_0}{2\pi}b)^\zeta \text{ with } \zeta = -1 - \frac{23_1}{3_0^2} \quad (121)$$

If we introduce through $\tilde{\Pi}(Q/\mu_b) = (\frac{Q}{\mu_b})^4 \tilde{\Pi}(b)$ the Q -independent Borel amplitude $\tilde{\Pi}(b)$ we can recalculate from it a renormalization group improved scattering amplitude $\Pi(\alpha(Q))$ by:

$$\Pi(a(Q)) - \Pi(0) = \int_0^\infty e^{-b/a_* (Q)} \tilde{\Pi}(b) db \quad (122)$$

The singularity of $\Pi(b)$ at $b = 8\pi/\beta_0$ forbids the back-transformation 111 from the Borel function. If, unless we give a definite prescription how to circumvent it, e.g. by a principal value prescription. The arbitrariness in the prescription reflects itself in an arbitrary additional term proportional to $\exp[-\frac{8\pi}{\beta_0(Q)}] \sim (Q^2)^{-2}$, which in the QCD sum rule method (sect. 4.3) will be identified with a nonperturbative contribution to the Lorentz and gauge invariant operator product: $G_{\mu\nu}^F G_{\mu\nu}^F$ (see sect. 4.3).

$$D(Q^2) = \frac{8\pi^2}{N_c} Q^2 \frac{d}{dQ^2} \Pi_1(Q^2) \quad (123)$$

$$D(Q^2) = 1 - \frac{4}{\pi} P \int_0^\infty \frac{e^{-b/a_s(Q^2)}}{4 - \beta_0 b/(2\pi)} \quad (124)$$

calculated to two loops with a subsequent renormalization group improvement by replacing α_s by $\alpha_s(Q^2)$, i.e. $D(Q^2) = 1 + \alpha_s(Q^2)/\pi$ (solid line) and the result of a renormalization group improvement according to eq. 122 with a principal value prescription (dashed line) i.e.:

A more general treatment of the renormalons is based on a separation of the amplitude indicated in Fig. 8 : The full correlator is written as a sum of contributions which contain only internal gluon momenta $k' > \Delta$, (Fig. 8, shaded amplitude) which is connected by soft gluon lines to the lower (unshaded) amplitude for which no restrictions on internal momenta are made. By means of functional integration this separation can be made independent of perturbation theory. The part containing no soft internal gluon lines does not pose difficulties with the Landau pole. Using renormalization group improvement one obtains from this separation in a more general way the result 121.

4 Models for Non-Perturbative QCD

4.1 Quark models

Quark models are much older than QCD and since the introduction of quarks by Gell-Mann (1964) and Zweig (1964) (there called *aces*) a lot of hadronic properties could be derived, at least qualitatively, from properties of the quarks. I shall not dwell on the history of the quark model but start with the time when it became truly respectable, i.e. after the discovery of the J/ψ -meson in 1974.

At this time QCD was already quite established, and it was realized that a system of two heavy quarks (quarkonium) could be treated like the positronium in quantum electrodynamics. Indeed in a low-lying quarkonium state the mean distance between two quarks is proportional to $1/mq$ and for sufficiently heavy quarks the running coupling constant is small enough in order to neglect the gluon-gluon interaction. The interaction between the heavy quark and antiquark can thus be described by the well-known Breit-Fermi potential (see e.g. Schwinger 1973)

$$\begin{aligned}
V_{BF} = & \frac{4}{3}\alpha_s \left(-\frac{1}{r} + \frac{1}{2r^3} \left[\frac{\vec{L}_1 \cdot \vec{s}_1}{m_1^2} - \frac{\vec{L}_2 \cdot \vec{s}_2}{m_2^2} + \frac{2}{m_1 \cdot m_2} (\vec{L}_1 \cdot \vec{s}_2 - \vec{L}_2 \cdot \vec{s}_1) \right] \right. \\
& + \frac{8\pi}{3m_1 m_2} \delta^{(3)}(\vec{r}) \vec{s}_1 \cdot \vec{s}_2 + \frac{3}{m_1 m_2 r^5} \left[(\vec{s}_1 \cdot \vec{r})(\vec{s}_2 \cdot \vec{r}) - \frac{r^2}{3} \vec{s}_1 \cdot \vec{s}_2 \right] \\
& \left. + \frac{\pi}{2} \left(\frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \delta^{(3)}(\vec{r}) + \frac{1}{2m_1 m_2} \left[\vec{p}_1 \frac{1}{r} \vec{p}_2 + (\vec{p}_1 \vec{r}) \frac{1}{r^3} (\vec{r} \cdot \vec{p}_2) \right] \right) \quad (125)
\end{aligned}$$

Here $\vec{r} = \vec{x}_1 - \vec{x}_2$ is the distance between the two quarks, $\vec{p}$, the momentum operator acting on quark i ($i = 1, 2$), $\vec{L}_i$ is the angular momentum operator $\vec{L}_i = [\vec{r} \times \vec{p}]$, $\vec{s}_i$ the spin operator and m_i the mass of quark i .

Even for a small coupling α , one cannot obtain the spectrum of the quarkonium systems directly from perturbation theory and here comes the main asset of (non-relativistic) quark models into play:

there exists a time honoured non-perturbative method, namely the Schrödinger equation. Of course there are many relativistic extensions of the simple nonrelativistic quark model and I shall quote some results of these models in later sections. Here I shall only report on some characteristic features of the nonrelativistic model. One generally uses the solutions of the Schrödinger equation with the central potential (here $-4/3 \alpha_s/r$) as a starting point for perturbation theory in terms of $O(1/m^2)$. An attempt to solve the equation with the full potential (125) leads to inconsistencies due to the parts of the potential which are attractive and more singular than $1/r$.

The spectra obtained in this way agree qualitatively with the observed charmonium ($c\bar{c}$) and bottomonium ($b\bar{b}$) spectra (see Fig. 1c) and give insight in the structure of the states, but there are important quantitative points of disagreement. They are more severe for charmonium than for bottomonium systems. This indicates that the description of the interaction by the Breit-Fermi potential is not sufficient and genuine non-perturbative effects of QCD have to be taken into account. These effects can be treated in the field theoretical framework of QCD with the help of the Wilson loop. The Wegner-Wilson loop was introduced Wegner (1971) as a non-local gauge invariant order parameter in statistical mechanics and its important role in QCD was established by Wilson (1974). I have shortly introduced it on the lattice already in sect. 3.1 and 3.2. Since it is a crucial concept I also present it in the Euclidean continuum [Brown and Weisberger 1979]. In order to make clear that we are dealing with Euclidean space time I use again capital letters for the coordinates.

We have already introduced in sect 2.5 the gauge invariant non-local operator (see 46).

$$\bar{\psi}(X')\phi(X,X')\psi(X) \quad (126)$$

where $\psi(X)$ is a quark field and $\phi(X, X')$ the string along a straight line

$$\phi(X, X') = P \exp \{ i g \int_0^1 \mathbf{B}_\mu(X + \lambda(X' - X)) \cdot (X' - X)_\mu d\lambda \} \quad (127)$$

(see 40 and the following discussion). Applying the operator 126 to the vacuum state of hadrons we obtain a gauge and Lorentz invariant state consisting of a quark at position X' and an antiquark at X . Its evolution is given by the Greens function written as a functional integral:

$$G(X, X'; Y, Y') = \int DB D\psi D\bar{\psi} e^{-S_{QCD}(\psi, \bar{\psi})} \cdot \bar{\psi}(X')\phi(X, X')\psi(X)(\bar{\psi}(Y')\phi(Y, Y')\psi(Y'))^* \quad (128)$$

The QCD-action S_{QCD} is quadratic in the fermion fields (see 10)

$$S_{QCD}^F = \int \bar{\psi}(X) \{ i \gamma_\mu (\partial_\mu + i g_s \mathbf{B}_\mu) - m \} \psi(X) dX + S_{YM} \quad (129)$$

where S_{YM} is the pure gauge Lagrangian: $S_{YM} = \frac{1}{4} G_{\mu\nu}^F G_{\mu\nu}^F$. Therefore the fermion integration in (128) can be performed yielding

$$G(X, X'; Y, Y') = \int DB e^{-S_{YM}[B]} \text{Det}[B] \text{tr} \{ (S(X', Y'; B) \phi(Y' Y') S(Y, X; B)) \} \quad (130)$$

Here $S(X, Y; B)$ is the inverse of the operator between the fermion fields:

$$\{ i \gamma_\mu (\frac{\partial}{\partial X_\mu} + i g_s \mathbf{B}_\mu) - m \} \cdot S(X, Y; B) = \delta(X - Y). \quad (131)$$

It is the propagator of a quark in the external colour field $\mathbf{B}_\mu$. $\text{Det}[B]$ is the functional determinant coming from the fermion integration (see 100. It describes internal fermion loops if expanded in a power series of g_s . Since it is one in the approximation we are going to make I shall not discuss it further.

Using $\gamma_4 = i\gamma_0$ one verifies easily that (up to terms $1/m$) the following expression is a solution to (131):

$$S(X, Y; B) = \phi(X, Y) \delta(\vec{X} - \vec{Y}) \cdot \left\{ e^{-m(X_4 - Y_4) \Theta(X_4 - Y_4)} \frac{1 + \gamma_0}{2} + e^{-m(Y_4 - X_4) \Theta(Y_4 - X_4)} \frac{1 - \gamma_0}{2} \right\} + O(1/m). \quad (132)$$

To leading order in $1/m$ we thus have:

$$G((\vec{X}, 0), (\vec{X}', 0); (\vec{Y}, T), (\vec{Y}', T)) \sim \delta(\vec{X} - \vec{Y}) \delta(\vec{X}' - \vec{Y}') e^{-2mT} \cdot \int DB e^{-S_{YM}[B]} \text{tr} \{ \phi(Y, X) \phi(X, X') \phi(X', Y') \phi(Y', Y) \} \quad (133)$$

= $\delta(\vec{X} - \vec{Y}) \delta(\vec{X}' - \vec{Y}') e^{-2mT} \text{tr} \left\{ \int DB e^{-S_{YM}[B]} P \exp \{ i g_s \oint \mathbf{B}_\mu(Z) dZ_\mu \} \right\}$ where the contour in the last integral is the rectangle with the corners $\{(\vec{X}, 0), (\vec{X}', 0), (\vec{Y}', T), (\vec{Y}, T)\}$. We shall abbreviate the curly bracket in 133 by $W[L]$:

$$W[L] = \int DB e^{-S_{YM}[B]} (P \exp \{ i g_s \oint \mathbf{B}_\mu(Z) dZ_\mu \}) \quad (134)$$

It is the continuum limit of the Wegner-Wilson loop introduced on the lattice (see eq. 83). On the other hand we know that the Euclidean Greens function behaves like e^{-ET} where E is the energy of the system. Hence we obtain:

$$E = - \lim_{T \rightarrow \infty} \frac{1}{T} \log \text{tr} W + 2m \quad (135)$$

The second term is the rest energy of the two quarks, hence we can identify the first term with the static potential:

$$V(\vec{X} - \vec{X}') = - \lim_{T \rightarrow \infty} \frac{1}{T} \log \text{tr} W[L]. \quad (136)$$

(see 106 for the corresponding expression on the lattice).

The rectangular Wegner-Wilson loop has been evaluated in strong coupling expansion [Wilson 1974] and in lattice calculations and one finds (for large R and T) the famous area law:

$$\text{tr} W[L] = \sigma \cdot T \cdot |\vec{X} - \vec{X}'| \quad (137)$$

This area law leads to the linear potential with the "string tension" σ :

$$V(\vec{X} - \vec{X}') = \sigma |\vec{X} - \vec{X}'| \quad (138)$$

In a similar though technically much more cumbersome way the terms proportional $1/m^2$ (there are no $1/m$ terms) can be extracted [Eichten and Feinberg 1981, Lucha *et al.* 1991]. The resulting potential is of the general form:

$$V(r) = V_0(r) + \left[\frac{1}{m_1^2} \vec{L}_1 \cdot \vec{s}_1 - \frac{1}{m_2^2} \vec{L}_2 \cdot \vec{s}_2 \right] \left(\frac{V'_0(r)}{2r} + \frac{V'_1(r)}{r} \right) + \left[\frac{1}{m_1 \cdot m_2} \vec{L}_1 \cdot \vec{s}_2 - \frac{1}{m_1 m_2} \vec{L}_2 \cdot \vec{s}_1 \right] \frac{V'_2(r)}{r} + \frac{\vec{s}_1 \cdot \vec{s}_2}{3 m_1 m_2} V_4(r) + \frac{1}{m_1 m_2} \left[(\vec{r} \cdot \vec{s}_1)(\vec{r} \cdot \vec{s}_2) - \frac{r^2}{3} (\vec{s}_1 \cdot \vec{s}_2) \right] \cdot \frac{V'_3(r)}{r^2} \quad (139)$$

The potential V_0 is the static potential treated above (see 136), the functions V'_1, V'_2, V'_3 and V_4 can be expressed as expectation values of the colour electric and magnetic fields inserted in a Wegner-Wilson loop.

Lorentz invariance yields a relation between the central and the spin-orbit potentials, derived by Gromes (1984)

$$V_2 - V_1 = V_0 \quad (140)$$

For a Coulomb potential one has:

$$V_0 = -\frac{4}{3} \frac{\alpha_s}{r}; V_1 = 0; V_2 = V_0; V_3 = V'' = -\frac{1}{r} V'; V_4 = 2\Delta V. \quad (141)$$

For a further discussion of more exact relations as well as for the application to phenomenology I refer to the review of Lucha, Schöberl and Gromes (1991) and the literature quoted there.

Baryons are a challenge for potential models since here we expect genuine many-body forces to play an important role. We confine ourselves here to $N_c = 3$ where an interpolating field for a baryon contains (at least) three quark fields.

A very simple picture is the so-called diquark picture. If we introduce the nonlocal operator

$$\Gamma_{\alpha\beta} \psi_\alpha^A(X_1) \psi_\beta^B(X_2) \phi_{AA'}(X_1, Y) \phi_{BB'}(X_2, Y) \cdot \epsilon_{AB' C} \quad (142)$$

it behaves from the point of view of the local $SU(3)$ gauge symmetry as a local antiquark operator at position Y . The Lorentz-structure is determined by the choice of $\Gamma_{\alpha\beta}$. A gauge invariant nonlocal baryon field can be constructed from the nonlocal operator (142) and a quark field at position X_3 connected to Y by a string $\phi(X_3, Y)$ (see §5.0). The diquark picture consists in replacing the nonlocal operator (142) by a local one. In this picture the baryons can be viewed as bound states of a quark and a diquark and treated analogously to the mesons consisting of a quark and an antiquark. The diquark picture is in some cases astonishingly successful (see e.g. [Goebel *et al.* 1993]).

In a less radical approach one starts from the genuine multilocal operator 50, i.e. an operator

$$\psi(X_1, X_2, X_3, Y) = \psi^A(X_1) \psi^B(X_2) \psi^C(X_3) \phi_{AA'}(X_1, Y) \phi_{BB'}(X_2, Y) \phi_{CC'}(X_3, Y) \epsilon_{A'B'C'} \quad (143)$$

where I have omitted spinor indices, since they play no role in the static limit which we are treating now.

A treatment analogous to the one following equation 130 leads to the static Greens function:

$$\begin{aligned} G((\vec{X}_3, 0), (\vec{X}_2, 0), (\vec{X}_1, 0), (\vec{Y}, 0), (\vec{X}_2, T), (\vec{X}_3, T), (\vec{Y}, 0)) \\ \sim e^{-3\sigma T} \delta(\vec{X}_1 - \vec{Y}) \delta(\vec{X}_2 - \vec{X}_3) \\ \delta(\vec{X}_3 - \vec{X}_2) W_{AA'}[C_1] W_{BB'}[C_2] W_{CC'}[C_3] \cdot \epsilon_{ABC} \epsilon_{A'B'C'} \end{aligned} \quad (144)$$

where $W[C_i]$ is given by 134 with the rectangular closed loop with the corners $\{(\vec{X}_1, 0), (\vec{X}_1, T), (\vec{Y}, T), (\vec{Y}, 0)\}$ (see Fig. 9).

If the area law holds also for these loops the Greens function is proportional to $\exp(-\sigma T \sum_{i=1}^3 |\vec{X}_i - \vec{Y}|)$.

The choice of the point Y is arbitrary but it is clear that the leading contribution to the functional integral comes from a choice of Y where the exponent is minimal. We thus obtain in that case for the static three-quark potential:

$$V_3(\vec{X}_1, \vec{X}_2, \vec{X}_3) = \min_Y \sigma \cdot \sum_{i=1}^3 |\vec{X}_i - \vec{Y}|. \quad (145)$$

This genuine 3-body potential can be very well approximated by a sum of two-body potentials:

$$V_3(\vec{X}_1, \vec{X}_2, \vec{X}_3) \approx 0.54 \cdot \sigma \sum_{i < j} |\vec{X}_i - \vec{X}_j| \quad (146)$$

Figure 9: Loops for a baryon in $SU(3)$

(see Dosch and Müller (1976) for further details).

Though quark bound state models are best justified for heavy quarks they have been applied successfully also to hadrons consisting of light quarks. In that case the spontaneous symmetry breaking of chiral symmetry is simulated by a hard one, namely by the introduction of the so-called constituent quark mass which may be orders of magnitudes larger than the mass occurring in the QCD-Lagrangian. For the up and down quarks it ranges from 200 to 400 MeV, the strange quark is 120 to 150 MeV heavier.

Since a lot of fantasy enters the potentials of light quarks it is in most cases not necessary to solve the Schrödinger equation but rather work with a simple and easily manageable test function the parameters of which are either fitted to experiment or adjusted in order to minimize the energy in a given potential. For many applications a Gaussian test function with only one free parameter, the width parameter w is used for the radial part

$$\phi_{test}(r) = \frac{4w^{3/2}}{\sqrt{\pi}} e^{-wr^2}. \quad (147)$$

The principal domain of constituent quark models is hadron spectroscopy. The fact that strange hadrons are heavier is in the quark model a consequence of the mass difference of the strange and non-strange quark. Another important feature is the strong hyperfine splitting, which according to the Breit fermi potential is given by

$$\Delta E_{hf} = \frac{\pi}{2} |\phi(0)|^2 \left\{ \frac{1}{m_1^2} + \frac{1}{m_2^2} \right\}. \quad (148)$$

It is not only observed in heavy mesons, but also in light ones ($\rho - \pi$) and baryons ($\Delta - N$). The $\pi - \rho$ mass difference is however delicate, since the π is so light because it is the Goldstone-boson of chiral symmetry.

In the following we shall come across further applications, especially in connection with electromagnetic and semileptonic decay form factors. Here I mention only one further application, namely that to Joe nergy hadron-hadron scattering.

The general potential 139 is valid only for quarks inside a hadron and local gauge invariance tells us that it is absolutely meaningless for quarks which belong to different colour singlets. Indeed we expect the interaction of hadrons to be dominated by the exchange of hadrons. At short distances however simple quark exchange or quark plus gluon exchange between the hadrons may lead to observable effects. Lieberman (1977) has explained the repulsive core in that way: If all 6 quarks of a nucleon-nucleon intermediate state strongly overlap the hyperfine interaction leads to a net repulsion. This has been verified in more sophisticated investigations [Bender and Dosch (1982), Faessler *et al.* (1982)]. Hadron exchanges are, however, very important for nucleon-nucleon scattering and have to be taken into account in order to get realistic phase shifts [Bräuer *et al.* (1985) and earlier literature quoted there].

The treatment of scattering of K^+ -mesons on nucleons has several methodological advantages over nucleon-nucleon scattering: (I) One-pion exchange is not possible, (II) the two-pion exchange is sup-

Figure 10: Theoretical evaluation [Bender et al. (1984)] of the $K^+ N$ S -wave phase shift for isospin $I = 0$ and $I = 1$ and experimental data, dots for $I = 0$ and crosses for $I = 1$ from Martin and Oades (1980)

pressed as compared to $N - N$ scattering. Therefore the simple quark and gluon exchange mechanism can be expected to explain already the main features of $K^+ - N$ scattering. This is indeed the case. The isospin $I = 1$ channel already receives important contributions from simple quark exchange whereas for the $I = 0$ channel the quark exchange must be accompanied by a gluon exchange Fig.10 shows the results of a theoretical evaluation of the S -wave phase shift of $K^+ - N$ scattering for two typical sets of parameters taken from hadron spectroscopy [Bender et al. (1984)].

4.2 Instantons

This section contains a very short overview on instantons and their applications. I shall start with a trivial exercise. Instead of a complicated functional integral we shall first consider the simple Riemann-integral:

$$I = \int_{-\infty}^{+\infty} F(x) e^{-\lambda(x)} dx \quad (149)$$

where $\lambda(x)$ is some positive definite function. If $\lambda(x)$ has a minimum at the position x_0 we may approximate it by

$$\lambda(x) = \lambda_0 + \frac{\lambda_2}{2}(x - x_0)^2 + \dots \quad \text{with } \lambda_0 = \lambda(x_0), \lambda_2 = \lambda''(x_0), \quad (150)$$

and obtain for I the approximate value

$$I \approx F(x_0) e^{-\lambda_0} \sqrt{\frac{2\pi}{\lambda_2}}. \quad (151)$$

If the function $\lambda(x)$ has several minima at positions $x_0^{(i)}$ an approximation for I is given by:

$$I \approx \sum_i F(x_0^{(i)}) e^{-\lambda_0^{(i)}} \sqrt{\frac{2\pi}{\lambda_2^{(i)}}} \quad \text{with } \lambda_0^{(i)} = \lambda(x_0^{(i)}), \lambda_2^{(i)} = \lambda''(x_0^{(i)}) \quad (152)$$

A similar situation may occur in the evaluation of a functional integral. If the action $S[\Phi]$ has a minimum for a field $\Phi_0(x)$ then this field gives a "classical" contribution to the functional integral analogous to 151:

$$\int F[\Phi] e^{-S[\Phi]} D\Phi \approx \text{const } F[\Phi_0] e^{-S[\Phi_0]} \quad (153)$$

Corrections going beyond the contribution 153 are quantum corrections. If the field $\Phi_0(x)$ leads to a minimum of the action $S[\Phi]$ it is a solution of the Euler-Lagrange equations for that action. Hence solutions of the classical equations of motion lead to classical contributions to the functional integral of the form 153. There exist such nontrivial classical solutions of non-Abelian gauge theories in a Euclidean space time continuum with finite action [Polyakov 1975, Belavin et al. 1975], the so called instantons. They are primarily solutions of a Yang-Mills theory based on the gauge group $SU(2)$, but since this group is a subgroup of any $SU(N)$ with $N \geq 2$ the solutions can be embedded in any $SU(N)$ gauge theory. Such a solution depends on an arbitrary point Z in Euclidean space time and a scale ρ :

$$G_{\mu\nu}^{clF} = -\frac{4}{g_s} \eta_{F\mu\nu} \frac{\rho^2}{(X - Z)^2 + \rho^2} \quad (154)$$

with the η -coefficients [t Hooft 1976]:

$$\eta_{F\mu\nu} = \epsilon_{F\mu\nu} \quad \text{for } \mu, \nu = 1, 2, 3; \eta_{F4\mu} = -\delta_{F\mu}; \eta_{F44} = 0. \quad (155)$$

ϵ_{ijk} is the totally antisymmetric tensor in 3 dimensions, note that the index F runs only from 1 to 3 since we are dealing with the (subgroup) $SU(2)$.

If $\eta_{F\mu\nu}$ is replaced by $\tilde{\eta}_{F\mu\nu}$ with

$$\tilde{\eta}_{F\mu\nu} = (-1)^{\delta_{\mu 4} + \delta_{\nu 4}} \eta_{F\mu\nu} \quad (156)$$

one obtains the so called anti-instanton solution. In an *Euclidean* space-time continuum one might think of these solutions as particles with size ρ and position Z . In the Minkowski space time continuum the instantons correspond not to particles but can be interpreted as contributions to the tunneling between different vacua. The action for the solution 154 is easily obtained to be:

$$S[G^{clF}] = \frac{-1}{4} \int G_{\mu\nu}^{clF} G_{\mu\nu}^{clF} d^4X = \frac{8\pi^2}{g_s}. \quad (157)$$

The instanton fields are self-dual, i.e.

$$\tilde{G}_{\mu\nu}^{clF} := \epsilon_{\mu\nu\alpha\beta} G_{\alpha\beta}^{clF} = G_{\mu\nu}^{clF}. \quad (158)$$

The topological charge is defined as

$$Q_I := \int \frac{g_s^2}{32\pi^2} G_{\mu\nu}^{clF} \tilde{G}_{\mu\nu}^{clF} d^4X. \quad (159)$$

An instanton has the topological charge +1, an anti-instanton -1, as can easily be calculated.

The contribution of a single instanton to the vacuum expectation value of the functional $F[G]$ of the field tensor $G_{\mu\nu}^{clF}$ is given according to 153 by

$$F[G^{clF}] \exp \left((-S[G^{clF}]) \right) = F[G^{clF}] \exp \left(-\frac{8\pi^2}{g_s} \right). \quad (160)$$

From 160 and 157 it is evident that the instantons give genuine nonperturbative contributions, since the exponential $\exp[-\frac{8\pi^2}{g_s^2}]$ cannot be expanded in a convergent power series of g_s and its asymptotic expansion in g_s is identical zero.

The superposition of two instanton solutions will not be a solution of the Euler-Lagrange equations, due to the nonlinearity of these equations for a non-Abelian gauge theory. If we consider to solutions with positions Z far apart, the superposition should however be a good approximation for the two

instanton solutions (with topological charge 2). In this approximation one gets the full contribution (corresponding to 152) by an independent integration over all instanton positions Z and sizes ρ . This is the so called dilute instanton gas approximation. Technically it is achieved by introducing the dimensionless measures d^4Z/ρ^4 and $d\rho/\rho$ (note that ρ is the only available scale). There must be an upper cutoff ρ_c in order to ensure that the instantons do not overlap and thus make the dilute gas approximation obsolete. In order to obtain the equivalent of the terms $\sqrt{2\pi}/\lambda_2$ in 152 a elaborate calculation is necessary [t Hooft 1976, Callen *et al.* 1978]. One finally obtains for $SU(3)$:

$$<F[G]> = 2 \int d^4Z d\rho \frac{1}{\rho^5} F[G^d] d_I(\rho) \quad (161)$$

with the instanton density function

$$d_I(\rho) = C \left\{ \frac{2\pi}{\alpha_s(1/\rho)} \right\}^6 \exp \left[-\frac{2\pi}{\alpha_s(1/\rho)} \right] \quad \text{with} \quad C \approx 0.06 \quad (162)$$

where we have replaced the coupling g_s in 160,157 by the running coupling given by 30:

$$\frac{2\pi}{\alpha_s(1/\rho)} = \frac{8\pi^2}{g_s^2(\rho)} = \frac{2\pi}{\alpha_s(1/\rho_c)} + \beta_0 \log\left(\frac{\rho_c}{\rho}\right). \quad (163)$$

For the pure gauge $SU(3)$ theory we are dealing with we have $\beta_0 = 11$.

As a small exercise for handling instantons in the dilute instanton gas approximation we shall now estimate the mean distance between instantons and the cutoff size ρ_c [Shifman *et al.* 1978]. It will turn out that the dilute gas approximation is inconsistent. The estimates are based on the so called gluon condensate. This is the nonperturbative contribution to the vacuum expectation value of two gluon fields, it plays an essential role in the sum rule method (see sect. 4.3).

$$<g^2 GG> := <g_s^2 G_{\mu\nu}^F(0) G_{\mu\nu}^F(0)>_{non-pert.} \quad (164)$$

Its numerical value in a pure gauge theory is $<g^2 GG> \approx 1 - 2 \text{ GeV}^4$ (see App. 2).

We first calculate the instanton and anti-instanton contribution to the gluon condensate assuming that the part ζ is generated by instantons and anti-instantons. Using 161, 162, and 154 we obtain:

$$\begin{aligned} \zeta <g^2 GG> &= \int g_s^2 G_{\mu\nu}^d F_{\mu\nu} G_{\mu\nu}^d d_I(\rho) \frac{1}{\rho^5} d^4Z d\rho \\ &= 2 \cdot 16\pi^2 \eta_F \eta_{F\mu\nu} C \int \frac{\rho^4}{[Z^2 + \rho^2]^4} \left\{ \frac{2\pi}{\alpha_s(1/\rho)} \right\}^6 \exp \left[-\frac{2\pi}{\alpha_s(1/\rho)} \right] \frac{1}{\rho^5} \Theta(\rho_c - \rho) d^4Z d\rho \\ &= C \frac{64\pi^2}{3} \pi^2 \exp \left[-\frac{2\pi}{\alpha_s(1/\rho_c)} \right] \rho_c^{-11} \int_0^{\rho_c} \rho^6 \left\{ \frac{2\pi}{\alpha_s(1/\rho)} \right\}^6 + \beta_0 \log\left(\frac{\rho_c}{\rho}\right) \rho^6 d\rho \\ &= C \frac{64\pi^2}{21\rho_c^4} \left\{ \frac{2\pi}{\alpha_s(1/\rho_c)} \right\}^6 \exp \left[-\frac{2\pi}{\alpha_s(1/\rho_c)} \right] \left(\sum_{k=0}^6 \frac{6!}{(6-k)!} \left(\frac{11\alpha_s(1/\rho_c)}{14\pi} \right)^k \right) \end{aligned} \quad (165)$$

where we have used 163, $\eta_F \eta_{F\mu\nu} \eta_{F\mu\nu} = 4$ and performed the integration over the position Z and the size ρ .

In Fig. 11 the result for $\rho_c(\zeta <g^2 GG>)^{1/4}$ is displayed as a function of $\alpha_s(1/\rho_c)$. The solid line is obtained from the full expression 166, for the dashed line only the term of the sum with $k=0$ has been taken into account. A plausible value for the strong coupling at the instanton cutoff ρ_c is $\alpha_s(1/\rho_c) \approx 1$, i.e. the product $\rho_c(\zeta <g^2 GG>)^{1/4}$ comes out to be larger than 4. We next estimate the average distance d_I between two instantons. From the topological charge 159 and the self-duality 158 we obtain for n dilute instantons in a volume V which is large as compared to the instanton size:

$$n_I = \int_V \frac{g_s^2}{32\pi^2} G_{\mu\nu}^d F_{\mu\nu} G_{\mu\nu}^d F_{\mu\nu} d^4X = \frac{g_s^2}{32\pi^2} G_{\mu\nu}^d F_{\mu\nu} G_{\mu\nu}^d F_{\mu\nu} \cdot V \quad (167)$$

Figure 11: $\rho_c(\zeta <g^2 GG>)^{1/4}$ as a function of the strong coupling at the instanton cut-off, $\alpha_s(1/\rho_c)$. For details see text

where we have introduced the averaged instanton field $G_{\mu\nu}^d F_{\mu\nu}$. If the fraction ζ of the gluon condensate is generated by instantons, we obtain the relation:

$$n_I/V = \frac{1}{32\pi^2 \zeta} <g_s^2 GG> \quad \text{or} \quad d_I(\zeta <g^2 GG>)^{1/4} = (32\pi^2)^{1/4} \approx 4.22, \quad (168)$$

i.e. the ratio of the instanton size ρ_c and the instanton distance d_I is by no means small as compared to one, and the dilute gas approximation seems indeed inconsistent.

Therefore a more promising picture is that of an instanton liquid [shuryak 1982, Diakonov and Petrov 1984, Shuryak 1993], and the literature quoted there, see also [Reinhardt 1993]. In this model the strong interaction of the (anti-)instantons between themselves are taken into account. I shall not dwell on this point in this review and refer to the quoted literature.

As has been noticed by t Hooft (1976) there are zero mode solutions for the Dirac equation in an instanton field, i.e. there are Dirac fields $\psi_0(x)$ which satisfy:

$$\gamma D_{ab}^d \psi_0^b = 0 \quad \text{where} \quad D_{ab}^d = i\partial_\mu \cdot \delta_{ab} + g_s B_{\mu\nu}^d F_{\mu\nu}^d \sigma_{ab}/2 \quad (169)$$

In a gauge where the potential has the form:

$$B_{\mu\nu}^d = \frac{2}{g_s} \eta_{F\mu\nu} \frac{(X-Z)_\nu}{X-Z)^2 + \rho^2} \quad (170)$$

the zero mode solution is given in the chiral basis of the γ -matrices (i.e. where γ_3 is diagonal) by:

$$\psi_0^b(x) = \frac{\rho}{\pi} \frac{1}{[(X-Z)^2 + \rho^2]^{3/2}} \phi^b \quad (171)$$

with the spinor $(\phi^b)^T = (0, 0, \epsilon_{ab}, \epsilon b/2)/\sqrt{2}$ where ϵ_{ab} is the antisymmetric tensor in two dimensions. The zero modes in an instanton field are left-handed, those in an anti-instanton field right handed.

If the fermion integration in the functional integral is performed, these zero modes make the instanton density proportional to the mass of the quarks and hence the latter vanishes for massless quarks. There is however another effect. The zero modes induce an effective of the form [t Hooft 1976]:

$$L_{eff} \sim (\bar{\psi} \psi_0)(\bar{\psi}_0 \psi) \quad (172)$$

where η is the field of a light quark and $\psi_0(X)$ the zero mode Dirac spinors (see 171). This leads to an effective coupling of right- to left-handed chiral spinors and hence to an effective quark mass which restores the instanton density to a finite value. So in addition to the problem of the interaction of the instanton among themselves one has another uncertainty due to the interaction with the quarks. We shortly come back to the effective Lagrangian 172 in connection with the sum rules for the pseudoscalar current and for baryons.

4.3 QCD Sum Rules

4.3.1 General Method and Two Point Functions

One of the most important tools in phenomenological non-perturbative QCD is the method of "QCD sum rules" invented in 1979 by M. Shifman, A. Vainshtein and V. Zakharov (1979a). Its basic concepts are well rooted in quantum field theory and it is a fully relativistic approach (in these two points it is superior to potential models) and very often it is quite simple and transparent in its application. On the other hand it is of limited accuracy and even a lot of analytical work does normally not allow one to improve precision beyond a certain point (depending on the process typically 10 to 30 %). Furthermore it is not a machinery which, once the assumptions are made, yields unique results, but it is rather based on self-consistency. So confinement is not a result of the treatment by QCD sum rules, but it has to be put into it. Apart from more epistemological problems this leads to a psychological disadvantage: groups using the same type of sum rules come sometimes to quite different conclusions. This makes the discussions among people working in the field very lively and not free of polemics which is sometimes used as an argument against the method.

A further limitation of the method stems from the fact that it evaluates Greens functions in the Euclidean region from which the observable quantities have to be extracted by extrapolation. Given the approximative nature of the method one can get only information on the ground state of a system with given quantum numbers and furthermore the more fields there are in the Greens function the more cumbersome and uncertain is the extraction of observables.

The fundamental concept of the method of QCD sum rules is the operator product expansion [Wilson 1969, 1971]. The basic idea is the separation of perturbative and non-perturbative effects by this method: The former are calculated, the latter parametrized by (a few) vacuum expectation values of composite operators. As was mentioned in section 2.6 on the operator product expansion the high frequency internal contributions (short distance effects) to an operator product are taken into account by the Wilson coefficients $C_n(x)$ (which depend on the process to be considered) whereas the contributions of low frequencies (long distance effects) are contained in the renormalized operators. These operators are universal, i.e. independent of the special product considered. As already mentioned in sect. 2.6 there was a lively discussion on the feasibility of such a procedure [David 1982, David 1984] but Novikov, Shifman, Vainshtein and Zakharov (1984a) have argued convincingly that for QCD the method is very well practically applicable. Further support has been given by a detailed discussion of the perturbative series by Mueller [Mueller 1985, Mueller 1993], see sect. 3.1. The method has been first applied to two-point functions of heavy mesons (quarkonia) and ordinary mesons [Shifman *et al.* 1979b, Shifman *et al.* 1979c]. In the discussion of charmonium it was especially successful, since it showed that the state at that time considered to be the singlet- S state of the charmed quark and its antiquark (η_c) could not be this one, since its mass was much lower than the one predicted by QCD sum rules. The state has indeed disappeared later and a new one showed up at the right mass. The method has been soon applied also to baryons [Chung *et al.* 1981, Ioffe 1981] and in the last time there has been a considerable activity to investigate systems composed of one heavy and one light quark. Since the method is a practical phenomenological one, I shall show its main features by discussing in some detail a specific example of the last mentioned kind, namely the two-point function of the pseudoscalar operator composed of a heavy (here beauty) and light (here up) quark. Although it might seem somewhat unfamiliar to you, I shall work again in an *Euclidean* space time continuum, therefore the coordinates are denoted by capitals.

Let us denote the spinor field of a b and u quark of colour C by $b_C^a(X)$ and $u_C^a(X)$, respectively (a is the Dirac-spinor index). The local field we are investigating is the composite field operator $\phi_B(X)$

$$\phi_B(X) := \bar{b}^C(X) \gamma_5 u^C(X). \quad (173)$$

This composite field has the quantum numbers of a positive pseudoscalar state with open beauty, it hence interpolates the B^+ -meson, but it couples also to all states with the same quantum numbers. be

they resonant or not. We denote by $\Pi_B^s(X^2)$ its two-point function

$$\Pi_B^s(X^2) = \langle \Omega | \phi_B(X) \phi_B^*(0) | \Omega \rangle. \quad (174)$$

and by $\tilde{\Pi}_B^s(P^2)$ its Fourier transform

$$\tilde{\Pi}_B^s(P^2) := \int d^4 X e^{-iP \cdot X} \Pi_B^s(X^2). \quad (175)$$

Insertion of intermediate states leads to the dispersion relation:

$$\tilde{\Pi}_B^s(P^2) = \frac{|\langle B | \phi_B(0) | \Omega \rangle|^2}{P^2 + m_B^2} - \frac{(P^2)^2}{\pi} \int_0^\infty \frac{\sigma_B^{s, \text{phen}}(s)}{s^2(s + P^2)} + \text{Polynomial in } P^2. \quad (176)$$

The dispersion integral represents the higher resonances and non-resonant contributions; we have introduced two subtractions.

The B -meson vacuum coupling $\langle B | \phi_B | \Omega \rangle$ is proportional to the leptonic decay constant of the B -meson which is defined through

$$\langle B | \bar{b}^C(X) \gamma^\mu \gamma_5 u^C(X) | \Omega \rangle =: i P_\mu f_B. \quad (177)$$

From this definition follows by taking the divergence

$$m_b < B | \phi_B | \Omega \rangle = m_B^2 f_B, \quad (178)$$

where m_B is the mass of the B -meson and m_b that of the b -quark.

If we knew $\Pi_B^s(P^2)$ near the pole at $P^2 = -m_B^2$ well enough, we could determine from that knowledge the mass of the B -meson (position of the pole) and the decay constant f_B (from the residue).

The evaluation of the two-point function Π_B^s by the sum rule technique proceeds in principle through the operator expansion, i.e. one expands the operator product $\phi_B(X) \phi_B^*(0)$ into a sum of local gauge and Lorentz invariant operators O_n , to wit

$$\phi_B(X) \phi_B^*(0) = \sum_n C_n^{(b)}(X) O_n. \quad (179)$$

The c -number coefficients $C_n^{(b)}(X)$ (Wilson coefficients) are distributions in X . Since we shall later analyze mostly the Fourier-transformed two-point function let us make also the dimensional considerations for that case, i.e. consider

$$\tilde{\Pi}_B^s(P^2) = \sum_n \tilde{C}_n^{(b)}(P^2) < \Omega | O_n | \Omega \rangle \quad (180)$$

Since 179 is a short distance expansion (X^2 has to be small), the Fourier-transform 180 is suited for large P^2 .

The canonical dimension (see App. 1) of the correlator $\Pi_B^s(P^2)$ is $D = 6 - 4 = 2$ (4 fermion fields and 1 Fourier-transform), hence the dimension of the Wilson-coefficient $\tilde{C}_n(P^2)$ is $D_n = 2 - D[O_n]$, where $D[O_n]$ is the dimension of the operator O_n . Hence for $P^2 \rightarrow \infty$ the Wilson coefficient must go like $(1/P^2)^{D_n/2}$. This behaviour will be modified in general by logarithmic terms in an asymptotically free theory like QCD.

Other operators must be composed of quark and/or gluon fields and hence have a dimension larger than one; Lorentz invariance, which allows only powers of P^2 , tells us, that only operators with even canonical dimension will contribute to 180. Thus in the deep Euclidean region ($P^2 \rightarrow \infty$) the unit

operator is dominant, but if we want to approach the physical region we need more and more power corrections of type $(\bar{p}\lambda)^{-D_n/2} < O_n >$ the closer we want to approach it. The general strategy of the sum rule technique is to include only a few power corrections and hope in that way to penetrate into the not so deep Euclidean region (typically about 5 to 1 GeV² below the lowest lying threshold) and thus get information about the lowest lying pole (here about m_B and f_B) by consistency: i.e. one assumes that there is a pole and fits its parameters by matching this ansatz to 180 in the not so deep Euclidean region. This makes the assertion made above clearer, that by the sum rule method one cannot predict the existence of poles (as potential models do, once the potential is given) but only (hopefully) determine the parameters once the existence of a pole is assumed.

The classical way to determine the Wilson coefficients is the following: We insert both sides of 179 into a set of Greens functions, evaluate these by integrating out the rapidly fluctuating fields occurring in the functional integral and then compare the coefficients.

Consider e.g.

$$< \Omega | \phi_B(X) \phi_B^*(0) | \Omega >_{HF} = \sum_n C_n^{(3)}(X) < \Omega | O_n | \Omega >_{HF}, \quad (181)$$

where the index HF means functional integration over the high frequency components of the fields only. Then on the r.h.s. only the unit operator $O_1=1$ contributes since by definition the operators O_n are supposed only to absorb the low frequencies, whereas the high frequency contributions are absorbed in the Wilson coefficients. In a somewhat sloppy but convenient way we can express this as

$$< \Omega | O_n | \Omega >_{HF} = < 0 | O_n | 0 >, \quad (182)$$

where $|0 >$ is the "perturbative vacuum state". The restrictions to high frequencies is essential if infrared singularities appear on the left-hand side; for infrared regular expressions we can practically forget the restriction.

From that we conclude that to lowest order in perturbation theory we have

$$C_1(X^2) = < \Omega | \bar{u}^C(X) \gamma^5 u^C(X) \bar{u}^D(0) \gamma^5 u^D(0) | \Omega >_{HF} = N_c \text{Tr} \{ S^u(X) \gamma^5 S^d(-X) \gamma^5 \}, \quad (183)$$

where S^u and S^d are the free propagators of an u - and d -quark, respectively. No infrared problems occur and we may skip the restriction to high frequencies, but we should not forget that this restriction is the basis for the application of perturbation theory.

In order to determine another Wilson coefficient, namely that of the operator $\bar{u}^C(0) u^C(0)$ we consider the Greens function

$$< \Omega | \bar{u}_a^D(V) \Phi_B(X) \Phi_B^*(0) u_a^D(W) | \Omega >_{HF} \quad (184)$$

Direct application of perturbation theory, i.e. here contraction of all quark fields, yields:

$$< \Omega | \bar{u}_a^D(V) \Phi_B(X) \Phi_B^*(0) u_a^D(W) | \Omega >_{HF} = S_B^D(W) \{ \gamma^5 S^d(-X) \gamma^5 \}_{ab} S_a^D(X - V). \quad (185)$$

Inserting for the product $\Phi_B(X) \Phi_B^*(0)$ the expansion 179 we obtain from the operator product $C_{aa}(X) \bar{u}_a^C u_a^C$ occurring on the left hand side of 179:

$$\begin{aligned} < \Omega | \bar{u}_a^D(V) \Phi_B(X) \Phi_B^*(0) u_a^D(W) | \Omega >_{HF} &= < \Omega | \bar{u}_a^D(V) C_{aa}(X) \bar{u}_a^C u_a^C \delta_a u_a^D(W) | \Omega >_{HF} \\ &= -C_{aa}(X) \delta_a S_B^D(W) S_a^D(-V) N_c. \end{aligned} \quad (186)$$

In order to determine $C_{aa}(X)$ from a comparison of 186 with 185 we have to project out from $(\gamma^5 S^d(-X) \gamma^5)_{ab}$ the term proportional to δ_{ab} and expand $S_a^D(X - V) = S_a^D(-V) + O(X)$ in order to isolate the most singular part in 185.

Figure 12: Graphs relevant for the operator product expansion of the two point function of a heavy light meson. Thick lines represent heavy quarks, thin ones light quarks, small wavy lines gluons and large wavy ones hadronic currents, open gluon or quark lines indicate condensation.

In that way we obtain:

$$C_{aa}(X) = -\frac{1}{4} \text{Tr} \{ \gamma^5 S^d(-X) \gamma^5 \} \quad (187)$$

which is proportional to m_u .

One can in principle in this way proceed and calculate all Wilson coefficients to any order in perturbation theory. But like so often, the devil inhibits the detail: Mixing of operators, conditions posed by equations of motion, seemingly appearing infrared singularities make the determination of Wilson coefficients a complicated and tedious business with many traps to fall in. Fortunately, good reason here helps laziness: The number of operators increases considerably with their dimension (there are 45 types of 4-quark operators e.g.) and since the vacuum expectation values of these operators are in principle all free parameters the evaluation of too many Wilson coefficients is of no phenomenological use. The number of parameters soon exceeds the number of quantities to be determined.

So if sum rules do not work with a few terms in the operator product expansion theoretical effort hardly improves the situation. It might however sometimes explain why they did not work if the Wilson coefficient of a complicated operator or higher order perturbative corrections are particularly large.

After these general warnings let me proceed in the investigation of our two-point function Π_B^2 .

For practical purposes it is more advisable to evaluate the Wilson coefficients with the help of graphs. To that end one draws all possible diagrams for the Greens function under consideration where also "open internal" lines are permitted (i.e. lines which literally end in the nothing). These open lines are translated into formulae as the corresponding field operators. I illustrate this procedure in Fig. 12

The large wavy lines are the operators ϕ_B , thick lines correspond to b -quarks, thin lines to u -quarks and small wavy lines to gluons. If you are missing some graphs (and you should) wait until the end of this discussion where I shall give reasons for the omission.

It is advisable to work first in X -space. Let me give as examples the graphs Fig. 12a), e) and m)

$$\begin{aligned} F'' &= -N_c \text{Tr} \{ S^u(X) \gamma^5 S^d(-X) \gamma^5 \} \\ F' &= \bar{u}_a^C(0) \{ \gamma^5 S^d(-X) \gamma^5 \}_{ab} u_a^D(X) \delta^{CD} \\ F''' &= \int dZ \bar{u}^C(0) \{ \gamma^5 S^d(-Z) \gamma^5 u_a A_a^C(Z) S^d(Z - X) \}_{ab} u_a^D(X) \end{aligned} \quad (188)$$

Now we take the vacuum expectation value and group a) and e) together:

$$< \Omega | F'' + F' | \Omega > = \{ \gamma^5 S^d(-X) \gamma^5 \}_{ab} \delta^{CD} S_a^D(X) \delta^{CD} - < \Omega | \bar{u}_a^C(0) u_a^D(X) | \Omega > \quad (189)$$

The factor in the square bracket can be interpreted easily: The first part, $S_a^D(X)$ is a short distance contribution to the quark propagator $< \Omega | u_a(0) \bar{u}_a(x) | \Omega >$ and the second part is a long distance (i.e.

It is easy to show that this gauge is very convenient in order to express the potential by the fixed tensor:

$$A_\mu^F(X) = \int_0^1 \lambda d\lambda X_\rho G_{\mu\rho}^F(\lambda X) = \frac{1}{2} X_\rho G_{\mu\rho}^F(0) + O(X^2). \quad (196)$$

Inserting 196 into 188 we obtain

$$F^\mu = -\frac{1}{2} \int d^4 Z \bar{u}_\alpha^C(0) [i\gamma^5 S^\delta(-Z) g_\delta \gamma_\mu Z_\rho G_{\mu\rho}^{CD}(0) S^\delta(Z-X) i\gamma^5]_{\alpha\delta} u_\rho^D(0), \quad (197)$$

where we have already replaced $u_\rho^D(x)$ by its value at 0 and used only the term proportional to X in 196.

Taking the vacuum expectation value we encounter the new condensate:

$$< \Omega | \bar{u}_\alpha^C(0) G_{\mu\rho}^{CD} u_\rho^D(0) | \Omega >. \quad (198)$$

Again using the invariance properties of the vacuum we must conclude that this expression transforms as a Dirac tensor in the indices α and δ and as an antisymmetric Lorentz tensor in μ and ρ . The only tensor of that kind we have to our disposal is $\sigma_{\mu\rho} := \frac{i}{2}(\gamma_\mu \gamma_\rho - \gamma_\rho \gamma_\mu)$ and hence we have:

$$< \Omega | \bar{u}_\alpha^C(0) G_{\mu\rho}^{CD} u_\rho^D(0) | \Omega > = \frac{1}{48} (\sigma_{\mu\rho})_{\alpha\delta} < \Omega | \bar{u}_\beta^C(0) (\sigma_{\mu\rho})_{\beta\gamma} G_{\nu\sigma}^{CD} u_\sigma^D(0) | \Omega >. \quad (199)$$

But graph m) is not the only one containing the operator $\bar{u}\sigma_{\mu\rho}G_{\mu\rho}u$, also the graph e) leads to it. In order to see that, we have to investigate the Taylor expansion 193 together with the equation of motion. The latter is for a massless quark:

$$i\gamma_\mu D_\mu u(X) = (i\gamma_\mu \partial_\mu D^{CD} + g_\mu \gamma_\mu B^{CD}) u^D(X) = 0 \quad (200)$$

In the chosen gauge 195 $B_\mu(0) = 0$ and therefore we have $D_\mu[0] = \partial_\mu$ and hence we may replace in the Taylor expansion 190 the usual derivative by the covariant one. From Lorentz covariance follows that $< \Omega | \bar{u}_\alpha^C(0) \partial_\mu u_\rho^D(0) | \Omega >$ must be proportional to $(\gamma_\mu)_{\alpha\delta}$ (we have no other Lorentz vector at our disposal). Contracting $< \Omega | \bar{u}_\alpha^C(0) \partial_\mu u_\rho^D(0) | \Omega >$ with γ_μ and using the equation of motion 200 we deduce:

$$< \Omega | \bar{u}_\alpha^C(0) \partial_\mu u_\rho^D(0) | \Omega > = 0 \quad (201)$$

The quadratic term in the Taylor expansion 190, however, leads to an operator of type 198. From $(\gamma_\mu D_\mu)(\gamma_\mu D_\mu)u = 0$ we deduce that in our gauge, expanding around $X = 0$, one obtains

$$\partial_\mu \partial_\mu u^C = -\frac{i}{2} \sigma_{\mu\nu} G_{\mu\nu}^{CD} u^D(0) \quad (202)$$

Hence the second term in the Taylor expansion 190 inserted in 189 yields the additional term

$$-\frac{i}{2} [\gamma^5 S^\delta(-X) \gamma^5]_{\alpha\delta} \delta^{CD} X^2 < \Omega | \bar{u}\sigma_{\mu\nu} G_{\mu\nu} u | \Omega > \delta^{CD}. \quad (203)$$

We denote the vacuum expectation value of the renormalized product of quark and gluon fields mixed condensate:

$$< \bar{u} g \sigma G u > := < \Omega | \{ \bar{u}^C \sigma_{\mu\nu} g_\delta G_{\mu\nu}^{CD} u^D \} | \Omega >. \quad (204)$$

low frequency) contribution. It is for that reason that I have written it with the curly brackets which indicate the extraction of the high frequency (i.e. singular short distance) part. Intuitively speaking a non-vanishing value of $< \Omega | \{ \bar{u}_\alpha^C(x) u_\rho^D(0) \} | \Omega >$ indicates that the physical vacuum contains low frequency virtual quark antiquark pairs and a quark can either travel directly or it may dive into the physical vacuum and show up at a possibly distant place again. Since the light quarks are really light, virtual pairs might indeed live long enough in order to travel a long distance. This picture tells us that we do not expect a similar situation for the heavy quarks, their high mass makes low frequency virtual pairs live much too short in order that such an effect can take place. The propagation of quarks is also disturbed by low frequency gluons as we shall see later.

Since the low frequency part is by definition regular for $X = 0$ we may use a Taylor expansion for it (which of course would be completely senseless for the full product because of the short distance singularity).

Hence we have

$$< \Omega | \{ \bar{u}_\alpha^C(0) u_\rho^D(X) \} | \Omega > = < \Omega | \{ \bar{u}_\alpha^C(0) u_\rho^D(0) \} | \Omega > + X^\mu < \Omega | \bar{u}_\alpha^C(0) \partial_\mu u_\rho^D(0) | \Omega > + \dots \quad (190)$$

In evaluating the first term of 190 we use colour and Lorentz invariance of the vacuum:

$$< \Omega | \{ \bar{u}_\alpha^C(0) u_\rho^D(0) \} | \Omega > = \frac{1}{4N_c} \delta_{\alpha\delta} \delta^{CD} < \bar{u}u > \quad (191)$$

with the quark condensate $< \bar{u}u >$:

$$< \bar{u}u > := < \Omega | \{ \bar{u}_\alpha^C(0) u_\alpha^C(0) \} | \Omega >. \quad (192)$$

So we arrive at a first expression for the pseudoscalar Greens function:

$$\Pi_B^5(X) = (\gamma^5 S^\delta(-X) \gamma^5)_{\alpha\delta} [N_c S_{\alpha\delta}^B(X) - \frac{1}{4} \delta_{\alpha\delta} < \bar{u}u >] + \dots \quad (193)$$

We see that this expression agrees with that obtained from 183 and 187.

The dots indicate further corrections due to high frequency contributions leading to a power series in α_s and further corrections due to the low frequency contributions, expressed through vacuum expectation values of higher dimensional operators. Although this is by no means the end of a reasonable calculation let us stop here for a moment the evaluation of Wilson coefficients (since this we have done effectively) and look at 193 in the more familiar Fourier-transformed form:

$$\tilde{\Pi}_B^5(P^2) = \int \frac{d^4 Q}{(2\pi)^4} \frac{\text{Tr } \gamma Q (\gamma(P-Q) - m_b)}{Q^2 [(P-Q)^2 + m_b^2]} - \frac{m_b < \bar{u}u >}{P^2 + m_b^2} + \dots \quad (194)$$

where we have set the mass of the u-quark to zero.

The first term is ultraviolet divergent and must be renormalized (we can avoid that by taking twice a derivative of $\tilde{\Pi}_B^5$ with respect to P^2). The second term blows up if we approach the physical region ($-P^2 \geq m_b^2$). Before we proceed with a phenomenological analysis we shall add some more terms to the operator product expansion. To that purpose let us discuss the diagram m) where the heavy quark interacts with a soft gluon and the light quark dives into the vacuum. In order to investigate such a diagram true miracles can be achieved using a convenient gauge, namely the Fock-Schwinger gauge [Fock 1937, Schwinger 1951]. This gauge is defined by the condition:

$$X_\mu A_\mu^F(X) = 0 \quad A_\mu^F(0) = 0. \quad (195)$$

Figure 13: Graphs not contributing to the vacuum expectation value of the operator product expansion of the heavy-light two-point function; same symbols as preceding figure.

Putting these pieces together, we obtain the following contribution to the Fourier-transform of the correlator proportional to the mixed condensate

$$\tilde{\Pi}_B^3(P^2) \langle \bar{u} \sigma G u \rangle = -\frac{1}{2} \left(\frac{m_B^2}{(P^2 + m_B^2)^2} - \frac{m_B^2}{(P^2 + m_B^2)^3} \right) \langle \bar{u} g \sigma G u \rangle. \quad (205)$$

The terms with two gluon lines ending in the vacuum (graphs j), k), l) are treated in a way similar to that applied to m), yielding a contribution proportional to the "Gluon condensate"

$$\langle g^2 G G \rangle := \langle \Omega | \{ g_{\mu\nu}^2 G_{\mu\nu}^F G_{\mu\nu}^F \} | \Omega \rangle \quad (206)$$

namely

$$\tilde{\Pi}_B^3(P^2) G_2 = \frac{1}{48\pi^2} \frac{1}{P^2 + m_B^2} \langle g^2 G G \rangle. \quad (207)$$

The next contributions would be proportional to the vacuum expectation value of 4 quark operators.

I had already mentioned that there is no direct correction of virtual heavy quark pairs in the vacuum to the heavy quark propagator. Therefore the diagram Fig. 13a) was omitted in Fig. 12. Sometimes you will find diagrams like Fig. 13 b) and c), but they are somewhat misleading, since they should not be evaluated using Feynman rules but stand for the mixed condensate term coming from the Taylor expansion of $\langle \Omega | \{ \bar{u}_\alpha(0) u_\beta(x) \} | \Omega \rangle$. All graphs which cannot lead to gauge or Lorentz invariant vacuum expectation values are of course omitted, like Fig. 13d)

The perturbative part of the pseudoscalar correlator is given in eq. 194 as an integral over internal momenta in the Feynman diagram. It is ultraviolet divergent and therefore has to be renormalized. For reasons which will become clear later we express the perturbative contribution as a twice subtracted dispersion integral:

$$\tilde{\Pi}_B^{3, \text{pert}}(P^2) = aP^2 + b + \frac{(P^2)^2}{\pi} \int_{m_B^2}^{\infty} \frac{\sigma_B^3(s)}{s^2(s + P^2 + i\epsilon)}. \quad (208)$$

To lowest order in perturbation theory one easily obtains:

$$\sigma_B^3(s) = \frac{N_c}{8\pi} \frac{(s - m_B^2)^2}{s}. \quad (209)$$

Nonperturbative corrections have already been evaluated in 193, 205 and 207:

$$\begin{aligned} \tilde{\Pi}_B^{3, \text{nonpert}}(P^2) = & \frac{-m_B \langle \bar{u} u \rangle}{P^2 + m_B^2} - \frac{1}{2} \left(\frac{m_B^2}{(P^2 + m_B^2)^2} - \frac{m_B^2}{(P^2 + m_B^2)^3} \right) \langle \bar{u} \sigma G u \rangle \\ & + \frac{1}{48\pi^2} \frac{1}{P^2 + m_B^2} \langle g^2 G G \rangle \end{aligned} \quad (210)$$

The "theoretical" expression for the two point function so far calculated is thus

$$\tilde{\Pi}_B^{3, th}(P^2) = \tilde{\Pi}_B^{3, \text{pert}}(P^2) + \tilde{\Pi}_B^{3, \text{nonpert}}(P^2) \quad (211)$$

with the expressions 208 and 211. The values for the condensates and the mass m_b are given and discussed in App. 2.

We could now proceed in principle in a straightforward way and tentatively equate:

$$\frac{d^2}{d(P^2)^2} \tilde{\Pi}^{3, th}(P^2) = \frac{-2m_B^2 f_B / m_b}{(P^2 + m_B^2)^3} \quad (212)$$

where on the right-hand side we have neglected the continuum contribution of 176 and used 178.

To state that procedure in a more general way, we equate a theoretical expression for a Greens function $\Pi^{th}(P^2)$ to a phenomenological one, $\Pi^{phen}(P^2)$. The latter can, at least in principle, be expressed in terms of observable quantities. The theoretical Green's function contains perturbative and non-perturbative contributions. The latter ones are parametrized through condensates which appear in coefficients of a power series in $1/P^2$ (here for good reasons arranged as power series in $1/(P^2 + m_B^2)$). The theoretical part can only be calculated in the Euclidean region, i.e. away from the physical thresholds ($-P^2 < m_B^2$). Hence we must use dispersion relations to continue the phenomenological Green's functions to the Euclidean region. From the dispersion integrals comes the name sum rules. The general structure of the sum rule is:

$$\begin{aligned} \frac{(P^2)^l}{\pi} \int_{s_0}^{\infty} \frac{\sigma^{th}(s) ds}{s^l(s + P^2)} &+ \sum_n \frac{C_n < O_n >}{(P^2 + m_n^2)^n} \\ &= \frac{R}{P^2 + m_n^2} + \frac{(P^2)^l}{\pi} \int_{s_0}^{\infty} \frac{\sigma^{phen}(s) ds}{s^l(s + P^2)} + \text{Polynomial in } P^2 \end{aligned} \quad (213)$$

Here σ^{th} is the imaginary part of the Feynman diagrams determining the purely perturbative part of the operator product expansion and the $< O_n >$ are proportional to the vacuum expectation values of composite operators in the operator product expansion, the coefficients C_n are calculated in perturbation theory, m_n stands for heavy quark mass. On the phenomenological side we have singled out the lowest lying state as a pole at the meson mass m_n . The phenomenological dispersive part $\sigma^{phen}(s)$ will be referred to in the following as the continuum. In order to separate the parameters of the lowest lying state we have to approach the pole term as near as possible in order to safely neglect the continuum contribution. But the nearer we want to come to the pole the more terms in the operator product expansion we have to take into account. But such a strategy would not only be extremely cumbersome, since the calculation of the Wilson coefficients is complicated, but also utterly useless, since the number of free parameters would increase very fast, as has been discussed above. For that practical reason we do not need to discuss the nature of the convergence of the operator product expansion.

There are many proposals to handle the sum rule 213 in an efficient way, but in this review I cannot discuss the virtues (and deficiencies) of the different approaches (see [Watson 1989] for a discussion). I shall stick to the procedure introduced originally by Shifman, Vainshtein and Zakharov (1979a), namely that of Borel improvement, since I think this is the most transparent, the most flexible and the most reliable method to exploit the sum rule 213. (The method of moments, also introduced by Shifman et al. is closely related to it). A Borel transformation applied to a function $\Pi(z)$ is generally given by (see sect.3.2)

$$\tilde{\Pi}(b) := \frac{1}{2\pi i} \int_{s_0 - i\infty}^{s_0 + i\infty} dz e^{-z/b} \Pi(z). \quad (214)$$

If the function $\Pi(z)$ has a cut in the complex plane from s_0 to infinity and the discontinuity across this cut is $\sigma(s)$, the Borel transformation is given by the Laplace transform of the discontinuity $\sigma(s)$:

$$B[\Pi](b) := \tilde{\Pi}(b) = \frac{1}{\pi} \int_{s_0}^{\infty} \sigma(s) e^{-s/b} ds. \quad (215)$$

Representing the imaginary part of the pole $(z + m^2 - i\epsilon)^{-n}$ as $(-1)^{n-1} \frac{\pi}{(n-1)!} \delta^{(n-1)}(z + m^2)$, one obtains the Borel transformation of a pole:

$$B\left\{\frac{1}{(z + m^2 - i\epsilon)^n}\right\}(b) = \frac{(-1)^{n-1} e^{-m^2/b}}{(n-1)! b^{n-1}}. \quad (216)$$

The factor $\frac{1}{(n-1)!}$ is the typical factor of the Borel improved series. If we apply the transformation to both sides of 213, the polynomials $P(P^2)$ disappear and we obtain, calling M^2 the variable of the Borel transformed function:

$$\begin{aligned} \frac{1}{\pi} \int_{s_A}^{\infty} \sigma^{th}(s) e^{-s/M^2} ds + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!(M^2)^{n-1}} e^{-m_Q^2/M^2} C_n < O_n > \\ = R e^{-m_Q^2/M^2} + \frac{1}{\pi} \int_{s_{phen}}^{\infty} \sigma^{phen}(s) e^{-s/M^2} ds \end{aligned} \quad (217)$$

where s_A is the threshold for the perturbative contribution ($s_A = m_b$ for a massless light quark) and s_{phen} the physical threshold ($s_{phen} = (m_M + 2m_\pi)^2$). The advantage of the Borel improvement is evident: The higher power corrections are suppressed by additional factors of $(n-1)!$ and on the phenomenological side the continuum is suppressed by an exponentially decreasing weight function. From 217 we get a sum rule for the residue by rearranging terms:

$$\begin{aligned} R = e^{(m_M^2 - m_Q^2)/M^2} \left(\frac{1}{\pi} \int_{s_Q}^{\infty} \sigma^{th}(s) e^{-(s-m_Q^2)/M^2} ds - \frac{1}{\pi} \int_{s_Q}^{\infty} \sigma^{phen}(s) e^{-(s-m_Q^2)/M^2} ds \right. \\ \left. + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!(M^2)^{n-1}} C_n < O_n > \right). \end{aligned} \quad (218)$$

The general expression may be adapted by inserting for $\sigma^{th}(s)$ the expression 209 and according to 211:

$$\begin{aligned} C_1 < O_1 > &= -m_b < \bar{u}u > + \frac{1}{48\pi^2} < g^2 G G > \\ C_2 < O_2 > &= -\frac{1}{2} m_b < \bar{g}u G u > \\ C_3 < O_3 > &= -\frac{1}{4} m_b^3 < \bar{g}u G u > \end{aligned} \quad (219)$$

The contribution of the 4-quark condensate can be found in the literature (see e.g. [Ball *et al.* 1991a]). If we knew nothing about the phenomenological dispersive part, $\sigma^{phen}(s)$, we might hope that the exponential suppression factor makes its contribution small and neglect it altogether. The result for R in that approximation is shown in Fig. 14a), solid line. R is a constant, i.e. theoretically it should be independent of M^2 , but as the curve shows, this is far from being the case. We understand why it is not a constant for the given approximations: For small values of M^2 the higher power corrections due to additional operators in the operator product expansion will become important and for large values of M^2 the suppression of the phenomenological continuum is not sufficiently strong. We can however improve the situation considerably by making an additional hypothesis, namely that of quark-hadron duality. This hypothesis states that if we smear the phenomenological continuum $\sigma^{phen}(s)$ over an interval of about 1 GeV it can be approximated by the theoretical one $\sigma^{th}(s)$. This is the quark hadron duality hypothesis. Since we are always more than a GeV below the physical threshold the smearing is done for us by the integral and we may assume:

$$\int_{s_{phen}}^{\infty} \sigma^{phen}(s) e^{-(s-m_Q^2)/M^2} ds = \int_{s_Q}^{\infty} \sigma^{th}(s) e^{-(s-m_Q^2)/M^2} ds \quad (220)$$

By this assumption we have introduced a new parameter, the "continuum threshold" s_0 , but also extended the domain of applicability of the sum rules considerably. The sum rule 218 becomes in this approximation:

$$R = e^{(m_M^2 - m_Q^2)/M^2} \tilde{R}(M^2) = \left(\frac{1}{\pi} \int_{s_Q}^{\infty} \sigma^{th}(s) e^{-(s-m_Q^2)/M^2} ds + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!(M^2)^{n-1}} C_n < O_n > \right) \quad (221)$$

Figure 14: a) The fitted residue for the pseudoscalar correlator and b) the quadratic mass gap $m_B^2 - m_Q^2$ as functions of the Borel parameter M^2 . solid line: $s_0 = \infty$, long dashes: $s_0 = (m_B + 0.64 \text{ GeV})^2$, short dashes: $s_0 = (m_B + 0.50 \text{ GeV})^2$

Unfortunately the parameter s_0 is not fundamentally related to QCD. Nevertheless it has turned out that the value $s_0 = (m_M + 0.5 \text{ GeV})^2$ yields reasonable results for most cases (Fig. 14a), short dashed line). This might be due to the fact that this is for most channels the position of the first radial excitation. As far as a more scientific choice of s_0 is concerned there are several approaches. The most daring one is the one of "onset of stability", it demands that this value of s_0 should be chosen, that the right-hand side of 221, i.e. $R(M^2)$ has a maximal interval of stability. This approach yields unique results, but it depends of course very much on the assumption that the approximation for the physical continuum through the theoretical one is reliable. The result for this approach is given in Fig. 14a). long dashed line, corresponding to $s_0 = (m_B + 0.64 \text{ GeV})^2$. A more conservative approach consists in a variation of s_0 in "reasonable limits", say from $(m_M + 0.4)$ to $(m_M + 0.7 \text{ GeV})^2$. The most conservative attitude [Narison 1989] is to take the value of R from the onset of stability as lower limit and the minimum of $R(M^2)$ for values of s_0 where this result becomes stable in s_0 as upper limit. It should be noted that this limit is no rigorous upper bound, since additional power corrections could influence this value. An important concept in analyzing these sum rules is the so-called sum rule window introduced by [Shifman *et al.* 1979b]: It admits only such a region of M^2 in which both the power corrections (lower limit for the window) and the influence of the continuum, represented by $\int_{s_Q}^{\infty} \sigma^{phen}(s) e^{-\frac{(s-m_Q^2)}{M^2}} ds$ are less than 50% (more conservatively 30%) of the total value.

The residue obtained from Fig. 14a) at M^2 values of about 2.5 GeV^2 yields a leptonic decay constant $f_B \approx 140 \text{ MeV}$. A major source of error for the determination of f_B is the uncertainty of the mass of the b -quark.

We can check consistency by constructing a sum rule for the mass m_M . It is obtained by forming the logarithmic derivative of 221 with respect to $1/M^2$. This yields:

$$m_{1/1}^2 - m_Q^2 = \frac{1}{\tilde{R}(M^2)} \int_{s_Q}^{s_0} s^{th}(s) (s - m_Q^2) e^{-(s-m_Q^2)/M^2} ds + \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-2)!(M^2)^{n-2}} e^{-m_Q^2/M^2} C_n < O_n > \quad (222)$$

The result for the left hand side of 222 is plotted in Fig. 14 b) for the same choices of the continuum threshold s_0 as in Fig. 14 a). The experimental value is $m_B^2 - m_\pi^2 = 5.8 \text{ GeV}^2$ (for $m_b = 4.7 \text{ GeV}$). Before we discuss the influence of radiative (i.e. perturbative) corrections we shall discuss the sum rule for R in the effective heavy quark theory, which can be easily obtained from 221 by the reparametrization:

$$s \equiv m_Q^2 + 2m_Q E, s_0 \equiv m_Q^2 + 2m_Q \Delta E, M^2 \equiv 2m_Q/\tau, m_{1/1}^2 \equiv m_Q^2 + 2m_Q \Delta m \quad (223)$$

and performing the limit $m_Q \rightarrow \infty$. In that way one obtains for the pseudoscalar correlator a sum rule

for the decay constant f_P for the pseudoscalar heavy meson:

$$f_P^2 m_P = e^{\Delta m_P} \left[\frac{3}{\pi^2} \int_0^{\Delta E} E^2 e^{-E^2} dE - < \bar{u}u > - \frac{1}{16} \tau^2 < g \bar{u} \sigma G u > \right] \quad (224)$$

We see that the right-hand side is independent of m_Q and therefore f_P should scale of $1/\sqrt{m_P}$ with the heavy quark mass. This is in full agreement with the quark model, since $\sqrt{m_P} \cdot f_P$ is proportional to the wave function of the origin in that model.

I shall not give a detailed discussion of perturbative corrections but only report the main results and refer to the original literature for details [Broadhurst and Grozin 1992, Bagan *et al.* 1992]. The two-loop corrections to the purely perturbative contribution (see graphs 12) have been calculated by Broadhurst (1981), the result is rather lengthy. The leading logarithms in $\log(m_Q^2/\mu^2)$ can be summed up to give pseudo-anomalous dimensions [Voloshin and Shifman 1988], the non-leading terms yield corrections of the order $10\alpha_s/\pi$, which is by no means small as compared to 1! There are good reasons to believe that they represent the "Coulombic" corrections due to gluon exchange and can be taken into account with reasonable accuracy [Bagan *et al.* 1992]. They lead to an increase of the value of f_B of about 25%.

The scale invariant decay constant $\hat{f}_{\text{rel}}$ of the heavy quark effective theory is related to the physical static decay constant f_P (i.e. the one correct in the limit $m_Q \rightarrow \infty$) by

$$\hat{f}_{\text{rel}} = f(m_P) \sqrt{m_P} \left(1 + \frac{\alpha_s(m_P)}{\pi} \left(\frac{2}{3} - \Gamma \right) (\alpha_s(m_P))^{w_0/6} \right) \quad (225)$$

with $w_0 = -4$, $\beta_1 = \frac{1}{2} - \frac{1}{3}\tau_1$, $\Gamma \approx -23$ for $N_c = 3$. This quantity has been determined by the sum rule method [Broadhurst and Grozin 1992, Bagan *et al.* 1992] to $\hat{f} = (0.32 \dots 0.43) \text{ GeV}^{3/2}$, it agrees with a determination from lattice gauge calculations [Abada *et al.* 1992, Bernard *et al.* 1992, Sommer *et al.* 1992]. If one evaluates f_B from that value one obtains a value which is about 20% larger than the one obtained by a calculation with finite masses, i.e. one is even for the b -meson still a good way away from the heavy quark limit.

Before we come back to the mesons consisting of a heavy and a light quark in connection with three-point functions I shall shortly discuss two-point functions of some other channels.

For the two-point functions of the vector and axial vector current one introduces the scalar functions $\Pi_V(P^2)$ and $\Pi_S(P^2)$ through:

$$\begin{aligned} \Pi_{\mu\nu}(P^2) &= \int d^4X e^{-iP \cdot X} < 0 | \bar{\psi}(X) \gamma_\mu \psi(X) \bar{\psi}(0) \gamma_\nu \psi(0) | 0 > \\ &= (P^2 \delta_{\mu\nu} - P_\mu P_\nu) \Pi_V(P^2) + P_\mu P_\nu \Pi_S(P^2) \end{aligned} \quad (226)$$

and a corresponding expression for the axial vector $\bar{\psi}(X) \gamma_\mu \gamma_5 \psi(X)$.

If the current is conserved this yields in momentum space $P_\mu \Pi_{\mu\nu}(P^2) = 0$ and hence $\Pi_S = 0$

The charmonium system, i.e. a system of a charmed quark and antiquark has been investigated in the original paper by Shifman, Vainshtein and Zakharov (1979b) and the analysis has been extended by Reinders *et al.* (1985). Since now both quarks are heavy and heavy quarks do not condensate directly in the vacuum, the graphs (e) - (j) and (m) of Fig. 12 are absent. The leading non-perturbative contribution is given by the gluon condensate, corresponding to graphs (j) - (l). The radiative corrections are not negligible.

The charmonium channel allows the analysis of states in different quantum numbers, namely the $1S_0$, $3S_1$, P_1 , $3P_0$, P_1 , and $3P_2$ state. The ground states are all accessible to the sum rule method and it is by no means trivial that all that information can be accommodated by only two parameters, namely the mass of the charmed quark and the gluon condensate (the value of α_s can be taken from other sources). In spite of the overdetermination the data allow a rather wide range of values for the gluon condensate and the charmed mass. The bottomonium channel is in principle very similar, but

Figure 13: Graphs yielding the contribution of the four quark condensate same symbols as before

there the radiative corrections become truly annoying. The reason for that is easily understood: due to the high mass of the b -quarks the radius of the system becomes small and therefore the colour-Coulomb potential becomes dominant. This is an advantage for potential models but a disadvantage for the sum rule method. One can try to improve the situation by using the propagator in the Coulomb field instead of the free propagator but this is technically very cumbersome. The difficulties in the bottomonium channel are reflected in the different values for the (pole) mass of the b -quark obtained by different methods: 4.8 GeV by [Voloshin and Zaitsev 1987] and 4.5 GeV by [Reinders 1988].

For the channel of currents composed of two light quarks the contributions of the quark condensate (i.e. graphs corresponding Fig. 12e) play no big role, since they are proportional to the light quark mass. There is however an important contribution from the 4-quark condensate due to the graphs of Fig. 15. The 4-quark condensates should in principle be treated as new condensates but this would lead to too many free parameters. Shifman, Vainshtein and Zakharov (1979b) proposed therefore to use the factorization hypothesis which is rigorous in the $N_c \rightarrow \infty$ limit. This hypothesis is discussed in App. 2. By it the 4-quark condensates are reduced to squares of two-quark condensates.

For the vector current $\bar{q}\gamma^\mu q$ the full discontinuity of its two-point function $\sigma^{\mu\nu}(s)$ is known from e^+e^- annihilation into hadrons

$$\frac{1}{\pi} \text{Im} \Pi_V(s) = \frac{s}{16\pi^2 \alpha_s} \sigma(e^+ + e^- \rightarrow \text{hadrons}) \quad (227)$$

where α is the fine structure constant of QED. This seems to be an ideal basis for the determination of condensates but unfortunately they are all strongly correlated and different analyses yield quite different results or at least quite different errors (see [Narison 1989] for a detailed discussion).

The 4-quark condensate is essential for the different spectral functions of the vector and axial vector current and hence for the mass difference of the ρ and A_1 -meson. In perturbation theory with massless quarks the vector and axial vector channel are degenerate. The mass difference between the up and down quark determines largely the $\rho - \omega$ -mixing [Shifman *et al.* 1979c].

In a theory with practically massless quarks it is by no means obvious that mesons and baryons, which are for $N_c = 3$ interpolated by operators with three quarks, have different masses. Here, as in the $\rho - A_1$ -difference the quark condensate plays an essential role, but in a more direct way. As baryonic currents one can choose

$$\Psi_1^{\lambda\lambda'} = u_a^\lambda [C \gamma_5]_{\alpha\beta} d_\alpha^\beta u_\gamma^{\lambda'} \epsilon_{ABC} \quad \Psi_2^{\lambda\lambda'} = u_a^\lambda C_{\alpha\beta} d_\alpha^\beta u_\gamma^{\lambda'} \epsilon_{ABC} \quad (228)$$

or any linear combination of these; C is the charge conjugation matrix for Dirac spinors.

The currents 228 are colour singlets and Dirac spinors. For the two-point function of baryonic currents we use the Källen-Lehmann representation [Källen 1972]:

$$\int d^4X < \Psi_1^{\lambda\lambda'}(X) \bar{\Psi}_2^{\lambda''\lambda'''}(0) > e^{-iP \cdot X} = F_1(-P^2) + F_2(-P^2) \gamma_5 P \quad (229)$$

The scalar functions $F(P^2)$ have a pole at the nucleon mass and the residues of the functions F_i fulfill the condition

$$(P^2 + m_N^2) F_1(m_N^2) m_N = (P^2 + m_N^2) F_2(m_N^2) \quad (230)$$

Figure 16: Graphs relevant for the operator product expansion of the two point function of a baryonic system. Thick lines represent heavy quarks, thin ones light quarks, small wavy lines gluons and large wavy ones hadronic currents.

The leading terms for the vacuum expectation value of the operator product expansion are given by the graphs of Fig. 16.

In the limit of massless quarks the graphs a) - d) contribute to the scalar function F_1 of 230, the graphs e) and f) to $F_2(P^2)$. Graph 16 d) receives thus its leading contribution from a non-perturbative term and the non-vanishing value of F_2 implied by relation 230 teaches us that there must be non-vanishing non-perturbative terms. Indeed in a world with massless quarks a vanishing quark condensate would imply either nucleons or parity degenerate nucleons; in both cases F_2 could be zero.

Baryon sum rules successfully explain the different masses of the ρ -meson and the nucleon [Chung *et al.* 1981, Chung *et al.* 1982, Ioffe 1981]. The mass difference between the nucleon and the Δ resonance is mostly due to the mixed condensate and perturbative corrections. Unfortunately, the results for the nucleon parameters depend somewhat on the choice of the current for the baryon. This indicates that higher condensates and/or perturbative corrections are important. The perturbative corrections are for the nucleon uncomfortably large [Jamin 1988]. A problem also exists in the evaluation of the flavour SU(3) breaking. An analysis of the $1/2^+$ octet as well as the $3/2^+$ decuplet [Dorsch *et al.* 1989] indicates that the mixed condensate of strange quarks, $\langle \bar{s}s g G s \rangle$, is larger than the corresponding quantity with non-strange light quarks, whereas non-diagonal sum rules come to the opposite conclusion (see App. 2).

Other channels are more controversial. There are conventional sum rule analyses for the scalar and pseudoscalar current with light quarks and also of the gluonic current

$$j_s = \frac{3\alpha_s}{4\pi} G_{\mu\nu}^a G_{\mu\nu}^a \quad (231)$$

The gluonic current interpolates glue-balls, i.e. hadrons consisting (mainly) of gluons as constituting particles. In refs. [Novikov *et al.* 1981] and [Geshkenbein and Ioffe 1980] the relevance of these sum rules has been questioned, mainly because of strong instanton effects (see sect 3.2, esp. 172) in these channels. I refer for a discussion of the conventional sum rules to [Narison 1989] and for the objections to [Novikov *et al.* 1981].

4.3.2 Three Point Functions

QCD sum rules for three-point functions have been applied to the evaluation of the electromagnetic formfactor of the pion by [Ioffe and Smilga 1982] and [Nesterenko and Radyushkin 1982]. Later this method was applied to several semileptonic decays. Here I shall mainly follow the systematic treatment given in [Ball 1992a] and [Ball *et al.* 1991a], where for the first time the dependence of the momentum transfer t for $t > 0$ was treated.

If one remembers the difficulties encountered in extracting observable information from the two-point function and the large uncertainty of the leptonic decay constant f_B one might think of the application of the sum rule method to three-point functions as a hopeless endeavour. It turns out, however, that the dependence of the leptonic decay constant on a variation of the QCD parameters is much stronger

than for the semileptonic formfactors, and if one is willing to make a few very plausible assumptions one can obtain rather precise results. At present QCD sum rules are the most promising tool in the calculation of semileptonic formfactors. Semileptonic decays are one of the most important sources for the determination of the Cabibbo-Kobayashi-Maskawa matrix elements in the standard model (see e.g. Nachtmann 1990), and therefore I shall treat these processes with some detail.

We consider the semileptonic decays:

$$M \rightarrow M' + \ell + \bar{\nu}_\ell \quad (232)$$

where the decaying meson shall consist of a heavy (c or b) and light (u, d or s) (anti-)quark. The meson in the final state, M' , can be a light-light or a heavy-light system. ℓ denotes a lepton (e or μ) and ν_ℓ the associated neutrino. There is ample experimental information on these decays (see e.g. [Ball *et al.* 1991a] for references). As a test for our understanding of the hadronic (i.e. non-perturbative) dynamics in those decays the decays $D \rightarrow K \ell \nu$ and $D \rightarrow K^* \ell \nu$ are especially suitable since the $c \rightarrow s$ transition is Cabibbo allowed and thus the CKM mixing matrix element V_{cs} is practically 1. Therefore this decays contain (practically) no unknown parameter of the weak interaction.

The matrix elements we want to calculate are: $\langle M' | J_\mu | M \rangle$ where J_μ is the weak hadronic current

$$J_\mu(x) = \bar{\psi}(x) \gamma_\mu (1 - \gamma^5) Q(x) \quad (233)$$

where $Q(x)$ describes the field of the heavy quark in M , $\psi(x)$ is the field of the decay product, it might be a light or a heavy quark field. The summation over the colour indices is not written explicitly. The decay amplitude is obtained from 233 by contracting it with the leptonic current $\bar{\psi}'(x) \gamma_\mu (1 - \gamma^5) \psi'(x)$ and multiplying with the weak decay constant.

The information on the hadronic matrix element of the weak current 233 is contained in the three-point function:

$$\Pi_\mu(P_M, P_{M'}) = \int d^4 X d^4 Y \exp[i P_M X - i P_{M'} Y] \langle j_{M'}(0) j_M^\dagger(Y) \rangle \quad (234)$$

where j_M and $j_{M'}$ are the currents which couple to the mesons: $j_M(y) = \bar{q}(x) i \gamma^5 Q(x)$ and $j_{M'}(y) = \bar{q}(x) O_{M'} \psi(x)$ where $Q(x)$ and $q(x)$ are heavy and light quark fields, respectively, $\psi(x)$ might be heavy or light. If the decaying meson is a pseudoscalar particle, $O_{M'}$ is $i \gamma^5$ and $O_{M'} = \gamma^5$ if the decay product is a vector meson. (For the interpolating field of the π and the K , however, one uses the pseudoscalar part of the axial vector operator, but I shall not treat this technical detail here.) In the following I shall concentrate on $O_{M'} = i \gamma^5$. Now the three-point function 234 is a quantity which can be evaluated in QCD and we shall do so in a moment.

In order to extract the matrix element $\langle M' | J_\mu | M \rangle$ from 234 we insert hadronic states between the hadronic and the weak currents and obtain in a symbolic notation:

$$\Pi_\mu(P_{M'}, P_M) = \sum_{Z_M, Z_{M'}} \frac{\langle \Omega | j_{M'} | Z_{M'} \rangle \langle Z_M | j_M | \Omega \rangle}{(Z_{M'}^2 + P_{M'}^2)(Z_M^2 + P_M^2)} \quad (235)$$

Z_M and $Z_{M'}$ are hadronic states coupling to the currents j_M and $j_{M'}$, respectively, and Z_M^2 and $Z_{M'}^2$ denotes their mass square. Here we have approximated all higher contributions through resonances with zero width. Let us introduce the variables

$$s = -P_M^2, u = -P_{M'}^2, t = -(P_M - P_{M'})^2 \quad (236)$$

The content of 235 is illustrated in Fig. 17 in the s, u -plane: The full dot represents the transition $M \rightarrow M'$ we are interested in, the open circles are transitions where at least one state is a radial excitation. We isolate the "ground state to ground state" transition and write in analogy to 176

$$\Pi_\mu(s, u) = \frac{\langle \Omega | j_{M'} | M' \rangle \langle M' | J_\mu | M \rangle \langle M | j_M | \Omega \rangle}{(u - m_{M'}^2)(u - m_M^2)} + \text{higher resonances} \quad (237)$$

Figure 17: Schematic plot of the singularities of the three-point function in the zero width approximation. The residue of the full circle is the "ground-state to ground-state" transition one wants to calculate, the open circles correspond to "higher resonance" contributions. The shaded area is accessible to theoretical calculation.

We recognize in the numerator of 237 besides the hadronic matrix element of the weak current $\langle M' | J_\mu | M \rangle$ the "vacuum transitions" $\langle M | J_\mu | 0 \rangle$ and $\langle 0 | J_\mu | M' \rangle$ which are related to the leptonic decay constants $f_M, f_{M'}$ and which have been discussed above (see 178 e.g.).

From 237 we will derive a sum rule for the product of these quantities and at the end have to divide through the leptonic decay constants. This division is a big advantage. If we express these decay constants again by sum rules the dependence on the QCD parameters (as for instance the Q -mass) becomes much weaker than for the simple sum rule due to cancellations in the numerator and the denominator.

The steps now to be performed are very similar to those discussed above for the two-point functions: We evaluate the three-point function 234 making use of the operator product expansion and compare it to the phenomenological expression 237 where again we have to apply tricks in order to suppress the contribution of the "higher states".

Before going into details let us first perform the invariant decomposition of the three-point function 234 and the matrix element $\langle M' | J_\mu | M \rangle$

$$\Pi_\mu(P_M, P_{M'}) = \Pi_+(s, u, t)(P_M + P_{M'})_\mu + \Pi_-(s, u, t)(P_M - P_{M'})_\mu \quad (238)$$

$$< M' | J_\mu | M \rangle = f_+(t)(P_M + P_{M'})_\mu + f_-(t)(P_M - P_{M'})_\mu \quad (239)$$

The contribution of $f_-(t)$ to the decay rate is negligible, since it gives a contribution proportional to the lepton mass. From parity follows that only the vector part of the weak current 233 contributes to that decay of a pseudoscalar to a pseudoscalar meson.

First we evaluate the invariant function Π_+ using the operator product expansion for the trilocal product in 234. The Wilson coefficients will now depend on X and Y , and the Fourier transforms of the leading condensates to lowest order perturbation theory can be obtained from the diagrams of Fig. 18

Diagram a) yields the perturbative contribution which will be discussed later. Diagram b) yields non-perturbative contributions proportional to $\langle \bar{q}q \rangle$ and $\langle \bar{q}g\sigma Gq \rangle$. Again for conceptual reasons it is best to work first in coordinate space. Diagram b) yields:

$$< \bar{q}(X) S^u(X) \gamma_\mu (1 - \gamma^5) S^q(-Y) q(Y) > \quad (240)$$

where again S^Q and S^u are the propagators of the heavy quark Q and the quark u . By a Taylor expansion of $\bar{q}(x)$ and $q(y)$ (see 190 ff.) we obtain the contribution of the quark condensate and a part of the one of the mixed condensate.

Figure 18: Graphs relevant for the operator product expansion of the three point function of a heavy light system. Thick lines represent heavy quarks, thin ones light quarks, small wavy lines gluons and large wavy ones hadronic currents of the mesons M and M' and the weak current J . The propagator from J to M' might be that of a light or heavy quark.

The general result for the operator product expansion for Π_+ is:

$$\Pi_+(s, u, t) = \frac{1}{\pi^2} \int \frac{\rho^h(s', u', t)}{(s' - s)(u' - u)} ds' du' + \sum_n C_n(s, u, t) < O_n > \quad (241)$$

where the first term is the perturbative contribution (eventually necessary subtractions are omitted for simplicity). We equate this expression to the phenomenological one, which in our case is:

$$\Pi^{phen}(s, u, t) = \frac{m_M^2 m_{M'}^2}{m_Q^2 m_p^2 s} f_M f_{M'} f_+(t) \frac{1}{s - m_M^2} \frac{1}{u - m_{M'}^2} + \text{higher resonances} \quad (242)$$

If ψ is a light pseudoscalar meson and one works with the axial vector current, the quark mass m_u does not appear. In Fig. 17 we also show the region which is accessible to the theoretical calculation. In order to suppress the contributions of the higher resonances as well as the higher condensates, it is advantageous to perform a Borel improvement (see 214) in both the variables s and u . By this transformation all terms which have no singularities both in s and u , as e.g. the graph of Fig. 18e) will be annihilated. But as in the case of two point functions this is not enough for the elimination of the higher resonances, so we approximate them by an extended duality ansatz

$$\frac{1}{\pi^2} \int \frac{\rho^h(s', u', t)}{(s' - s)(u' - u)} C(s', u') ds' du' = \text{contribution of higher resonances} . \quad (243)$$

Here $C(s', u')$ is some cut-off function, e.g.

$$C(s', u') = \Theta(s' - s_0) \Theta(u' - u_0) \quad (244)$$

After the Borel transformation in s and u , the sum rule for $f_+(t)$ becomes:

$$f_+(t) = \frac{m_Q m_u}{f_M f_{M'} m_M^2 m_{M'}^2} \exp[m_M^2/M_M^2 + m_{M'}^2/M_{M'}^2] \left(\frac{1}{\pi^2} \int \rho^h(1 - C(s', u')) \exp[-s'/M_M^2 - u'/M_{M'}^2] ds' du' + \sum_n \hat{C}_n(M_M^2, M_{M'}^2, t) < O_n > \right) \quad (245)$$

where $\hat{C}_n$ are the double Borel-transformed Wilson coefficients $C(s, u, t)$ in eq. 241. We have incorporated the continuum model 243 and divided through the amplitudes $\langle 0 | J_{M'} | M' \rangle$ and $\langle M | J_M | 0 \rangle$ in order to isolate $f_+(t)$. We see that we could have used any number of subtractions in the double dispersion integral of 241 without affecting the sum rule 245.

There is a very subtle point in the evaluation of the double spectral function $\rho(s', u', t)$ for $t > 0$, since the analytic structure of the three-point function is rather involved and so-called "non-Landau" singularities

have to be taken into account. This point has been clarified in [Ball 1992a], see also [Ball *et al.* 1991a], and after that the formfactor $f_+(t)$ can be evaluated by the sum rule 245 as long as t stays sufficiently far (ca. 0.5-1 GeV) below the QCD threshold $(m_Q + m_\psi)^2$ (see sect. 5)

The leptonic decay constants f_D and f_K occurring in the denominator of the sum rule 245 can be expressed again through sum rules of the type 221. Then it is natural to make the same approximations for the sum rule of the three-point function in the numerator and in the denominator, especially to choose the same continuum thresholds. In this way one can hope that some of the uncertainties introduced by the approximations as neglecting higher condensates and perturbative corrections, approximating the continuum contributions by the "duality assumption" 220 or by the errors at the QCD parameters, are cancelled. Partially this can be checked numerically, and one finds indeed that e.g. the strong dependence of f_B on the b -mass observed in the sum rule 221, does not occur for 245 if the heavy meson decay constant is also expressed by a sum rule. In the limit that the mass of the heavy quark goes to infinity (in our example the mass of the Q and the ψ quark), one can even prove these cancellations at a given kinematical configuration. From heavy quark-effective theory one also learns that the Borel parameters at which the three-point sum rules are to be evaluated should have twice the value of the corresponding one in the two-point function. Furthermore, the ratio of M^2 and M'^2 , the Borel parameters in the three-point function, should have the ratio: $M^2/M'^2 = m_Q/m_{Q'}$.

With all that information the analysis of three-point sum rules can be performed and yields indeed more stable results for the semileptonic decay from factors than the two-point sum rules.

Who believes that this is miraculous may be assured by the following considerations: In the language of the quark model the evaluation of the leptonic decay constant f_M requires the evaluation of the wave function at zero, since the heavy and light quark annihilate into the hadronic vacuum. For the semileptonic decay the final state is again a hadron, so we have only to calculate the overlap of two (in general different) wave functions. An incomplete knowledge of the potential will influence strongly the value of the wave function at the origin, but much less an overlap integral since the initial and final wave functions are affected by the change of the potential in a similar way.

I shall now discuss the concrete results of the analysis outlined above for the decay $D \rightarrow K\ell\nu$; i.e. we have Q representing the c -quark and ψ the s -quark. M is the D -meson and M' the K -meson. The analytic expressions are given in [Ball 1992a, Ball *et al.* 1991a] and I shall report here only the results in graphical form.

In Fig. 19a) $f_+(0)$ is given as a function of the Borel parameter $M^2 = M_D^2 = 1.8M_K^2$, $s_0^D = 6 \text{ GeV}^2$ for various values of the continuum threshold in the K -channel. We see that for small values of M^2 , ($M^2 < 2\text{GeV}^2$) evidently higher condensates become important, and therefore $f_+(0)$ becomes unstable, but for $M^2 > 2.5 \text{ GeV}^2$ we have acceptable stability. From an analysis of the two-point sum rule we conclude that $s_0^K = 1.7 \text{ GeV}^2$ is the preferred value. In Fig. 19 b) we show the single contributions to $f_+(0)$. We see that the stability is mostly due to a balance between the perturbative contribution Π^{pert} and the non-perturbative contribution due to the quark condensate ($\Pi^{(q)}$). The contribution of the mixed condensate ($\Pi^{(g)}$) is smaller, those of the gluon and four-quark condensate are negligible.

The analysis of the semileptonic decays into vector mesons follows the same lines. The information in this case is richer since the vector meson in the final state can have three different polarizations and we have several scalar functions for the description of the matrix element of the weak current. Using a normalization similar to that of [Bauer *et al.* 1985] we write:

$$\begin{aligned} \langle K^* | J_\mu | D \rangle &= -i(m_D + m_{K^*}) A_1(t) \epsilon_\mu^{(\lambda)} + \frac{i A_2(t)}{(m_D + m_{K^*})} (\epsilon^{(\lambda)} \cdot p_D) (p_D + p_{K^*})_\mu \\ &+ \frac{i A_3(t)}{(m_D + m_{K^*})} (\epsilon^{(\lambda)} \cdot p_D) (p_D - p_{K^*})_\mu + \frac{2V}{(m_D + m_{K^*})} \epsilon_\nu^{\mu\sigma} \epsilon_\nu^{(\lambda)} p_{D\sigma} p_{K^*\sigma}. \end{aligned} \quad (246)$$

The vector part of the weak current contributes to the amplitude $V(t)$, the weak axial current to the amplitudes $A_1(t)$, $A_2(t)$, and $A_3(t)$. The contribution of $A_3(t)$ gives a contribution to the decay amplitude which is proportional to the mass of the leptons and can therefore be neglected. The values

Figure 19: a) The semileptonic decay form factor $f_+(0)$ of the decay $D \rightarrow K\ell\nu$ as function of the Borel parameter M^2 . b) Different contributions to $f_+(0)$, see text

		$B(D^0 \rightarrow K^{*-})$	$B(D^+ \rightarrow K^{*0})$	Γ_L/Γ_T	Γ_+/ Γ_-	Γ^*/Γ
<i>Experiment</i>						
Bai <i>et al.</i> 1991	e	3.4 ± 0.9	$5.3^{+1.3}_{-1.7}$	$0.5^{+1.1}_{-0.3}$		$1.0^{+0.3}_{-0.2}$
Anjos <i>et al.</i> 1989	e	3.8 ± 1.1	4.4 ± 1.2	$1.8^{+0.7}_{-0.7}$	$0.15^{+0.10}_{-0.08}$	0.50 ± 0.16
Kodama 1991	μ	2.3 ± 0.9		1.2 ± 0.2		
Adamovich <i>et al.</i> 1991	μ			$0.6^{+0.6}_{-0.4}$		0.51 ± 0.24
Crawford <i>et al.</i> 1991	e	3.7 ± 0.9				
	μ	3.3 ± 1.0				
Albrecht <i>et al.</i> 1991	e	4.4 ± 1.4	4.2 ± 0.7			
	μ	4.3 ± 1.7				
<i>Lattice Gauge Calculations</i>						
Crisafulli <i>et al.</i> 1989		4.0 ± 1.7				
Lubicz <i>et al.</i> 1991		2.1 ± 0.3	5.5 ± 2.0	1.7 ± 0.6		1.1 ± 0.3
<i>Quark models</i>						
Bauer <i>et al.</i> 1985		3.5	10.1	0.89	0.28	1.14
Isgur and Scora 1989		3.5	9.4	1.11	0.29	1.06
<i>QCD Sum Rules</i>						
Ball <i>et al.</i> 1991		2.7 ± 0.6	4.0 ± 1.6	0.86 ± 0.06	0.09 ± 0.02	0.50 ± 0.15

Table 1: Experimental and theoretical values for branching ratios of the decays $D \rightarrow K^{(*)}\ell\nu$. $\Gamma(D \rightarrow K^{(*)}\ell\nu)$, $\Gamma^* = \Gamma(D \rightarrow K^{(*)}\ell\nu)$. The indices L and T, + and - refer to the polarization and helicity of the $K^{(*)}$ meson. e and μ indicate semileptonic and semimuonic decays respectively.

of the decay amplitudes are determined by the sum rule method in the same way as sketched above for the the amplitude $f_+(t)$ of the decay $D \rightarrow K\ell\nu$.

In Table 1 I quote experimental results, together with the results of different theoretical approaches.

The experimental values underwent some changes in the last years at the edges of the errors, but the situation seems to stabilize. The oldest results are those of quark models. They give generally quite good results, there only failure is the ratio of the total decay rates $\Gamma(D \rightarrow K^{(*)}\ell\nu)/\Gamma(D \rightarrow K\ell\nu)$ which comes out too large by a factor 2. QCD sum rules are in agreement with all features of the decays, lattice gauge calculations have still very large errors but the ratio for $K^* \rightarrow K$ decay from [Lubicz *et al.* 1991] yields a wrong ratio outside the quoted errors of the calculation.

The t -dependence of the semileptonic formfactors is discussed in sect. 5.

Suggested reading for the sum rule method: There exist a monography on sum rules by S. Narison (1989), and a reader edited by M. Shifman (1992). For technical help the book of P. Pascual and R. Tarrach (1984) is very useful.

4.4 The Model of the Stochastic Vacuum

4.4.1 The Model

In this section I introduce a simple model for the treatment of the low frequency contributions which has been introduced by Yuri Simonov and myself several years ago [Dosh 1987, Dosh and Simonov 1987, Simonov 1988]. In some sense it can be viewed as an extension of the sum rule method. There nonperturbative effects are described by (static) condensates, e.g. the gluon condensate $\langle g^2 GG \rangle := \langle \Omega | G_{\mu\nu}^a(0) G_{\mu\nu}^a(0) | \Omega \rangle$. As we have seen in sect. 4.3 in a simple-minded derivation of sum rules non-local condensates like

$$\langle \Omega | G_{\mu\nu}^a(0) G_{\mu\nu}^a(X) | \Omega \rangle \quad (247)$$

can occur in intermediate steps (see 189 for an analogous expression of the quark condensate). These are evaluated by a Taylor expansion (see 190 in order to make contact with the well-established theoretical framework of the operator product expansion. Intuitively, the nonlocal condensates are expected to be functions which fall off with a typical hadronic scale (ca. 1 GeV). If the two points 0 and X in 247 are separated by several units of the hadronic scale the matrix element will be very small. If in the evaluation of hadronic matrix elements these distances become relevant, a Taylor series with few terms is in general not adequate to describe this fast decrease. Therefore the model of the stochastic vacuum goes in a different direction. It is based on the idea that the low frequency contributions to the functional integral in quantum field theoretical Greens functions can be approximated by a simple stochastic process. In that way the expected decrease of nonlocal expectation values of fields at distant points can be incorporated directly by introducing a finite correlation length.

Let me express that idea in a somewhat more formal language. Let

$$\prod \mathcal{D}\tilde{\phi}_i(K) e^{-S[\tilde{\phi}]} \quad (248)$$

be the measure of a quantum field theory with fields $\tilde{\phi}_i(K)$ in Euclidean momentum space. The measure is split into low and high frequency parts

$$\prod \mathcal{D}\tilde{\phi}_i(K) = \prod_{|K| \leq \mu} \mathcal{D}\tilde{\phi}_i(K) \prod_{|K| > \mu} \mathcal{D}\tilde{\phi}_i(K). \quad (249)$$

In an asymptotically free theory like QCD the integration over the high frequency components can be accounted for by perturbation theory. Little is known analytically about the functional measure at low frequencies. As we have seen in sect. 3.2 on renormalons the infrared properties of a non-Abelian gauge theory like QCD most probably introduce genuine nonperturbative ingredients. We have to assume that nature has managed to regularize the infra-red problems of QCD (after all we do observe hadrons) and that therefore it is reasonable to approximate the contributions of the low frequency components of the fields by a simple stochastic process, the most simple being a Gaussian stochastic process described by one correlator.

Before I come to that restriction I shall attack the problem in a more general way, namely: by the cumulant expansion. I first give a short recapitulation of that expansion. Be $\xi(t)$ a stochastic variable, depending on one (or several) parameter(s) t . It shall be distributed according to some distribution which fixes the expectation values

$$\langle \xi(t) \rangle, \langle \xi(t_1) \xi(t_2) \rangle, \langle \xi(t_1) \xi(t_2) \xi(t_3) \rangle \quad \text{etc.} \quad (250)$$

The cumulants or linked clusters are defined by:

$$\begin{aligned} \langle P \exp \int \xi(t) dt \rangle &= \exp \left[\int \langle \xi(t) \rangle dt + \frac{1}{2!} \int \int \langle \xi(t_1) \xi(t_2) \rangle dt_1 dt_2 \right. \\ &\quad \left. + \frac{1}{3!} \int \int \int \langle \xi(t_1) \xi(t_2) \xi(t_3) \rangle dt_1 dt_2 dt_3 + \dots \right]. \end{aligned} \quad (251)$$

The path ordering prescription (see 41) on the left-hand side has to be introduced if the stochastic variable $\xi(t)$ are non commuting.

For commuting stochastic variables $\xi(t)$ an expansion of the left-hand side of (251) yields

$$\begin{aligned} \langle \xi(t) \rangle &= \langle \xi(t) \rangle \\ \langle \xi(t_1) \xi(t_2) \rangle &= \langle \xi(t_1) \xi(t_2) \rangle - \langle \xi(t_1) \rangle \langle \xi(t_2) \rangle \\ \langle \xi(t_1) \xi(t_2) \xi(t_3) \rangle &= \langle \xi(t_1) \xi(t_2) \xi(t_3) \rangle - \langle \xi(t_1) \rangle \langle \xi(t_2) \rangle \langle \xi(t_3) \rangle \\ &\quad - \langle \xi(t_1) \xi(t_2) \rangle \langle \xi(t_3) \rangle - \langle \xi(t_1) \xi(t_3) \rangle \langle \xi(t_2) \rangle - \langle \xi(t_2) \xi(t_3) \rangle \langle \xi(t_1) \rangle \\ &\quad \text{etc.} \end{aligned} \quad (252)$$

This can be checked directly by expanding the exponential and resumming the expectation values. A centered Gaussian process is a process where only cumulants with $n = 2$ occur, i.e. all the expectation values (250) are determined by the correlator with $n = 2$:

$$\begin{aligned} \langle \xi(t_1) \rangle &= 0 & \langle \xi(t_1) \xi(t_2) \xi(t_3) \rangle &= 0 \\ \langle \xi(t_2) \xi(t_2) \rangle &= \langle \xi(t_1) \xi(t_2) \rangle \\ \langle \xi(t_1) \xi(t_2) \xi(t_3) \xi(t_4) \rangle &= \langle \xi(t_1) \xi(t_2) \rangle \langle \xi(t_3) \xi(t_4) \rangle \\ &\quad + \langle \xi(t_1) \xi(t_3) \rangle \langle \xi(t_2) \xi(t_4) \rangle + \langle \xi(t_1) \xi(t_4) \rangle \langle \xi(t_2) \xi(t_3) \rangle \\ &\quad \text{etc.} \end{aligned} \quad (253)$$

I shall now show that the assumption that the contributions of the low frequency fields can be described by a functional integral (i.e. stochastic process) with a *converging* cluster expansion leads already to an area law of the Wilson loop and hence to linear confinement for static sources (see sect. 3.1.2 and sect. 4.1)

For that we consider the Wegner-Wilson loop in a pure gauge theory (see 84 and 134) and sect. 4.1)

$$W[C] = \int P \exp \left[ig \int_{\mathcal{B}} B_\mu(Z) dz_\mu \right] e^{S_{YM}[\mathcal{B}]} \mathcal{D}B \quad (254)$$

where S_{YM} is the part of the QCD Lagrangian containing only gauge fields. (see 129) As always occurs with approximations in gauge theories a control of gauge invariance is essential. If that control is missing one is never sure whether the output of a calculation is significant or a gauge artefact. So we do not deal with correlators of the gauge potentials B_μ , but rather of the field strength $G_{\mu\nu}$.

A very desired result of the method of the stochastic vacuum is that even in the simplest approximation there is a crucial difference between Abelian gauge theories like quantum electrodynamics and non-Abelian ones like QCD, but nevertheless I shall start for pedagogical reasons with an Abelian theory and discuss afterwards shortly the additional physical ingredients and technical complications in a non-Abelian gauge theory.

If we want to describe the expectation values entering the cluster expansion through field strength tensors, we have to apply Stokes theorem to the W-loop 254. From this theorem we obtain

$$\int_C B_\mu(Z) dz_\mu = \int_{\mathcal{F}} G_{\mu\nu}(Z) d\sigma_{\mu\nu}(Z) \quad (255)$$

where $\mathcal{F}$ is an area whose border is the loop C , $d\sigma_{\mu\nu}(Z)$ ($\mu < \nu$) is the surface element of $\mathcal{F}$ at point Z , and $G_{\mu\nu}$ is the Abelian field strength tensor $G_{\mu\nu}(X) = \partial_\mu B_\nu(X) - \partial_\nu B_\mu(X)$. We thus rewrite (254) as

$$W[C] = \int \mathcal{D}B_\mu e^{-S_{YM}[\mathcal{B}]} \exp \left[ig \int_{\mathcal{F}} G_{\mu\nu}(Z) d\sigma_{\mu\nu}(Z) \right]. \quad (256)$$

We shall now concentrate only on the low frequency contributions and denote in the following

$$\langle \cdot \rangle_D = \int \prod_{|K| < \mu} dB_\mu e^{-S_{YM}[\mathcal{B}]}. \quad (257)$$

The low frequency contributions to the W-loop are given by the cluster expansion:

$$\begin{aligned}
W[C]_B &= \langle \exp[ig \int_{\mathcal{F}} G_{\mu\nu}(Z) d\sigma_{\mu\nu}(Z)] \rangle_B \\
&= \exp \left[-\frac{g^2}{2!} \int_{\mathcal{F}} \langle G_{\mu\nu}(Z) G_{\kappa\lambda}(Z') \rangle \gg d\sigma_{\mu\nu}(Z) d\sigma_{\kappa\lambda}(Z') \right] \\
&\quad + \frac{g^4}{4!} \int_{\mathcal{F}} \langle G_{\mu\nu}(Z) G_{\kappa\lambda}(Z') G_{\eta\zeta}(Z'') G_{\rho\sigma}(Z'') \rangle \gg d\sigma_{\mu\nu}(Z) d\sigma_{\kappa\lambda}(Z') d\sigma_{\eta\zeta}(Z'') d\sigma_{\rho\sigma}(Z'') + \dots
\end{aligned} \tag{258}$$

Here we have used that due to the Lorentz invariance of the vacuum we have $\langle G_{\mu\nu} \rangle = 0$ and (in an Abelian theory) $\langle G_{\mu\nu} G_{\kappa\lambda} G_{\eta\zeta} \rangle = 0$.

We first discuss the expansion qualitatively. $\langle G(Z)G(Z') \rangle$ is supposed to fall off fast if $|Z - Z'| > \lambda$. By $|\mathcal{F}|$ we denote the area of the surface $\mathcal{F}$ encompassed by the loop C . If we keep Z fixed and if $|\mathcal{F}| \gg \lambda^2$ the integration over $d\sigma_{\kappa\lambda}(Z')$ will lead to an expression of the order λ^2 . The remaining integration over $d\sigma_{\mu\nu}(Z)$ is unconstrained, leading to a factor proportional to $|\mathcal{F}|$. We thus obtain

$$\int_{\mathcal{F}} \langle G_{\mu\nu}(Z) G_{\kappa\lambda}(Z') \rangle d\sigma_{\mu\nu}(Z) d\sigma_{\kappa\lambda}(Z') \sim \langle G^2 \rangle \lambda^2 |\mathcal{F}|. \tag{259}$$

In a similar way we can treat the 4-cluster yielding a contribution $\sim \langle G^4 \rangle \lambda^6 |\mathcal{F}|$. In that way all terms of the cluster expansion are proportional to the area $|\mathcal{F}|$ and we can write

$$W[C]_B = \exp \left[- \left(\Sigma < G^2 > c_{2n} \lambda^{2n-1} \right) |\mathcal{F}| \right] \tag{260}$$

where the constants $c_{2n} < G^{2n}$ depend on the details of the cluster expansion, but the area $|\mathcal{F}|$ is a common factor. In order to obtain this result the application of the cumulant expansion is essential, since the cumulants decrease if *any* pair of arguments becomes distant.

Before I discuss this result further let me investigate the leading cumulant in more detail. Lorentz and translational invariance allows as most general form of the correlator:

$$\begin{aligned}
\langle G_{\mu\nu}(X) G_{\rho\lambda}(X') \rangle &= \frac{1}{12} \langle GG \rangle \cdot (\delta_{\mu\rho} \delta_{\nu\lambda} - \delta_{\mu\lambda} \delta_{\nu\rho}) D(Z^2) \cdot \kappa \\
&\quad + \left[\left(\frac{1}{2} \frac{\partial}{\partial Z_\mu} (Z_\rho \delta_{\nu\lambda} - Z_\lambda \delta_{\nu\rho}) + \frac{1}{2} \frac{\partial}{\partial Z_\nu} (Z_\lambda \delta_{\mu\rho} - Z_\rho \delta_{\mu\lambda}) \right) D_1(Z^2) (1 - \kappa) \right]
\end{aligned} \tag{261}$$

where $\langle GG \rangle$ is the vacuum expectation value of the operator $G_{\mu\nu}(0)G_{\mu\nu}(0)$. Inserting this expression into the cluster expansion we obtain from the leading cluster (i.e. the 2-cluster)

$$\langle W[C] \rangle_B = \exp \left[-\kappa < g^2 G_{\mu\nu} G_{\mu\nu} \rangle \lambda^2 \cdot |\mathcal{F}| \cdot K \cdot \left(1 + O\left(\frac{1}{\mathcal{F}}\right) \right) \right] \tag{262}$$

where the constant K depends on the shape of the scalar function $D(Z^2)$. It is important to note that the result is proportional to κ , the correlator $D_1(Z^2)$ of eq. 261 does not contribute to the term proportional to the area of the loop.

The assumption of a convergent cluster expansion thus leads to the area law of the W-loop if the tensor structure with D in 261 does not vanish. Thus we seemingly obtain linear confinement quite naturally also for Abelian gauge theories.

But in Abelian gauge theory we also have the homogeneous Maxwell equations

$$\partial_\rho \epsilon_{\mu\nu\sigma} G_{\mu\nu} = 0 \tag{263}$$

and a simple calculation shows that these imply for 261 that $\kappa = 0$, i.e. we have *no* area law (and hence no confinement) in an Abelian gauge theory since the relevant correlator vanishes.

Figure 20: Derivation of the non-Abelian Stokes theorem by deformation of the path W from $X_{(1)}$ to $X_{(2)}$

Now let us turn to non-Abelian theories. The first and most welcome difference is that due to the self-coupling of the gluons eq. 263 is no longer valid but we have

$$\partial_\rho \epsilon_{\mu\nu\sigma} G_{\mu\nu}^G = -ig \int_{FGH} (A_\mu^G A_\nu^H) \cdot \epsilon_{\rho\mu\nu\sigma} \neq 0. \tag{264}$$

Hence there is no reason for $\kappa = 0$ and the area law occurs naturally.

After the importance of the correlator $D(Z^2)$ has been stressed by Dosch and Simonov (1988) it has been calculated on the lattice by Di Giacomo and Panagopoulos (1992) using the cooling technique. Their results showed unambiguously that this tensor structure is present and even dominant ($\kappa \approx 0.74$). This special role of the typically non-Abelian correlator $D(Z^2)$ is gratifying, but all other points are much more cumbersome in a non-Abelian gauge theory. The field strength tensors are no longer gauge invariant (see 16) and in order to form gauge invariant correlators we have to transport the colour content of all fields to a reference point Y . This is done by the string operator $\phi(X, Y)$ (see 40)

$$G_{\mu\nu}(X, Y) = \phi^{-1}(X, Y) G_{\mu\nu}(X) \phi(X, Y). \tag{265}$$

Under a local gauge transformation all the parallel transported field strength tensors transform with the gauge transformation at the point Y :

$$G_{\mu\nu}(X, Y) \rightarrow U(Y) G_{\mu\nu}(X, Y) U^\dagger(Y). \tag{266}$$

The Stokes theorem has to be replaced by its non-Abelian version [Bralic 1980, Simonov 1988] By deforming the path as indicated in Fig. 20 we express the line integral $\oint_C B_{\mu\nu}(Z) dZ^\mu$, where the closed path C goes from W to $X_{(1)}$ to $X_{(2)}$ and back to W into the surface integral $\int_{\mathcal{F}} G_{\mu\nu}(Z, W) d\sigma^{\mu\nu}(Z)$. We use that for a sufficiently small surfaces we have

$$\oint_C B_{\mu\nu} dZ^\mu = \int_{\mathcal{F}} G_{\mu\nu} d\sigma^{\mu\nu} + O(|\mathcal{F}|^2). \tag{267}$$

and obtain in this way:

$$P \exp \left[\oint_C -ig A_\mu(Z) dZ^\mu \right] = P_s \exp \left[-ig G_{\mu\nu}(Z, W) d\sigma^{\mu\nu}(Z) \right] \tag{268}$$

where P_s denotes surface ordering according to Fig. 20.

The correlator $\langle g^2 G_{\mu\nu}^F(X, Y) G_{\kappa\lambda}^F(X', Y) \rangle_B$ depends in general also on the reference point Y . In applications generally the assumption is made that it can be approximated by an expression depending only on the difference $X - X'$. Indeed in the limit $X - X' = 0$ this is the case and even if for large

distances the assumption might not be well justified, these contributions are small anyhow due to the finite correlation length. Also the cluster expansion is more delicate for the case of a non-Abelian theory but I refer here only to the discussion in Dösch and Simonov (1988) and the literature quoted there.

In the most restricted and phenomenologically most useful form of the model of the stochastic vacuum all higher cumulants are neglected, i.e. the low frequency contributions of the functional integration over the gluon fields are approximated by a Gaussian stochastic process characterized by the correlator with the same structure as 261:

$$\begin{aligned} < G_{\mu\nu}^f(X, Y) G_{\kappa\lambda}^f(X', Y') > = \frac{\delta^2 G}{12(N_c^2 - 1)} < GG > \cdot \left\{ (\delta_{\mu\kappa} \delta_{\nu\lambda} - \delta_{\mu\lambda} \delta_{\nu\kappa}) D(Z^2) \cdot \kappa \right. \\ & \left. + \left[\frac{1}{2} \frac{\partial}{\partial Z_\mu} (Z_\kappa \delta_{\nu\lambda} - Z_\lambda \delta_{\nu\kappa}) + \frac{1}{2} \frac{\partial}{\partial Z_\nu} (Z_\lambda \delta_{\mu\kappa} - Z_\kappa \delta_{\mu\lambda}) \right] D_1(Z^2) (1 - \kappa) \right\}. \end{aligned} \quad (269)$$

4.5 Applications

4.5.1 Static potentials

From the area law of the Wegner-Wilson loop one obtains, taking only into account the 2-cumulant (Gaussian process) given in (269) by an easy calculation

$$\begin{aligned} V_0(\tau) &= \frac{1}{24N_c} < g_s^2 GG > \left[\tau \int_0^\tau dp \int_{-\infty}^\infty d\tau D(\tau^2 + \rho^2) \right. \\ & \left. + \int_0^\tau p dp \int_{-\infty}^\infty d\tau (-D(\rho^2 + \tau^2) + \frac{1}{2} D_1(\rho^2 + \tau^2)) \right] \end{aligned} \quad (270)$$

The extraction of the spin-dependent terms, which are of order $1/m_Q$, m_Q being the heavy quark mass is more tedious. From phenomenology we know that the spin-dependent contributions of the confining part of the quark-antiquark potential must be specifically different from the one of a Coulomb potential. Very good results are obtained assuming that the confining potential leads to the same spin-dependent forces as a scalar exchange (as opposed to the vector exchange) (see Schnitzer 1978 and Gromes 1984). It has been shown Schiestl and Dösch (1988) and Simonov (1989) that the structure $D(Z^2)$ which is responsible for linear confinement, leads indeed to that result. Again the spin-dependent potentials can be expressed through the correlators D and D_1 leading for the equal mass case to [Simonov (1989)]:

$$\begin{aligned} V_1'(\tau) &= -\frac{1}{24N_c} < g_s^2 GG > \int_0^\tau dp \int_{-\infty}^\infty d\tau \left[(\tau - \rho) \{ D(\tau^2 + \rho^2) + D(\rho^2 + \tau^2) + \rho^2 \frac{\partial}{\partial \rho^2} D_1(\rho^2 + \tau^2) \} \right. \\ & \left. + \tau \rho^2 \frac{\partial}{\partial \tau^2} D_1(\rho^2 + \tau^2) \right] \\ V_2'(\tau) &= \frac{1}{24N_c} < g_s^2 GG > \left[\int_0^\tau dp \int_{-\infty}^\infty d\tau \frac{\rho}{\tau} (D(\tau^2 + \rho^2) + D_1(\rho^2 + \tau^2) + \rho^2 \frac{\partial}{\partial \rho^2} D_1(\rho^2 + \tau^2)) \right. \\ V_3(\tau) &= -\frac{1}{24N_c} < g_s^2 GG > \tau^2 \int_{-\infty}^\infty d\tau \frac{\partial}{\partial \tau^2} D_1(\tau^2 + \tau^2) \\ V_4(\tau) &= \frac{1}{24N_c} < g_s^2 GG > \int_{-\infty}^\infty d\tau \{ 3D(\tau^2 + \tau^2) + 3D_1(\tau^2 + \tau^2) + 2\tau^2 \frac{\partial}{\partial \tau^2} D_1(\tau^2 + \tau^2) \} \end{aligned} \quad (271)$$

The relation of the potentials V_i given here with the spin dependent ones is given in sect. 4.1, 139. The model of the stochastic vacuum has also been extended to the treatment of light quarks by Simonov (1991) using the proper time formalism for the representation of Greens functions. Using saddle-point methods Simonov finds a (state dependent) effective mass of the quarks of a few hundred MeV, very suggestive of a constituent mass. Unfortunately this mass appears in a chirally symmetric theory, thus leading to an (unobserved) parity doubling of states. There are strong indications that the attractive hyperfine structure can cure that deficiency.

4.5.2 Soft High Energy Scattering

The correlator in eq. 269, specifying the Gaussian process that approximates non-perturbative effects of QCD, is the starting point for an evaluation of observables in soft high-energy scattering, i. e. high energy scattering with small momentum transfer. In this application the basic entity is the expectation value of two Wegner-Wilson loops. The model of the stochastic vacuum is even pushed further, since there the leading term is an expectation value of a product of four field strength tensors, which is in the Gaussian approximation reduced to the two-point correlator by means of factorization

In a general analysis on soft high energy scattering Nachtmann (1991) has evaluated the quark-quark scattering amplitude using the eikonal approximation for the interaction of the quarks with the gluon field. In a first step, we follow the same approach, and consider the scattering amplitude of a single quark in an external field A_μ . If the energy of the quark is very high and the background field has only a limited frequency range, the quark moves on an approximately straight light-like line and the eikonal approximation can be applied.

Along its path Γ , the quark picks up the eikonal phase V (which here is a unitary $N_c \times N_c$ matrix)

$$V = P \exp \left[-ig \int_\Gamma A_\mu(z) dz_\mu \right]. \quad (272)$$

Here A_μ is again the Lie-Algebra valued potential and P denotes path ordering (see 41). The phase factor for an antiquark is obtained by complex conjugation.

From the scattering amplitudes for single quarks in the background field, we obtain the non-perturbative quark-quark scattering amplitude by functional integration, with respect to the background field, of the product of the two scattering amplitudes. More specifically, consider two quarks travelling along the light-like paths Γ_1 and Γ_2 given by

$$\Gamma_1 = (x^0, \bar{b}/2, x^3 = x^0) \text{ and } \Gamma_2 = (x^0, -\bar{b}/2, x^3 = -x^0), \quad (273)$$

corresponding to quarks moving with velocity of light in opposite directions, with an impact vector $\bar{b}$ in the $x^1 - x^2$ -plane (referred to in the following as the transverse plane). Let $V_{1,2}(\pm \bar{b}/2)$ be the phases picked up by the quarks along these paths

$$V_{1,2}(\pm \bar{b}/2) = P \exp \left[-ig \int_{\Gamma_{1,2}} A_\mu(z) dz^\mu \right]. \quad (274)$$

Then according to [Nachtmann 1991] the scattering amplitude for two quarks with momenta p_1, p_2 and colours c_1, c_2 leading to two quarks of momenta p_3, p_4 and colours c_3, c_4 is given by

$$T_{c_1 c_2 c_3 c_4}(s, t) = \bar{u}(p_3) \gamma^\mu(p_1) \bar{u}(p_2) \gamma_\mu(p_4) V, \quad (275)$$

where

$$V = i \langle Z_{c_1}^{-2} \rangle_A < \int d^2 \bar{b} e^{-i\bar{q} \cdot \bar{b}} \{ [V_1(-\frac{\bar{b}}{2}) - 1]_{c_1 c_3} [V_2(+\frac{\bar{b}}{2}) - 1]_{c_2 c_4} \} >_A. \quad (276)$$

Here $\langle \rangle_A$ denotes functional integration over the background field; $\bar{q}$ is the momentum transfer $(p_1 - p_3)$ projected on the transverse plane. Of course the approximation makes sense only if $|\bar{q}| \ll |\bar{p}|$. The quantity Z_{c_i} is the fermion wave-function renormalization constant in the eikonal approximation, given by

$$Z_{c_i}[-4] = \frac{1}{N_c} \text{tr}[V^1(0)] = \frac{1}{N_c} \text{tr}[V^2(0)]. \quad (277)$$

Figure 21: Wegner-Wilson loops formed by the paths of quarks and antiquarks inside two mesons. The impact vector $\vec{b}$ is the distance vector between the middle lines of the two loops. $\vec{R}_1$ and $\vec{R}_2$ are the vectors in the transverse plane from the middle lines to the quark lines of meson 1 and 2 respectively. The front lines of the loops guarantee that the mesons behave as singlets under local gauge transformations.

The subtraction of the unit operator from the phase-matrices V is due to the transition from the S to the T -operator.

In the limit of high energies we have helicity conservation

$$\bar{u}(p_3) \gamma^\mu u(p_1) \bar{u}(p_4) \gamma_\mu u(p_2) \xrightarrow{s \rightarrow \infty} 2s \delta_{\lambda_1 \lambda_3} \delta_{\lambda_2 \lambda_4} \quad (278)$$

where λ_i are the helicities of the quarks and $s = (p_1 + p_2)^2$. In the following we can thus ignore the spin degrees of freedom.

The scattering amplitude eq.275 is explicitly gauge dependent and the cautioning remarks made above apply here. But we know that, in meson-meson scattering, for each quark there is an antiquark moving on a nearly parallel line. Furthermore, the meson must be a colour singlet state under local gauge transformations. To construct such a colourless state we have to parallel-transport the colour content from the quark to the antiquark (or vice versa) in the same way as discussed in Sect. 2.5 for the field-strength tensor. Since this parallel-transport of the colours is made by a Schwinger string $\phi(x_q, x_{\bar{q}})$ (see eq.40), we obtain for the meson a Wilson loop whose light-like sides are formed by the quark and antiquark paths, and front ends by the Schwinger strings (see Fig. 21). The direction of the path of an antiquark is effectively the opposite of that of a quark, so that the loop has indeed a well defined internal direction. The resulting loop-loop amplitude is now specified not only by the impact parameter, but also by the transverse extension vectors of the loops. We thus introduce the loop-loop scattering amplitude

$$\begin{aligned} J(\vec{b}, \vec{R}_1, \vec{R}_2) &= \frac{1}{N_c} \text{tr} W_1(0, \vec{R}_1) >_A < \frac{1}{N_c} \text{tr} W_2(0, \vec{R}_2) >_A \\ &< \frac{1}{N_c} \text{tr} [W_2(-\frac{\vec{b}}{2}, \vec{R}_2) - 1] \frac{1}{N_c} \text{tr} [W_1(-\frac{\vec{b}}{2}, \vec{R}_1) - 1] >_A, \end{aligned} \quad (279)$$

where the closed loop ∂S_1 of the W-loop $W_1(-\vec{b}/2, \vec{R}_1)$ is a rectangle whose long sides are formed by the quark path $\Gamma_1^q = (x_0, \vec{b}/2 + \vec{R}_1, x_3 = x_0)$ and the antiquark path $\Gamma_1^{\bar{q}} = (x_0, \vec{b}/2 - \vec{R}_1, x_3 = x_0)$ and whose front sides are formed by lines from $(T, \vec{b}/2 + \vec{R}_1, T)$ to $(T, \vec{b}/2 - \vec{R}_1, T)$ for large positive and

negative T (we will then take the limit $T \rightarrow \infty$). $W_2(\vec{b}/2, \vec{R}_2)$ is constructed analogously. The first factor in eq.279 is the loop renormalization that replaces the quark field renormalization in eq.276.

Our next aim is to perform the functional integration over A by applying the model of the stochastic vacuum discussed in the preceding section. Since the correlator is given in terms of the parallel-transported field tensor $F_{\mu\nu}(x, w)$, we have first to express the line integrals $\int A_\mu dz^\mu$ through integrals over the field tensor. This is done with the help of the non-Abelian Stokes-theorem 268. Since the reference point w in the correlator 269 must be the same for both fields we have to choose a common reference point for both traces in the product

$$< \text{tr} [W_1(-\frac{\vec{b}}{2}, \vec{R}_1) - 1] \text{tr} [W_2(\frac{\vec{b}}{2}, \vec{R}_2) - 1] >_A \quad (280)$$

We choose the point w in the most symmetric way and then the surface emerging from the loop ∂S_1 is formed by the sliding sides of a pyramid with the loop ∂S_1 as basis and the point with coordinates w as apex; the same holds for ∂S_2 .

Before it can be applied to high-energy scattering, the model of the stochastic vacuum must be translated from Euclidean space-time, in which it is naturally formulated, to the Minkowski continuum. Unfortunately we cannot go the other way and continue eq.279 to the Euclidean continuum, which would be the safe way from the point of view of the functional integration. However the Wilson loops occurring in eq.279 have light-like sides which would shrink to a point if continued to a space time continuum with Euclidean metric. We think that this is a serious obstacle for all attempts to evaluate soft high-energy amplitudes numerically on a lattice.

Since we cannot adapt the scattering amplitude to the Euclidean world, we have to go the other way and adapt the model of the stochastic vacuum to the Minkowski world. We are fully aware that this is by no means a trivial step and, pending a better analytical understanding of non-perturbative effects, we have to let the experiment decide on its justification. Thus we must translate the correlation function 269 to the Minkowski world. This is obvious for the tensor structure, where we just substitute $\delta_{\mu\nu}$ by $-g_{\mu\nu}$, etc., but simple choices for the correlation functions like $\exp(-z^2/a^2)$ or $\exp(-\sqrt{z^2}/a)$ cannot be continued in a meaningful way. Therefore, we must look for correlation functions $D(z^2/a^2)$, $D_1(z^2/a^2)$ which fall off for negative z^2 values (corresponding to Euclidean distances), and whose Fourier transforms exist in Minkowski metric, since these will enter essentially in the scattering amplitudes. We write for the correlator, in terms of the Fourier transforms

$$\begin{aligned} g^2 \langle F_{\mu\nu}^A(x, w) F_{\rho\sigma}^B(y, w) \rangle_A &= \frac{\delta^4 B}{N_c^2 - 1} \frac{1}{12} (g^2 F F) \int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \{ (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\nu\rho}) \kappa i \tilde{D}(k^2) \\ &+ (-g_{\sigma\sigma} k_\mu k_\rho + g_{\sigma\rho} k_\mu k_\sigma - g_{\mu\rho} k_\nu k_\sigma + g_{\mu\sigma} k_\nu k_\rho) (1 - \kappa) i \tilde{D}_1(k^2) \}, \end{aligned} \quad (281)$$

where

$$\tilde{D}(k^2) = \int d^4 z D(z^2/a^2) e^{ikz/a} \quad \text{and} \quad i \tilde{D}_1(k^2) = \frac{d}{d(k^2)} \int d^4 z D_1(z^2/a^2) e^{ikz/a}. \quad (282)$$

After this choice is made, all functional integrations can be performed, in principle. The quantities $W_{1,2}$ in eqs.(279) can be expressed as surface integrals, according to eq.255. The exponential being expanded, the expectation value can be calculated according to eq.281 and the factorization in a Gaussian process.

I shall not describe the details of the rather tedious evaluation but rather quote some of the principal results. [Krämer and Dosch 1991], [Dosch and Ferreira 1993]. In an Abelian model (i.e. $\kappa = 0$, only the structure D_1 present) the "scattering amplitude" $J(b, R_1, R_2)$ is independent of R_1 and R_2 as soon as $|R_i| \gg \lambda$, the correlation length of the correlator. The typically non-Abelian structure D , however, leads to a dependence which increases with R_1 and R_2 even if $|R_i| \gg \lambda$. This can be interpreted in the following way:

In an Abelian theory we have only a quark-(anti)quark interaction, whereas in a non-Abelian theory the same correlator which causes confinement gives also rise to a (non-perturbative) string-string interaction.

Figure 22: a) Correlator used in the model of the stochastic vacuum for high energy scattering [Dosch and Ferreira 1993] (solid line) and the one from the lattice calculation [DiGiacomo and Panagopoulos 1992] (dashed line) with $\Lambda_{\text{Lattice}} = 4.4 \text{ MeV}$. The arrows indicate the range inside which lattice data exist. b) Relation between the total cross section and B , the logarithmic slope of the elastic cross section at zero momentum transfer for proton-antiproton scattering. Stars are experimental results. Solid line: prediction of Dosch and Ferreira (1993), dashed line: relation obtained from the Regge amplitude with Pomeron intercept $\alpha(0) = 1.0808$ and slope $\alpha'(0) = 0.25$ [Donnachie and Landshoff 1992]

From the string-string amplitude one comes to a hadron-hadron scattering amplitude by averaging over a transversal wave function $\psi_H(\vec{R})$, which for a meson is simply chosen as a Gaussian

$$\psi_H(\vec{R}) = \sqrt{\frac{2}{\pi S_H}} e^{-R^2/S_H}$$

For the confining correlator the following form is chosen [Krmer and Dosch 1991, Dosch and Ferreira 1993]

$$D((z^2/\lambda^2)) = \int \frac{d^4 k}{(2\pi)^4} e^{ikz/\lambda} \bar{D}(k^2) \bar{D}(k^2) = \frac{27\pi^4}{4 \cdot (k^2 - \frac{9\pi^2}{64})^4}$$

The parameters of the model, the gluon condensate and the correlation length can be adapted to the scattering parameters and a consistent choice is

$$\lambda = 0.35 \text{ fm}, \kappa < g^2 G G > = 1.7 \text{ GeV}^4, S_p = 0.84 \text{ fm}. \quad (283)$$

These parameters yield a string tension (slope of the linear confining potential) $\sigma = 0.18 \text{ GeV}^2$ in full agreement with phenomenological values. Taking the ratio of the hadronic radii S_H equal to the ratio of the electromagnetic radii one can evaluate the pion-nucleon total cross section without additional parameters and one obtains: $\sigma_{\pi N}/\sigma_{NN} = 0.66$. This ratio is here a consequence of the different radii of proton and pion and not of quark additivity [Levin and Frankfurt 1965]. The model also yields the correct flavour dependence of total cross sections at high energies. The correlator $\kappa < g^2 G G > D(Z^2)$ obtained with the parameter set 283 can in the Euclidean region be compared with the result obtained from lattice calculations from DiGiacomo and Panagopoulos (1992). The result is displayed in Fig. 22a)

If the radius parameter S_H is eliminated, the model yields a parameter-free relation between the total cross section and the logarithmic slope B of the elastic cross section at zero momentum transfer.

Fig. 22b) shows that this prediction is in remarkable agreement with experiment (and Regge phenomenology). Concluding one can remark, that the model of the stochastic vacuum does not only yield a good description of soft high energy scattering, but also relates the parameters of hadron spectroscopy to high energy scattering and its fundamental ingredient, the correlator 269, in an excellent agreement with results from lattice gauge theory.

5 Formfactors

The knowledge of the dependence of formfactors on the momentum transfer t is of great theoretical and practical interest and a severe test for theoretical models. For large negative values of t perturbation theory is applicable and the hard line counting rule [Lepage and Brodsky 1977] should apply. It leads to a behavior of the form factors like $1/|t|^{(n-1)}$ where n depends on the number of quarks in the hadron. In non-relativistic quark models the formfactor is the Fourier transform of the wave functions. For a meson consisting of a quark and an antiquark with a wave function ψ depending on the relative distance $\vec{r}$ one learns in elementary quantum mechanics that the formfactor is given by:

$$F(t) = \int d^3 x \psi_M^*(\vec{x}) \psi_M(\vec{x}) \exp(-i\vec{q} \cdot \vec{x}) \quad (284)$$

Here ψ_M and ψ_M^* are the wave functions of the meson in the initial and final state respectively with momenta p and p' , $\vec{q} = \vec{p}' - \vec{p}$ is the momentum transfer vector in the "brick-wall" frame, i.e. $t = (p' - p)^2 = -(\vec{p}' - \vec{p})^2 = -\vec{q}^2$. For small values of momentum transfer the formfactor is determined by the mean radius of the system; for radially symmetric states we obtain immediately from 284:

$$F(t) = 1 + \frac{t}{6} \int \psi_M^*(r) r^2 \psi_M(r) 4\pi r^2 dr + O(t^2) \quad (285)$$

For large values of $|t|$ the behaviour of the formfactor is very sensitive to the form of the wave functions at large distances. Furthermore relativistic effects become important. In Fig. 23 I display the experimental values of the electromagnetic form factor of the pion together with various theoretical results. Fig. 23a) shows the result of a non-relativistic quark model [Jagur *et al.* 1989]. The curve with $\kappa = 1$ is the direct result of the model, an ad-hoc correction term was introduced in order to bring the results of the non-relativistic quark model to better agreement with the data ($\kappa = 0.7$). In ref. [Gromes 1992] a covariant Bethe-Salpeter wave function of the pion was introduced which for quark mass going to infinity becomes the solution of a Salpeter equation with Coulomb potential. It was also constructed in such a way as to have the correct analytic properties in a world with asymptotic quarks. Once the form is chosen, it has only two parameters, the quark mass ($m_q = 0.314 \text{ GeV}$) and another less intuitive constant. The agreement with the experimental values is remarkable, especially at large momentum transfer (Fig. 23 b). The result of a classical application of sum rules to three point functions [Ioffe and Smilga 1982] is given in Fig. 23 c). The physical region in the t -channel starts at $t = 4m_q^2 \approx 0$ (note that in the field-theoretical method of QCD sum rules we work with the well defined Lagrangian mass) and therefore the formfactor can only be evaluated for $t < -1 \text{ GeV}^2$ (-0.5 GeV^2 if one is optimistic). There is also a limitation for large negative values of t . The reason for that is the following: In the expansion of the non-perturbative contributions not only power corrections like $\text{const}/(q^2)^n$ show up, but also those of the form $t/(q^2)^n$ where q is the four-momentum of one of the hadronic channels. Eventually these terms should organize in such a way as not to lead to a blow-up of the formfactor, but since we work with a truncated series, this can only happen in a finite range of t -values. Just think of the exponential function e^t , a truncated series of it will eventually increase (in modulus) also for large negative values of t . This effect becomes clearly visible around $Q^2 = -t \approx 3 \text{ GeV}^2$ in Fig. 23 c). In [Nesterenko and Radyushkin 1982] the $(t$ -independent) terms proportional to the condensates which lead to too large values of the formfactor in [Ioffe and Smilga 1982] for large t -values are suppressed by letting the Borel-parameter M^2 go to infinity

Figure 24: Some graphs relevant for the operator product expansion of the two point function of a baryonic system in an external electromagnetic field. Solid lines represent quarks, small wavy lines photons and large wavy ones interpolating baryonic fields.

of formfactors is a relic from the time of the analytic S-matrix: The vector dominance hypothesis. Though it is not directly related to QCD, it is quite often used as a practical tool even in QCD inspired models and furthermore there has been lately an interesting discussion on the compatibility of this hypothesis with QCD [Jaffe 1990, Jaffe and Mende 1992] to which I shall come back later.

If we insert in the three-point function 234 systems of hadronic states not between the hadronic currents and the weak current, but between the product of the hadronic currents and the weak current we obtain in the approximation that all intermediate states are zero width resonances :

$$\Pi_\mu(p_M, p_{M'}) = \sum_{Z_V} \int d^4x d^4y \exp[i(p_M x - p_{M'} y)] \langle 0 | j_M(x) j_M(y) | Z_V \rangle \langle Z_V | j_\mu(0) | 0 \rangle > \frac{1}{m_V^2 - t} + \text{eventual subtraction terms} \quad (287)$$

Here Z_V signifies particles with masses m_V and the same quantum numbers as the current J_μ ; t is the momentum transfer: $t = (p_M - p_{M'})^2$. The vector dominance hypothesis postulates that subtraction terms are absent and in its practical application that the lowest state is sufficient to give a good approximation to the sum 287. Hence the formfactor is given in this approximation universally by:

$$F(t)/F(0) = \frac{1}{1 - t/m_V^2} \quad (288)$$

where m_V is the mass of the lowest meson with the quantum numbers of the current. This meson must be a vector or axial vector meson whose flavour content is determined by that of the current. For the electromagnetic current it is the ρ and ω meson, for the current 233 it is a vector and axial vector meson with the flavour of an Q and anti- ψ quark.

In Fig. 23e) the pole dominance hypothesis is applied to the electromagnetic formfactor of the π -meson. Fig. 24 f) shows the hadronic graph and e) the comparison with experiment using in 288 the observed mass of the ρ -meson: $m_V = m_\rho = .768$ GeV. We remark a good agreement between experimental values and the parameter free results of the hypothesis.

For the weak current the negative values of t are accessible through scattering, whereas in semileptonic decays the physical, i.e. kinematically allowed region goes from $t = 0$ (more precisely square of the sum of the lepton masses) to the square of the difference of the hadron masses:

$$0 \leq t \leq (m_{H'} - m_H)^2 \quad (289)$$

A large part of this region is accessible to the treatment with QCD sum rules. If in the semileptonic decay the quark m_1 decays into m_2 , we can use sum rules for t values up to 0.5 to 1 GeV² below the perturbative threshold at $(m_1 + m_2)^2$. There is however a delicate point connected with the double dispersion relation for the perturbative contribution, 241. The evaluation of the triangular diagram Fig. 18 a) is very simple, but in order to subtract the contribution of the higher states by the continuum model using quark hadron duality (see 243) it is necessary to represent it as a double dispersion integral. A straightforward application of the Cutkosky and Landau rules [Cutkosky 1960, Landau 1959] leads to singularities in the physical region and therefore in the older literature the semileptonic decay formfactors were only investigated at $t = 0$. The analytic structure of the three-point amplitude $\Pi(s, u, t)$ for fixed

Figure 23: Electromagnetic formfactor of the π -meson calculated in different models: a) [Isur et al. 1989], b) [Gromes 1992], c) [Ioffe and Smilga 1982], d) [Nesterenko and Radyushkin 1982], e) vector dominance model, f) hadronic diagram for vector dominance.

corresponding to an evaluation in the very deep Euclidean region. In this way the results depend of course crucially on the continuum model 220 and all universal non-perturbative contributions have been excluded. A way out of this dilemma is to make a specific ansatz for the large distance behaviour of the non-local condensate [Mikhailov and Radyushkin 1989] [Shuryak 1989] [Ball et al. 1991a] $\langle \bar{q}(x)q(y) \rangle$ occurring through diagram 18b), and not to perform the Taylor expansion which leads to the classical operator product expansion (see 190). The price one has to pay is however quite high, since one leaves the solid ground of the operator product expansion.

Magnetic moments of nucleons, i.e. magnetic formfactors at zero momentum transfer, have been calculated since the first days of the quark model and the qualitative agreement with experiment has been a strong support for this model. In the sum rule approach the usual operator product expansion cannot be expected to give reliable results since for momentum transfer zero one sits directly at the theoretical threshold. In order to overcome this difficulty Ioffe and Smilga (1983) have considered the operator product expansion of two nucleonic currents in a static external electromagnetic field. In the presence of such a field Lorentz symmetry is broken and condensates which are forbidden by the Lorentz invariance of the vacuum may now occur. A particularly important example is the tensor condensate

$$\langle \bar{q} \sigma_{\mu\nu} q \rangle = e F_{\mu\nu} \chi < \bar{q} q \rangle \quad (286)$$

Here e and $F_{\mu\nu}$ are the electromagnetic coupling and field strength tensor respectively and the constant χ is the magnetic susceptibility of the vacuum. It measures the response of the quark pairs in the vacuum to an external field. Its value has now been settled at $\chi \approx -6$ GeV⁻² (see e.g. [Balitsky et al. 1985]). It describes the indirect interaction of the electromagnetic field with the nucleonic current (Fig. 24 c)) which has to be taken into account besides the direct ones (Fig. 24 a, b)). For extensions and variations of this approach see e.g. [Braun and Filyanov 1989]. A very useful tool for estimating the t -dependence

Figure 25: Importance of non-Landau singularities for the double dispersion relation of the triangular diagram. For details see text.

Figure 26: Formfactors of semileptonic decays as function of t as calculated by the sum rule analysis of [Ball *et al.* 1991a]. For details see text.

$t > 0$ as function of s and u is indeed rather intricate. It has been carefully investigated in ref. [Ball 1992a, Ball *et al.* 1991a, Röschenhahler 1993] see also [Nakanishi 1960]. Besides the usual Landau-singularities also so-called non-Landau singularities contribute to the double spectral function $\rho^H(s, u, t)$ and the inclusion of these parts is essential. In Fig. 25 a) the support of the double spectral function in the $s - u$ -plane is displayed for a fixed value of $t > 0$. The light gray area is the region whose boundaries are determined by the Landau conditions [Landau 1959], the dark gray area is the part due to non-Landau singularities. Figure 25 b) shows the importance of this part: The solid curve is the correct result, the dotted line is obtained if the non-Landau singularities are omitted. In Fig. 26 we show the results of the analysis [Ball 1992a, Ball *et al.* 1991a] for the formfactors of the decays $D \rightarrow K e \nu$ and $D \rightarrow K^* e \nu$. The formfactors of decay into a K meson, $f_+(t)$ and the amplitude $V(t)$ of the decay into the K^* (see 239 and 247) are both due to the vector part of the weak current. They show a t -dependence (Fig. 26a) and 26b)) which can be described very well by the vector dominance hypothesis (diamonds). The resulting pole masses are 1.81 GeV for $f_+(t)$ and 1.95 GeV for $V(t)$. The t -dependence of the amplitude $f_+(t)$ has been measured and a pole fit yielded [Bai *et al.* 1991]: $m_V = 1.8^{+0.3}_{-0.2}$ GeV. The mass of the vector particle with the right quantum numbers, the D^* -meson is: $m_{D^*} = 2.01$ GeV. It can also be seen from Fig. 26a,b) that the sum rule is very stable over the full range of t , since the formfactor vary little with a change of the Borel-parameter M^2 . The amplitudes $A_1(t)$ and $A_2(t)$, which are determined by the axial part of the weak current show a very unspecific

Figure 27: Semileptonic decay formfactor for the Decay $B \rightarrow \pi \ell \nu$ calculated with different models. For details see text.

t -dependence and could not be fitted by the pole formula 288 with a reasonable mass m_V . It has been tested that the truncation of the series in the operator product expansion which leads to a completely unrealistic t -dependence for large negative values of t , does not influence the results shown in Fig. 26, since a hybrid treatment with a non local quark condensate yields very similar results. Evidently the first term in the Taylor expansion of $\langle \bar{q}(x)q(y) \rangle$ (see 190) which leads to the mixed condensate $\langle \bar{q}q G q \rangle$ is sufficient in the range of t -values covered by semileptonic decays, to approximate the coordinate dependence of the nonlocal condensate.

The decay $B \rightarrow \pi \ell \nu$ is particularly interesting since it is an important source of information on the transition element from a b to a u -quark. In this transition the kinematically allowed t range is particularly large, it extends from 0 to 26 GeV². Quark models are generally believed to be only reliable in a limited kinematical region, the essentially non-relativistic model of [Isgur *et al.* 1989, Isgur and Wise 1990] at maximum momentum transfer, the relativistic model of [Bauer *et al.* 1985] at zero momentum transfer. In ref. [Bauer *et al.* 1985] the pole formula 288 with the observed mass of the B^* -meson (5.325 GeV) was used to extrapolate from $t = 0$ over the full range of t . Isgur and Wise (1990) however claimed that the B^* - π -coupling depends strongly on the momentum transfer and therefore the pole formula 288 is not applicable to that decay. In Fig. 27 their result is displayed. The long dashed curve is the result of their quark model, the short dashed one is the contribution of the vector pole, i.e. the B^* -meson where the t -dependent coupling $g_{B^*\pi}$ has also been estimated in an extended quark model allowing for creation of quark pairs. In Fig. 27 we also show the result of the sum rule analysis of Ball *et al.* (1991b) which can cover the range $0 \leq t \leq 20$ GeV² (circles). In contradiction to the analysis of the two point sum rules, where the existence of a pole was a presupposition of the analysis, here no assumption was made about the analytical structure in the t -channel. Nevertheless the calculated formfactor shows a behaviour which can be fitted very well by a pole formula (solid line) with a pole mass $m_V = 5.25$ GeV which is only slightly below the mass of the B^* -meson.

In the vector dominance hypothesis the t dependence of the formfactors is determined by the mass of the appropriate vector meson, whereas in the quark model it is determined by the wave function. For mesons consisting of two heavy quarks the non-relativistic quark potential model should give good results and the formfactor calculated in this model should be reliable. This is in contradiction to the vector dominance model. To see that let us consider a meson consisting of two quarks with mass m_q . The fitting vector meson has the mass $m_V = 2m_q(1 + O(\alpha_s))$ which according to 288 leads to a slope of the formfactor $F'(0) = 1/(4m_q^2)$, whereas the form factor obtained from the quark model is determined by the mean square radius $\langle r^2 \rangle > F'(0) = \frac{2}{3}$. The mean square radius of the (effectively Coulombic) heavy meson is given by:

$$\langle r^2 \rangle = \int \psi_{VV}^*(r) r^2 \psi_{VV}(r) 4\pi r^2 dr \approx \frac{27}{4\alpha_s^2 m_q^2}. \quad (290)$$

Since with increasing mass the coupling α_s goes to zero due to asymptotic freedom, the slope of the formfactor obtained from the quark model is much larger than that obtained from vector dominance.

Figure 28: Anomalous and normal thresholds for the deuteron formfactor

In a field theory where the quantum fields correspond to asymptotic particles this phenomenon can be understood in terms of anomalous thresholds [Karplus et al. 1959]. These singularities are branch points in the complex t -plane; they occur for lightly bound systems and are much nearer to the physical region than the poles due to the insertion of the vector meson states 287. The dramatic effects of anomalous thresholds for the analogous case of the deuteron are displayed in Fig. 28. I show the position of the pole of the formfactor of the deuteron due to a vector state consisting of a nucleon (mass m_N) and an anti-nucleon (the equivalent of the vector meson consisting of two heavy quarks), the ρ pole, and the anomalous threshold whose position depends on the mass of the deuteron m_d :

$$t_a = 4m_N^2 - \frac{1}{m_N^2}(m_d^2 - 2m_N^2)^2. \quad (291)$$

The distance of the physical region to the anomalous threshold is only one hundredth of the distance to the two-nucleon threshold! In a confining theory these anomalous thresholds cannot really occur, since quarks do not exist as free particles. The same holds off course for normal singularities, but there the singularities of QCD simulate the hadronic poles and cuts (quark-hadron duality). The question how to treat the cut starting from an anomalous threshold in QCD and if quark hadron duality can be applied to it is an old one [Dosch et al. 1979]. It has been solved in a very realistic model by Jaffe [Jaffe 1990, Jaffe and Mende 1992]. The hadronic singularities of mesons consisting of very heavy quarks behave in such a way, that it is economic to approximate them by a cut starting from the anomalous threshold. The problem has been investigated in the framework of QCD sum rules [Hirsch 1992]. In that case the continuum threshold (see 220) leads indeed to a slope of the formfactor which is much larger than the one induced by a pole (or a cut) at $4m_N^2$. In principle the problem can be investigated in semileptonic decays of quarkonia into quarkonia. The formfactors of decays like $B_c \rightarrow \eta_c$ and $B_c \rightarrow J/\psi$ have been investigated by Bagan et al. [1994]. The slope of the formfactor as obtained from the sum rule analysis is indeed by more than a factor two larger as it is expected to be from vector dominance.

If in a meson with one light and one heavy quark the mass of the heavy one tends to infinity one can again apply the machinery of heavy quark effective theory. Due to the spin independence all semileptonic (and electromagnetic) formfactors of pseudoscalar and vector mesons can be expressed by a single function, the Isgur-Wise function ξ_{IW} (see [Wise 1991]). It depends only on the product of the four-velocities of the mesons in the final and initial state:

$$y \equiv v \cdot v' = \frac{p \cdot p'}{m m'}. \quad (292)$$

Here p , p' , m , and m' are the momenta and masses of the mesons respectively; these quantities coincide up to terms of order $1/m$ with those of the heavy quark.

For the decay of the pseudoscalar heavy meson one obtains:

$$\lim_{m, m' \rightarrow \infty} \frac{1}{\sqrt{m m'}} \langle p' | V_\mu | p \rangle = \xi_{IW}(y)(v + v')_\mu. \quad (293)$$

For the decay of the pseudoscalar meson into a vector meson with polarization ϵ the result is:

$$\lim_{m, m' \rightarrow \infty} \frac{1}{\sqrt{m m'}} \langle p' | (\bar{v}' - A) | p \rangle = i \xi_{IW}(y) \{ -(1+y) \epsilon_\mu^* + 2(\epsilon^* \cdot v) v'_\mu - \epsilon_{\mu\nu\rho\sigma} \epsilon^{*\nu} v^\rho v'^\sigma \}. \quad (294)$$

These formulae can be understood very easily in a nonrelativistic quark model, but it should be noted that their validity does not depend on such a model. The symmetry is restricted to certain kinematical regions, there are even regions where the symmetry breaking terms increase with the heavy quark mass [Ball et al. 1993]. The Isgur-Wise function ξ_{IW} has the value 1 for maximum momentum transfer $y = 1$. It is this requirement which leads to the restrictions on the choice of the continuum threshold and the Borel parameters as discussed in sect. 4.3. The Isgur-Wise function has been evaluated by QCD sum rules [Ball 1992b, Neubert 1991, Blok and Shifman 1993] and in lattice gauge theory [Mandula and Ogilvie 1992, Bernard et al. 1993, Booth et al. 1993, all quoted after Mackenzie 1993]. In the sum rule method special care must be taken for the appropriate choice of the continuum model and the cut-off function $C(s, u)$ in eq. 243. Blok and Shifman (1993) have shown in a special model that an inadequate choice of the cutoff function leads to admixtures of p-wave contributions which distort the result. Radiative corrections [Bagan et al. 1993, Neubert 1993] are found to be of less importance than in the case of the leptonic decay constant.

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Appendix 1

In this appendix I collect some very simple things, which are nevertheless quite important. In a relativistic quantum field theory it is natural to choose the units in such a way that the velocity of light, c , and the Planck action $\hbar$ is equal to one, i.e. to express all quantities in units of $\hbar$ and c . In the rationalized Gauss system we have thus only one unit left, which we may choose as mass, as length, as momentum or as energy. Since $c = 1$, mass, momentum, and energy, which are related by powers of velocity, are measured in the same unit, which we might call energy. This is the reason why we generally speak in particle physics of masses in units of GeV. The length scale is related to the mass scale by powers of c and $\hbar$ and therefore we can also express lengths in units of energy, and we obtain from $\hbar c = .197 \text{ GeV} \cdot \text{fm}$

$$1 \text{ fm} := 10^{-15} \text{ m} = (0.197 \text{ GeV})^{-1} \quad (295)$$

Since the wave number of a particle with momentum p is given by $p/\hbar$ this relation has also an important physical meaning. It tells us which momentum transfer (wavelength) we need, in order to reach a certain spatial resolution, namely with a momentum transfer $|q|$ (in GeV) we reach a spatial resolution $\Delta_s = 0.2/|q|$ (in fm). The canonical dimension of an object is per definition its power in units of energy. Mass has the dimension 1, length the dimension -1, electric charge has dimension zero (remember $e^2/\hbar c = 1/137$, i.e. dimensionless). We can also determine the dimension of field operators, either by using the canonical commutation relations or by remembering that the action is "dimensionless" (unit of the action is $\hbar$). Consider e.g. the kinetic term in the action for a scalar field

$$\int d^4x \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \quad (296)$$

The dimension of length is -1, hence the integration measure brings dimension -4 and the two derivatives +2, and therefore the scalar field has to have the dimension +1 in order that the above given contribution of the kinetic energy to the action is dimensionless. In the same way one finds for the dimension of a vector field also +1 and for that of a spin $\frac{1}{2}$ field the dimension $+\frac{3}{2}$.

The canonical dimension of an operator and its modification by quantum corrections plays an essential role for the behaviour of expectation values under scale transformations (see sect. 2.5).

Finally one can introduce a natural unit for the mass by Newton's gravitational constant G_N . From $G_N = 6.672 \cdot 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} = 6.707 \cdot 10^{-39} \hbar (\text{GeV}/c^2)^{-2}$ we conclude that a natural unit for the mass

is given by $m_p^2 \cdot G_N = 1$ or

$$m_p = 1.22 \cdot 10^{19} \text{ GeV} \quad (297)$$

This is called the Planck mass, since Planck was the first to propose this choice of natural units. One can ask why elementary particles are so much lighter than the Planck mass. The answer to this question might be given by the chiral symmetry (see sect. 2.4). This symmetry forbids the presence of mass terms for fermions. Indeed, as compared to the Planck mass even the expected mass of the top (ca. 150 GeV) is negligible and thus the chiral symmetry is only weakly broken. The corresponding length scale given by $m_p \cdot l_p = 1$ (in units $\hbar/c$) is

$$l_p = 1.6 \cdot 10^{-20} \text{ fm} \quad (298)$$

At these distances the quantum gravitational effects become strong and can no longer be neglected in a fundamental theory.

Appendix 2

The QCD parameter Λ enters as an integration constant which determines the value of the running coupling as solution of the RG equation (see 30). It is not a directly observable quantity and depends on the regularization scheme and the number of flavours chosen. It is convenient to use only the flavours which can be produced at a given scale (i.e. $n_f = 4$ for $\mu < m_b$, the mass of the b-quark). The relation between the values of Λ for different flavours is given by the condition that $\alpha(\mu)$ varies continuously in the transition region. The value of Λ has still rather large errors since it is determined via the strong coupling α_s , which depends on it logarithmically. The current value for Λ_{QCD} with $n_f = 4$ flavours in the $\overline{MS}$ scheme is

$$\Lambda_{n_f=4,\overline{MS}} = 0.26 \pm 0.05 \text{ GeV} \quad (299)$$

The Λ parameter enters as a quantity with dimension even in a pure gauge theory or a gauge theory with massless fermions; this is a consequence of the necessity to regularize the theory.

The first determination of the light quark masses was made by Leutwyler (1974). He used chiral symmetry and the $SU(6)$ symmetry which relates the pseudoscalar with the vector mesons. The chiral symmetry is certainly respectable, but this is not the case for the $SU(6)$ symmetry, which mixes internal degrees of freedom (spin) with external ones (flavour $SU(3)$). He found that small quark masses were really small, $m_u + m_d \approx 11 \text{ MeV}$. One has to see this small number on the background of the so-called constituent masses of the quark models which are around 300 MeV. The value for the mass difference stays on more solid grounds, since it can be deduced from observable data using only chiral symmetry to be

$$\frac{m_d - m_u}{m_d + m_u} = 0.28 \pm 0.03 \quad (300)$$

The value of $(m_u + m_d)$ has been determined by sum rule analyses of the pseudoscalar correlator (see [Narison 1989] and literature quoted there). The following value for the running mass at 1 GeV in the $\overline{MS}$ scheme has been obtained:

$$(\bar{m}_u + \bar{m}_d) = (14.0 \pm 1.1) \text{ MeV} \quad (301)$$

For the single masses one obtains thus

$$\bar{m}_u = 5.2 \text{ MeV} \quad \bar{m}_d = 9.2 \text{ MeV} \quad (302)$$

with errors of around 10 %. The direct use of the pseudoscalar correlator has been questioned because of supposedly large instanton effects (see e.g. [Novikov *et al.* 1981, Geshkenbein and Ioffe 1980]), but the results obtained from that correlator seem consistent.

Since the first days of the quark model [Zweig 1964], the mass difference between the strange and the light quarks was estimated to be of the order of 100-200 MeV. This was deduced from the mass difference between the corresponding strange and nonstrange hadrons. Determinations of the strange quark mass from QCD sum rules yield for the running $\overline{MS}$ mass at 1 GeV values between 110 and 170 MeV, so a conservative estimate is:

$$\bar{m}_s = 140 \pm 30 \text{ MeV} \quad (303)$$

For the heavy quarks it is in most applications advantageous to use the pole mass (defined as in quantum electrodynamics) rather than the running mass in the $\overline{MS}$ scheme (see 34). For the determination of the heavy quark masses the quarkonia systems are in principle an ideal source of information. Unfortunately the treatment of the Coulomb interaction is delicate, and different groups come from the same set of data to different results with non-overlapping errors (see [Voloshin and Zaitsev 1987] and [Reinders 1988]). We give the following conservative values for the pole masses:

$$m_{\text{pole},c} = 1.4 \pm 0.1 \text{ GeV}, \quad m_{\text{pole},b} = 4.65 \pm 0.15 \text{ GeV} \quad (304)$$

This range of values is also obtained from lattice calculations of the bottomonium system [NRQCD 93]. Though the top mass plays no decisive role in non-perturbative QCD, let me just quote that the experimental lower bound of it is 113 GeV, and from radiative corrections in the standard model of electroweak interactions one expects a value around 160 GeV.

I now come to QCD parameters which should follow dynamically from the parameters of the QCD Lagrangian and have indeed been partially calculated in lattice gauge calculations. These parameters are vacuum expectation values of gauge and Lorentz-invariant operator products (see sect. 2.5), they are called condensates in analogy to solid states physics.

The existence of the (light) quark condensate (see 192) was known long before QCD was developed, since it is the indicator for the spontaneous breaking of chiral symmetry. It is related to the quark masses, the pion decay constant f_π and the pion mass through a relation obtained by Gell-Mann, Oakes and Renner (1968)

$$(m_u + m_d) \langle \bar{u}u \rangle + \langle \bar{d}d \rangle = m_\pi^2 f_\pi \quad (305)$$

Due to isospin invariance one expects that $\langle \bar{u}u \rangle$ and $\langle \bar{d}d \rangle$ agree up to less than 1% (see also [Narison 1989]), and one then obtains in the $\overline{MS}$ scheme at 1 GeV:

$$\langle \bar{q}q \rangle := \langle \bar{u}u \rangle = \langle \bar{d}d \rangle = -(23)^3 \text{ GeV}^3 \quad (306)$$

This value agrees well with independent determinations from QCD sum rules for baryons [Ioffe 1981, Chung *et al.* 1981] and lattice calculations.

The condensate of the strange quark is definitely smaller than that of the nonstrange ones. This has been first established from sum rules for the baryon octet and decuplet [Chung *et al.* 1981] and has been confirmed in many other investigations (see [Narison 1989] and literature quoted there). A conservative estimate for the ratio is

$$\frac{\langle \bar{s}s \rangle}{\langle \bar{u}u \rangle} = 0.65 \pm 0.15 \quad (307)$$

This value is interesting since we learn from low energy theorems [Novikov *et al.* 1984a] that the derivative

$$\frac{\partial}{\partial m_q} \langle \bar{q}q \rangle < 0 \quad \text{for} \quad m_q = 0 \quad (308)$$

i.e. one expects for the strange condensate a larger absolute value than for the nonstrange one. On the other hand one expects less and less condensation the heavier the quarks are, since the virtuality increases. Evidently the strange quark with its mass around 150 MeV is already governed by the latter mechanism. For heavy quarks one expects no direct condensation, but only via annihilation into gluons. The mixed condensate $\langle \bar{q}q Gq \rangle$ (see 204 has been determined from baryon sum rules [Ioffe 1981, Dosch *et al.* 1989], and most stringently by a sum rule analysis of the mass difference of the B and B^+ mesons [Narison 1988]. The ratio to the light quark condensate is (again at 1 GeV):

$$\langle g_s \bar{q}q Gq \rangle / \langle \bar{q}q \rangle \approx 0.8 \text{ GeV}^2 \quad (309)$$

with an accuracy of better than 10%. For the strange mixed condensate controversial results exist. Baryon sum rules [Dosch *et al.* 1989] and also a reanalysis of [Belyaev and Ioffe 1983] indicate that the strange mixed condensate is larger than the nonstrange one, whereas the analysis of a correlator especially sensitive to the ratio $\langle g_s \bar{q}q Gq \rangle / \langle \bar{q}q \rangle$ [Khatysimovsky *et al.* 1987, Benecke and Dosch 1992] yield for it a value smaller than one and similar to the ratio $\langle \bar{s}s \rangle / \langle \bar{q}q \rangle$. Presumably the smaller value is more reliable since higher power corrections might not be well under control in the baryon sum rules.

Since there are many independent choices of gauge and Lorentz-invariant operators constructed from four-quark fields, a rough estimate is normally made via the factorization hypothesis. Be q_a^a a light quark of colour A and flavour a , α is the Dirac index, then the factorization hypothesis states:

$$\begin{aligned} \langle \bar{q}_a^A q_b^B \bar{q}_c^C q_d^D \rangle &= \frac{\delta^{AB} \delta^{CD} \delta_{ab} \delta_{cd}}{N_c^2} \langle \bar{q}_a^A q_b^B \rangle \langle \bar{q}_c^C q_d^D \rangle < \frac{\delta^{AB} \delta_{ab} \delta_{cd}}{N_c^2} \langle \bar{q}_a^A q_b^B \rangle \langle \bar{q}_c^C q_d^D \rangle < \frac{\delta^{AB} \delta_{ab} \delta_{cd}}{N_c^2} \langle \bar{q}_a^A q_b^B \rangle \langle \bar{q}_c^C q_d^D \rangle \\ &= - \frac{\delta^{AB} \delta^{CD} \delta_{ab} \delta_{cd}}{N_c^2} \langle \bar{q}_a^A q_b^B \rangle \langle \bar{q}_c^C q_d^D \rangle < \frac{\delta^{AB} \delta_{ab} \delta_{cd}}{N_c^2} \langle \bar{q}_a^A q_b^B \rangle \langle \bar{q}_c^C q_d^D \rangle < \frac{\delta^{AB} \delta_{ab} \delta_{cd}}{N_c^2} \langle \bar{q}_a^A q_b^B \rangle \langle \bar{q}_c^C q_d^D \rangle \end{aligned} \quad (310)$$

This hypothesis is correct in the limit $N_c \rightarrow \infty$. For finite N_c it can only be correct at one special renormalization scale, since operator mixing to order α_s violates factorization (see [Jamin and Kremer 1986]).

The most accurate determination of the gluon condensate $\langle g^2 GG \rangle$ (see 206 comes from sum rule analyses of the charmonium system. In their original paper on sum rules, Shifman Vainshtein and Zakharov [1979b] obtained the value

$$\langle g^2 GG \rangle = 0.47 \text{ GeV}^4 \quad (311)$$

This analysis was extended by Reinders *et al.* (1985) by taking into account all observed ground states of given quantum numbers in the charmonium system. This yields the following range of values

$$0.51 \text{ GeV}^4 \leq \langle g^2 GG \rangle \leq 0.79 \text{ GeV}^4 \quad (312)$$

Other analyses [Bertlmann *et al.* 1988, Geshkenbein 1990, Marrow *et al.* 1987] yield considerable higher bounds for the gluon condensate. The main uncertainties are the error in the pole mass of the charmed meson, radiative corrections and the contributions of higher condensates. The last point is the most difficult to control. Models indicate that they could increase considerably the value of the gluon condensate (see e.g. [Dosch and Lisenfeld 1989]). A conservative estimate for the gluon condensate is

$$\langle g^2 GG \rangle = (0.95 \pm 0.45) \text{ GeV}^4 \quad (313)$$

Quantity	Value	Rp	γ_0	Remarks
Λ_{QCD}	250 MeV	/	/	4 flavours, $\overline{MS}$
$\bar{m}_u$	5 MeV	1 GeV	4	running mass, $\overline{MS}$
$\bar{m}_d$	9 MeV	1 GeV	4	running mass, $\overline{MS}$
$\bar{m}_s$	140 MeV	1 GeV	4	running mass, $\overline{MS}$
$\bar{m}_c$	1.4 GeV	/	/	pole mass
$\bar{m}_b$	4.7 GeV	/	/	pole mass
$\bar{m}_t$	> 113 GeV	/	/	not seen, theoretical value ca. 160 GeV
$\langle g^2 GG \rangle$	$0.95 \pm 0.45 \text{ GeV}^4$	-	0	error estimate conservative, good arguments for value around 0.6 GeV^4
$\langle \bar{q}q \rangle$	-0.013 GeV^3	1 GeV	-4	
$\langle \bar{s}s \rangle$	$0.7 < \bar{q}q >$	1 GeV	-4	
$\langle \bar{q}q Gq \rangle$	$0.8 \text{ GeV}^2 < \bar{q}q >$	1 GeV	-2	
$\langle g_s \bar{q}q Gs \rangle$	$(0.8 \text{ or } 1.2) \text{ GeV}^2 < \bar{s}s >$	1 GeV	-2	controversial, better arguments for smaller value
$\langle \bar{q}q \bar{q}q \rangle$	-	-	-	Factorization is normally applied, might lead to results which are by a factor 2 to 3 too small

Table 2: QCD parameters. The values are only indicative, for a discussion of the errors see text. Rp is the renormalization point of the scale dependent quantities in the $\overline{MS}$ scheme. γ_1 is the first coefficient of the anomalous dimension for $N_c = 3$. The scaling behaviour of the quantity to one loop is given by: $Q(\mu_1) = Q(\mu_2)(\alpha_s(\mu_1)/\alpha_s(\mu_2))^{\gamma_1/\beta_0}$ with $\beta_0 = 11 - 2n_f/3$.

Novikov *et al.* (1984a) have argued convincingly against the larger values (say larger than 0.8 GeV^4) stating that the analyses yielding large upper bounds for the gluon condensate either rely on only one specific channel or are based on particular models. A non-diagonal sum rule analysis of the matrix element $\langle 0 | \bar{A}_\mu | \pi \rangle$ with

$$\bar{A}_\mu = \frac{1}{2} \delta s^\dagger \gamma_\mu \delta \gamma_5 \bar{F}_0 \delta u \quad (314)$$

that is particularly sensitive to the gluon condensate, yields also a low value.

There is, however, another theoretical difficulty. In a world without light quarks the value of the gluon condensate could be considerably larger. A low energy theorem derived by Novikov *et al.* (1981) relates the change of the gluon condensate with (light) quark masses to the quark condensate

$$\frac{d}{dm_q} \langle g^2 GG \rangle = - \frac{96\pi^2}{\beta_0} \langle \bar{q}q \rangle, \quad (315)$$

where $\langle \bar{q}q \rangle$ is the light quark condensate and β_0 is the first coefficient of the Gell-Mann-Low function (see 30). The factor is huge, for $SU(3)$ it is $96\pi^2/\beta_0 \approx 100$. Since the quark condensate $\langle \bar{q}q \rangle$ is negative, an increase of the quark mass leads to an increase of the gluon condensate. In a world of pure gauge fields the gluon condensate would thus have larger value. This *pure gauge* value is obtained from the *empirical* one, which refers to a world with three light flavours, by taking all masses to infinity, i.e. by integrating eq. 315 up to a value of m_q where the quarks decouple. The authors of the theorem estimate that the pure gauge value is about two to three times higher than the empirical value.

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