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Abstract

We examine the problem as to how accurately hadronic distribution amplitudes can be reconstructed from given sum rule moments. It is found that the expansion coefficients of wave functions are uniquely determined by the moments, but additionally are highly sensitive to small changes in the moments. As a consequence, the inherent uncertainty in such moments leads to a considerable indefiniteness of the wave functions. Thus, it seems that the widely used procedure of constructing wave functions from QCD sum rule moments becomes questionable.

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Nowadays, quantum chromodynamics is the generally accepted theory to describe strong interaction phenomena, but, unfortunately, purely perturbative QCD is applicable only in the region of very high momentum transfers. A well-known example for the currently used factorization ansatz into hard and soft (non-perturbative) contributions is the hard-scattering approach to hadronic form factors [1,2]. It has the following form (in the notation of [1,3]):

$$F(Q^2) = \int [dy, dx] \phi^*(y, \tilde{Q}_y^2) T_H(y, x; Q^2) \phi(x, \tilde{Q}_x^2), \quad (1)$$

where $\phi(x)$ is the quark distribution amplitude containing the non-perturbative content of the hadron under consideration. For the time being, we shall confine our attention to the case of the pion for sake of simplicity.

The pion wave function ϕ_π can be expanded into an infinite series of Gegenbauer polynomials $C_i^{3/2}(\zeta)$ [4] which are the eigenfunctions of the evolution equation [1]:

$$\phi_\pi(\zeta, Q^2) = \phi_{as}(\zeta) \sum_{i=0}^{\infty} c_i \left(\frac{\alpha_S(Q^2)}{\alpha_S(\mu_0^2)} \right)^{\gamma_i} C_i^{3/2}(\zeta) \quad . \quad (2)$$

ϕ_{as} is the asymptotic wave function remaining in the formal limit $Q^2 \rightarrow \infty$. If the true function ϕ_π were known, all c_i could be calculated from the orthogonality relation of the Gegenbauer polynomials. As ϕ_π is unknown one must follow another way.

It is possible from sum-rule techniques to obtain the moments $\langle \zeta^n \rangle$ of ϕ_π which are related to the pion distribution amplitude via

$$\langle \zeta^n \rangle := \int_{-1}^1 d\zeta \zeta^n \phi_\pi(\zeta) \quad , \quad (3)$$

with $\langle \zeta^0 \rangle = 1$. These moments were first calculated from QCD sum rules up to 4th order by Chernyak and Zhitnitsky [5], who also gave a simple model for ϕ_π approximately reproducing these moments.

So what information can be obtained about the expansion coefficients c_i of the distribution amplitude ϕ_π when only a limited number of moments is given?

From the well-known property of orthogonal polynomials like the Gegenbauer polynomials, $\langle C_i^{3/2} | \zeta^n \rangle = 0$ for all $i > n$, we get the following relationship between the c_i and $\langle \zeta^i \rangle$:

$$\begin{aligned} c_0 &= \langle \zeta^0 \rangle \\ c_1 &= \sqrt{5} \cdot \langle \zeta^1 \rangle \\ c_2 &= \frac{\sqrt{7} \cdot \sqrt{2} \cdot (-\langle \zeta^0 \rangle + 5 \cdot \langle \zeta^2 \rangle)}{4} \\ c_3 &= \frac{\sqrt{3} \cdot \sqrt{10} \cdot (-3 \cdot \langle \zeta^1 \rangle + 7 \cdot \langle \zeta^3 \rangle)}{4} \\ c_4 &= \frac{\sqrt{11} \cdot \sqrt{5} \cdot (\langle \zeta^0 \rangle - 14 \cdot \langle \zeta^2 \rangle + 21 \cdot \langle \zeta^4 \rangle)}{8} \\ c_5 &= \frac{\sqrt{91} \cdot (5 \cdot \langle \zeta^1 \rangle - 30 \cdot \langle \zeta^3 \rangle + 33 \cdot \langle \zeta^5 \rangle)}{8} \\ c_6 &= \frac{\sqrt{7} \cdot \sqrt{5} \cdot (-5 \cdot \langle \zeta^0 \rangle + 135 \cdot \langle \zeta^2 \rangle - 495 \cdot \langle \zeta^4 \rangle + 429 \cdot \langle \zeta^6 \rangle)}{32} \\ &\vdots \end{aligned} \quad (4)$$

In other words: the lower order part of the wave function is *uniquely fixed by the finite number of moments of the same order*. The higher order polynomials in the expansion remain completely undetermined! For a finite order polynomial there is no freedom left since its coefficients are determined by moments of the same order.

To determine how accurately the procedure described above works in practical calculations, we choose a *model* wave function f for the pion, e.g. see [3,6],

$$\begin{aligned} f(\zeta) &:= N_f \cdot \exp\left(-a \cdot \left(\frac{1}{1+\zeta} + \frac{1}{1-\zeta}\right)\right) \\ &= N_f \cdot \exp\left(-2a \cdot \frac{1}{1-\zeta^2}\right), \end{aligned} \quad (5)$$

where $a = 0.1$ and N_f such that $\int_{-1}^{+1} f(\zeta) d\zeta = 1$. This model allows us to calculate exact moments of any order, so that we can test the reliability of a finite polynomial expansion constructed from a finite number of moments.

To preserve the similarity with practical applications, for example the calculations of Mikhaïlov and Radyushkin [7], we confine ourselves to moments up to 6th order. We furthermore denote an n^{th} -order expansion in Gegenbauer polynomials (at $Q^2 = \mu_0^2$) as $p_n(\zeta)$.

As a next step, we simulate a situation analogous to QCD sum rules where the moments of the true distribution amplitude are not known exactly, but only with an error of approximately 10–20 % [8,9]. To this end, we impose errors δ_i with alternating sign on the exactly known moments $\langle \zeta^i \rangle$ of our model function f , where $|\delta_i| \approx 5\text{--}20\%$. From these moments

FIG. 1. (a) Polynomial approximations $p_6^{\pm\delta}(\zeta)$ of 6th order calculated from the moments of f which have been distorted by errors of $\pm 5\%$ and $\mp 5\%$ in comparison to the exact function $f(\zeta)$. The polynomial expansions do not resemble the model f , and, of course, the situation gets even worse with larger errors. (b) Form factors calculated from the polynomials $p_6^{\pm\delta}$ of the distorted moments in comparison to the 6th order approximation p_6 from the exact moments. Normalization and hard-scattering amplitude T_H were taken from the “real pion” case, $m_g = 323$ MeV [10]. Furthermore, the widely used procedure of calculating form factors with $m_g = 0$ and $\alpha_S(Q^2) \equiv \bar{\alpha}_S = 0.3$ does not help us here. The deviations among the form factors remain exceedingly large.

with associated errors, an approximate polynomial $p_6^{\pm\delta}$ can be constructed via eqs. (4) and the properties of $p_6^{\pm\delta}$ can then be compared to those of the original f .

The results are very surprising, as can be seen from Fig. 1, where $f(\zeta)$ is compared to the polynomials with distorted moments, $p_6^{\pm\delta}(\zeta)$. We see very strong oscillations over the whole range of the interval $[-1; +1]$, which has little to do with the original function! In Fig. 1, also the corresponding form factors are plotted, and again large differences from the “correct” behaviour are observed. Additionally, the moments used for the construction of the test polynomials $p_6^{\pm\delta}$ are compared in Table I to those of the original function f and of Mikhaïlov and Radyushkin [7]. Observing that the moments of our model function f are lying very close to those of ref. [7], we have another striking example: The polynomial $p_6^{(\text{MR})}$ constructed *directly* from the MR-moments via eqs. (4) shows a very different behaviour from that of f . It looks similar to $p_6^{\mp 5\%}$.

TABLE I. Moments $\langle \zeta^n \rangle$ of Mikhaïlov and Radyushkin, the exact model function f and the test polynomials $p_6^{\pm 5\%}$ and $p_6^{\mp 5\%}$.

n	MR	f	$p_6^{\pm 5\%}$	$p_6^{\mp 5\%}$
0	1	1.0000000	1.0000	1.0000
2	0.25 ± 0.01	0.2488002	0.2612	0.2364
4	0.13 ± 0.01	0.1184642	0.1125	0.1244
6	0.07 ± 0.02	0.0693753	0.0728	0.0659

In the last part of this paper, we shall examine the question of whether the results for the pion case can be generalized to the nucleon, i.e. the question of how accurately the nucleon distribution amplitude is determined by the QCD sum-rule moments of Chernyak, Ogloblin and Zhitnitsky [11]. In particular, we examine the convergence property of a polynomial expansion determined directly from the moments (this problem arises also in the pion case).

Since we are dealing with a three-particle system, the quark distribution amplitude of the nucleon, $\phi_N(x_1, x_2, x_3)$, is defined on the triangle $x_1 + x_2 + x_3 = 1$. This implies that there are *two* independent variables, e.g. x_1, x_3 and consequently $x_2 = 1 - x_1 - x_3$. The nucleon distribution amplitude ϕ_N can be expanded into a complete set of orthogonal polynomials defined on the triangle, the so-called Appell polynomials $A_i(x_1, x_2, x_3)$ [1], in complete analogy to the pion in eq. (2). The moments of the nucleon wave function are defined by:

$$\langle x_1^i x_2^j x_3^k \rangle := \int [dx] x_1^i x_2^j x_3^k \cdot \phi_N(x_1, x_2, x_3) \quad . \quad (6)$$

Inserting the triangle condition $x_2 = 1 - x_1 - x_3$ we find the following relation between moments of different order:

$$\langle x_1^i x_2^j x_3^k \rangle = \langle x_1^{i+1} x_2^j x_3^k \rangle + \langle x_1^i x_2^{j+1} x_3^k \rangle + \langle x_1^i x_2^j x_3^{k+1} \rangle, \quad (7)$$

that is to say, the moments are not independent, in contrast to the pion case.

Choosing a polynomial expansion up to 3rd order, we find, similarly to the pion, that the ten expansion coefficients c_i can be *uniquely* determined by the overall normalization of

$\phi_N(x)$ and, for example, the nine 3rd order moments of [11]. The remaining moments have to be expressed as functions of the independent ones according to eq. (7). We have to pay attention to this fact when varying the independent moments so as to keep *all* moments within their error bounds! As the relationship between coefficients and moments becomes quite cumbersome for the nucleon, we give only an illustrating example:

$$c_5 = \left(42\langle x_1^2 x_2^0 x_3^1 \rangle - 21\langle x_1^2 x_2^1 x_3^0 \rangle + 21\langle x_1^0 x_2^3 x_3^0 \rangle + 42\langle x_1^1 x_2^0 x_3^2 \rangle - 21\langle x_1^0 x_2^1 x_3^2 \rangle - 3 \right) / \sqrt{5}. \quad (8)$$

Once again we find large coefficients with different signs in front of the moments.

We now present our results for the nucleon case.

First, we created different sets of moments (A , B , C) by varying the moments of Chernyak, Ogloblin and Zhitnitsky [11] within the given error bounds (see Table II). Via

TABLE II. Moments for three different nucleon model quark distribution amplitudes that are compatible with the moments obtained by Chernyak, Ogloblin and Zhitnitsky from QCD sum rules. The given sets of moments A , B and C fulfill the consistency relation eq. (7). Many more sets can be found yielding distribution amplitudes with different properties. Moment $\langle 111 \rangle$ was not given in [11], but it can, of course, be computed from ϕ .

Moments $\langle n_1 n_2 n_3 \rangle$					
$n_1 n_2 n_3$	COZ		A	B	C
0 0 0	1		1	1	1
1 0 0	0.560	$+0.06$ -0.02	0.560	0.545	0.573
0 1 0	0.192	$+0.008$ -0.012	0.199	0.209	0.194
0 0 1	0.229	$+0.021$ -0.029	0.242	0.247	0.234
2 0 0	0.350	$+0.07$ -0.03	0.354	0.332	0.364
0 2 0	0.084	$+0.004$ -0.019	0.0760	0.0763	0.0770
0 0 2	0.109	$+0.011$ -0.019	0.107	0.107	0.0980
1 1 0	0.090	$+0.01$ -0.01	0.0968	0.103	0.0948
1 0 1	0.102	$+0.008$ -0.012	0.109	0.110	0.114
0 1 1	-0.03 ... 0.03		0.0258	0.0294	0.0218
3 0 0	0.236	$+0.014$ -0.026	0.250	0.238	0.260
0 3 0	0.032	$+0.008$ -0.004	0.0320	0.0359	0.0400
0 0 3	0.052	$+0.004$ -0.004	0.0560	0.0576	0.0560
2 1 0	0.045	$+0.004$ -0.004	0.0490	0.0466	0.0490
2 0 1	0.050	$+0.005$ -0.006	0.0550	0.0478	0.0550
1 2 0	0.035	$+0.002$ -0.008	0.0370	0.0370	0.0300
1 0 2	0.041	$+0.002$ -0.004	0.0430	0.0429	0.0430
0 2 1	-0.004 ... 0.007		0.00700	0.00338	0.00700
0 1 2	-0.005 ... 0.008		0.00800	0.00669	-0.00100
1 1 1	—		0.0108	0.0193	0.0158

relations like eq. (8), the expansion coefficients c_i can be *uniquely* determined from the corresponding moments, and thus we get different 3rd order model wave functions ϕ_N^A , ϕ_N^B , ϕ_N^C . Like the pion case, the graphs of the distribution amplitudes exhibit strong differences between the functions all reproducing the moments within the allowed range. The differences persist in the form factors calculated in the hard-scattering formalism.

Second, and most importantly, we discuss the convergence properties of a model distribution amplitude determined by increasing the order of the moments step by step. To this end, we constructed a sequence of wave functions $\phi_N^{A,1}$, $\phi_N^{A,2}$, $\phi_N^{A,3}$ from the set A of moments, where the number denotes the order of expansion. These functions are plotted in Fig. 2. With increasing order of expansion, the oscillations become stronger, which indicates that the moments do not give a converging expansion as it should be. The size of the form factors of these functions show a strong dependence on the order of expansion of our model (values for $Q^4 F_1^p/\text{GeV}^4$ with an expansion up to 1st order: 0.1, 2nd order: 1, 3rd order: 4). This also contradicts the expectation that higher orders should yield an increasingly refined approximation to the “true” distribution amplitude of the nucleon.

FIG. 2. Comparison of distribution amplitudes ϕ_N of different orders obtained from set of moments A via relations like eq. (8). We recall that, due to the one-to-one relation between moments and coefficients, a polynomial of 3rd order is the solution with the lowest possible degree reproducing given moments up to 3rd order exactly. Since the higher order polynomials do not contribute to the lower order moments the moment values up to 1st, 2nd, ... order yield the coefficients of Appell polynomial approximations of 1st, 2nd, ... order. No convergence can be seen at orders available today.

We conclude that polynomial models for ϕ_π (Gegenbauer polynomials) and ϕ_N (Appell polynomials) cannot be determined reliably from QCD sum rule moments because these unavoidably have errors. The reason for this unreliability lies in the extreme sensitivity of the expansion coefficients c_i to small variations of the moments, which may lead to a completely artificial behaviour. On the other hand, this cannot be avoided because of the *unique* relationship between coefficients and moments.

As a consequence, we point out that all existing polynomial models of hadronic wave functions constructed from QCD sum rules *alone* have to be considered unphysical. Since the false oscillations of the distribution amplitudes also show up in convolution integrals like form factors, these quantities are not suited for comparison with experiments.

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