Natural Gauge Hierarchy in SO(10)

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Abstract

It is shown that a natural gauge hierarchy and doublet-triplet splitting can be achieved in SO(10) using the Dimopoulos-Wilczek mechanism. Artificial cancellations (fine-tuning) and arbitrary forms of the superpotential are avoided, the superpotential being the most general compatible with a symmetry. It is shown by example that the Dimopoulos-Wilczek mechanism can be protected against the effects of higher-dimension operators possibly induced by Planck-scale physics. Natural implementation of the mechanism leads to an automatic Peccei-Quinn symmetry. The same local symmetries that would protect the gauge hierarchy against Planck-scale effects tend to protect the axion also. It is shown how realistic quark and lepton masses might arise in this framework. It is also argued that "weak suppression" of proton decay can be implemented more economically than can "strong suppression", offering some grounds to hope (in the context of SO(10)) that proton decay could be seen at Superkamiokande.
1 Introduction

In this paper we present a realistic and natural supersymmetric grand-unified theory (SUSY-GUT) based on the gauge group SO(10). By a natural SUSY-GUT we mean one that satisfies the following three conditions. (1) The doublet-triplet splitting (or “2/3 splitting”) of the Higgs multiplet does not involve artificial cancellations or “fine-tuning” of parameters; (2) the superpotential has the most general form allowed by some symmetry principle; and (3) local symmetry prevents the appearance in the effective sub-Planck-scale theory of higher dimension operators that would disrupt the gauge hierarchy.

In a previous paper\(^1\) we explored an elegant mechanism, proposed originally in 1981 by Dimopoulos and Wilczek,\(^2\) which achieves the 2/3 splitting without artificial cancellations or fine tuning. There we showed that this mechanism makes possible a simple suppression of proton decay coming from the dimension-five operators mediated by the exchange of the color-triplet higgsinos, while at the same time preserving the wonderfully successful SUSY-GUT prediction\(^3\) of \(\sin^2 \theta_W\).

The Dimopoulos-Wilczek (DW) mechanism, which accomplishes all this, calls for the group SO(10) rather than SU(5) or flipped SU(5)\(^4\) \([SU(5) \times U(1)]\). SO(10) has long been regarded as an especially attractive gauge group for grand unification\(^5\) for a number of other reasons as well. (a) It unifies a whole family into a single irreducible representation. (b) It is automatically anomaly free. (c) It completes each family with a right-handed neutrino. And (d), it automatically builds in “matter parity”, since quarks and leptons are in spinors and Higgs fields are in tensors of SO(10). [Matter parity is a \(Z_2\)
subgroup of the $Z_4$ center of SO(10), whereas the centers of SU(5) and $E_6$ do not contain matter parity.]

In Reference 1 it was not shown that the DW mechanism is natural in the strong sense which is used in this paper and which is defined above. While it was shown that $2/3$ splitting is achieved without artificial cancellations (condition 1), the superpotentials studied in Ref. 1 were not the most general allowed by some symmetry (condition 2); rather, certain terms were simply left out. This is certainly natural in the weaker sense that the non-renormalization theorems of supersymmetry would prevent such omitted terms from being induced by radiative corrections, but it is nevertheless arbitrary. In section 2 we present an SO(10) model which satisfies the first two naturalness conditions. This model is quite simple (simpler than those studied in Ref. 1) and perhaps even a minimal SO(10) SUSY-GUT.

We treat the problem of studying the third naturalness condition in section 3. The reason we treat this separately is that it is unknown whether Planck-scale physics does in fact induce all possible higher-dimension operators (not forbidden by local symmetry) in the effective sub-Planck-scale lagrangian, and, if so, how large their coefficients might be. Conceivably there could be some tremendous suppression that would make it unnecessary to worry about these effects at all. In section 3, however, we will make the most pessimistic assumption that such operators are suppressed only by the dimensionally appropriate powers of the Planck mass: that is, that they are as large as they can be. Even under this assumption it is shown that a straightforward extension of the model of section 2 which is only slightly more complicated can prevent any disruption of the gauge hierarchy by Planck-
scale physics.

In the models of both sections 2 and 3 the symmetries imposed to make the gauge hierarchy natural are closely akin to Peccei-Quinn symmetries\(^\text{(7)}\) and in fact lead automatically to the existence of an invisible axion. This is not completely surprising, since the terms that have to be prevented are those that would produce a large $\mu$-parameter, and it is well-known that a $\mu$-parameter can be prevented by a Peccei-Quinn symmetry.\(^\text{(8)}\) However, we find it significant that a natural gauge hierarchy in SO(10) may not only require the existence of an axion but that in fact the local symmetry that may be needed to protect the hierarchy from Planck-scale-induced higher-dimensional operators may protect the axion from such effects as well.\(^\text{(9)}\) This will be discussed in section 4.

Finally, there is the question of realistic quark and lepton masses. The question is how to avoid the “bad” predictions of SO(10), namely that the up and down quark mass matrices are proportional ($m^0_t : m^0_c : m^0_u = m^0_l : m^0_c : m^0_d$) and that $m^0_\mu = m^0_s$ and $m^0_\tau = m^0_d$. The prediction $m^0_\tau = m^0_\tau$ is, of course, the “good” one; and so the obvious and common suggestion is that higher order effects disturb the relations involving the lighter generations. It turns out that this suggestion is easy to implement in the context of the ideas discussed here. In fact these ideas lend themselves very well to a promising approach to the quark and lepton mass puzzle that has already been advocated in the literature.\(^\text{(10)}\) This will be discussed in section 5. In section 6 we will summarize our results and conclusions.
Consider a supersymmetric SO(10) model with chiral superfields in the following representations: three 16's of quarks and leptons (F_I, I = 1, 2, 3), and Higgs fields that are in two 10's (T_1, T_2), three 45's (A, A', A''), a 54 (S), a 126 (C) and a 126 (C). Let there also be a Z_3 discrete symmetry under which the fields transform as in Table I. Then the most general SO(10)×Z_3-invariant, renormalizable superpotential is given by

\[ W = \sum_i W_i \]

\[ W_1 = \sum_{I,J} \lambda_{IJ} F_I F_J T_1 + \sum_{I,J} \lambda'_{IJ} F_I F_J \tilde{C} \]

\[ W_2 = \lambda_1 T_1 A T_2 + M_1 (T_2)^2 + \lambda_2 T_2 S T_2 \]

\[ W_3 = M_A A' + \lambda_3 S A A' + M_S S^2 + \lambda_4 S^3 \]

\[ W_4 = M''_A A''^2 + \lambda_5 S A''^2 + M_C \tilde{C} C + \lambda_6 \tilde{C} A'' C \]

\[ W_5 = \lambda_7 A A' A''. \]

We assume that the dimensionless couplings, \( \lambda_i \), that appear in \( W \) are of order one and that the mass parameters are of order \( M_{GUT} \approx 10^{16}\, GeV \). (As will be seen shortly, however, \( M_2 + \lambda_2 \langle S \rangle \) must be slightly less than \( M_{GUT} \) – about \( 10^{15}\, GeV \) – in order to suppress proton decay.) Consequently, the VEVs of A, A', A'', S, C, and C will be of order \( 10^{16}\, GeV \).

The function of each term in \( W \) is easily understood. \( W_1 \) is to give mass to the quarks and leptons: \( F F T_1 \) gives Dirac mass to the light fermions and \( F F \tilde{C} \) gives Majorana masses to the right-handed neutrinos. (Of course, by themselves the terms displayed in \( W_1 \) would not give a realistic pattern of
masses, as noted in the introduction. Some attractive and economical ideas for improving this situation are considered in section 5.)

$T_1$ contains the light Higgs doublets, $H$ and $H'$, of the supersymmetric standard model as well as their color-triplet partners, $H_3$ and $H'_3$, that must be made superheavy. This is the well-known $2/3$ splitting referred to in the introduction, which here is accomplished by the Dimopoulos-Wilczek sector, $W_2$. The VEV of the adjoint, $A$, is assumed to have the DW form, $\langle A \rangle = \text{diag}(a, a, 0, 0) \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus the color-triplets in $T_1$ and $T_2$ have a mass matrix of the form

$$
(3(T_1), 3(T_2)) \begin{pmatrix} 0 & \lambda_1 a \\ -\lambda_1 a & M_2 + \lambda_2 \langle S \rangle \end{pmatrix} \begin{pmatrix} 3(T_1) \\ 3(T_2) \end{pmatrix},
$$

which makes all of them superheavy, as $M_2$, $a$, and $\langle S \rangle$ are all of or near the GUT scale, $M_{\text{GUT}} \simeq 10^{16} \text{GeV}$. The doublets, on the other hand, have a mass matrix (ignoring weak-scale effects) of the form

$$
(2(T_1), 2(T_2)) \begin{pmatrix} 0 & 0 \\ 0 & M_2 + \lambda_2 \langle S \rangle \end{pmatrix} \begin{pmatrix} 2(T_1) \\ 2(T_2) \end{pmatrix},
$$

so that the 2 ($\equiv H$) and the 2 ($\equiv H'$) in $T_1$ are light, as required. Higgsino-mediated proton decay happens through the diagram in Fig. 1. This gives a proton-decay amplitude proportional to $(M_2 + \lambda_2 \langle S \rangle)/(\lambda_1 a)^2$, as can be seen directly by inverting the mass matrix in Eq. (2). Thus the proton-decay rate from colored-higgsino exchange has a suppression factor $[(M_2 + \lambda_2 \langle S \rangle)/(\lambda_1 a)^2$ which allows a comfortable consistency with present experimental limits if $M_2 + \lambda_2 \langle S \rangle$ is about $10^{15} \text{GeV} = 10^{-1} M_{\text{GUT}}$. This is what is called “weak suppression” of proton decay in Ref. 1.
The color-triplet higgsinos in the $126$ multiplet can, in general, also induce proton decay, with an amplitude proportional to the neutrino Majorana mass matrix. However, this does not happen in the present model, since $Z_3$ prevents terms such as $C^2S$ and since the color-triplet in $C$ does not mix with the color-triplets in $T_1$ or $T_2$.]

The purpose of $W_3$ is to give the required DW form to $\langle A \rangle$ and $\langle A' \rangle$. If $\langle S \rangle \equiv \text{diag}(s, s, s, -\frac{3}{2}s, -\frac{3}{2}s) \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\langle A \rangle \equiv \text{diag}(a, a, a, b, b) \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then $F_A = 0$ gives

\begin{align*}
(M_A + \lambda_3 s) a &= 0, \\
(M_A - \frac{3}{2}\lambda_3 s) b &= 0.
\end{align*}

One solution is $b = 0, a \neq 0$, and $s = -M_A/\lambda_3$. The $F_A = 0$ equation then forces $b' = 0$ where $\langle A' \rangle \equiv \text{diag}(a', a', a', b', b') \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ so that $\langle A' \rangle$ also has the DW form. The magnitude of the combination $aa'$ is fixed by the $F_S = 0$ equation, but the individual magnitudes of $a$ and $a'$ are not determined until SUSY is broken, when the soft terms $|A|^2$ and $|A'|^2$, whose coefficients are both of order $M_{\text{GUT}}^2$ will set $a \sim a' \sim M_{\text{GUT}}$.

The form of $W_3$ differs in a very significant way from the form considered by Srednicki in Ref. 11 and adopted in Ref. 1. The Srednicki form is $S^3 + M_S S^2 + SA^2 + M_A A^2$. Here $A^2$ has been replaced by $AA'$. The reason for this is the necessity of ruling out an $M_{\text{GUT}}(T_1)^2$ term, which would destroy the gauge hierarchy by making all the Higgs doublets superheavy. (This is just a $\mu$-term with $\mu$ of order $M_{\text{GUT}}$.) The connection with the form of $W_3$ is seen through Fig. 2. The point of Fig. 2 is not that this diagram itself is large - in
fact, due to the non-renormalization theorems of SUSY, it will be suppressed by the SUSY-breaking scale and not endanger the gauge hierarchy. Rather, the point is that if such a diagram can be drawn it implies that no symmetry of the theory forbids a \( (T_1)^2 \) term, and therefore, notwithstanding the non-renormalization theorems that allow this term to be omitted, it is unnatural in the sense of our second condition to leave it out. The vertex \( T_1 A T_2 \) that appears in Fig. 2 is a necessary ingredient of the DW mechanism, and the term \( M_3(T_2)^2 \) is required if there are not to be two pairs of light Higgs doublets, which would be very bad for \( \sin^2 \theta_W \).\(^{(1)} \) Thus any hope of preventing the \( (T_1)^2 \) term by symmetry requires the vertex \( MA^2 \) which appears in Fig. 2 not to exist. Hence our replacement of it by \( MAA' \).

The effect of having only the combination \( AA' \) appear, required to forbid \( (T_1)^2 \), is that the model has an “accidental” Peccei-Quinn symmetry under which \( A \to e^{i\alpha} A, A' \to e^{-i\alpha} A', T_1 \to e^{-i\alpha} T_1 \), and \( F_1 \to e^{i\alpha/2} F_1 \). The resulting axion is invisible, as \( f_a = \langle A \rangle \sim M_{\text{GUT}} \). This will be discussed further in section 4.

The \( W_4 \) sector generates \( SU(5) \)-singlet VEVs for \( A', \bar{C}, \) and \( C \) that break the rank of \( SO(10) \) down to that of the standard model and gives right-handed-neutrino masses. \( W_3 \) and \( W_4 \) together break \( SO(10) \to SU(3) \times SU(2) \times U(1) \).

As explained in Ref. 1, the term \( \lambda_7 A A' A'' \) couples the \( (A, A', S) \) and \( (A'', \bar{C}, C) \) sectors together in such a way as to prevent goldstone bosons without destabilizing the DW form of \( \langle A \rangle \) and \( \langle A' \rangle \).

The model given in Eq. (1) is close to being a minimal realistic and natural \( SO(10) \) SUSY-GUT. It breaks \( SO(10) \) completely to the standard
model, avoids extra light fields (goldstone or otherwise) that would disturb sin$^2 \theta_W$, gives natural 2/3 splitting, allows higgsino-mediated proton decay to be suppressed, and produces quark and lepton masses and the see-saw mechanism for neutrino masses. It is hard to imagine doing all this with a smaller set of fields or fewer couplings. At least one 45 of Higgs fields is needed to break SO(10) and do 2/3 splitting. Because of the antisymmetry of 45 one needs two 10’s to write the term $10_145 10_2$. (As emphasized in Ref. 1 such a doubling is probably necessary in any case to suppress proton decay.) Getting the DW form of VEVs is done most simply with an auxiliary 54. Breaking the rank of SO(10) requires at least either $\mathbf{T20} + 26$ or $\overline{\mathbf{T20}} + 16$. And the tripling of adjoint representations serves two important purposes: allowing the $AA'A''$ term that couples different sectors together without destabilizing VEVs and allowing a Peccei-Quinn-type symmetry to keep the doublets $H$ and $H'$ light.

Certainly this model is not unique. For example, there are other ways to achieve the DW form of VEVs or to break SO(10). But other choices do not seem to lead to models that are simpler with respect to the number of fields or the number of terms in $W$.

From an experimental point of view a very important question is whether higgsino-mediated proton decay can be “strongly suppressed” (in the terminology of Ref. 1) in a fully natural way. The answer is yes, but it appears that a model of considerably greater complexity is required. We have already had to use three adjoints to get VEVs in the DW form. (Note that because we were forced to have terms $AA'$ and $SAA'$ to prevent $(T_1)^2$, the VEVs of $A$ and $A'$ are both compelled to have the DW form.) To get a VEV in the
upside-down DW form, $diag(0, 0, 0, b, b) \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, required for “strong suppression” of proton decay would involve at least two more adjoints for a total of five.

On the grounds of economy it is justified to say that in SO(10) proton decay is more likely to be suppressed “weakly” (ie. numerically by factors of order one) than “strongly”. This holds out some hope that proton decay mediated by higgsinos can be seen experimentally.

3 The Problem of Higher-Dimension Operators

As noted in the introduction, it is not known whether Planck-scale physics necessarily induces all possible higher dimension operators allowed by local symmetry into the effective theory below the Planck scale, and if they do how large the coefficients of such operators would be. If such effects are negligible then the relatively simple SUSY-GUT presented in the previous section is adequate (except for realistic quark and lepton masses, which will be dealt with in section 5).

In this section we will assume the “worst”, namely that every higher-dimension operator allowed by local symmetry is present suppressed only by dimensionally appropriate powers of the Planck mass. We will show by explicit example that the Dimopoulos-Wilczek mechanism can give a natural gauge hierarchy even under this assumption. This will involve identifying the dangerous higher-dimensional operators, exhibiting a local symmetry that rules them out, and showing that this symmetry can satisfy anomaly
constraints. The symmetry used in the model constructed in this section is a single local U(1) for simplicity of analysis. A combination of symmetries, either continuous or discrete, could also play the same role.

There are two kinds of higher-dimensional operators that endanger the gauge hierarchy, those that contain \((T_1)^2\) or \(T_1 \cdot T_2\) and directly give a mass term to the light doublets, and those that destabilize the DW form of the VEV of \(A\). In a model with only two 10's of Higgs fields, such as that in section 2, it is very difficult to prevent the first kind of operator by symmetry. The reason can be seen by considering Fig. 3. This diagram uses vertices \(T_1AT_2\) and \(M(T_2)^2\) that must exist in a satisfactory model with only two 10's; the first for the DW mechanism, and the second to avoid an extra pair of light doublets. The resulting operator is not itself dangerous, as it involves the contraction \(T_1 \cdot A \cdot A \cdot T_1\) and thus, because of the DW form of \(\langle A\rangle\), does not give mass to the doublets in \(T_1\). However, if this term were allowed by local symmetry, then so also would be \((T_1 \cdot T_1)\text{tr}(A \cdot A)/M_{Pl}\) which, by assumption, would be induced by Planck-scale physics and would give a superlarge mass of order \(\mu \sim M_{GUT}^2/M_{Pl}\) to the doublets in \(T_1\). This problem could be alleviated somewhat if the mass of \(T_2\) came from a cubic term, \(N(T_2)^2\) where \(N\) is a singlet superfield, instead of an explicit mass term, \(M(T_2)^2\). Then if the superpotential contained \(N^3\) or \(MN\bar{N}\) a local symmetry would allow \((T_1 \cdot T_1)\text{tr}(A \cdot A)N^2/M_{Pl}^3\) or \((T_1 \cdot T_1)\text{tr}(A \cdot A)\bar{N}/M_{Pl}^3\) giving \(\mu\) to be of order \(M_{GUT}^3/M_{Pl}^3 \sim 10^7\text{GeV}\) or \(M_{GUT}^3/M_{Pl}^3 \sim 10^{10}\text{GeV}\), respectively; in either case too large.

One could imagine an R-symmetry that would prevent such terms. For example, consider \(R \times Z_3\), where under \(R\) all superfields transform as \(\phi \rightarrow -\phi\)
and $W \rightarrow -W$, and under $Z_3$ all superfields transform as $\phi \rightarrow e^{2\pi i/3}\phi$ and $W \rightarrow W$. Then $N(T_2)^2$ and $T_1 AT_2$ would be allowed but no higher-dimensional term would be allowed until $d = 9$. However, we have found no satisfactory model where the $T_1^2 A^2$ terms are forbidden using R-symmetries.

Another approach which we have found to lead to a fully realistic scenario involves the existence of three 10’s, which will be denoted $T_1$, $T_2$, and $T_3$. Suppose there are terms $T_1 AT_2$, $QT_2 T_3$, and $P(T_3)^2$, where $P$ and $Q$ denote singlet superfields with VEVs $\sim M_{GUT}$ (later, for the sake of economy, we will identify $Q$ with $P$ in our illustrative model), and $\langle A \rangle$ has the DW form. Then the $3 \times 3$ mass matrix for the color triplets in $T_i$ has rank 3, while the mass matrix for the doublets has rank 2, with the massless doublet being in $T_1$. The point of this straightforward generalization of the model of section 2 is that the analogue of the diagram of Fig. 3, which is shown in Fig. 4, gives an operator of at least dimension 7: $(T_1 \cdot A \cdot A \cdot T_1)\bar{Q}^2 P$. (It is assumed there is a field $\bar{Q}$ that has coupling $\bar{Q}Q$. If there is no such field then Fig. 4 will give an operator of dimension even greater than 7.) Again, it is not this operator itself which of interest, for given the DW form of $\langle A \rangle$ and the way it is contracted, this operator contributes nothing to $\mu$. But if such a term is allowed then so is $(T_1)^2 \text{tr}(A)^2 \bar{Q}^2 P/M_{Pl}^4$, which gives $\mu \sim M_{GUT}^5/M_{Pl}^4 \sim 10$ TeV. Given that this is a very crude dimensional estimate, this is satisfactorily close to the weak scale.

Contributions to $(T_1 \cdot T_2)$, being off-diagonal in the mass matrix (see Eq. (3)), need only be suppressed to $O(1/M_{Pl}^2)$ to prevent $\mu$ from being larger than $O(1/M_{Pl}^2)$. Thus dimension-four operators like $T_1 \cdot T_2 \text{tr}(A \cdot A^\nu)$ must be forbidden by local symmetry. Given that the term $T_1 AT_2$ must exist (for
the DW mechanism), it must be that $A''$ transforms non-trivially under the local symmetry.

Finally, there are the operators that destabilize the DW form of $\langle A \rangle$. If $\langle A \rangle \sim M_{\text{GUT}} \cdot \text{diag}(1, 1, 1, \epsilon, 0) \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then, since $A$ appears off-diagonally in the mass matrix of the $T_i$, $\mu(T_i)^2$ is induced with $\mu \sim \epsilon^2 M_{\text{GUT}}$. Thus $\epsilon$ must be $\epsilon \left( \frac{M_{\text{GUT}}}{M_{\text{Pl}}} \right)^2$, and the destabilizing term for $\langle A \rangle$ must be at least fifth order in superfields.

A dangerous term would be $(A \cdot A' \cdot \bar{C} \cdot C)$. From the fact that under $SU(2)_L \times SU(2)_R \times SU(4)_C \subset SO(10)$, $\langle \bar{C} \rangle \sim (1, 3, 10)$, $\langle C \rangle \sim (1, 3, \overline{10})$, and $\langle A' \rangle \sim (1, 1, 15)$, it follows that this term acts as a linear term for the $(1, 3, 1)$ component of $A$. (The DW form is in $(1,1,15)$.)

Armed with this information, one can write down an $SO(10) \times U(1)$ model with the fields given in Table II, that avoids all higher-dimension operators dangerous to the gauge hierarchy. The form of the superpotential (including certain relevant higher-dimension operators) is

$$W = F_i F_j T_1 + F_i F_j \bar{C} + T_1 A T_2 + \bar{P} T_2 T_3 + P(T_3)^3 + M_{\text{GUT}} A A' + S A A' + M_{\text{GUT}} S^2 + S^3 + \bar{P} \bar{C} C + \bar{C} A'' C + (A'')^2 (P^2 / M_{\text{Pl}}) + A A' A'' (P / M_{\text{Pl}}) + m_P \bar{P} P + (\bar{P} P)^2 / M_{\text{Pl}},$$

where each term has a dimensionless coefficient which has not been written.

The structure of Eq. (5) is easily understood by comparison with Eq.
Since $A''$ has to transform non-trivially (as noted above, to forbid $T_1 \cdot T_1 \text{tr}(A \cdot A'')$) the terms $AA'A''$ and $MA'\mu$ have now to come from higher-dimension operators (which, by assumption, are present). Similarly, because the singlet, $P$, has a non-trivial $U(1)$ charge, higher-dimension operators must be taken into account if it is to get a VEV. Since $\langle P \rangle \sim \langle \overline{P} \rangle \sim (m_P M_{Pl})^{1/2}$, it must be that $m_P$ is $\sim M_{GUT}^2/M_{Pl}$; this in turn implies that $\langle \overline{C} \rangle \sim \langle C \rangle \sim M_{GUT}^2 M_{Pl}^{1/2} \sim M_{GUT}/30$. [It should be noted that the soft SUSY-breaking terms, $|\overline{C}|^2$, $|C|^2$, et c., will insure that $\langle \overline{C} \rangle \sim \langle C \rangle$ and $\langle \overline{P} \rangle \sim \langle P \rangle$.]

There are three anomaly conditions to be satisfied: $SO(10) \times U(1)$, $\text{gravity} \times U(1)$, and $U(1)^3$. Since there are only two unknowns, $p$ and $q$ (see Table II), these equations are over-determined. But it is clear that additional gauge singlets will contribute to the $\text{gravity} \times U(1)$ and $U(1)^3$ anomalies without in any way affecting the issues discussed up to this point. For example, fields $N_i + \overline{N}_i$ with charges $n_i$ and $(-n_i + p)$ can get mass from $\overline{N} N \overline{P}$ and contribute $+p$ to the $\text{gravity} \times U(1)$ anomaly and $3n_i^2 p - 3n_i p^2 + p^3$ to the $U(1)^3$ anomaly while leaving the $SO(10)^2 \times U(1)$ condition unaffected. The addition of such fields allows all the anomaly conditions to be satisfied. There could, of course, be also additional $SO(10)$-non-singlet fields.

4  Peccei-Quinn Symmetries and Axions

It is apparent that the model exhibited in the last section, like that of section 2, has an “accidental” Peccei-Quinn symmetry under which $A \rightarrow e^{i\alpha} A$, $A' \rightarrow e^{-i\alpha} A'$, $T_1 \rightarrow e^{-i\alpha} T_1$, and $F_1 \rightarrow e^{i\alpha/2} F_1$. In fact, every vector $(p, q)$, where $p$ and $q$ are defined by Table II, corresponds to the generator of a $U(1)$, which
we can denote by $U(1)_{(p,q)}$, so that there are two linearly independent $U(1)$ symmetries. One linear combination is anomaly free, by construction, and is local. The other $U(1)$ has an $SO(10)^2 \times U(1)$ (and thus an $SU(3)_C^2 \times U(1)$) anomaly and qualifies as a Peccei-Quinn symmetry.

The $SO(10)^2 \times U(1)$ anomaly-cancellation condition can be written as $mp + nq = 0$, where $m$ and $n$ are integers that depend on the particle content of the model. Then the local $U(1)$ is just $U(1)_{(n,-m)}$. Each operator will have a charge under $U(1)_{(p,q)}$ of $Mp + Nq$, where $M$ and $N$ depend on the charges of the fields of which it is composed. If $M = N = 0$ then the operator trivially is invariant under the full $U(1) \times U(1)$. If $(M,N) \propto (m,n)$ then the operator will be invariant under the local $U(1)$ but not under the global $U(1)$, that is, the Peccei-Quinn symmetry. Under the assumption of the previous section such an operator will be induced by quantum gravity in the effective superpotential below the Planck scale, and, if it has dimension $D$ and is composed entirely of fields that have VEVs of order $M_{GUT}$, then it will contribute $\Delta m_a \sim M_{GUT}^{D-3}/M_{Pl}^{D-4}$ to the axion mass. To solve the strong CP problem by the Peccei-Quinn mechanism this must be less than the QCD-instanton-generated axion mass, which in this case is given by $m_{a}^{QCD} \sim \Lambda_{QCD}/f_{a} \sim \Lambda_{QCD}^2/M_{GUT}$, and thus $D$ must be greater than 14. In the model constructed in the previous section, the lowest-dimension operator that contributes to the axion mass will be of dimension $D = m + n$ where $m$ and $n$ are defined as above and are normalized to be relatively prime integers.

To take a concrete example, suppose that the quarks and leptons, $F_I$, of the model of section 3 have charge $\frac{1}{2}(\frac{3}{2}p + q)$ so that they get mass directly from the term $\sum_{I,J} \lambda_{I,J} F_I F_J T_I$, and that there are no other $SO(10)$-
non-singlet fields in the theory except those listed in Table II. Then the $SO(10)^2 \times U(1)$ anomaly-cancellation condition turns out to be $31p + 2q = 0$. Consequently, the lowest dimension operators respecting $U(1)_{\text{local}}$ that break $U(1)_{\text{PQ}}$ are of dimension 33. For example, one such is $A^{12} F^{31}$. (See Table II.)

It is apparent that the charge assignments and thus the dimension of the smallest PQ-violating operators are dependent upon details of the model. To take another example, if the quark and lepton masses came instead from a higher-dimension operator like $F_i F_j T_i A^p / M_{\text{GUT}}$ then the charge of $F_i$ would be $\frac{1}{7}(\frac{5}{7}p + q)$, the anomaly condition would be $17p + q = 0$, and the lowest-dimension PQ-violating operator would have $D=18$.

Charges such as these seem rather bizarre. But this model, wherein the local symmetry that controls the Planck-scale physics is just a single $U(1)$, is presented merely as an illustration. The true theory could have, instead of a single $U(1)$, a product of several discrete or continuous symmetries. Then even with the fields having smaller charges the smallest operator that respects all the local symmetries but violates $U(1)_{\text{PQ}}$ could be of high dimension as required.

The important lesson that the illustrative examples teach is that the same local symmetry that protects the gauge hierarchy from the (possible) effects of quantum gravity will also tend to preserve the axion solution to the strong CP problem. It should be emphasized that the axion decay constant, $f_a$, is of order $M_{\text{GUT}} \sim 10^{16} \text{GeV}$, and thus there is a potential “axion energy problem” for cosmology. However, in inflationary scenarios this problem is not necessarily real. But laboratory searches for axionic dark matter
would have a difficulty due to the small cross-sections.

5 Realistic Quark and Lepton Masses

The simple Yukawa term that appears in eq.(1), $\sum_{i,j} \lambda_{ij} F_i F_j T_i$, gives the relations at the GUT scale $M_{\text{lepton}} = M_{\text{down quark}} \propto M_{\text{up quark}}$, and is therefore not satisfactory. In order that these characteristic SU(5) and SO(10) predictions be modified it is necessary that the light fermion mass matrices feel the effects of the breaking of the grand-unified symmetries. The simplest possibility is that higher-dimensional operators contribute to them; for instance, $F_i F_j T_i \tilde{A}/M$, where $\tilde{A}$ is some SO(10)-non-singlet field that has a VEV of order $M_{\text{GUT}}$. If $M = M_{\text{Pl}}$ this term yields contributions to the quark and lepton mass matrices which are too small to cure the problem. (We are assuming that $\tan \beta \cong m_t/m_b$, which would be typical of a simple SO(10) model, in which case the corrections to the lepton and down-quark matrices from gravity would be at most of order $10^{-3} m_t$ and $10^{-3} m_b$.) Thus it would appear that $M$ must be of order $M_{\text{GUT}}$, which implies that these operators are induced not by Planck-scale physics but by integrating out fields with mass $O(M_{\text{GUT}})$ appearing in the effective sub-Planck-scale theory. In other words, new fields must be introduced into the theory.

A most economical possibility is to introduce a vectorlike pair consisting of a family and an anti-family, $F + \bar{F} = 16 + \overline{16}$. Then diagrams like that shown in Fig. 5 become possible. (In that figure, $M$ could be replaced by the VEV of a Higgs field in the product $16 \times \overline{16} = 1 + 45 + 210$, and/or $\tilde{A}$ could be replaced by a singlet, depending on the model.) What makes this
such an economical and compelling idea is that it not only allows SU(5) and
SO(10)-breaking effects to be introduced into the light quark and lepton mass
matrices, but it also suggests a qualitative explanation for various observed
features of those masses. In fact the kind of diagram shown in Fig. 5 is the
basis of a model of light-fermion masses that has already appeared in the
literature.$^{(10)}$

To understand this idea in more detail, consider adding to the model of
section 3 the fields $F + \bar{F}$. And imagine that there is an adjoint Higgs field, $\hat{A}$,
whose VEV points in a direction $Q$ which is a non-trivial linear combination
of $B$-$L$ and $Y$ (weak hypercharge). ($Q$ is, of course, a generator of SO(10).)
Let $F$, $\bar{F}$, $F_1$, and $\hat{A}$ be assigned $U(1)_{local}$ charges such that the diagram in
Fig. 5 exists, but that the cubic term $F_1 F_2 T_1$ is forbidden. Then the effective
mass term that arises from Fig. 5 is

$$\kappa \sum_{I,J} f^c(F_1) \left\{ \hat{b}_I \hat{a}_J \bar{Q} + \bar{Q} \hat{a}_I \hat{b}_J \right\} f(F_J) \cdot \langle T_1 \rangle. \tag{6}$$

The Yukawa couplings, $a_I$ and $b_I$, as shown in Fig. 5, are those that couple $F_1$
to $\bar{F}$ and $F$: $\sum_I a_I F_I \bar{F} \hat{A} + \sum_J F_1 F_2 T_1$. And the hatted quantities in Eq. (6)
are simply defined by $\hat{a}_I \equiv a_I / |a|$ and $\hat{b}_I \equiv b_I / |b|$. The magnitudes of the
couplings and the superheavy VEVs that appear in Fig. 5 are all combined
in the factor $\kappa$. $f^c(F_1)$ and $f(F_J)$ are the anti-fermion and fermion that
are contained in $F_1$. For the down quark matrix, for instance, $f^c(F_1) = d^c_j$
and $f(F_1) = d_j$. Note that the Yukawa couplings, $a_I$ and $b_I$, are vectors in
“family space” rather than $3 \times 3$ matrices as is the $\lambda_{IJ}$ of Eq. (1). Thus
a “factorized” form of the mass matrices results.$^{(14)}$ If there were only one
such factorized term the mass matrices would be rank-1, but as there are two
terms in Eq. (6) each mass matrix is rank-2, thus explaining the extreme
lightness of the first generation (which, of course, is assumed to get mass
from some other, perhaps higher order, operator.)

One may, without loss of generality, choose the basis in family space so
that \( \hat{a}_I = (0, 0, 1) \) and \( \hat{b}_I = (0, \sin \theta, \cos \theta) \). Then for the \( 3 \times 3 \) mass matrix
of fermions of type \( f \) (\( f = \ell^-, d, \) or \( u \)) one has simply

\[
M^{(f)} = \kappa \cdot v^{(f)} \cdot \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & Q_f \sin \theta \\
0 & Q_f \sin \theta & (Q_f + Q_{\ell^-}) \cos \theta
\end{pmatrix}
\]

where \( v^{(f)} \) is defined to be \( v \) for \( f = u \) and \( v' \) for \( f = d \) or \( \ell^- \). Aside
from the fact that they have rank 2 (thus explaining the lightness of \( e^- \), \( u \),
and \( d \)) these matrices have other significant features. If \( \sin \theta \) is somewhat
small, then the second generation has mass of order \( \frac{1}{4} \tan^2 \theta \) times that of the
third generation. Thus \( \sin \theta \sim \frac{1}{3} \) would be sufficient to explain the hierarchy
between the two heavier generations. But most interesting is that the form
of these matrices explains why \( m_\ell^0 \approx m_r^0 \) while \( m_u^0/m_d^0 \), \( m_s^0/m_b^0 \), and \( m_c^0/m_t^0 \)
are all different. For, neglecting effects of order \( \sin^2 \theta \),

\[
\frac{m_\ell^0}{m_r^0} = \frac{Q_d + Q_{\ell^-}}{Q_{\ell^-} + Q_{\ell^+}} = 1,
\]

while

\[
\frac{m_u^0}{m_d^0} \approx \frac{Q_d Q_{\ell^-}/(Q_d + Q_{\ell^-})}{Q_{\ell^-} Q_{\ell^+}/(Q_{\ell^-} + Q_{\ell^+})} = \frac{Q_d Q_{\ell^-}}{Q_{\ell^-} Q_{\ell^+}} \neq 1.
\]

The second equality in Eq. (8) follows from the fact that the same Higgs
field, $H'$, couples to $d^c d$ and to $\ell^\pm \ell^-$, and thus $Q_d + Q_{d^c} = Q_{\ell^-} + Q_{\ell^+} = -Q(H')$ for any generator $Q$; whereas $Q_d Q_{d^c} \neq Q_{\ell^-} Q_{\ell^+}$ in general.

We see that the simple diagram of Fig. 5 can explain why $m^0_u = m^0_d$ is the good relation, while the other SU(5) and SO(10) relations are not good, why the first generation is extremely light compared to the second and third, and can give “Fritzschian” relations between the mass ratios and mixing angles. But obviously this is not the whole story, for some other diagram is needed to give non-zero masses to the first generation. Also, the field denoted $\tilde{A}$ cannot be one of the adjoints $A$, $A'$, or $A''$ of the model of section 3, for $Q$ cannot be either purely $B - L$ or purely $X$ ($X$ being the generator of the $U(1)$ in $SU(5) \times U(1) \subset SO(10)$). For $Q = B - L$ would give a vanishing $(3,3)$ element in Eq. (7), and $Q = X$ would give $m^0_u \cong m^0_\mu$. Nevertheless, the basic approach proposed in Ref. 10 and reviewed here clearly has many attractive features and seems quite compatible with the Dimopoulos-Wilczek framework that we have elaborated in the previous sections. It should be emphasized that a great advantage of SO(10) for approaching the problem of understanding the quark and lepton masses and mixings is that SO(10) relates all the types of fermions – up quarks, down quarks, and leptons – and can provide a natural explanation of the smallness of the KM angles.

6 Conclusions

The Dimopoulos-Wilczek mechanism has been shown to be a completely natural way to achieve doublet-triplet splitting and a gauge hierarchy. Not only can a Higgs sector be constructed for SO(10) which implements this
mechanism,\cite{1,15} but it can be done so in such a way that the superpotential is the most general consistent with some symmetry. Moreover, even if Planck-scale physics induces higher-dimension operators suppressed only by the dimensionally appropriate powers of the Planck mass, it has been shown that local symmetry can protect the DW mechanism from disruption by these operators. It would appear that an invisible axion is an automatic consequence of the DW mechanism if it is implemented in a fully natural way, and that whatever local symmetries may be necessary to protect the gauge hierarchy from Planck-scale effects also tend to protect the axion. Finally, it has been shown how realistic and predictive schemes for quark and lepton masses can be obtained within the framework of the DW mechanism.

There are, of course, other attractive schemes of unification. Each has strong and weak points. SUSY-SU(5) can be made natural if 2/3 splitting is done using the “missing-partner mechanism”. The cost is the introduction of the somewhat high-rank Higgs representations, $50 + \overline{50} + 75$. The main virtue of SU(5) is that it is the smallest grand-unified group. Flipped SU(5) (really $SU(5) \times U(1)$) has the great virtues that the missing-partner mechanism can be implemented in a beautiful and economical way, and that only small representations are required. The main drawbacks are that the group is not simple, so that the great accuracy of the $\sin^2 \theta_W$ prediction has a less straightforward explanation, as has the relation $m_0^0 = m_{\tau}^0$. Another intriguing possibility is the group $SU(3) \times SU(3) \times SU(3)$,\cite{16} which is suggested by some superstring scenarios.

The good points of SO(10) unification are many and well-known. Some of them have been mentioned in the introduction and in section 5 of this
paper. It would seem that a fully natural 2/3 splitting and gauge hierarchy in SO(10) requires the Dimopoulos-Wilczek mechanism. We have shown that this natural and realistic.

Finally, we have argued that “weak suppression” of higgsino-mediated proton decay can be achieved in much simpler models than “strong suppression” (in the terminology of Ref. 1), giving some reason to expect, in the context of SO(10), that proton decay can be seen experimentally.
References


Table I: The particle content of the $SO(10) \times Z_3$ model discussed in section 2. Here $\omega \equiv e^{2\pi i/3}$.

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<th>$T_1$</th>
<th>$T_2$</th>
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<th>$A'$</th>
<th>$A''$</th>
<th>$S$</th>
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<td>54</td>
<td>$\overline{126}$</td>
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Table II: The particle content of the $SO(10) \times U(1)$ model discussed in section 3.

<table>
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<th>$T_3$</th>
<th>$A$</th>
<th>$A'$</th>
<th>$A''$</th>
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<td>$126 \cdot \overline{126}$</td>
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Figure Captions:

Fig.1: A graph showing how in the model of section 2 proton decay is mediated by color-triplet higgsinos. The amplitude is proportional to \((M_2 + \lambda_2 \langle S \rangle)/M_{GUT}\), which can be made small. This is “weak suppression” in the terminology of Ref. 1.

Fig.2: A graph showing that if a term \(M A^2\) exists in the superpotential then symmetry will allow a \((T_1)^2\) term to exist as well.

Fig.3: A graph showing that if a \(M(T_2)^2\) term exists in the superpotential then symmetry will allow a \(T_1 \cdot A \cdot A \cdot T_1\) term. The related term \(T_1 \cdot T_1 \text{tr}(A^2)\) is dangerous to the gauge hierarchy.

Fig.4: The analogue of Fig. 3 in a certain model with three \(10\)'s of Higgs fields, showing that it is not necessary for terms containing \((T_1)^2\) to arise at less than seventh order in superfields.

Fig.5: A graph that can induce a higher-dimension operator that contributes to light quark and lepton masses. These operators involve GUT-symmetry breaking and so can give realistic mass relations.