Two-dimensional SU(N) Gauge Theory on the Light Cone

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Two-dimensional SU(N) gauge theory is accurately analyzed with the light-front Tamm-Dancoff approximation, both numerically and analytically. The light-front Einstein-Schrödinger equation for mesonic mass reduces to the 't Hooft equation in the large \( N \) limit, \( g^2 N \) fixed, where \( g \) is the coupling constant. Hadronic masses are numerically obtained in the region of \( m^2 \ll g^2 N \), where \( m \) is the bare quark (\( q \)) mass. The lightest mesonic and baryonic states are almost in valence. The second lightest mesonic state is highly relativistic in the sense that it has a large 4-body (\( qq\bar{q}\bar{q} \)) component in addition to the valence (\( q\bar{q} \)) one. In the strong coupling limit our results are consistent with the prediction of the bosonization for ratios of the lightest
and second lightest mesonic masses to the lightest baryonic one. Analytic solutions to the lightest hadronic masses are obtained, with a reasonable approximation, as $\sqrt{2Cm(1 - 1/N^2)}^{1/4}$ in the mesonic case and $\sqrt{CmN(N - 1)(1 - 1/N^2)}^{1/4}$ in the baryonic case, where $C = (g^2N\pi/6)^{1/2}$. The solutions well reproduce the numerical ones. The $N$- and $m$-dependences of the hadronic masses are explicitly shown by the analytical solutions.
1. INTRODUCTION AND SUMMARY

Two-dimensional SU($N$) quantum chromodynamics (QCD($N$)$_2$) is a good model for studying ideas and tools which might be feasible in analyses of QCD in 3+1 dimensions. ’t Hooft introduced the model to test the power of the $1/N$ expansion[1]. He summed planar diagrams which dominate the leading order in the expansion and derived an equation. The ’t Hooft equation is valid in the large $N$ limit, $g^2 N$ fixed, where $g$ is the coupling constant. The mass spectrum of the equation reveals a nearly straight "Regge trajectory”.

The large $N$ limit corresponds to the weak coupling one, since $g^2 N$ is fixed. The $1/N$ expansion then works in the weak coupling regime ($g \ll m$, $m$ being the bare quark mass), but not in the strong coupling one, because it is almost impossible to calculate higher-order terms in the expansion. For this reason, QCD($N$)$_2$ in the strong coupling regime has been studied with some other methods so far. Nevertheless, the dynamics is not understood well in the region. The bosonization predicts ratios of meson masses to a baryon mass[2], but they are valid only in the strong coupling limit. The lattice calculation has given a low-energy spectrum of SU(2), but their accuracy is very poor [3].

The discretized light-cone quantization (DLCQ) has been proposed as a useful tool for computing the hadronic mass[4–8]. The mass obtained with DLCQ is a function of $K$, the parameter which characterizes the discretization of the total light-cone momentum, $P$. One then has to take the large $K$ limit to get a physical mass containing no unphysical parameter, but the convergence is very slow for large $g$[5]. Increasing $K$ demands a lot of numerical efforts, so that a reasonably large $K$ was not taken in calculations done so far for strong coupling[6].
Recently the light-front Tamm-Dancoff (LFTD) approximation [9] has been proposed as one of the alternative non-perturbative tools to the lattice gauge theory. In the standard equal-time field theory the vacuum state is a sea of an infinite number of constituents (quarks (q) and gluons in QCD) and it is then unlikely that the vacuum and hadronic states are well described with a finite number of constituents. In fact such a truncation of the Fock space, i.e. the Tamm-Dancoff approximation[10], causes some serious problems[9]. Such problems do not appear in light-front field theory[11], owing to the fact that the vacuum on the light cone is trivial[9]. The LFTD approximation is the Tamm-Dancoff approximation applied to light-front field theory. Both LFTD and DLCQ are based on light-front field theory, but LFTD might be more reliable than DLCQ for strong coupling [12].

On the other hand, LFTD includes its own problems; (a) Non-perturbative renormalization, (b) the relation between spontaneous symmetry breaking and the triviality of the vacuum (what is called, the "zero-mode" problem) and (c) recovery of rotational symmetry. These essential problems has not been settled yet. They, however, would not appear in two-dimensional models such as QCD(N)_2. Coleman’s theorem[13] says that symmetry breaking of global continuous symmetry does not occur in two dimensions. We then do not face the problem (b). The problem (a) and (c) do not exist, from the outset, in two dimensions.

In this paper, we study QCD(N)_2 in the region of g^2N \gg m^2 with LFTD; the region corresponds to the strong coupling region (g \gg m) for small N, and for large N it covers not only the strong coupling region but also the medium and weak coupling ones. We first derive the light-cone Hamiltonian, P^-, and the Einstein-Schrödinger (ES) equation, 2\not{P}P^-|\Psi> = M^2|\Psi>, for hadronic mass and wave func-
tion, $M$ and $\Psi$, in the framework of light-front field theory [11]. As the Tamm-Dancoff approximation, the mesonic wave function with SU($N$) symmetry is truncated to 2-body ($q\bar{q}$) and 4-body ($qq\bar{q}\bar{q}$) components, and the baryonic state to the $N$-body ($q^N$) one. Inclusion of the 4-body state is essential to obtain the results (ii) and (iii) mentioned in the next paragraph. The ES equation is numerically solved by diagonalizing $P^-$ within a space spanned by a finite number of basis functions. All tools needed for this calculation are prepared by our previous work [14] for the massive Schwinger model.

Our main results are summarized as follows.

(i) $P^-$ involves a term proportional to $Q^2/\eta$, where $Q^2$ is the Casimir operator of SU($N$) and $\eta$ is an infinitesimal constant. The term enforces confinement, restricting finite energy solutions to color singlets. This is a well-known property of QCD ($N$)$_2$.

(ii) The ’t Hooft equation is compared with the ES equation which couples the 2-body wave function with the 4-body ones. The couplings are of order $1/\sqrt{N}$, so that the 2-body sector of the ES equation is decoupled from the 4-body one in the large $N$ limit. The 2-body sector, as expected, reduces to the ’t Hooft equation in the limit. The 4-body sector does not generate any bound state in the limit.

(iii) Numerical solutions of the ES equation yield two mesonic and one baryonic bound states for small $N$. Masses of the three states tend to zero as $m^{0.5}$ in the massless limit ($m \rightarrow 0$), as expected from PCAC [14]. Ratios of the two mesonic masses to the baryonic one at small $m$ consist with the predictions [2] of the bosonization within $\sim 7\%$ error. The lowest mesonic and baryonic states are almost in valence. The second mesonic state is highly relativistic in the sense that it has the $q\bar{q}$ and $qq\bar{q}\bar{q}$ components with almost the same magnitude. The existence of such a rela-
tivistic state is unpredictable from the diagramatic consideration based on the $1/N$ expansion, since the ’t Hooft equation as a result of the consideration generates only 2-body states.

(iv) Assuming that the lightest mesonic and baryonic states are in valence, we can obtain approximate solutions to their masses as $\sqrt{2Cm(1 - 1/N^2)^{1/4}}$ in the mesonic case and $\sqrt{CmN(N - 1)(1 - 1/N^2)^{1/4}}$ in the baryonic case, where $C = (g^2N \pi/6)^{1/2}$. The assumption assures that the approximate solutions are accurate in the region of $g^2N \gg m^2$. They show the $N$-dependence of the masses explicitly in the whole range of $N$. The leading order in $1/N$ is $O(N^0)$ for the mesonic mass and $O(N)$ for the baryonic one, as predicted by some works [15–17] based on the expansion. A theoretical surprise is that the next-to-leading order is not $O(N^{-1})$ but $O(N^{-2})$ for the mesonic mass. This makes the ’t Hooft solution more reliable in the case of the lightest mesonic mass.

We derive the light-cone Hamiltonian and the result (i) in sec. 2.1 and the ES equation for hadronic mass and the result (ii) in sec. 2.3. In sec. 2.2 the color-singlet states of meson and baryon are constructed within the truncated Fock space. We first present, in sec. 2.4, the approximate solutions to the lightest hadronic masses and the result (iv). The approximate but analytic solutions are convenient to see the $N$- and $m$-dependences of the masses explicitly. Accuracy of the approximate solutions are tested in sec. 3. Numerical methods and the result (iii) are also presented there. Section 4 is devoted to discussions. Appendices are collections of lengthy expressions.

2. LIGHT-FRONT TAMM-DANCOFF APPROXIMATION

2.1. Light-cone Hamiltonian
The Lagrangian density of QCD for interacting quark and gauge fields, \( \psi \) and \( A_\mu^a \) (\( a = 1 \) to \( N^2 - 1 \)), is

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi, \tag{2.1}
\]

where \( D_\mu = \partial_\mu - igA_\mu^a T^a \) and \( F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf_{abc} A_\mu^b A_\nu^c \) for the generator \( T^a \) and the structure constant \( f_{abc} \) of SU(\( N \)). Light-front field theory [11] starts with the introduction of light-cone coordinates, \( x^\mu = (x^+, x^-) \equiv ((x^0 + x^1)/\sqrt{2}, (x^0 - x^1)/\sqrt{2}) \);

for any other vector, \( V^\pm = (V^0 \pm V^1)/\sqrt{2} \). (We take the same notations and conventions as in Ref. [14].) The equations of motion are

\[
i\sqrt{2} \partial_- \psi_L = m\psi_R,
\]

\[
i\sqrt{2} \partial_+ \psi_R = m\psi_L - \sqrt{2}gA^- \psi_R,
\]

\[
\partial_-^2 A^+ = \sqrt{2}g \psi_R \gamma^+ T^a \psi_R,
\]

\[
-\partial_+ \partial_- A^- = \sqrt{2}g \psi_R \gamma^+ T^a \psi_L + gf_{abc} A^b \partial_- A^c^-.
\]

for the light-cone gauge, \( A^a_+ = 0 \), where \( \psi = (\psi_R, \psi_L)^T \). The first and third equations do not involve the time derivative \( (\partial_+) \) and are therefore just constraints which determine \( \psi_L \) and \( A^- \) in terms of \( \psi_R \). Thus, \( \psi_L \) and \( A^- \) are not independent variables and not subject to a quantization condition. The constraints are then solved with the inverse derivative operator \( \partial_-^{-1} \),

\[
\psi_L(x^-) = -i\frac{m}{\sqrt{2}} \frac{1}{\partial_-} \psi_R
\]

\[
= -i\frac{m}{2\sqrt{2}} \int dy^- \delta(x^- - y^-) \psi_R(y^-), \tag{2.3}
\]

\[
A^a_-(x^-) = \sqrt{2}g \frac{1}{\partial_-} \psi_R \gamma^+ (x^-) T^a \psi_R(x^-), \tag{2.4}
\]

where \( \delta(x) \) is 1 for \( x > 0 \) and -1 for \( x < 0 \). The only independent variable \( \psi_R \) is quantized by an anticommutation relation at the equal light-cone time \( x^+ = y^+ \),

\[
\{\psi_R(x), \psi_R\gamma^+(y)\}_{x^+ = y^+} = \frac{1}{\sqrt{2}} \delta_{ij} \delta(x^- - y^-). \tag{2.5}
\]

\[77\]}
The use of the light-cone coordinates and light-cone gauge thus reduces the number of independent variables. This is an advantage of light-front field theory. The energy-momentum vectors commute mutually and are therefore the constants of motion.

The time component (light-cone Hamiltonian) is

\[ P^- = -\frac{i m^2}{2 \sqrt{2}} \int dx^- dy^- \psi_R \frac{1}{2} \partial_x^- \psi_R(y^-) \]

\[ -\frac{g^2}{2} \int dx^- j^a(x^-) \frac{1}{\partial_x^+} j^a(x^-), \tag{2.6} \]

and the spatial one (light-cone momentum) is

\[ P^+ = i \sqrt{2} \int dx^- \psi_R \frac{1}{\partial_y^-} \partial_y^- \psi_R(x^-). \tag{2.7} \]

The field \( \psi_R \) is expanded at \( x^+ = 0 \) in terms of free waves [18], each with momentum \( k^+ \),

\[ \psi_{iR}(x^-) = \frac{1}{2^{1/4}} \int_{-\infty}^{\infty} \frac{dk^+}{2\pi \sqrt{k^+}} [b_i(k^+) e^{-ik^+ x^-} + d_i^\dagger(k^+) e^{ik^+ x^-}], \tag{2.8} \]

with

\[ \{b_i(k^+), b_j^\dagger(l^+)\} = \{d_i(k^+), d_j^\dagger(l^+)\} = 2\pi k^+ \delta_{ij} \delta(k^+ - l^+), \tag{2.9} \]

The color current, \( j^{a+} \equiv \sqrt{2} : \psi_R \frac{1}{\partial_x^-} \psi_R : \), is normal-ordered with respect to the creation and annihilation operators. The charge is then

\[ Q^a = \int dx^- j^{a+} = \sum_{i,j} (T^a)_{ij} \int_{-\infty}^{\infty} \frac{dk^+}{2\pi k^+} [b_i(k^+) b_j(k^+) - d_i^\dagger(k^+) d_j(k^+)]. \tag{2.10} \]

The last term in \( P^- \) can be rewritten with the standard Fourier transform [19],

\[ \int dx_1^- j^{a+}(x^-) \frac{1}{\partial_x^+} j^{a+}(x^-), \]

\[ = \frac{1}{4} \int dx_1^- dx_2^- dy^- j^{a+}(x^-) e(x^- \cdot y^- \cdot x_2^-) j^{a+}(x^-), \tag{2.11} \]

\[ = \frac{1}{2} \int dx_1^- dx_2^- j^{a+}(x^-)[x_1^- \cdot x_2^-] j^{a+}(x^-) - \frac{1}{4\eta} \sum_{a=1}^{N^2-1} Q^a Q^a + O(\eta), \]

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where use has been made of

\[\epsilon(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk \left( \frac{1}{k + i\eta} + \frac{1}{k - i\eta} \right) e^{ikx}, \quad (2.12)\]

\[|x| = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dk \left[ \frac{1}{(k + i\eta)^2} + \frac{1}{(k - i\eta)^2} \right] e^{ikx}. \quad (2.13)\]

The term \(Q^2/4\eta (Q^2 \equiv \sum Q^a Q^a)\) enforces confinement, restricting finite eigenvalues to the color-singlet \(Q^2 = 0\) subspace.

The Hamiltonian is expressed with the creation and annihilation operators,

\[P^- = P^-_{\text{free}} + P^-_{\text{self}} + P^-_0 + P^-_2, \quad (2.14)\]

\[P^-_{\text{free}} = \frac{m^2}{8\pi^2} \sum_{i=1}^{N} \int_0^\infty \frac{dk}{k^2} [b_i \dagger(k) b_i(k) + d_i \dagger(k) d_i(k)],\]

\[P^-_{\text{self}} = \frac{g^2}{8\pi^2} \sum_{a=1}^{N^2-1} \sum_{i,j,k,l} (T^a)_{ij} (T^a)_{kl} \int_0^\infty \frac{dk_1}{k_1} \delta_{j,k} [b_i \dagger(k_1) b_i(k_1) + d_i \dagger(k_1) d_i(k_1)]\]

\[\times \int_0^\infty dk_2 \left[ \frac{1}{(k_1 - k_2)^2} - \frac{1}{(k_1 + k_2)^2} \right],\]

\[P^-_0 = \frac{g^2}{8\pi^3} \sum_{a=1}^{N^2-1} \sum_{i,j,k,l} (T^a)_{ij} (T^a)_{kl} \int_0^\infty \prod_{m=1}^4 \frac{dk_m}{\sqrt{k_m}} \delta(k_1 + k_2 + k_3 - k_4)\]

\[\times \left\{ [b_i \dagger(k_1) b_k \dagger(k_2) b_l(k_3) b_j(k_4) + d_j \dagger(k_1) d_i \dagger(k_2) d_k(k_3) d_l(k_4)] \frac{1}{2(k_1 - k_4)^2}\]

\[-b_i \dagger(k_1) d_k \dagger(k_2) d_k(k_3) b_j(k_4) \frac{1}{(k_1 - k_4)^2} + b_i \dagger(k_1) d_j \dagger(k_2) d_k(k_3) b_l(k_4) \frac{1}{(k_1 + k_2)^2}\right\},\]

\[P^-_2 = \frac{g^2}{8\pi^3} \sum_{a=1}^{N^2-1} \sum_{i,j,k,l} (T^a)_{ij} (T^a)_{kl} \int_0^\infty \prod_{m=1}^4 \frac{dk_m}{\sqrt{k_m}} \delta(k_1 + k_2 + k_3 - k_4)\]

\[\times [b_i \dagger(k_1) b_k \dagger(k_2) d_i \dagger(k_3) b_j(k_4) + b_i \dagger(k_1) d_k \dagger(k_2) b_l(k_3) b_j(k_4)\]

\[+ d_j \dagger(k_1) d_i \dagger(k_2) b_k \dagger(k_3) d_i(k_4) + d_j \dagger(k_4) b_i(k_3) d_k(k_2) d_i(k_1)],\]

where the integration stands for the Cauchy’s principal-value one. The Hamiltonian does not involve any term having the creation operators only or the annihilation ones only. This indicates that the Fock vacuum is an eigenstate of \(P^-\), i.e. the true
vacuum. The property of the Hamiltonian stems from the conservation of the total light-cone momentum. Each particle must have either zero or a positive momentum, as shown in Eq. (2.8). The creation or the annihilation of particles, each with positive $k^+$, breaks the conservation. An exception is the zero mode ($k^+ = 0$): Only the mode can make the true vacuum non-trivial without breaking the conservation. The mode is thus responsible for non-trivial structure of vacua such as spontaneous symmetry breaking. In the present model, however, the mode is prohibited as long as $m 
eq 0$, because the mass term in $P_{\text{free}}^{-}$ enforces the eigenstate of $P^{-}$ to vanish at $k^+ = 0$ [20].

There appears a force, $b_i^\dagger(k_1)d_i^\dagger(k_2)b_j(k_3)d_j(k_4)$, in $P_{0}^{-}$, after the summation is made over $a$. The force is considered to be induced by the so-called annihilation diagrams where a $q$-$\bar{q}$ pair annihilates into an instantaneous gluon at a vertex while another pair is created at the second vertex. Further discussion will be made in sec. 4.

2.2. Hadronic color-singlet states

The conserved color charges $Q^a$ ($a = 1 \sim N^2 - 1$) are generators of SU($N$). These can be recombined into $N - 1$ operators being mutually commutable and $N(N - 1)/2$ pairs of raising and lowering operators. Whenever these operators act on color-singlet states, the value is always zero. Using the condition, one can easily construct color-singlet states of meson and baryon,

$$ |\Psi_{\text{meson}} \rangle = |\text{meson}\rangle_2 + |\text{meson}\rangle_4, \quad \text{(2.15)}$$

$$ |\text{meson}\rangle_2 = \frac{1}{\sqrt{N}} \int_0^P \frac{dk_1dk_2}{2\pi\sqrt{k_1k_2}} \delta(P - k_1 - k_2) \psi_2(k_1, k_2) \sum_{m=1}^{N} b_m^\dagger(k_1)d_m^\dagger(k_2)|0\rangle, \quad \text{(2.16)}$$
\[
|\text{meson}\rangle_4 = \frac{1}{\sqrt{2N(N+1)}} \int_0^P \prod_{i=1}^4 \frac{dk_i}{\sqrt{2\pi k_i}} \delta(P - \sum_{i=1}^4 k_i) \\
\times \left\{ \psi_A(k_1, k_2, k_3, k_4) \sum_{m=1}^N b_m \dagger(k_1) d_m \dagger(k_2) b_m \dagger(k_3) d_m \dagger(k_4) \\
+ \sqrt{\frac{N+1}{N-1}} \psi_S(k_1, k_2, k_3, k_4) \right\} \left|0\right\rangle,
\]

\[ (2.17) \]

\[
|\Psi\text{\_baryon}\rangle = \int_0^P \delta(P - \sum_{i=1}^4 k_i) \psi_b(k_1, k_2, \cdots, k_N) \prod_{i=1}^N \frac{dk_i}{\sqrt{2\pi k_i}} b_i \dagger(k_1) \cdots b_N \dagger(k_N) \left|0\right\rangle,
\]

\[ (2.18) \]

where the wave functions have some symmetries for the interchange of two momenta,\n
\[
\psi_A(k_1, k_2, k_3, k_4) = -\psi_A(k_3, k_2, k_1, k_4) = -\psi_A(k_1, k_4, k_3, k_2),
\]

\[ (2.19) \]

\[
\psi_S(k_1, k_2, k_3, k_4) = \psi_S(k_3, k_2, k_1, k_4) = \psi_S(k_1, k_4, k_3, k_2),
\]

\[ (2.20) \]

\[
\psi_b(\cdots, k_i, \cdots, k_j, \cdots) = \psi_b(\cdots, k_j, \cdots, k_i, \cdots)
\]

\[ (2.21) \]

for any \( i \) and \( j \). The color-singlet states are expanded in terms of the number of quarks and antiquarks, and truncated to the 2- and 4-body components in the case of the mesonic state and to the \( N \)-body one in the case of the baryonic state. The \( Q^a \)'s do not couple the truncated space with the remainder, so they keep proper commutation relations between them within the truncated space. The truncation, i.e. the Tamm-Dancoff approximation [10], thus does not break the SU(\( N \)) symmetry.

This is an advantage of LFTD. In the equal-time quantization, the corresponding charge operators contain terms having \( b_i d_j \) or \( b_i \dagger d_j \dagger \) which combine the truncated space with the remainder.

2.3. Light-front Einstein-Schrödinger (ES) equation for hadronic mass
The ES equation for hadronic mass is $2 \mathcal{P} \mathcal{P}^- |\Psi\rangle = M^2 |\Psi\rangle$ in the light-front form, where $\mathcal{P}^+$ has been replaced by its eigenvalue $\mathcal{P}$ as a constant of motion. In the equation, $\mathcal{P}$ can be scaled out by changing variables $k_i$ into their fractions $x_i = k_i/\mathcal{P}$: The 2- and 4-body wave functions, $\psi_2(k_1, k_2)$ and $\psi_4(k_1, k_2, k_3, k_4)$, are also replaced by $\psi_2(x_1, x_2)$ and $\psi_4(x_1, x_2, x_3, x_4)/\mathcal{P}$ in $|\Psi_{\text{meson}}\rangle$, while $\psi_b(k_1, \cdots, k_N)$ by $\psi_b(x_1, \cdots, x_N)/\mathcal{P}^{N/2-1}$ in $|\Psi_{\text{baryon}}\rangle$. Left-multiplying the rescaled equation by individual basis of the truncated Fock space leads to a set of coupled equations for the wave functions. In the mesonic case, the equations are lengthy and then presented in Appendix A. In the equations, couplings between the 2- and 4-body sectors are of order $1/\sqrt{N}$ in the $1/N$ expansion, where $g^2N$ is fixed. In the large $N$ limit, the 2-body sector is then decoupled from the 4-body one, and the 2-body sector tends to the ’t Hooft equation. ( This discussion is even clearer in the matrix representation of the coupled equations in Appendix C. ) This conclusion is not changed by further inclusion of the 6-body states, since the resultant equations involve no direct coupling between the 2- and 6-body sectors. According to numerical calculations done in sec. 3, the decoupled 4-body sector does not produce any bound state. All the bound states in the large $N$ limit thus appear as two-body states.

To see the behavior of the mesonic mass in the massless limit ($m/g \to 0$), we integrate Eq. (A.1) over $x$,

$$M^2 \int_0^1 dx \psi_2(x, 1-x) = m^2 \int_0^1 dx \left( \frac{1}{x} + \frac{1}{1-x} \right) \psi_2(x, 1-x),$$

where all interaction terms have been completely canceled to each other. The equation shows that $M = 0$ and/or $\int \psi_2 dx = 0$ at $m = 0$. The first condition says that $M = 0$ for the ground state, and the second one that all excited states giving positive $M$ are orthogonal to 1. Only a state orthogonal to all excited states is the ground
state, so $\psi_2 = 1$ for the ground state. When $\psi_2 = 1$, all couplings between the 2- and 4-body sectors vanish, so that $\psi_A = \psi_S = 0$. It turns out that $M = 0$ and $\psi_2 = 1$ and $\psi_A = \psi_S = 0$ for the ground state. This fact suggests that the ground state has small 4-body components in the region of $m^2 \ll g^2 N$. This will be supported by numerical tests in sec. 3.

The equation for the baryonic wave function $\psi_b$ is

$$\begin{align*}
M^2 \psi_b(x_1, x_2, \cdots, x_N) &= \left( m^2 - \frac{N^2 - 1}{2N} g^2 \right) \left( \sum_{i=1}^{N} \frac{1}{x_i} \right) \psi_b(x_1, x_2, \cdots, x_N) \\
&- \frac{N + 1}{2N} g^2 \int_0^1 dy_1 \int_0^{x_1} dy_2 \sum_{i>j}^{N} \frac{\delta(x_i + x_j - y_1 - y_2)}{(x_i - y_1)^2} \psi_b(x_1, \cdots, y_1, \cdots, y_2, \cdots, x_N),
\end{align*}$$

(2.23)

where $\sum_{i=1}^{N} x_i = 1$ and the $i$-th and $j$-th arguments of $\psi_b$ in the second term have been replaced by $y_1$ and $y_2$, respectively. Again, the equation reduces to Eq. (2.22), when it is integrated over all $x_i$. Equation (2.22) is still derivable, even if the truncated space is extended up to the $(N + 2)$-body state $\psi_{N+2}$. This is explicitly shown in Appendix A for the case of SU(2). Hence, the baryonic mass as well as the mesonic one vanishes in the massless limit, as far as the ground state is concerned. The baryonic wave function is then $\psi_b = 1$ and $\psi_{N+2} = 0$. Just like the mesonic case, this implies that $\psi_{N+2}$ nearly equals to 0 in the region of $m^2 \ll g^2 N$. For this reason, the $(N + 2)$-body component will be neglected in numerical calculations done in sec. 3.

2.4. Approximate solutions to the lightest hadron masses

The 2-body sector of the coupled equations for mesonic wave functions agrees
with the corresponding one in the massive Schwinger model, except for the factor $N(1 - 1/N^2)$ irrelevant to the following statement. ’t Hooft [1] and Bergknoff [20] showed in the massive Schwinger model that the term in $P^-$ involving $m$ enforces $\psi_2(0) = \psi_2(1) = 0$. Following their analysis, we can determine the behavior of $\psi_2$ near $x = 0$ and 1 as $[x(1-x)]^\beta$ with

$$2\pi m^2/[g^2 N(1 - 1/N^2)] - 1 + \pi \beta \cot(\pi \beta) = 0, \quad (2.24)$$

where it is assumed that $2\pi m^2/[g^2 N(1 - 1/N^2)] \ll 1$. In the massless limit $[x(1-x)]^\beta$ tends to 1, that is, the exact solution at $m = 0$, because of $\beta = 0$ there. This strongly implies that $[x(1-x)]^\beta$ is a good approximation to $\psi_2(x)$ at all $x$, as long as $2\pi m^2/[g^2 N(1 - 1/N^2)] \ll 1$. This will be supported by numerical tests in sec. 3. The same discussion can be made for baryon. Inserting $\psi_2(x) = [x(1-x)]^\beta$ or $\psi_b = [x_1 x_2 \cdots x_N]^\beta$ into Eq.(2.22), one can obtain $M$ of the lightest state in the region of $g^2 N(1 - 1/N^2) \gg m^2$ as

$$\sqrt{2C m \sqrt{1 - \frac{1}{N^2}}} \quad (2.25)$$

for the mesonic case and

$$\sqrt{C m N(N-1) \sqrt{1 - \frac{1}{N^2}}} \quad (2.26)$$

for the baryonic case, where $C = (g^2 N \pi/6)^{1/2}$. The approximate solutions show $m$- and $N$-dependences of $M$ explicitly.

In the $1/N$ expansion of the solutions, the leading order is $O(N^0)$ for the mesonic mass and $O(N)$ for the baryonic mass, as expected from topological considerations [13–17]. As an interesting result, the next-to-leading order is not $O(N^{-1})$ but $O(N^{-2})$ for the mesonic mass. This makes the $1/N$ expansion more reliable.
especially for the lightest mesonic mass. This is not the case for other mesonic states and the lightest baryonic one. The approximate solutions also indicate that the hadronic masses behave like $m^{0.5}$ for small $m$. This behavior is also seen in the massive Schwinger model with two flavors[14]. As a result of the behavior, the "pion" decay constant becomes really a constant, indicating that PCAC is a valid concept even for the toy model. This is true also for the present model.

3. NUMERICAL METHOD AND RESULTS

3.1. Basis functions

The truncated ES equations for hadron masses are numerically solved with the variational method: The wave functions are expanded in terms of basis functions, and the coefficients of the expansion are determined by diagonalizing $P^-$ in the space spanned by the basis functions. All tools needed for computations are shown in Ref. [14].

A reasonable choice of the basis functions is

$$\psi_2(x, 1-x) = \sum_{n=0}^{N_2} a_n f_n(x),$$

$$\psi_A(x_1, x_2, x_3, x_4) = \sum_{n=0}^{N_4} b_n G_n(x_1, x_2, x_3, x_4),$$

$$\psi_S(x_1, x_2, x_3, x_4) = \sum_{n=0}^{N_4} c_n G_n(x_1, x_2, x_3, x_4),$$

$$\psi_B(x_1, \cdots, x_N) = \sum_{n=0}^{N_N} d_n F_n(x_1, \cdots, x_N)$$

with

$$f_n = \begin{cases} [x(1-x)]^{\beta+n} \\ [x(1-x)]^{\beta+n}(2x-1) \end{cases},$$

$$-15-$$
\[ G_n = \begin{cases} (x_{1234})^{\beta}(x_{13}^{-1})^{n_1}(x_{24}^{+})^{n_2}(x_{24}^{-})^{n_3} \\ (x_{1234})^{\beta}(x_{13}^{+})^{n_1}(x_{24}^{-})^{n_2}(x_{24}^{+})^{n_3} x_{13}^{+} \end{cases} \] (3.6)

for all \( N \) and

\[ F_n(x, 1 - x) = [x(1 - x)]^{\beta+n} \] (3.7)

for \( \text{SU}(2) \) and

\[ F_n(x_1, x_2, x_3) = \begin{cases} (x_{1234})^{\beta}S[(x_{12}^{-1})^{n_1}(x_{12}^{+})^{n_3}] \\ (x_{1234})^{\beta}S[(x_{12}^{+})^{n_1}(x_{12}^{-})^{n_3}] \end{cases} \] (3.8)

for \( \text{SU}(3) \), where \( \sum x_i = 1 \) for each basis function, \( x_{ij}^{\pm} = x_i \pm x_j \) and \( S \) is the symmetrizer. The subscript \( n \) of \( G_n \) stands for a set \( (n_1, n_2, n_3) \), and for other functions analogously. As already discussed in sec. 2.4, \( f_0 \) ( \( F_0 \) ) is a good approximation to the exact \( \psi_2 \) ( \( \psi_n \) ). Each type of basis functions forms a complete set, when the upper limit \( N_n(n = 2, 4, b) \) of the summation is infinite. The \( G_n \)’s are constructed from the set \( \{(x_{1234})^{\beta}x_{12}^{+}x_{13}^{n_1}x_{23}^{n_2}x_{34}^{n_3} \} \) which obviously form a complete set. First, it is transformed into \( \{(x_{1234})^{\beta}(x_{13}^{-1})^{n_1}(x_{24}^{+})^{n_2}(x_{24}^{-})^{n_3} \} \). Next, the factor \( (x_{13}^{+})^{n_2}(x_{24}^{+})^{n_3} \) in the set is expanded in terms of \( (x_{13}^{+}x_{24}^{+})^{n} \) and \( (x_{13}^{+}x_{24}^{+})^{n}x_{13}^{+} \), where \( x_{13}^{+} + x_{24}^{+} = 1 \). (See Appendix B.) The final form is Eq.(3.6), in which the number of summations has been reduced from 4 to 3. This is a merit of this form. Another merit is that the symmetry for the interchange of \( x_1 \) and \( x_3 \) ( \( x_2 \) and \( x_4 \) ) is easily imposed on \( \psi_4 \) by taking either even or odd \( n_1 \) ( \( n_3 \) ). Similar consideration is made for \( f_n \) and \( F_n \).

3.2. Numerical results

In general, \( M \) calculated with the variational method depends on \( N_{\alpha} \) (\( \alpha = 2, 4, b \)) which characterizes a size of the space spanned by the basis functions, unless the space is large enough to yield an accurate \( M \). In the present calculation, the
space would be sufficiently large, since the dependence is very weak, owing to the
effective choice of basis functions. This is shown in Fig. 1 for the case of SU(2)
meson. Fig. 1(a) represents the \( N_2 \)-dependence of the lightest mass \( (M_1) \) and the
second lightest one \( (M_2) \), while Fig. 1(b) does their \( N_4 \)-dependence. Hereafter, \( m \)
and \( M \) are presented in units of \( \sqrt{g^2 N/2\pi} \). In the case of \( m = 10^{-4} \), \( M_1 \) and \( M_2 \)
converge at \( (N_2, N_4) = (4, 4) \). For the baryonic mass \( (M_b) \), convergence is seen at
\( N_b = 2 \). Our full-fledged calculations are then done with \( (N_2, N_4) = (4, 4) \) for the
mesonic case and with \( N_b = 2 \) for the baryonic one.

The \( m \)-dependence of hadron masses obtained with full-fledged calculations
is shown in Table 1 and Fig. 2 for both SU(2) and SU(3). There are two mesonic
and one baryonic bound states in the range \( m < 0.1 \). The hadronic masses behave
like \( m^{0.5} \) as \( m \to 0 \), as predicted by the approximate solutions ( Eqs. (2.25) and
(2.26)) to \( M_1 \) and \( M_b \). Ratios \( M_1/2M_b \) and \( M_2/2M_b \) at small \( m \), say \( m = 10^{-4} \), are
0.4585 and 0.8122 for SU(2) and 0.2886 and 0.5722 for SU(3), while the corresponding
results, \( \sin\left(\frac{\pi n}{2(N_b-1)}\right) \) (\( n = 1, 2 \)), of the bosonization in the massless limit are 0.5000
and 0.8660 for SU(2) and 0.3090 and 0.5877 for SU(3). The two types of results are
identical within error of \( \sim 10\% \) for SU(2) and \( \sim 5\% \) for SU(3). In general, our
calculations of the baryonic mass are relatively inaccurate compared with those of
the mesonic masses, since the truncated Fock space is smaller in the baryonic case
than in the mesonic one; to be precise, only the valence \( (N\text{-body}) \) state is included
in the baryonic Fock space, while both the valence \( (q\bar{q}) \) and the 4-body \( (qq\bar{q}\bar{q}) \) state
are included in the mesonic space. In the SU(2) case, \( M_1 \) is reduced by \( \sim 10\% \) by
extending the Fock space from the 2-body subspace to the 2-body plus 4-body one.
It is very likely that such a reduction takes place also for \( M_b \), since the ES equation
for $M_b$ is very similar to that for $M_1$ in the SU(2) case, as shown in Appendix A. The 
$\sim 10\%$ error for SU(2) thus might come from the fact that the $(N + 2)$-body state 
is not included in the baryonic Fock space. In the SU(3) case, on the other hand, 
$M_1$ is not changed by the extension. It is then likely that $M_b$ is also unchanged by 
a similar extension of the Fock space from the $N$-body subspace to the $N$-body plus 
$(N + 2)$-body one. Hence, our calculations might be accurate for the SU(3) baryonic 
mass. An unsettled problem is what causes the $\sim 5\%$ error for SU(3). This will be 
discussed in sec. 4.

Unfortunately, our results can not be compared with those of DLCQ in Ref. [6] ; hadronic masses presented by some figures and a table in the paper are inconsistent.

For the lightest mesonic state of SU(2), the probability ($P_2$) of being in the 
2-body state is much larger than that ($P_4$) in the 4-body state; $P_2 = 98.3\%$ and 
$P_4 = 1.7\%$ at $m = 10^{-4}$. The lightest mesonic state is odd under charge conjugation, 
because the 2-body component is symmetric under $x_1 \leftrightarrow x_2$. The second lightest 
state is, on the other hand, highly relativistic in the sense that $P_2 \sim P_4$; $P_2 = 42.9\%$ 
and $P_4 = 57.1\%$ at $m = 10^{-4}$ for SU(2). This state is even under charge conjugation, 
because the 2-body piece is antisymmetric under $x_1 \leftrightarrow x_2$.

The approximate solutions (Eqs.(2.25) and (2.26)) to $M_1$ and $M_b$ are com-
pared with numerical ones obtained with the full-fledged calculations, in two cases of 
SU(2) and SU(3). For SU(2), the approximate $M_1$ agrees with the approximate $M_b$, 
as shown in Eqs. (2.25) and (2.26). They are depicted by the common dashed line 
in Fig. 3(a), and compared with the numerical solutions for $M_1$ (solid line) and for 
$M_b$ (dot-dashed line). The numerical and approximate solutions consist with each 
other at $m < 0.001$ for $M_1$ and at $m < 0.03$ for $M_b$. For SU(3) in Fig. 3(b), the
approximate solutions well reproduce the numerical ones, for both \( M_1 \) and \( M_b \), at \( m < 0.1 \). The agreement would be seen also at \( N \) larger than 3; this is true at least for \( M_1 \) (see Fig. 4). The \( N \)-dependence of \( M_1 \) and \( M_b \) is thus obtained accurately with the approximate solutions, as long as \( m^2 \ll 1 \).

The \( N \)-dependence of \( M_1 \) and \( M_2 \) is shown in Fig. 4(a) for the case of \( m = 10^{-4} \). \( N \) is varied widely from 2 to \( \infty \). The approximate solution to \( M_1 \) (dashed line) well simulates the numerical solution (solid line). As expected from the weak \( N \)-dependence of the approximate \( M_1 \), \( M_1 \) at small \( N \) is close to that at \( N \to \infty \) (the lightest ’t Hooft mass). The second mass is below the threshold \( (2M_1) \) for \( N = 2 \) and 3, but not for \( N \geq 4 \). The second mesonic state is thus bound only for small \( N \) such as 2 and 3. For \( N \to \infty \), on the other hand, it becomes a 4-body unbound state absent in the ’t Hooft solution. The 1/\( N \) expansion thus works well for the first mesonic mass, but not for the second mass.

The \( N \)-dependence of \( M_b \) is shown in Fig. 4(b). The approximate solution to \( M_b \) (the solid line) well reproduces the numerical result (the dashed line) for \( N = 2 \) and 3. If the mass is expanded in 1/\( N \) and truncated to the leading, as in Ref. [15–17], the mass should be proportional to \( N \). The ratio of \( M_b \) at \( N = 3 \) to that at \( N = 2 \) is, however, 1.807 and larger than 3/2. This indicates that the higher-order terms are not negligible for such small \( N \).

Figure 5 shows the \( m \)-dependence of mesonic masses in the large \( N \) limit. In this limit, the 2- and 4-body sectors of the ES equations are decoupled with each other, so that all states appear as either 2- or 4-body state. Obviously, all the 2-body states (the ’t Hooft solutions) are bound. In Fig. 5, on the other hand, the lightest 4-body state is above the threshold \( (2M_1) \), where \( M_1 \) is the mass of the lightest 2-
body bound state. All the 4-body states are thus unbound. This is understandable from the statement [13] based on the $1/N$ expansion that two mesons (two $q\bar{q}$ pairs) do not interact in the limit.

4. DISCUSSIONS

Three unsettled problems are discussed.

(1) As shown in sec. 3.2, our results for $M_1/2M_b$ and $M_2/2M_b$ deviate from the predictions of the bosonization [2] by $\sim 5\%$ in the case of SU(3). A problem in the comparison is that the two types of results are obtained at different $m$; the results of the bosonization at $m = 0$ and ours at $m = 10^{-4}$. In the bosonization, the color-singlet bosonic field is coupled with the non-singlet fields. The non-singlet fields are, however, neglected in the bosonic form of the Hamiltonian. The neglect seems to induce an error of $O(M_S/M_{NS})$, where $M_S$ ($M_{NS}$) is a mass of the states generated by the singlet (non-singlet) field. We intuitively think that the non-singlet fields can generate singlet states as a result of their superposition, but it is not clear from the bosonic form of the Hamiltonian which does not possess the SU($N$) symmetry explicitly. The $M_{NS}$ seems to be $O(1)$ independently of $m/g$, since the non-singlet fields have mass terms of $O(1)$. The $M_S$, on the other hand, tends to zero in the massless limit ($m/g \to 0$). The result of the bosonization is then correct in the massless limit. At $m = 10^{-4}$, it has an error of $O(M_S/M_{NS}) = O(M_1/M_{NS}) = 10^{-2}$. This error is a possible origin of the discrepancy between the two types of results.

(2) Our truncated Fock space consists of the 2- and 4-body states in the mesonic case. Further inclusion of 6-body states would produce the third mesonic bound state in the strong coupling region, because the bosonization [2] predicts for
SU($N$) meson that there appear $2N-1$ massless bound states in the strong coupling limit, and because DLCQ [6] concludes that there are many massless mesonic states in the limit and the $n$-th state consists of $n$ components from 2-body to $2n$-body. The conclusion of DLCQ also implies that the second mesonic state could be described accurately within the present truncated Fock space, as long as $g^2N/m^2 \gg 1$.

(3) In the Schwinger model, the axial symmetry is anomalous and the light-cone Hamiltonian involves a force induced by the so-called annihilation diagrams where a $q$-$\bar{q}$ pair annihilates into an instantaneous gluon at a vertex while another pair is created at the second vertex [14]. The force generates a term $2 \int_0^1 dx \psi(x, 1-x)$ in the ES equation for $\eta$ (iso-singlet) mass, but not for $\pi$ (iso-triplets) mass. The term splits the $\eta$ mass from the $\pi$ mass; especially in the chiral limit, the $\eta$ mass keeps a finite value, but the $\pi$ mass vanishes. In the present model, on the other hand, the axial symmetry is not anomalous, but the annihilation force is still in $P^-$. In this case, the force does not generate any term which makes $\eta$ massive in the chiral limit. Thus, the $\eta$-$\pi$ splitting (the U(1) problem) may not be resolved simply as a matter of the annihilation force.

Throughout this work, we conclude that LFTD is a powerful tool for computing non-perturbative quantities such as hadronic masses. LFTD might be more useful than the $1/N$ expansion and the bosonization which are valid only in a particular situation such as the large $N$ and $g/m$ limits.
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Appendix A: EINSTEIN-SCHRÖDINGER EQUATIONS

A set of coupled integral equations is obtained by applying the Hamiltonian (2.14) to the states (2.15) \( \sim \) (2.18). For the 4-body wave functions, \( \sum_{i=1}^{4} x_i = 1 \). It reads, for SU(N) mesons,

\[
M^2 \psi_2(x, 1-x) = \left( m^2 - \frac{N^2 - 1}{2N} g^2 \right) \left( \frac{1}{x} + \frac{1}{1-x} \right) \psi_2(x, 1-x) \\
- \frac{N^2 - 1}{2N} g^2 \int_{0}^{1} dy \frac{\psi_2(y, 1-y)}{(x-y)^2} \\
+ \frac{(N-1)\sqrt{2(N+1)}}{2N} g^2 \int_{0}^{1} dy_1 dy_2 dy_3 \\
\times \left\{ \psi_A(x, y_1, y_2, y_3) \frac{\delta(y_1 + y_2 + y_3 - (1-x))}{(y_3 - (1-x))^2} \\
- \psi_A(y_1, y_2, y_3, 1-x) \frac{\delta(y_1 + y_2 + y_3 - x)}{(y_1 - x)^2} \right\} \\
+ \frac{(N+1)\sqrt{2(N-1)}}{2N} g^2 \int_{0}^{1} dy_1 dy_2 dy_3 \\
\times \left\{ \psi_S(x, y_1, y_2, y_3) \frac{\delta(y_1 + y_2 + y_3 - (1-x))}{(y_3 - (1-x))^2} \\
- \psi_S(y_1, y_2, y_3, 1-x) \frac{\delta(y_1 + y_2 + y_3 - x)}{(y_1 - x)^2} \right\},
\]
\[ M^2 \psi_A(x_1, x_2, x_3, x_4) = \left( m^2 - \frac{N^2 - 1}{2N} \frac{g^2}{\pi} \right) \left( \sum_{i=1}^{4} \frac{1}{x_i} \right) \psi_A(x_1, x_2, x_3, x_4) \\
+ \frac{(N - 1) \sqrt{2(N + 1)} \, g^2}{8N} \, \frac{1}{\pi} \times \left\{ \frac{1}{(x_2 + x_3)^2} - \frac{1}{(x_3 + x_4)^2} \right\} \psi_2(x_1, 1 - x_1) \\
+ \frac{1}{(x_3 + x_4)^2} - \frac{1}{(x_1 + x_4)^2} \right\} \psi_2(1 - x_2, x_2) \\
+ \frac{1}{(x_1 + x_4)^2} - \frac{1}{(x_1 + x_2)^2} \right\} \psi_2(x_3, 1 - x_3) \\
+ \frac{1}{(x_1 + x_2)^2} - \frac{1}{(x_2 + x_3)^2} \right\} \psi_2(1 - x_4, x_4) \right\} \\
+ \frac{N - 1 \, g^2}{2N} \int_0^1 dy_1 \, dy_2 \left\{ \frac{\delta(x_1 + x_3 - y_1 - y_2)}{(x_1 - y_1)^2} \psi_A(y_1, x_2, y_2, x_4) \\
+ \frac{\delta(x_2 + x_4 - y_1 - y_2)}{(x_2 - y_1)^2} \psi_A(x_1, y_1, x_3, y_2) \right\} \\
+ \frac{N - 1 \, g^2}{4N} \int_0^1 dy_1 \, dy_2 \left\{ \frac{\delta(x_1 + x_2 - y_1 - y_2)}{(x_1 + x_2)^2} \psi_A(y_1, y_2, x_3, x_4) \\
- \frac{\delta(x_1 + x_4 - y_1 - y_2)}{(x_1 + x_4)^2} \psi_A(y_1, y_2, x_3, x_2) \\
- \frac{\delta(x_3 + x_2 - y_1 - y_2)}{(x_3 + x_2)^2} \psi_A(y_1, y_2, x_1, x_4) \\
+ \frac{\delta(x_3 + x_4 - y_1 - y_2)}{(x_3 + x_4)^2} \psi_A(y_1, y_2, x_1, x_2) \right\} \\
- \frac{(N - 1)(N + 2) \, g^2}{4N} \int_0^1 dy_1 \, dy_2 \left\{ \frac{\delta(x_1 + x_2 - y_1 - y_2)}{(x_1 - y_1)^2} \psi_A(y_1, y_2, x_3, x_4) \\
- \frac{\delta(x_1 + x_4 - y_1 - y_2)}{(x_1 - y_1)^2} \psi_A(y_1, y_2, x_3, x_2) \\
- \frac{\delta(x_3 + x_2 - y_1 - y_2)}{(x_3 - y_1)^2} \psi_A(y_1, y_2, x_1, x_4) \\
+ \frac{\delta(x_3 + x_4 - y_1 - y_2)}{(x_3 - y_1)^2} \psi_A(y_1, y_2, x_1, x_2) \right\} \right\}
\[ M^2 \psi_s(x_1, x_2, x_3, x_4) \]
\[ = \left( m^2 - \frac{N^2 - 1}{2N} \frac{g^2}{\pi} \right) \left( \sum_{i=1}^{4} \frac{1}{x_i} \right) \psi_s(x_1, x_2, x_3, x_4) \]
\[ + \frac{(N + 1)\sqrt{2(N - 1)}}{8N} \frac{g^2}{\pi} \]
\[ \times \left\{ \left[ \frac{1}{(x_2 + x_3)^2} + \frac{1}{(x_3 + x_4)^2} \right] \psi_2(1, x_1) \right. \]
\[ - \left[ \frac{1}{(x_3 + x_4)^2} + \frac{1}{(x_1 + x_4)^2} \right] \psi_2(1, x_2) \]
\[ + \left[ \frac{1}{(x_1 + x_4)^2} + \frac{1}{(x_1 + x_2)^2} \right] \psi_2(1, x_3) \]
\[ - \left[ \frac{1}{(x_1 + x_2)^2} + \frac{1}{(x_2 + x_3)^2} \right] \psi_2(1, x_4) \right\} \]
\[ - \frac{N + 1}{2N} \frac{g^2}{\pi} \int_0^1 dy_1 dy_2 \left\{ \delta(x_1 + x_3 - y_1 - y_2) \frac{1}{(x_1 - y_1)^2} \psi_s(y_1, x_2, y_2, x_4) \right. \]
\[ + \delta(x_2 + x_4 - y_1 - y_2) \frac{1}{(x_2 - y_1)^2} \psi_s(x_1, y_1, x_3, y_2) \left. \right\} \]

\(-24\)
\[
+ \frac{N + 1}{4N} \frac{g^2}{\pi} \int_0^1 dy_1 dy_2 \left\{ \frac{\delta(x_1 + x_2 - y_1 - y_2)}{(x_1 + x_2)^2} \psi_S(y_1, y_2, x_3, x_4) \\
+ \frac{\delta(x_1 + x_4 - y_1 - y_2)}{(x_1 + x_4)^2} \psi_S(y_1, y_2, x_3, x_2) \\
+ \frac{\delta(x_3 + x_2 - y_1 - y_2)}{(x_3 + x_2)^2} \psi_S(y_1, y_2, x_1, x_4) \\
+ \frac{\delta(x_3 + x_4 - y_1 - y_2)}{(x_3 + x_4)^2} \psi_S(y_1, y_2, x_1, x_2) \right\} \\
- \frac{(N + 1)(N - 2)}{4N} \frac{g^2}{\pi} \int_0^1 dy_1 dy_2 \left\{ \frac{\delta(x_1 + x_2 - y_1 - y_2)}{(x_1 - y_1)^2} \psi_S(y_1, y_2, x_3, x_4) \\
+ \frac{\delta(x_1 + x_4 - y_1 - y_2)}{(x_1 - y_1)^2} \psi_S(y_1, y_2, x_3, x_2) \\
+ \frac{\delta(x_3 + x_2 - y_1 - y_2)}{(x_3 - y_1)^2} \psi_S(y_1, y_2, x_1, x_4) \\
+ \frac{\delta(x_3 + x_4 - y_1 - y_2)}{(x_3 - y_1)^2} \psi_S(y_1, y_2, x_1, x_2) \right\}
\]

\[ - \frac{\sqrt{(N + 1)(N - 1)}}{4N} \frac{g^2}{\pi} \int_0^1 dy_1 dy_2 \times \left\{ \frac{1}{(x_1 + x_2)^2} \psi_A(y_1, y_2, x_3, x_4) \right\} \left\{ \frac{1}{(x_1 - y_1)^2} \chi \right\} + \frac{N}{(x_1 + y_1)^2} + \frac{N}{(x_1 - y_1)^2} \]  

\[ + \frac{1}{(x_1 + x_4)^2} \psi_A(y_1, y_2, x_3, x_2) \left\{ \frac{1}{(x_1 - y_1)^2} \chi \right\} + \frac{N}{(x_1 - y_1)^2} \] 

\[ + \frac{1}{(x_3 + x_2)^2} \psi_A(y_1, y_2, x_1, x_4) \left\{ \frac{1}{(x_3 - y_1)^2} \chi \right\} + \frac{N}{(x_3 - y_1)^2} \] 

\[ + \frac{1}{(x_3 + x_4)^2} \psi_A(y_1, y_2, x_1, x_2) \left\{ \frac{1}{(x_3 - y_1)^2} \chi \right\} + \frac{N}{(x_3 - y_1)^2} \right\} . \]  

The baryonic state is truncated up to the \((N + 2)\)-body component. The \((N + 2)\)-body component is constructed for SU(2) as

\[
|\text{baryon}\rangle_{N+2} = \frac{1}{2} \int_0^\mathcal{P} \prod_{i=1}^4 \frac{dk_i}{\sqrt{2\pi k_i}} \delta(\mathcal{P} - \sum_{i=1}^4 k_i) \psi_M(k_1, k_2, k_3, k_4) \\
\times \left[ \tilde{b}_1^{k_1}(k_1) \tilde{b}_2^{k_2}(k_2) \tilde{b}_3^{k_3}(k_3) \tilde{d}_1^{k_4}(k_4) - \tilde{b}_1^{k_1}(k_1) \tilde{d}_2^{k_2}(k_2) \tilde{b}_3^{k_3}(k_3) \tilde{d}_1^{k_4}(k_4) \right] |0\rangle,
\]

\[ -25 - \]
with the symmetry

$$\psi_M(k_1, k_2, k_3, k_4) = -\psi_M(k_3, k_2, k_1, k_4),$$  \hspace{1cm} (A.5)

in addition to the $N$-body component in Eq. (2.18). The $(N+2)$-body wave function, 
$$\psi(k_1, k_2, k_3, k_4),$$ is antisymmetric under the interchange of $k_1$ and $k_3$, because the
operator, $\hat{b}_i^1(k_1)\hat{b}_j^1(k_2)\hat{b}_k^1(k_3)d_j^1(k_4)$, is antisymmetric under the interchange. The wave
function can be classified with irreducible representations of the symmetric group,

$$\psi(k_1, k_2, k_3, k_4) = \psi_a(k_1, k_2, k_3, k_4) + \psi_n(k_1, k_2, k_3, k_4) + \psi_M(k_1, k_2, k_3, k_4),$$  \hspace{1cm} (A.6)

where

$$\psi_a(k_1, k_2, k_3, k_4) = S_{123}\psi(k_1, k_2, k_3, k_4),$$

$$\psi_n(k_1, k_2, k_3, k_4) = A_{123}\psi(k_1, k_2, k_3, k_4),$$

$$\psi_M(k_1, k_2, k_3, k_4) = \frac{1}{3}[2\psi(k_1, k_2, k_3, k_4) - \psi(k_2, k_3, k_1, k_4) - \psi(k_3, k_1, k_2, k_4)],$$  \hspace{1cm} (A.7)

where $S_{123}$ is the symmetrizer and $A_{123}$ is the antisymmetrizer of momenta $k_1, k_2$ and
$k_3$. Only the mixed symmetry $\psi_M$ can survive under the condition that $Q^2|\Psi| > 0$. The
$(N+2)$-body wave function can be constructed straightforwardly for arbitrary
$N$. The coupled equations for SU(2) baryon are then

$$M^2 \psi_b(x, 1-x) = \left(m^2 - \frac{3g^2}{4\pi}\right)\left(\frac{1}{x} + \frac{1}{1-x}\right)\psi_b(x, 1-x) - \frac{3g^2}{4\pi}\int_0^1 dy \frac{\psi_b(y, 1-y)}{(x-y)^2}$$

$$+ \frac{g^2}{2\pi} \int_0^1 dy_1 dy_2 dy_3$$

$$\times \left\{ \delta(y_1 + y_2 + y_3 - x) \psi_M(y_1, 1-x, y_2, y_3) \left[ \frac{1}{(x-y_1)^2} + \frac{1}{(x-y_2)^2} \right] \right\} - 2\delta(y_1 + y_2 + y_3 - x) \psi_M(1-x, y_1, y_2, y_3) \left[ \frac{1}{2(x-y_1)^2} + \frac{1}{(x-y_2)^2} \right],$$

(A.8)
\[ M^2 \psi_M(x_1, x_2, x_3, x_4) \]
\[ = \left( m^2 - \frac{3g^2}{4\pi} \right) \left( \sum_{i=1}^{4} \frac{1}{x_i} \right) \psi_M(x_1, x_2, x_3, x_4) \]
\[ + \frac{g^2}{2\pi} \left\{ \frac{1}{2} \left[ \frac{1}{(x_3 + x_4)^2} - \frac{1}{(x_1 + x_4)^2} \right] \psi_b(x_2, 1 - x_2) \right. \]
\[ - \left. \frac{1}{(x_2 + x_4)^2} + \frac{1}{(x_3 + x_4)^2} \right] \psi_b(x_1, 1 - x_1) \]
\[ + \left[ \frac{1}{(x_2 + x_4)^2} + \frac{1}{(x_1 + x_4)^2} \right] \psi_b(x_3, 1 - x_3) \}
\[ - \frac{g^2}{2\pi} \int_0^1 dy_1 dy_2 \]
\[ \times \left\{ \delta(x_1 + x_2 - y_1 - y_2) \left[ \frac{1}{2(x_1 - y_1)^2} + \frac{1}{(x_1 - y_2)^2} \right] \psi_M(y_1, y_2, x_3, x_4) \right. \]
\[ + \delta(x_2 + x_3 - y_1 - y_2) \left[ \frac{1}{2(x_2 - y_1)^2} + \frac{1}{(x_2 - y_2)^2} \right] \psi_M(x_1, y_1, y_2, x_4) \]
\[ - \frac{1}{4} \delta(x_1 + x_3 - y_1 - y_2) \left[ \frac{1}{2(x_1 - y_1)^2} - \frac{1}{(x_1 - y_2)^2} \right] \psi_M(y_1, x_2, y_2) \}
\[ + \frac{g^2}{2\pi} \int_0^1 dy_1 dy_2 \]
\[ \times \left\{ \delta(x_1 + x_4 - y_1 - y_2) \left[ \frac{1}{4(x_1 + x_4)^2} - \frac{1}{(x_1 - y_1)^2} \right] \psi_M(y_1, x_2, x_3, y_2) \right. \]
\[ + \delta(x_3 + x_4 - y_1 - y_2) \left[ \frac{1}{4(x_3 + x_4)^2} - \frac{1}{(x_3 - y_1)^2} \right] \psi_M(x_1, x_2, y_1, y_2) \]
\[ + \delta(x_2 + x_4 - y_1 - y_2) \left[ \frac{1}{4(x_2 + x_4)^2} + \frac{1}{2(x_2 - y_1)^2} \right] \psi_M(x_1, y_1, x_3, y_2) \]
\[ + \delta(x_1 + x_4 - y_1 - y_2) \left[ \frac{1}{4(x_1 + x_4)^2} + \frac{1}{2(x_1 - y_1)^2} \right] \psi_M(x_2, y_1, y_3, y_2) \]
\[ + \delta(x_3 + x_4 - y_1 - y_2) \left[ \frac{1}{4(x_3 + x_4)^2} + \frac{1}{(x_3 - y_1)^2} \right] \psi_M(x_1, y_1, x_2, y_2) \}
\[ . \]  

Again, integrating Eq. (A.8) over \( x \) leads to Eq. (2.22).

Appendix B: BASIS FUNCTIONS

-27-
The function \( \{x^m(1-x)^n\} \) can be expanded in terms of \([x(1-x)]^k\) and \([x(1-x)]^l(1-2x)\), because

\[
x^m(1-x)^n = 2^{-m-n}\sum_{i=0}^{|m-n|} |m-n| C_i \sum_{j=0}^{i} \frac{4^j}{j!} (1-x)^{\min(m,n)+j}(1-2x).
\]

**Appendix C: MATRIX EIGENVALUE EQUATIONS**

The following eigenvalue equations of matrix form are obtained from the coupled equations (Eqs. (A.1) to (A.3)) by sandwiching them with individual 2- and 4-body basis functions.

\[
M^2 \left( \begin{array}{ccc}
A^{(1)} & 0 & 0 \\
0 & B^{(1)} & 0 \\
0 & 0 & B^{(1)}
\end{array} \right) \left( \begin{array}{c}
a \\
b \\
c
\end{array} \right) = \left( \begin{array}{ccc}
H_{11} & H_{12} & H_{13} \\
H_{21} & H_{22} & H_{23} \\
H_{31} & H_{32} & H_{33}
\end{array} \right) \left( \begin{array}{c}
a \\
b \\
c
\end{array} \right),
\]

where

\[
H_{11} = \left( m^2 - \frac{N^2 - 1}{2N} \frac{g^2}{\pi} \right) A^{(2)} - \frac{N^2 - 1}{2N} \frac{g^2}{\pi} A^{(3)},
\]

\[
H_{22} = 2 \left( m^2 - \frac{N^2 - 1}{2N} \frac{g^2}{\pi} \right) \left( B^{(2)} + B^{(3)} \right) + \frac{N - 1}{N} \frac{g^2}{\pi} B^{(4)}
\]

\[
- \frac{(N-1)(N+2)}{N} \frac{g^2}{\pi} B^{(5)} + \frac{N - 1}{2N} \frac{g^2}{\pi} \left( B^{(6)} + B^{(7)} \right),
\]

\[
H_{33} = 2 \left( m^2 - \frac{N^2 - 1}{2N} \frac{g^2}{\pi} \right) \left( B^{(2)} + B^{(3)} \right) + \frac{N + 1}{N} \frac{g^2}{\pi} B^{(4)}
\]

\[
- \frac{(N+1)(N-2)}{N} \frac{g^2}{\pi} B^{(5)} - \frac{N + 1}{2N} \frac{g^2}{\pi} \left( B^{(6)} + B^{(7)} \right),
\]

\[
H_{12} = H_{21}^T = \frac{(N-1)\sqrt{2(N+1)}}{2N} \frac{g^2}{\pi} \left( C^{(1)} - C^{(2)} \right),
\]

\[
H_{13} = H_{31}^T = \frac{(N+1)\sqrt{2(N-1)}}{2N} \frac{g^2}{\pi} \left( C^{(1)} - C^{(2)} \right),
\]

\[
H_{23} = H_{32}^T = -\frac{\sqrt{(N+1)(N-1)}}{N} \frac{g^2}{\pi} \left( B^{(4)} + NB^{(5)} \right).
\]
\[ A^{(1)}_{kl} = \int_0^1 dx f_k(x) f_l(x), \]
\[ A^{(2)}_{kl} = \int_0^1 dx \frac{f_k(x) f_l(x)}{x(1-x)}, \]
\[ A^{(3)}_{kl} = \int_0^1 dy dy \frac{f_k(x) f_l(y)}{(x-y)^2}, \]
\[ B^{(1)}_{kl} = \int_{(4)} G_k(x_1, x_2, x_3, x_4) G_l(x_1, x_2, x_3, x_4), \]
\[ B^{(2)}_{kl} = \int_{(4)} G_k(x_1, x_2, x_3, x_4) \frac{1}{x_1} G_l(x_1, x_2, x_3, x_4), \]
\[ B^{(3)}_{kl} = \int_{(4)} G_k(x_1, x_2, x_3, x_4) \frac{1}{x_2} G_l(x_1, x_2, x_3, x_4), \]
\[ B^{(4)}_{kl} = \int_{(6)} G_k(x_1, x_2, x_3, x_4) \frac{1}{(x_1 + x_2)^2} G_l(y_1, y_2, x_3, x_4), \]
\[ B^{(5)}_{kl} = \int_{(6)} G_k(x_1, x_2, x_3, x_4) \frac{1}{(x_1 - y_1)^2} G_l(y_1, y_2, x_3, x_4), \]
\[ B^{(6)}_{kl} = \int_{(6)} G_k(x_1, x_2, x_3, x_4) \frac{1}{(x_1 - y_1)^2} G_l(x_1, y_1, x_3, y_2), \]
\[ B^{(7)}_{kl} = \int_{(6)} G_k(x_1, x_2, x_3, x_4) \frac{1}{(x_2 - y_1)^2} G_l(x_1, y_1, x_3, y_2), \]
\[ C^{(1)}_{kl} = \int_{(4)} f_k(x_1) \frac{1}{(x_2 + x_3)^2} G_l(x_1, x_2, x_3, x_4), \]
\[ C^{(2)}_{kl} = \int_{(4)} f_k(1 - x_4) \frac{1}{(x_2 + x_3)^2} G_l(x_1, x_2, x_3, x_4), \]
\[ \int_{(4)} = \int_0^1 \prod_{i=1}^4 d x_i \delta \left( \sum_{i=1}^4 x_i - 1 \right), \]
\[ \int_{(6)} = \int_0^1 \prod_{i=1}^4 d x_i d y_1 d y_2 \delta \left( \sum_{i=1}^4 x_i - 1 \right) \delta (x_1 + x_2 - y_1 - y_2), \]
\[ \int_{(6)}' = \int_0^1 \prod_{i=1}^4 d x_i d y_1 d y_2 \delta \left( \sum_{i=1}^4 x_i - 1 \right) \delta (x_1 + x_3 - y_1 - y_2), \]
\[ \int_{(6)}'' = \int_0^1 \prod_{i=1}^4 d x_i d y_1 d y_2 \delta \left( \sum_{i=1}^4 x_i - 1 \right) \delta (x_2 + x_4 - y_1 - y_2). \]

These integrals with no $N$-dependence can be calculated analytically with the formulae collected in Ref. [14].
References

[11] An extensive list of references on light-front physics by A. Harindranath (light.tex) is available via anonymous ftp from public.mps.ohio-state.edu under the subdirectory tmp/infolight.


TABLE I. Calculated masses of the SU(2) and SU(3) hadronic bound states $M_1$, $M_2$ and $M_b$ are tabulated for various values of the quark mass $m$. Here all the masses are given in units of $\sqrt{g^2N/2\pi}$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>$N = 2$</th>
<th></th>
<th>$N = 3$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$M_1$</td>
<td>$M_2$</td>
<td>$M_b$</td>
<td>$M_1$</td>
</tr>
<tr>
<td>0.0001</td>
<td>0.01626</td>
<td>0.02880</td>
<td>0.01773</td>
<td>0.01849</td>
</tr>
<tr>
<td>0.0005</td>
<td>0.03635</td>
<td>0.06414</td>
<td>0.03964</td>
<td>0.04134</td>
</tr>
<tr>
<td>0.0010</td>
<td>0.05141</td>
<td>0.09067</td>
<td>0.05608</td>
<td>0.05846</td>
</tr>
<tr>
<td>0.0050</td>
<td>0.11512</td>
<td>0.20325</td>
<td>0.12569</td>
<td>0.13053</td>
</tr>
<tr>
<td>0.0100</td>
<td>0.16317</td>
<td>0.28858</td>
<td>0.17825</td>
<td>0.18443</td>
</tr>
<tr>
<td>0.0500</td>
<td>0.37414</td>
<td>0.66580</td>
<td>0.40748</td>
<td>0.41486</td>
</tr>
<tr>
<td>0.1000</td>
<td>0.54915</td>
<td>0.97461</td>
<td>0.59164</td>
<td>0.59895</td>
</tr>
</tbody>
</table>
Figure Captions

FIG.1 Masses of the lightest \((M_1)\) and the second lightest \((M_2)\) SU(2) mesons are shown as a function of one of the parameters which characterize the size of the space spanned by the basis functions (see eqs. (3.1) - (3.3)); (a) \(N_2\) is varied while \(N_4\) is fixed, (b) \(N_4\) is varied while \(N_2\) is fixed. Here the masses are presented in units of \(\sqrt{g^2 N/2 \pi}\).

FIG.2 Masses of the lowest two mesonic \((M_1\) and \(M_2\)) and one baryonic \((M_b)\) bound states, obtained with the full-fledged calculations, are shown in units of \(\sqrt{g^2 N/2 \pi}\) as a function of the quark mass \(m\); (a) for SU(2) and (b) for SU(3). They are graphed with solid lines. The 2-body decay threshold, \(2M_1\), is also shown by the dashed line.

FIG.3 Numerical and approximate masses of the lowest mesonic \((M_1)\) and baryonic \((M_b)\) states are shown in units of \(\sqrt{g^2 N/2 \pi}\) as a function of the quark mass \(m\). (a) For the SU(2) case, the numerical solutions to \(M_1\) and \(M_b\) are graphed with the solid and dot-dashed lines, respectively, while the approximate ones to both of them are degenerate and therefore graphed with a dashed line. (b) For the SU(3) case, the numerical and the approximate solutions are graphed with the solid and dashed lines, respectively, for both \(M_1\) and \(M_b\).

FIG.4 Numerical solutions to (a) the second lightest mesonic and (b) the lightest baryonic masses, calculated for SU\((N)\) by adopting a quark mass \(m = 10^{-4}\), are shown by the solid lines as a function of \(N\) in comparison to that of the lightest meson. The approximate solutions, which are available only for the lightest meson and baryon, are shown by the dashed lines in (a) and (b), respectively. The 2-body
decay threshold, $2M_1$, is also shown by the dot-dashed line in (a). Here all the masses are given in units of $\sqrt{g^2N/2\pi}$.

FIG. 5 Masses of the lowest four 2-body and a 4-body SU($N$) mesonic states in the large $N$ limit are shown as a function of the quark mass $m$ by the solid lines and the dot-dashed line, respectively. The 2-body decay threshold, $2M_1$, is also shown by the dashed line. Note that all the masses are given in units of $\sqrt{g^2N/2\pi}$. 