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ABSTRACT

A method is presented to derive hadronic gas equation of state incorporating the effect of finite size of baryons in a thermodynamically consistent way. The model contains a more realistic picture of incorporating the effect of excluded volume unlike earlier methods. The results obtained by our calculation and those by the thermodynamically inconsistent methods used in many earlier works are compared.

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1. INTRODUCTION

In the study of a phase transition from the hot hadronic gas (HG) to the quark gluon plasma (QGP) it is necessary to find the equation of state (EOS) of HG and QGP. Numerous attempts in this direction have been made [1-6]. In many of these calculations the QGP is described as an ideal gas of pointlike quarks and gluons. To make these calculations more realistic perturbative QCD corrections have also been applied [6-8]. On the other hand in many classical models the HG is described, for simplicity, as consisting of free hadron resonances except for a short range hard-core repulsion among baryons. This is taken into account in the EOS of HG by assigning the baryons (antibaryons) a hard-core volume resulting in a finite-size correction [9-14]. This results in an excluded volume effect. The effect of finite baryon sizes becomes important when considering a very hot and dense gas of hadrons. In almost all the earlier works the quantities for finite-size particles were obtained "in terms of the point-particle quantities." Thus the thermodynamical parameters there i.e. β^0 and μ^0 refer to the pointlike particles. The thermodynamic quantities were modified by different multiplicative factors to take into account the hard-core repulsion among hadrons. The modifications involved energy and

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particle number densities. For example, Hagedorn's correction gives [15,16]

$$n_i = n_i^0 / (1 + \epsilon^0 / 4B) \quad (1)$$

where n_i is the density of i^{th} type of finite-size baryons, n_i^0 and ϵ^0 are the particle number and energy density of "pointlike particles." B is the bag constant. Another modification gives [9-14]

$$n_i = n_i^0 / (1 + \sum_j n_j^0 V_j) \quad (2a)$$

$$P_i = P_i^0 / (1 + \sum_j n_j^0 V_j) \quad (2b)$$

$$\epsilon_i = \epsilon_i^0 / (1 + \sum_j n_j^0 V_j) \quad (2c)$$

where n_j^0 is the density of j^{th} "pointlike particles" and V_j the hard-core volume of the corresponding "finite-size particles." In different models n_j^0 has different meaning e.g. baryon and antibaryon corrections are separated and n_j^0 involves only baryons or antibaryons [12]. Similarly in some models baryons and antibaryons are mixed [11] or only the net baryon density is used [14]. However equations (1) and (2) are thermodynamically

inconsistent i.e. they do not satisfy the fundamental relations viz., $n = (\lambda/V) \partial \ln Z / \partial \lambda$ etc. Attempts have been made to make these equations thermodynamically consistent. It has been proposed [17] that the thermodynamical consistency of (2) requires that the chemical potential μ , for the evaluation of the quantities for finite-size particles, be replaced by $\tilde{\mu} = \mu - V P^{\text{excl}}(\beta, \mu)$, where P^{excl} is the pressure with the effect of excluded volume taken into account and is given by $P^{\text{excl}}(\beta, \mu) = P(\beta, \mu)$ with P as the pressure of pointlike particles with chemical potential μ . In this calculation the excluded volume is obtained by multiplying the number of "pointlike particles" with the finite eigen volume V_i of the corresponding finite-size species. However, this is not justified since pointlike particles do not result in excluded volume and hence any calculation to evaluate the amount of excluded volume cannot be done using the number of pointlike particles. Although the method used in ref. [17] appears to be mathematically justified but it is not physically justified.

2. EQUATION OF STATE OF FINITE-SIZE HADRONS

In order to find the EOS of finite-size particles with the thermodynamic parameters referring only to them we develop a

method where we first evaluate the partition function for the finite-size particles. Thus the EOS derived will be by the very definition, thermodynamically consistent. To obtain the excluded volume we take into account the number of "finite-size particles" present in the system which is justified. This method can be applied to a HG with "any number of finite-size components" present. The inconsistency of eqn (2) will also become evident in our method.

To obtain the partition function we write for a given finite-size baryon component (labelled 1), using the integral representation

$$\ln Z = \frac{g_1}{6\pi^2 T} \int_0^\infty \frac{dk \cdot k^4}{(k^2 + m_1^2)^{1/2}} \frac{1}{\{\lambda_1^{-1} \exp[\beta E(k; m_1)] + 1\}} \quad (3)$$

Similarly one can write for the corresponding antibaryon component (except that $\lambda_1 = 1/\lambda_1$). We use 'Boltzmann approximation' since this will simplify the treatment greatly. Using symbol 1 for the momentum space integral we can write

$$\ln Z_1 = \int_{V_1}^{V - \sum_j N_j V_j} dV' \cdot I_1 \cdot \lambda_1 \quad (4)$$

In the above expression the fugacity $\lambda_1 (= \exp(\mu_1/T))$, where μ_1 is the chemical potential of the given baryon component) appears as a multiplicative factor. Here the lower limit of the volume integral is V_1 which is the hard-core volume of the given baryon species, V is the total volume of the system and $\sum_j N_j V_j$ is the total "average excluded volume" with N_j being the mean number of j^{th} "finite-size baryon" species and V_j its hard core volume. This gives in the large volume limit (i.e. for $V_1/V \rightarrow 0$)

$$\ln Z = V(1 - \sum_j n_j V_j) I_1 \lambda_1 \quad (5)$$

where n_j is the number density of j^{th} baryon species. By definition the number density of the given baryon species (labelled 1) is

$$n_1 = \frac{\lambda_1}{V} \left. \frac{\partial \ln Z_1}{\partial \lambda_1} \right|_{T, V} \quad (6)$$

we get using (5)

$$n_1 = \lambda_1 \frac{\partial}{\partial \lambda_1} \left\{ (1 - \sum_j n_j V_j) I_1 \lambda_1 \right\} \quad (7)$$

For a single component system or a system with only one component having a hard-core volume V_0 say e.g., a HG consisting of pions and nucleons only, the density of the component having finite size can be written as (dropping subscript 1)

$$n = (1 - nV_0) I \lambda - \lambda^2 I V_0 \frac{\partial n}{\partial \lambda}$$

We thus obtain a differential equation for n

$$I V_0 \lambda^2 \frac{\partial n}{\partial \lambda} + (1 + IV_0 \lambda) n - I \lambda = 0 \quad (8)$$

The above differential equation can be solved to obtain

$$n = \frac{1}{V_0} \frac{\int_0^\lambda \exp(-1/IV_0 \lambda') d\lambda'}{\lambda \exp(-1/IV_0 \lambda)} \quad (9)$$

The partial pressure p_1 (= $T \ln Z_1/V$) can be obtained by using (9) and (5). We get

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$$p_i = \lim_{V \rightarrow \infty} \frac{T \ln Z_i}{V} = (1 - nV_i) T I_1 \lambda_1 \quad (10)$$

For the multicomponent system we will get from (7) (say for ith)

$$n_i = (1 - \sum_j n_j V_j) I_1 \lambda_1 - I_1 \lambda_1^2 \frac{\partial}{\partial \lambda_1} \left(\sum_j n_j V_j \right) \quad (11)$$

The above (set of) differential equations are difficult to solve analytically or numerically because they contain several partial derivatives. To overcome this we put $\sum_j n_j V_j = R$ in the equation (11) to write

$$n_1 = (1 - R) I_1 \lambda_1 - I_1 \lambda_1^2 \frac{\partial R}{\partial \lambda_1} \quad (12a)$$

$$n_2 = (1 - R) I_2 \lambda_2 - I_2 \lambda_2^2 \frac{\partial R}{\partial \lambda_2} \quad (12b)$$

.....

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multiplying above equations by $V_1, V_2, V_3, \dots$ respectively we get by adding

$$R = (1 - R) \sum_j I_j V_j \lambda_j - \sum_j I_j V_j \lambda_j^2 \frac{\partial R}{\partial \lambda_j} \quad (13)$$

In order to obtain solution of the above equations we use the "Method of Parametric Line." We assume that the lambdas move along a curve in the parametric space and $\lambda_j \equiv \lambda_j(t)$, where t is a parameter of the curve in lambda space such that

$$\frac{d\lambda_j(t)}{dt} = I_j V_j \lambda_j^2$$

Integration gives

$$\lambda_j(t) = \frac{1}{(a_j - I_j V_j t)}$$

The given equation then takes a convenient form

$$f(t) R + \frac{dR}{dt} = h(t) \quad (14)$$

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where $f(t) = [1 + \sum_j I_j V_j \lambda_j(t)]$ and $h(t) = \sum_j I_j V_j \lambda_j(t)$.

The above differential equation can be solved for R . Once R (and also $\partial R / \partial \lambda_j$) becomes known we can use equations (11a), (11b), ..., etc. to calculate $\eta_j, \eta_2, \dots$, etc. Knowing which we can obtain $\ln Z_1, \ln Z_2, \dots$, etc. and thus calculate other quantities also e.g., pressure, entropy density, energy density etc. of finite-size particles. It is to be stressed that all these quantities will be thermodynamically consistent since we are able to evaluate the "partition function of a system consisting of finite-size particles." Thus no further correction is required to make them thermodynamically consistent. The solution of (14) is as follows

$$R = \frac{1}{G(t)} \int_{-\infty}^t \left[\frac{1}{-t'} + \frac{I_2 V_2}{(a_2 - I_2 V_2 t')} + \frac{I_3 V_3}{(a_3 - I_3 V_3 t')} + \dots \right] G(t') dt' \quad (15)$$

where

$$G(t) = \frac{\exp(t)}{t \cdot (a_2 - I_2 V_2 t) \cdot (a_3 - I_3 V_3 t) \dots}$$

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and the upper limit t is negative; $t = -1/\lambda_1 I_1 V$, by setting $a_1 = 0$.

In the equation (8) for a single baryon component if we ignore the term containing $\partial n/\partial \lambda$ we obtain a simple solution

$$n = \frac{n_0}{1 + n_0 V_0}$$

where $n_0 = I\lambda$ is the density of pointlike particles. This result is same as the one obtained from (2a) for a single component system but is obviously thermodynamically inconsistent.

3. RESULT AND DISCUSSION

In the following we will further analyse and compare our result with those obtained by equation (2), for a single component system, when λ is large.

In the limit $\lambda \rightarrow \text{large}$ we have $n_0 (=I\lambda) \rightarrow \text{large}$ and we get from (2a) that $n \sim 1/V_0$. In this limit the integral in equation (9) $\sim \lambda$ and we again find that the number density $n \sim 1/V_0$. Thus equations (2a) and (9) give similar results in the large λ limit which one expects from physical arguments also. This shows that

the number density obtained by the thermodynamically inconsistent method is very close to that obtained by the thermodynamically consistent method in the large λ limit.

To analyse the behaviour of the pressure function in this limit we approximate the integrand in (9) upto first order. We get

$$n(\lambda \rightarrow \text{large}) \approx \frac{1}{V_0} \left[1 - \frac{1}{IV_0} \frac{\ln^2}{\lambda} \right] \quad (16)$$

The pressure using equation (10) comes to be

$$P_{\text{thermist}}(\lambda \rightarrow \text{large}) \approx \frac{T}{V_0} \ln \lambda \approx \frac{\mu}{V_0} \quad (17)$$

For the case of thermodynamically inconsistent correction we have from (2b) (in the Boltzmann approximation) $P_0 = Tn_0$. In the limit $\lambda \rightarrow \text{large}$ we thus obtain

$$P_{\text{thermist}}(\lambda \rightarrow \text{large}) \approx \frac{T}{V_0} \quad (18)$$

One can see the difference between the two results (17) and (18). In the case of thermodynamically inconsistent correction the pressure saturates for large λ values and is equal to T/V_0 .

However, the pressure in our case becomes "independent of T " for very large λ and rises linearly with the chemical potential μ .

In Fig. 1 we have plotted the variation of nucleon number density with the baryon chemical potential (μ_b) at fixed temperature $T = 200$ Mev for the cases of (i) without finite-size correction (ii) thermodynamically inconsistent correction using (2a) and (iii) thermodynamically consistent correction using our method. We find that for small μ_b the difference between the results of the case (ii) and (iii) is small. Similarly for large μ_b (or λ_b) this difference is again small. It is significant only for intermediate values of μ_b . If we plot the net baryon density the curves for $\mu_b < 400$ Mev become slightly more steeper while there is almost no change in the curves at larger μ_b 's. This is obvious since at large μ_b the antinucleon density $\ll$ nucleon density hence the net baryon density $\approx$ the nucleon density.

However, the situation is different in Fig. 2 where we have plotted the variation of pressure with μ_b . We find very significant difference between the values obtained by our method and that from the thermodynamically inconsistent method. While at small μ_b the curves for the case (ii) and (iii) are close, the pressure in our case rises more rapidly than the case (ii). For

example at $\mu_b = 800$ Mev (which corresponds to a baryon rich system) the pressure in our case is about $130 \text{ Mev}/\text{fm}^3$ while for the case (ii) it is only about $84 \text{ Mev}/\text{fm}^3$. In a phase transition model from HG - QGP the critical parameters of the phase transition i.e. T_c and μ_c [13], obtained by employing the Gibbs criteria, will be affected greatly by this. Thus the critical conditions under which the system can negotiate a phase transition will be significantly altered.

In summary we have given a method to obtain EOS of finite-size hadrons in a thermodynamically consistent manner. The excluded volume is obtained by considering the number of all the "finite-size particles" in the system which is physically justified unlike the method used earlier [17]. While as a result of thermodynamically consistent correction the particle number density is only slightly modified the values of pressure are significantly affected. This leads to the expectation that the critical parameters in the phase transition models will be altered significantly, if one uses our method for the finite-size correction. In some limiting cases only, the method of ref. [17] gives similar results as obtained from the partition function in equation 5.

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FIGURE CAPTIONS

Fig. 1 :

Variation of nucleon number density in a HG, with no other type of baryons, with respect to baryon chemical potential (μ_b) at fixed temperature $T = 200$ MeV is shown. The three cases are of (i) pointlike nucleons (ii) thermodynamically inconsistent correction for finite-size and (iii) thermodynamically consistent correction for finite-size indicated by dashed, dashed-dotted and solid curves respectively. For case (ii) and (iii) we have chosen the hard-core nucleon radius as 0.8 fm.

Fig 2 :

Variation of pressure for a gas of nucleons and antinucleons with μ_b at fixed temperature $T = 200$ MeV. The three cases are same as in Fig. 1.

Fig.1

Fig.2