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CHAOS IN THE EINSTEIN EQUATIONS - CHARACTERIZATION AND IMPORTANCE?

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Abstract. Is it possible to define what we could mean by chaos in a space-time metric (even in the simplest toy-model studies) ? Is it of importance for phenomena we may search for in Nature ?

1. INTRODUCTION

Theoretical physics would not die out even if we had *already* found the "master plan", the "master law" (T.O.E.) for how the Universe in which we live is constructed. Namely, given the knowledge of the basic laws of physics (on each level on the Quantum Staircase, say) it is still a major project to try to deduce their *complex* consequences, i.e. to find out the complex ways in which matter (and their interactions) organizes itself into living and nonliving forms.

The word "chaos" is somewhat of an unlucky choice since we may too easily associate it with something which is structureless. Chaotic systems do not lack structure. On the contrary, chaotic systems exhibit far more interesting and richer structure in their dynamical behavior than integrable systems.

However, chaotic systems have an aspect of unpredictability and "simulated" disappearance of information: Information walks down to the small scales and is replaced by noise walking up from the small scales. I.e. chaos pumps *information* up and down the chain of decimals in the phase space coordinates! (due to the exponential amplification of uncertainties in the specification of the initial state).

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Many physical systems, governed by *non-linear* equations of motion, will exhibit chaos. By now, examples are known from all disciplines in physics.

Due to the highly non-linear *self-interaction* of the gravitational and non-Abelian gauge fields, the time evolution of (generic) configurations of "gauge fields" and "gravitational fields" will be non-integrable even without any coupling to material bodies. For example, as evidenced by the simplest toy-model studies, we expect that the Einstein equations - in scenarios involving *strong* field strength, where the non-linearity of the equations is important (e.g. probing the Einstein equations near space-time singularities) - will exhibit *chaotic solutions* (rather than integrable ones) if not too much symmetry is imposed on the field configuration. The same is true for non-Abelian gauge fields. While on one hand this is *not very surprising* (the Einstein and Yang-Mills equations are highly non-linear theories and one of the lessons from chaos theory has been that *even the simplest* non-linear equations usually exhibit "chaos") there are, on the other hand, several issues of interest:

1. Can "standard indicators of chaos" be used to characterize the "metric chaos"? In other words: Are there some *deeper problems* in characterization of chaos in this context compared to other chaotic physical models?
2. What is the *structure* of this (non-dissipative) chaos? How does chaos "look like" in simple toy-models of the classical Yang-Mills equations and the classical Einstein equations?
3. Is it of any *physical significance* that field configurations of the fundamental forces at the classical (or semi-classical) level may exhibit chaotic, irregular non-integrable solutions?

Let us turn our attention to the gravitational field. It is well known that individual orbits of test particles (bodies) in a given gravitational field can exhibit "chaos". This is the case for the motion of test particles in Newton's theory of gravitation (e.g. "chaotic" orbits of individual stars in the potential generated by the other stars in a galaxy) and also in general relativity: Cf. e.g., the study by G. Contopoulos of periodic orbits and chaos around two black holes; G. Contopoulos (1990), the study of chaos around a single black hole, L. Bombelli and E. Calzetta (1992) and the study of chaotic motion of test particles around a black hole immersed in a magnetic field, V. Karas *et al* (1992). Also an extended object like a (cosmic) string may jump chaotically around the equatorial plane of a black hole, cf. A.L. Larsen (1993).

Because of their nonlinearity, the Einstein field equations however permit space-time to be curved ("gravity generates gravity") - even in the absence of any nongravitational energy - and the dynamical evolution of the spacetime metric " $g_{\mu\nu}(x)$ " itself is governed by *highly nonlinear* equations and may allow solutions of the chaotic type for the "metric" field itself.

If we explicitly write down (cf. Kip S. Thorne (1985)) the vacuum Einstein equations $G_{\mu\nu} = 0$ as differential equations for the "metric density" $g^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu}$ we have a very complicated set of partial differential equations for $g^{\mu\nu}$,²

²Note, that $\bar{g}_{\mu\nu}$ (the inverse of $g^{\mu\nu}$) is a highly nonlinear algebraic function of $g^{\mu\nu}$. Repeated indices are to be summed, commas denote partial derivatives, e.g. $\bar{g}_{\mu\beta}^{\alpha\beta} \equiv \partial g^{\alpha\beta} / \partial x^\mu$. The coordinate system has been specialized so the metric is in "deDonder gauge", $\bar{g}_{\alpha\beta}^{\alpha\beta} = 0$ (see K.S. Thorne (1985)).

$$\begin{aligned} \frac{\partial^2 \tilde{g}^{\alpha\beta}}{\partial x^\mu \partial x^\nu} &= \tilde{g}_{\mu}^{\alpha\sigma} \tilde{g}_{\nu}^{\beta\mu} + \frac{1}{2} \tilde{g}^{\alpha\beta} \tilde{g}_{\lambda\mu} \tilde{g}_{\nu}^{\lambda\mu} \tilde{g}_{\sigma}^{\mu\sigma} \\ &+ \tilde{g}_{\lambda\mu} \tilde{g}^{\nu\sigma} \tilde{g}_{\nu}^{\alpha\lambda} \tilde{g}_{\sigma}^{\beta\mu} - \tilde{g}^{\alpha\lambda} \tilde{g}_{\mu\nu} \tilde{g}_{\sigma}^{\beta\nu} \tilde{g}_{\lambda}^{\mu\sigma} - \tilde{g}^{\beta\lambda} \tilde{g}_{\mu\nu} \tilde{g}_{\sigma}^{\alpha\nu} \tilde{g}_{\lambda}^{\mu\sigma} \\ &+ \frac{1}{8} (2\tilde{g}^{\alpha\lambda} \tilde{g}^{\beta\mu} - \tilde{g}^{\alpha\beta} \tilde{g}^{\lambda\mu}) (2\tilde{g}_{\nu\sigma} \tilde{g}_{\sigma}^{\nu\lambda} - \tilde{g}_{\mu\sigma} \tilde{g}_{\sigma}^{\mu\nu}) \tilde{g}_{\lambda}^{\alpha\sigma} \tilde{g}_{\mu}^{\beta\sigma}. \end{aligned} \quad (1)$$

The left hand side of (1) is a kind of curved spacetime wave operator “ $\square$ ” acting on $\tilde{g}^{\alpha\beta}$ (giving a propagation effect of the gravitational degrees of freedom) whereas the right hand side is a sort of “stress-energy pseudotensor” for the gravitational field which is quadratic in the first derivatives of $\tilde{g}^{\alpha\beta}$ and acts as the source for “ $\square$ ” $\tilde{g}^{\alpha\beta}$.

At first, general relativity, i.e., Einstein’s theory of gravitation, is not even a dynamical theory in the usual sense. It does not, from the very beginning, provide us with a set of parameters (describing the gravitational degrees of freedom) evolving in “time”. “Time” loses here its absolute meaning as opposed to the classical dynamical theories where the “Newtonian time” is taken for granted. The division between space and time in general relativity comes through foliating the space-time manifold $\mathcal{M}$ into spacelike hypersurfaces Σ_t . The metric $g_{\mu\nu}$ on $\mathcal{M}$ induces a metric g_t on Σ_t and can be parametrized in the form

$$g_{\mu\nu} = \begin{pmatrix} N_i N^i - N^2 & N_j \\ N_i & g_{ij} \end{pmatrix}$$

bringing the metric on the 3+1 form³

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \quad (2)$$

where N and N_i are called lapse function and shift vector respectively. (Cf. e.g., MTW §21.4)

*Only after splitting the space-time into space and time (the 3+1 ADM splitting) we yield the possibility to treat the evolution of the metric under the governing Einstein equations as a dynamical system on somewhat equal footing as other dynamical systems which have some (physical) degrees of freedom evolving in “time”.*⁴

It is of interest to our discussion on the dynamics of the Einstein equations (chaotic or not) to know that one may show that the Einstein equations indeed admit a well posed *initial value* formulation, so the Einstein equations do *determine* the evolution of the metric (up to gauge transformations) *uniquely* from given initial conditions and the solutions admit a (*nearly* *stability* *criteria* which establishes that the solutions depend

³Albert Einstein taught us to treat space and time on an equal footing in a four-dimensional spacetime manifold and build up a space-time covariant formulation of the theory, yet to deal with dynamics we manifestly have to break the space-time covariance of the formulation.

⁴Compare, e.g., with the characterization of turbulence in connection with fields like Navier-Stokes flows or $St/(2)$ Yang Mills fields. In these cases there are no such fundamental problems or ambiguities as concerns the fields being in certain well defined space points and evolving in a well defined “external” time t . Such fields evolve in a flat Minkowski metric (if they are not coupled to gravitational fields) and no “mixing” between space and time concepts (up to “stiff” Lorentz transformations) occurs.

continuously on the initial data. (Cf., e.g., discussion in MTW §21.9, S.W. Hawking and G.F.R. Ellis (1973), sec.7, and R.M. Wald (1984), sec.10.

Given the complexity of the Einstein equations (1) by comparison, e.g., with the Navier-Stokes equations, it is not surprising that we at present have only little understanding of the dynamics of the Einstein equations involving strong gravitational fields.

From the observational side not many observable phenomena which involve the gravitational field need the full non-linear Einstein equations and in “daily life” gravity often the *linearized* equations suffice.⁵ Among the three classic tests of Einstein’s theory only the precession of the perihelia of the orbits of the inner planets tests a non-linear aspect of the Einstein equations.

To find scenarios where the full non-linearities of the Einstein equations are important one has to search among astrophysical and cosmological phenomena far removed from “daily life” gravity.

Among non-linear phenomena in general relativity (geons, white and black holes, wormholes, cf. Kip S. Thorne (1985), solitons, cf. G.W. Gibbons (1985)) the formation of spacetime singularities is one of the most remarkable phenomena which appears in nonlinear solutions of the *classical* Einstein equations under a variety of circumstances, cf. the singularity theorems by S.W. Hawking and R. Penrose (1970).

The singularity theorems of Hawking and Penrose tell us, however, practically nothing about the structure of such spacetime singularities. What do they look like?

1.1. Highly symmetric gravitational collapses as a laboratory for testing ideas about how to characterize “chaos” in a general relativistic context.

To capture a nonlinear aspect of the Einstein equations it is natural to probe them in scenarios involving strong field strength, e.g. near curvature singularities.

Without some restrictions of symmetry imposed on the spacetime metric $g_{\mu\nu}(x)$ the Einstein equations are intractable (though considerable progress has been made in numerical relativity of solving cases with little symmetry).

We shall consider a simple example and use it to test ideas and concepts about chaos. (If we are not able to agree upon how chaos should be defined in this simple example we may very well give up all hope to develop indicators of chaotic behavior in more complicated examples of spacetime metrics).

Thus, we restrict, for simplicity, attention to toy-model metrics with *spatially homogeneous* three-dimensional space-like slices (hypersurfaces). Then the gravitational fields are the same at every point on each of the surfaces of homogeneity and one may thus represent these fields via *functions of time* only!

More explicitly, spatially homogeneous 3-geometries are 3-manifolds on which a three-dimensional Lie group acts transitively. On the 3-manifold this symmetry is encoded in the existence of three linearly independent spacelike Killing vectors ξ_i , $i =$

⁵Note, however, that *nonlinear effects* of the Einstein equations may show up to be important even in regions where one would imagine the linearized equations to be sufficient. For example, one has a *nonlinear* effect in the form of a permanent displacement of test masses (of the same order of magnitude as the *linear* effects) after the passage of a gravitational wave train – even when the test masses are placed at arbitrary distances from the gravitational wave source. See D. Christodoulou (1991).

1, 2, 3, satisfying the Lie algebra $[\xi_i, \xi_j] = C_{ij}^k \xi_k$ where C_{ij}^k are the structure constants of the Lie algebra.⁶

The particular collapse ("big crunch"), the mixmaster collapse, we shall consider has the same non-Abelian isometry group $SU(2)$ on the three-space (i.e. same topology $\sim R \times S^3$) as the compact FRW-collapse but it contains three scale-factors a, b, c instead of just one. A 'freely falling astronomer' who falls into the spacetime singularity of the 'big crunch' will experience a growing tidal field, in which he is *compressed* along two directions and *stretched* (expanded) in one direction, the directions being permuted infinitely many times in a *not-predictable* way.

The possibility of chaos has been investigated only for very few toy model studies of spacetime metrics! Whether one should expect the Einstein equations to generate "chaos" in generic cases (for strong gravitational fields, i.e., high curvatures) the answer is absolutely: Nobody knows!

The paper is organized as follows: In sec.2 we describe various aspects of the mixmaster gravitational collapse. Not surprisingly, a collapse to a spacetime singularity is prevented if one includes matter with negative energy and pressure. In that case, however, the behavior of the spacetime metric is very interesting, very irregular and highly unpredictable, and oscillations of the three-volume occur. (Due to the negative pressure and energy density the attraction of matter turns into an unphysical repulsion preventing the "universe" from collapsing). If the metric is evolved according to the vacuum Einstein equations, the dynamics has, after some transient, a monotonically declining three-volume and the degrees of freedom of the spacetime metric is fast attracted into an interesting self-similar, never ending oscillatory behavior on approach to the big crunch singularity. This may be understood as a never ending sequence of short bounces against a potential boundary generated by the three-curvature scalar (^3R) on the three-space. This scattering potential becomes, to a very good approximation, infinitely hard when the metric approaches the singularity, and the collapse dynamics may, in that limit, be captured by a set of simple algebraic transition rules (maps), for example the so-called "Farey map", which is a strongly intermittent map (this map has, as sub-map, the Gauss map which is well known in chaos theory and which has positive Kolmogorov entropy).⁷

In sec.3 we describe the problem - inherent to general relativity - of transferring standard indicators of chaos, in particular the spectrum of Lyapunov exponents, to the general relativistic context, since they are highly gauge dependent objects. This fact was pointed to and emphasized in S.E. Rugh (1990 a,b). I can only moderately agree that this observation was arrived at independently by J. Pullin (1990). By referring to a specific gauge (the Poincaré disc) Pullin misses the point (in my opinion). No "gauge" is better than others. One should try to develop indicators which capture chaotic properties of the gravitational field ("metric chaos") in a way which is invariant under spacetime diffeomorphisms - or prove that this can not be done! (H.B. Nielsen and S.E. Rugh). This program of research is still in its infancy.

⁶The classification of three dimensional Lie-algebras dates back to L. Bianchi (1897) and the spatially homogeneous metrics are therefore often referred to as Bianchi metrics. See e.g. M.A.H. MacCallum (1979, 1983).

⁷Results obtained in the first part of sec.2 were also arrived at by D. Hobill *et al* in completely independent investigations.

In sec.4 and sec.5 some (even) more wild speculations are offered concerning the generality and applicability of the concept of "metrical chaos" etc. One would like to argue - but it is not easy - that *non-integrability* of the Einstein equations, is a generic phenomenon when considering scenarios involving really strong gravitational fields, e.g. near Planck scales where the gravitational field should be treated quantum mechanically. Whether there are implications of "metrical chaos" on the quantum level is a question which not possible to address since no good candidate for a theory of quantum gravity is known. In the context of the Wheeler-DeWitt equation (which however involves arbitrariness, cf. e.g. the factor ordering problem) one may address this question for the mixmaster gravitational collapse. This is beautifully illustrated in the "Poincaré disc gauge" (which I describe shortly) and was already considered by Charles W. Misner twenty years ago. The mixmaster collapse dynamics is however so special (the scattering domain of the Poincaré disc tiles the disc under the action of an "arithmetic group") that its quantization exhibits ungeneric features, relative to more generic Hamiltonian models (of similar low degree of dimensionality) studied in the discipline of "Quantum Chaos"⁸. This illustrates, once again, that the mixmaster gravitational collapse is a very beautiful, yet very special, example of chaos (algebraic chaos). However, for our purpose, to use it as a toy-model to investigate the applicability of indicators of chaos in the general relativistic context, it serves as a good starting point.

It is interesting whether "metrical chaos" (not yet defined) has potential applications for phenomena occurring in Nature. Certainly, non-integrability (i.e. lack of first integrals relative to the number of degrees of freedom) and non-linear effects may show up even in scenarios involving rather weak fields (cf. e.g. D. Christodoulou (1991)).

Considering the possibility of the early universe to be described by the mixmaster metric, we note in sec.6 that, according to the Weyl curvature hypothesis (of R. Penrose), which suggests that the Weyl curvature tensor should vanish at the initial singularity (at "big bang"), the mixmaster metric has too big Weyl curvature to be implemented in our actual Universe at Planck scales, say. The Guth/Linde inflationary phase may modify this viewpoint.

In sec.6 also some more general reflections are put forward concerning the "chaotic cosmology" concept by Charles W. Misner *et al* which attempt at arriving at our present Universe from (almost) arbitrary initial conditions.

⁸"Quantum Chaos", or what Michael Berry has named "Quantum 'Chaology'", cf. e.g. M. Berry (1987), investigates the semi-classical or quantum behavior characteristic of Hamiltonian systems whose classical motion exhibits chaos.

2. THE MIXMASTER GRAVITATIONAL COLLAPSE GIVES A "HINT" OF THE SORT OF COMPLEXITY ("METRICAL CHAOS") ONE MAY HAVE FOR SOLUTIONS TO EINSTEIN EQUATIONS.

The mixmaster gravitational collapse is a very famous⁹ gravitational collapse (a "big crunch") which generalizes the compact FRW collapse and which gives us a "hint" of the sort of complexity ("metrical chaos") one should expect for gravitational collapses which have more degrees of freedom than the simple (integrable) FRW-collapse.

We imagine that a "3+1" split has been performed, splitting the spacetime manifold into the topological product of a line (the "time" axis) and the three-dimensional spacelike hypersurfaces Σ_t (the dynamical degrees of freedom are the *spatial* components of the metric, the induced metric g_{ij} on Σ_t , which evolves in the "time" parameter t). In fact, we shall operate in a "synchronous reference frame" (Landau and Lifshitz §97 and MTW, §27.4) which brings the spacetime metric (2) on the very simple form $ds^2 = -dt^2 + g_{ij}dx^i dx^j$.

2.1. What are the degrees of freedom in this toy model ?

The symmetry-ansatz for the metric¹⁰ is:

$$ds^2 = -dt^2 + \gamma_{ij}(t)\omega^i(x)\omega^j(x) \quad , \quad \gamma_{ij}(t) = \text{diag}(a^2(t), b^2(t), c^2(t)) \quad (3)$$

The spacetime has the topology $\mathbf{R} \times S^3$ (product of a time axis and the compact three-sphere). The three-space is invariant under the $G=SU(2)$ group, as expressed by the $SU(2)$ invariant one-forms $\omega^i(x)$, $i = 1, 2, 3$ which satisfy $d\omega^i = \epsilon^i_{jk}\omega^j \wedge \omega^k$ where ϵ^i_{jk} is the completely antisymmetric tensor of rank 3. In terms of Euler angle coordinates (ψ, θ, ϕ) on $SU(2)$ which take values in the range $0 \leq \psi \leq 4\pi$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ (see also MTW (1973), p.808) we have

$$\begin{aligned} \omega^1 &= \cos \psi \, d\theta + \sin \psi \, \sin \theta \, d\phi \\ \omega^2 &= \sin \psi \, d\theta - \cos \psi \, \sin \theta \, d\phi \\ \omega^3 &= d\psi + \cos \theta \, d\phi \end{aligned} \quad (4)$$

Written out in terms of the coordinate differentials $d\psi$, $d\theta$, $d\phi$ we get for the line element of the spacetime metric (3)

$$\begin{aligned} ds^2 &= -dt^2 + c(t)^2 d\psi^2 + (a(t)^2 \cos^2 \psi + b(t)^2 \sin^2 \psi) d\theta^2 \\ &+ \left\{ \sin^2 \theta (a(t)^2 \sin^2 \psi + b(t)^2 \cos^2 \psi) + c(t)^2 \cos^2 \theta \right\} d\phi^2 \\ &+ (a(t)^2 - b(t)^2) \sin 2\psi \, \sin \theta \, d\theta \, d\phi + 2c(t)^2 \cos \theta \, d\psi \, d\phi \end{aligned} \quad (5)$$

This is the toy-model spacetime metric (with $a(t)$, $b(t)$, $c(t)$, the three scale-functions, as degrees of freedom) which we want to evolve on approach to the "big crunch" space-time singularity where the three-volume of the metric collapses to zero.

⁹ Cf. e.g., Ya B. Zel'dovich and I.D. Novikov (1983), esp. §22; C.W. Misner, K.S. Thorne and J.A. Wheeler (1973), esp. §30 or L.D. Landau and E.M. Lifshitz (1975), esp. §116-119. See, also, J.D. Barrow (1982).

¹⁰ If the metric is coupled to matter, e.g. perfect fluid matter, the assumption of diagonality of $\gamma_{ij}(t)$ is a *simplifying* ansatz. If no non-gravitational matter is present, the vacuum Einstein equations will automatically make the off-diagonal components of γ_{ij} vanish for a space with invariance group $G = SU(2)$, see e.g. Bogoyavlensky (1985), p. 34.

The space is closed and the three-volume (Landau and Lifshitz p.390) of the compact space is given by

$$V = \int \sqrt{-g} \, d\psi \, d\theta \, d\phi = \int \sqrt{\gamma} \, \omega_1 \wedge \omega_2 \wedge \omega_3 = 16\pi^2 \, abc \quad (6)$$

When $a = b = c = R/2$ the space reduces to a space of constant positive curvature with radius of curvature $R = 2a$, which is the metric of highest symmetry on the group space $SU(2)$. The volume (6) then reduces to the three-volume $V = 2\pi^2 R^3(t)$ of the compact (isotropic) *Robertson-Walker* space. If we couple the gravitational field to perfect fluid matter, the cosmological model with the ansatz (3) for the metric is an anisotropic generalization of the well known compact FRW model. It has different "Hubble-constants" along different directions in the three-space. One may also interpret the metric (3) as a closed FRW universe on which is superposed circularly polarized gravitational waves with the longest wavelength that will fit into a closed universe (cf. D.H. King (1991) and references therein).

Since the metric (3) is spatially homogeneous, the full non-linear Einstein equations for the metric are a set of ordinary (non-linear) differential equations. To see this more explicitly, introduce in place of the quantities a, b, c , their logarithms $\alpha = \ln a$, $\beta = \ln b$, $\gamma = \ln c$ and a new time variable $\tau = \int dt/abc$ in place of the proper (synchronous) time t appearing in the metric (3), cf. Landau and Lifshitz (1975), §116-119. With the inclusion of a perfect fluid matter source, the space-space components of Einstein's equations then reads

$$\begin{aligned} 2\alpha_{,\tau\tau} &= \frac{d^2}{d\tau^2} (\ln a^2) = (\dot{\beta}^2 - \dot{c}^2)^2 - a^4 + 8\pi G(\rho - p) a^2 b^2 c^2 \\ 2\beta_{,\tau\tau} &= \frac{d^2}{d\tau^2} (\ln b^2) = (c^2 - a^2)^2 - b^4 + 8\pi G(\rho - p) a^2 b^2 c^2 \\ 2\gamma_{,\tau\tau} &= \frac{d^2}{d\tau^2} (\ln c^2) = (a^2 - b^2)^2 - c^4 + 8\pi G(\rho - p) a^2 b^2 c^2 \end{aligned} \quad (7)$$

and the time-time component reads

$$(\alpha + \beta + \gamma)_{,\tau\tau} - 2(\alpha_{,\tau}\beta_{,\tau} + \alpha_{,\tau}\gamma_{,\tau} + \beta_{,\tau}\gamma_{,\tau}) = -4\pi G(\rho + 3p) a^2 b^2 c^2 \quad (8)$$

The quantities p and ρ denote the pressure and the energy density of the fluid.

Adopting the standard ansatz that a barotropic equation

$$p = (\gamma - 1)\rho \quad (9)$$

(γ constant) relates the two quantities one may easily show that the equations of motion (7),(8) have a first integral

$$\dot{I} = I - 8\pi G\rho a^2 b^2 c^2 = 0 \quad (10)$$

where

$$I = \alpha_{,\tau}\beta_{,\tau} + \alpha_{,\tau}\gamma_{,\tau} + \beta_{,\tau}\gamma_{,\tau} - \frac{1}{4}(a^4 + b^4 + c^4) + \frac{1}{2}(a^2 b^2 + a^2 c^2 + b^2 c^2) \quad (11)$$

To be in accordance with the full set of Einstein equations the solution should have $\dot{I} = 0$. The dynamical equations for the compact FRW cosmology is recovered in the case of $a = b = c = R/2$.

Important astrophysical examples of the barotropic equation (9) are $\gamma = 1, 4/3, 2$ corresponding to the cases of “dustlike” matter, “radiation” matter and “stiff matter” respectively. The energy density scales with the volume of the space as $\rho \sim V^{-\gamma}$, see also e.g. Landau and Lifshitz (1975) or Kolb and Turner (1990).

One may show (cf. also Landau and Lifshitz (1975), p.390) that sufficiently near the singularity the perfect fluid matter terms (appearing on the right hand side of the equations (7) and (8)) may be neglected if the equation of state $p \leq 2/3\rho$. Thus, it is sufficient to investigate the “empty space equation” (the vacuum Einstein equations) $R_{\mu\nu} = 0$ even if “dust” ($\gamma = 1$ in equation (9)) and “radiation” fluids ($\gamma = 4/3$) are included in the mixmaster big crunch collapse. One says that the mixmaster cosmology is “curvature dominated” in the region sufficiently near the space-time singularity! In physical terms this means that sufficiently near the space-time singularity the self-interaction of the gravitational field completely dominates the dynamical evolution and contributions from (non-gravitational) matter may be neglected in the study. This conclusion clearly does not apply to the case of a “stiff matter” fluid (where $p = \rho \sim (abc)^{-2}$) and - of course - does not a priori apply to sources of other physical origin. Such other material sources (non-Abelian Yang Mills fields, etc.) might very well couple to and significantly alter the dynamical structure of the gravitational collapse.

Note, that in the reversed time-direction, when the mixmaster spacetime metric (3) is evolved away from its singularity (i.e. as the volume $V = 16\pi^2 abc$ of the space increases) the matter terms gradually become more important and eventually dominate the dynamical evolution of the mixmaster metric (3) and the matter terms may lead to isotropization (though, not fast enough to explain the observed isotropy today, cf. e.g. Doroshkevich, Lukash and Novikov (1973) and Lukash (1983)).

Despite the fact that this metric is widely known, it is remarkable that only quite recently (cf. X. Lin and R.M. Wald (1990)) it has been rigorously proven to recollapse (this is also true in the “vacuum” case, i.e. when no perfect fluid, or any other matter source, is included in the model). So, we know that the mixmaster spacetime metric has *two* spacetime singularities (like the compact FRW cosmology): A “big bang” and a “big crunch”!

2.2. The three-volume of the mixmaster space-time metric cannot oscillate if evolved according to the vacuum Einstein equations

The three-volume $V = 16\pi^2 abc$ of the model-universe cannot oscillate. This can be shown in several (not truly independent) ways:

One may derive this fact directly from the governing set of differential equations (see S.E. Rugh (1990a) and S.E. Rugh and B.J.T. Jones (1990)). We note, that statements on monotonicity of the three-volume are equivalent whether given in t or in τ time, since $dt = abc \, d\tau$ and $abc > 0$. The property of $\ln V$ being a *concave* function (negative second derivative) does not translate from t to τ time. Below we show that $\ln V$ is a concave function in the t time variable (but not in τ time).

Neglecting, for notational convenience, the factor $16\pi^2$ in the expression for the three-volume, we have $\ln V \equiv \ln a + \ln b + \ln c \equiv \alpha + \beta + \gamma$, and the R_{00} equation for the mixmaster metric reads

$$\frac{1}{2}(\alpha + \beta + \gamma)_{\tau\tau} \equiv \frac{1}{2}(\ln V)_{\tau\tau} \equiv \alpha_{\tau}\beta_{\tau} + \alpha_{\tau}\gamma_{\tau} + \beta_{\tau}\gamma_{\tau} \quad (12)$$

From the definition $\partial_t = (abc)^{-1}\partial_{\tau} = V^{-1}\partial_{\tau}$, one arrives at $\partial_t^2 = V^{-2}\{\partial_{\tau}^2 - (\ln V)\partial_{\tau}\}$ and hence

$$\begin{aligned} V^2 \partial_t^2 (\ln V) &= V \partial_{\tau}^2 V - (\partial_t V)^2 = (\ln V)'' - ((\ln V)')^2 \\ &= 2(\alpha'\beta' + \alpha'\gamma' + \beta'\gamma') - (\alpha' + \beta' + \gamma')^2 \\ &= -\alpha'^2 - \beta'^2 - \gamma'^2 \leq 0 \end{aligned} \quad (13)$$

It follows that $(\ln V)'$ and therefore the volume itself, V , can have *no local minimum* (where we should have $\dot{V} = 0, \ddot{V} > 0$). As a corollary it follows that volume oscillations are not possible: After a transient (eventually passing a maximum in the three-volume), the volume of the three-space should decrease *monotonically* as the mixmaster gravitational collapse evolves towards the “big crunch” singularity.

We may, also, consider the Raychaudhuri equation, A. Raychaudhuri (1955), which governs an equation for the relative volume expansion $\Theta = \partial_t(\log\{V\text{volume}\})$ with respect to the coordinate t time. The equation was independently discovered by Lev Landau and A. Raychaudhuri (see S.W. Hawking & G.F.R. Ellis (1973), p.84) and is derived from the Einstein equations for a spacetime metric in a synchronous reference frame coupled to co-moving perfect fluid matter. The general form of Raychaudhuri's equation is:

$$\left(\begin{array}{c} \text{expansion} \\ \text{derivative} \end{array} \right) = - \left(\begin{array}{c} \text{energy density} \\ \text{term} \end{array} \right) - \left(\begin{array}{c} \text{shear} \\ \text{term} \end{array} \right) - \left(\begin{array}{c} \text{expansion} \\ \text{term} \end{array} \right) + \left(\begin{array}{c} \text{vorticity} \\ \text{term} \end{array} \right)$$

$$\dot{\Theta} = -R_{\mu\nu}u^{\mu}u^{\nu} - \sigma^2 - \frac{1}{3}\Theta^2 + 2\Omega^2. \quad (14)$$

A dot denotes derivation with respect to the time t . The term $R_{\mu\nu}u^{\mu}u^{\nu} = 4\pi G(\rho + 3p)$ refers to the *co-moving* perfect fluid source (the nongravitational matter) included in the cosmological model, whereas σ^2 and Ω^2 denote the “shear” and “vorticity” scalars contracted from the shear and vorticity tensors of the metric field (see, also, S.W. Hawking and G.F.R. Ellis (1973)). One may show that the metric (3) has no vorticity $\Omega^2 = 0$. For the quantity

$$\mathcal{G} \equiv \sqrt[3]{abc} \propto \sqrt[3]{V\text{volume}},$$

which is related to the relative volume expansion Θ as

$$\Theta = \frac{d/dt\{V\text{volume}\}}{V\text{volume}} = 3\frac{\dot{\mathcal{G}}}{\mathcal{G}}, \quad \dot{\Theta} + \frac{1}{3}\Theta^2 = 3\frac{\ddot{\mathcal{G}}}{\mathcal{G}},$$

one obtains, after a little algebra, the equation

$$\ddot{\mathcal{G}} = \frac{d^2/dt^2\{\sqrt[3]{abc}\}}{\sqrt[3]{abc}} = -\frac{4\pi G}{3}(\rho + 3p) - \frac{2}{3}\sigma^2 \leq 0. \quad (15)$$

Thus, provided ρ and p are not negative, $\ddot{\mathcal{G}}$ is always non-positive implying that $\mathcal{G}$ cannot have a local minimum (where one should have $\dot{\mathcal{G}} = 0$ and $\ddot{\mathcal{G}} > 0$). I.e., according to Raychaudhuri's equation, the three-volume for any cosmological model without vorticity $\Omega = 0$ (our diagonal mixmaster toy model collapse (3) belongs to this class) cannot be oscillatory, but can only have one maximum like the FRW cosmology. In the empty case, $p = \rho = 0$, the conclusion applies equally well.

2.3. Volume oscillations as a “probe” on numerical solutions

The fact that solutions to the vacuum Einstein equations should not have oscillating three volumes is no surprise. The idea, however, is to use the property of “no oscillations” as a “probe” to examine the validity of some previous investigations which were done on this model. In the references Zarddecki (1983), Francisco and Maitas (1988), and in fact in Barrow (1982, 1984) and Barrow and Silk (1984), the volume behavior of the depicted evolutions is not in accordance with the conclusion arrived at here. They have such oscillations and can therefore not be proper solutions in agreement with the Einstein equations for positive or zero non-gravitational energy densities. In fact, some of these models effectively included *stiff* matter with *negative* mass densities. (cf. S.E. Rugh (1990a) and S.E. Rugh and B.J.T. Jones (1990)). That is, one may easily show¹¹ that vacuum solutions which fail to satisfy the first integral constraint $I = 0$, with I given in (11), correspond to the inclusion of “stiff matter” (i.e. a perfect fluid with equation of state “ $p = \rho$ ”) with energy density

$$p = \rho = (8\pi G)^{-1} \frac{I}{a^2 b^2 c^2}.$$

The character of the solutions depends on the sign of I .

For $I < 0$ the negative mass densities create a *repulsion effect*, which causes oscillations in the three volume $V = 16\pi^2 abc$. In such solutions a typical evolution of the parameters $a(t)$, $b(t)$, $c(t)$ of the metric (3) will be like in fig.1 where the scale functions - or rather their logarithm's $\alpha = \log a$, $\beta = \log b$, $\gamma = \log c$ - are plotted against the standard (Landau and Lifshitz, Vol. II, §118) time variable $\tau = \int dt/abc$.

One may also display the “degree of anisotropy” of the spacetime metric in the anisotropy variables (ADM variables, see later)

$$\tilde{\beta} = (\beta_+, \beta_-) = \left(\frac{1}{6} \log \left(\frac{ab}{c^2} \right), \frac{1}{2\sqrt{3}} \log \left(\frac{a}{b} \right) \right). \quad (16)$$

$\tilde{\beta} = \vec{0}$ means no anisotropy, while a huge $|\tilde{\beta}|$ means that our model (this is the case for proper solutions near the singularity, cf. fig.2) is very anisotropic. In terms of this parameterization the metric (3) reads, cf., e.g., MTW (1973),

$$ds^2 = -dt^2 + e^{-2\tilde{\beta}_i} (e^{2\tilde{\beta}_j})_{ij} \omega^i(x) \omega^j(x) \quad (17)$$

where $\Omega = -\frac{1}{3} \log(abc)$ and β_j is the traceless, diagonal matrix with the diagonal elements $\beta_+ + \sqrt{3}\beta_-$, $\beta_+ - \sqrt{3}\beta_-$ and $-\beta_+$. The evolution of the metric is decomposed in *expansion* (volume change parametrized by Ω) and *anisotropy* (shape change parametrized by $\tilde{\beta} = (\beta_+, \beta_-)$). The trajectories of the “anisotropy” of the (erratic volume oscillating) mixmaster model, which correspond to the sketched solutions above, fig.1, is displaying in fig.3 (the C_{in} symmetry is apparent and is of course expected from the symmetry under the interchange $a \leftrightarrow b \leftrightarrow c$ of the scale factors in the metric (3)).

¹¹In the case of “stiff matter”, $p = \rho$, i.e. $\gamma = 2$ in (9), and we have $\rho \sim V^{-2}$. The 11, 22, 33-components of the Einstein field equations (7) attain the same form as the vacuum equations. The exception is the 00-component of the Einstein equations (8) and thereby the first integral constraint (10). Since the energy density scales with the three volume as $\rho \sim V^{-2}$ we put $\rho = \rho_0(abc)^{-2}$ which gives $I = 8\pi G \rho_0$. For the vacuum Einstein equations we ought to have $I \equiv 0$. However, if initial data fail to satisfy the zero density constraint, the corresponding solutions act as if we had included “stiff matter” with $p_0 = \rho_0 = (8\pi G)^{-1} I \neq 0$.

Figure 1. A typical evolution of the logarithmic scale functions $\alpha = \log a$, $\beta = \log b$, $\gamma = \log c$ (the thick curve is $\alpha = \log a$) as a function of τ -time, $\tau = \int dt/abc$, is rather interesting if we select initial conditions which have $I < 0$ and thus effectively introduces stiff matter with negative energy density. Apparently the evolution of the scale functions of the metric (3) is highly irregular.

The mixmaster equations pass the Painlevé test

It is interesting that Contopoulos *et al* (1993) recently have performed a Painlevé analysis on the set of mixmaster equations (7), for $\rho = p = 0$, and find that the set of mixmaster equations pass the Painlevé test. Apparently, this analysis does not utilize the additional information from the first integral constraint $I = 0$. Thus the equations of motion for the mixmaster space-time metric also pass the Painlevé test for the case $I < 0$ which, according to Contopoulos *et al*, is a strong indication that the trajectories corresponding to fig.1 and fig.3 (as well as fig.4 and fig.5 for $I = 0$) are integrable, i.e. that two additional *constants of motion* (*symmetries*) besides the Hamiltonian can be found. If an integrable system could produce an orbit like the one in fig.3 it would be surprising. We have previously searched for such additional integrals.¹² Our results, so far, indicate the lack of existence of such additional integrals in the equations of motion for the case $I < 0$.

2.4. The mixmaster gravitational collapse evolved towards the spacetime singularity under the governing vacuum Einstein equations

The dynamics of the spacetime metric (3) will be very complicated (though not as complicated as in figures 1, 2 and 3 above) when evolved according to the vacuum Einstein equations $R_{\mu\nu} = 0$. The deterministic evolution of the scale factors $a(t)$, $b(t)$, $c(t)$ and their first derivatives is described by the set of six coupled, first order, differential equations (7) (we put $p = \rho = 0$) constrained by the first integral (10). Due to the scale

¹²F. Christiansen, H.H. Rugh and S.E. Rugh, unpublished investigations

Figure 2. Behavior of the logarithm of the three-volume $V = 16\pi^2 abc$ as a function of the τ -time $\tau = \int dt/abc$ corresponding to the behavior of the scale functions in fig.1. Due to the effective inclusion of negative energy densities (see text) the solutions do not satisfy the reasonable conditions on the energy momentum tensor required for the Hawking-Penrose singularity theorems to apply. The metric does not evolve towards a singularity: The negative mass density creates a repulsion effect and, as a result, the three-volume oscillates in time.

Figure 3. Solutions corresponding to fig.1 but mapped out in the anisotropy variables $\vec{\beta} = (\beta_+, \beta_-)$. Not surprisingly, this behavior is reflected in positive values of the maximal characteristic Lyapunov exponent.

invariance of the Einstein equations there are four degrees of freedom in the problem. We may distinguish between solutions which are axisymmetric (an integrable case) and solutions without any axisymmetry.

With rotational invariance about one axis¹³ ($a=b$, say) in the $SU(2)$ homogeneous 3-space, we yield for the metric (3) the special case of the Taub spacetime metric of the form

$$ds^2 = -dt^2 + a^2(t)((\omega^1)^2 + (\omega^2)^2) + c^2(t)(\omega^3)^2,$$

and in this axisymmetric case the vacuum equations admit the *exact* solution (given in A.H.Taub (1951))

$$a^2 = b^2 = \frac{p \cosh(2pr + \delta_1)}{2 \cosh^2(pr + \delta_2)}, \quad c^2 = \frac{2p}{\cosh(2pr + \delta_1)}, \quad (18)$$

where p, δ_1, δ_2 are constants. However, this axisymmetric solution is *unstable* with respect to small perturbations in the parameter space of scale functions ($a, b, c, \dots$), see also C.W.Misner (1969). For a numerical investigation of the Taub solutions, see C. Behr (1962).

The dynamical behavior for a typical mixmaster gravitational collapse (without additional symmetries) is displayed in fig.4 and fig.5, where the three axes (a, b, c) are followed as a function of the standard Landau and Lifshitz time coordinate $\tau \equiv \int dt/abc \rightarrow \infty$. Near the singularity where the scale functions collapse to zero $a, b, c \rightarrow 0$ we prefer to display the logarithm $\alpha = \log a, \beta = \log b, \gamma = \log c$ of the scale functions. The "anisotropy" of the toy-model grows without limit on approach to the big crunch singularity.¹⁴

We have selected a set of reference initial conditions as in A. Zardecki (1983), but have adjusted the value of c' to make the first integral vanish to *machine precision*. (Such an adjustment is indeed necessary. Cf. discussions in S.E. Rugh (1990a) and D. Hobill *et al* (1991)). This yields the starting conditions

$$\begin{aligned} a &= 1.85400.., \quad b = 0.438500.., \quad c = 0.085400.. \\ a' &= -0.429200.., \quad b' = 0.135500.., \quad c' = 2.964843279..... \end{aligned} \quad (19)$$

All integrations have been performed by a standard *fourth order Runge Kutta* algorithm, and each calculation takes less than one CPU-minute.

We distinguish between two qualitatively different phases of the evolution of the spacetime metric (3) towards the "big crunch" singularity:

Transient behavior: The model cosmology trajectory "catches up" its initial conditions, and eventually passes a maximum in three-volume. The volume

¹³Note that the FRW solution, being invariant under rotations about any axis in the $SU(2)$ homogeneous 3-space (i.e. isotropic) is not obtained from the Taub solution by putting $a=b=c$. The Taub solution is a *vacuum* solution whereas the FRW solution is a solution with matter (a perfect fluid) in the model.

¹⁴If the anisotropy variables β_i calculated according to (16), are mapped out corresponding to the long τ time behavior of the α, β, γ variables depicted in fig.5, they will be pieces of straight line segments corresponding to bounces from outward expanding (almost infinitely steep) potential walls. It is like a game of billiards played on a triangular shaped table with outward expanding table boundaries, with a ball which moves faster than the outward expanding boundaries (cf., also, displayed figures of this behavior in S.E. Rugh (1990a) and B.K. Berger (1990)).

then begins to decrease monotonically. (Fig.4)¹⁵

Asymptotic behavior near the singularity: The evolution towards the "big crunch" singularity is fast attracted¹⁶ into an infinite sequence of oscillations of the three scale functions (fig.5). We may identify the Kasner segments (Kasner epochs) between each "bounce", and the combinatorial model by Belinskii *et al* describes well the transitions from one Kasner segment to the next. (Table 1).

Extracting lower dimensional signals from the gravitational collapse

In an ever expanding phase space (cf., also, discussions later) it is natural to try to "project out" some lower dimensional (compact) signal¹⁷, say, which captures some of the recurrent "chaotic" properties of the model. The dynamical behavior of the toy-model gravitational collapse is indeed in certain aspects chaotic, as captured by the parameter

$$u = \frac{\alpha' + \beta' + \gamma' - (\min\{\alpha', \beta', \gamma'\} + \max\{\alpha', \beta', \gamma'\})}{\min\{\alpha', \beta', \gamma'\}} \quad (20)$$

Extracting this parameter "u" (the so-called "Lifshitz-Khalatnikov" parameter "u" $\in \mathbf{R}$ ", see also Landau and Lifshitz, §116-119) from the displayed trajectories we get table 1. We find that the gravitational collapse is *extrinsically well* described by the *BKL*-combinatorial model which is summarized below, following O.I.Bogoyavlensky and S.P.Novikov (1973). "*BKL*" refers to the originators of this combinatorial description of the behavior of the scale functions: V.A. Belinskii, I.M. Khalatnikov and E.M. Lifshitz. See also I.M. Khalatnikov *et al.* (1985) and references therein.¹⁸

With the parameterization

$$(p_1(u), p_2(u), p_3(u)) = (-u, 1+u, u(1+u))/(1+u+u^2)$$

the *BKL*-piecewise approximation of the scale functions a, b, c by the power law functions (Kasner epochs) $t^{p_1}, t^{p_2}, t^{p_3}$, $p_i = p_i(u)$, is (on approach to the spacetime singularity) described by the sequence of states

$$(u_0, \sigma_0) \rightarrow (u_1, \sigma_1) \rightarrow (u_2, \sigma_2) \rightarrow \dots \quad (21)$$

where the *BKL*"Kasner state" transformation ("alternation of Kasner epochs") is given by

$$\begin{cases} (u, \sigma) \rightarrow (u-1, \sigma\sigma_1) & (2 \leq u < \infty) \\ (u, \sigma) \rightarrow (1/(u-1), \sigma\sigma_1\sigma_2) & (1 \leq u \leq 2) \end{cases} \quad (22)$$

¹⁵If we want a more realistic model cosmology at this stage, matter terms should be included here, since the omission of matter contributions for the mixmaster toy-model cosmology is only justified when we are sufficiently near the "big crunch" singularity.

¹⁶Note that the sequence of oscillations of the scale functions, as shown in fig.5, is an attractor for almost any initial condition, but not initial conditions with axisymmetry.

¹⁷It is not uncommon to "probe" chaos in dynamical flows which take place in higher dimensional phase spaces by extracting lower dimensional time-signals from the flow. (E.g. probing turbulence and complexity in Navier-Stokes hydrodynamical flows by measuring time signals of a temperature probe, placed at some given space point in the fluid). Extraction of a single physical variable $\xi(t)$ with "chaotic" behavior (a "time" signal or a discrete map ξ_n) mirrors "chaos" in the full phase flow.

¹⁸The accordance with the *BKL*-combinatorial model for the dynamical evolution of the mixmaster collapse has been further investigated in the work of B.K. Berger, cf., e.g., Berger (1993). For a description of the more complete 4-parameter map, derived by Belinskii *et al.*, see also I. Chernoff and J.D. Barrow (1983).

Figure 4. Transient evolution of the three scale functions of the mixmaster space-time metric governed by the vacuum Einstein equations with starting conditions (19). We have displayed the "transient" τ time interval $0 \leq \tau = \int dt/abc \leq 1$. (From S.E. Rugh (1990 a)). The three-volume has a maximum at $\tau \approx 1/5$ and then decreases monotonically (turning into an oscillatory behavior of the scale functions as shown in fig.5) towards the big crunch singularity. (It is interesting, and surprising, if this displayed behavior should turn out to be integrable, as suggested by Contopoulos *et al* (1993)).

Figure 5. Asymptotic evolution of the three scale functions (typical figures from a numerical experiment described in S.E. Rugh (1990a)). The axes $\alpha = \ln a$, $\beta = \ln b$, $\gamma = \ln c$ of the toy-model gravitational collapse, given by the metric (3), is followed in τ -time ($\tau = \int dt/abc$) towards the "big crunch" singularity under the governing vacuum Einstein equations. The spacetime singularity is reached in *finite* t time, but in τ time the dynamical evolution is stretched out to infinity $\tau \rightarrow \infty$. The three-volume of the space shrinks monotonically to zero, the anisotropy of the model grows without limit and the Weyl curvature $C^2 = C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ also diverges on approach to the spacetime singularity.

$\tau = \int dt/abc$	u	$1/u$
1	5.564816	
6	4.564816	
16	3.564815	
27	2.564816	
48	1.564816	
96	1.770488	0.5648160
305	1.297878	0.7704884
4700	3.357077	0.2978782
50000	2.357077	
180000	1.357077	
500000	2.800516	0.3570771
3000000	1.800510	
19000000	1.249229	0.8004938
75000000	4.013535	0.2491569

Table 1. This table summarizes the evolution of numerically extracted values of the Lifshitz-Khalatnikov parameter “ u ” corresponding to solutions of the vacuum Einstein equations as those depicted in fig.5. The description of the evolution of the “ u ” parameter in terms of the “Farey map” or “Gauss map” is in complete agreement with this table. From S.E. Rugh (1990a), p.98. (The τ -time values offered in the table correspond to τ -times when the collapse orbit is well beyond a bounce and has reached a new straight line behavior (“Kasner epoch”) in fig.5.

the “Kasner state” being described by the pair

$$(u, \sigma) : \sigma = \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}, \sigma_{12} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \sigma_{23} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad (23)$$

and σ denoting the permutation of the three Kasner axes.

2.5. The (compressed and stretched) astronomer who falls into the space-time singularity

A ‘freely falling astronomer’ who falls into the mixmaster spacetime singularity will experience a tidal field, in which he is *compressed* along two directions and *stretched* (expanded) in one. The directions of these gravitationally induced stresses are permuted infinitely many times (in a not-predictable way!) on approach to the space time singularity. A picture in Kip S. Thorne (1985) sketches the tidal gravitational forces felt by an observer (an astronomer) who falls into the singularity. Such tidal forces are produced by spacetime curvature. The astronomer, who in this example makes up the tidal field instrument (see, e.g., MTW, p.400-404), feels, in his local inertial frame, tidal accelerations given by the equation of geodesic deviation, $d^2\xi_i/dt^2 = R_{jio} \xi^t$, where ξ is the separation vector between two freely falling test particles (two reference points in

the body of an astronomer, falling freely along geodesics - if we neglect internal elastic forces in the body of the observer (justified, if the spacetime curvature is big?). There is in the local inertial frame of the infalling observer a preferred choice of coordinate axes ($i = 1, 2, 3$) which diagonalizes the tidal field. In terms of these “principal axes” the component R_{ioo} produces, according to the equation of geodesic deviation, $\ddot{\xi}_i/\xi_i = -R_{ioo}$, a tidal *compression* or *stretching* along direction “ i ”, depending on whether R_{ioo} is positive or negative.

One may swindle a bit and write down the Riemann curvature components of the Kasner-segments as if they were given by the Riemann curvature of the Kasner metric (instead of the Riemann tensor components of the spacetime metric (3)). The Riemann tensor components of the Kasner metric reads $R_{ioo} = -p_i(p_i - 1)t^{-2}$, $i = 1, 2, 3$, and we get, in terms of the Lifshitz-Khalatnikov parameter $u \in \mathbf{R}$, the following expressions for the tidal stresses:

Along the two axes of compression

$$\frac{\ddot{\xi}}{\xi} = -\frac{u(u+1)}{(1+u+u^2)^2} t^{-2} \quad \text{resp.} \quad \frac{\ddot{\xi}}{\xi} = -\frac{u^2(u+1)}{(1+u+u^2)^2} t^{-2}.$$

Along the axis of expansion

$$\frac{\ddot{\xi}}{\xi} = +\frac{u(u+1)^2}{(1+u+u^2)^2} t^{-2} > 0.$$

The tidal stresses grow up like $\sim t^{-2}$ where t denotes the *finite* time distance (as measured by the infalling observer) until the spacetime singularity is reached. The successive shifts $u \rightarrow u - 1$ & C in the parameter “ u ”, are governed by the combinatorial model (22). During each “era” (comprised of cycles $u \rightarrow u - 1$ until u reaches the value $1 \leq u \leq 2$) one of the principal axes experiences a continual tidal compression while the other two principal axes oscillate between compression and stretch. At a change of “era” (i.e. when $u \rightarrow 1/(u - 1)$ for $1 \leq u \leq 2$) there is a change in the role of the axes (another of the three principal axes experiences a continual tidal compression while the other two axes oscillate). Note, that “real” *physical* quantities, measurable by *tidal field instruments* exhibit chaotic oscillations which condense infinitely on approach to the spacetime singularity at $t \rightarrow 0$.

The astronomer may feel a little “worried”, being compressed like this in two directions and stretched (expanded) in one. The directions of these gravitationally induced stresses are even permuted infinitely many times on approach to the spacetime singularity.

2.6. The “Farey map” with “strong intermittency” encodes the “per bounce” dynamics of the mixmaster gravitational collapse

The *unpredictability* of the mixmaster gravitational collapse does not originate from the oscillations of the scale functions (as described by $u \rightarrow u - 1$) *within* a given major cycle (an “era” of oscillations), but rather from the *shifts* between major cycles (described by $u \rightarrow 1/(u - 1)$, $1 \leq u \leq 2$). These shifts give rise to the (highly chaotic) Gauss map, cf. e.g. J.D. Barrow (1982). The Gauss map is well known from “chaos theory” (cf., e.g., R.M. Corless *et al* (1990) and references therein) and acts as a *left shift* on the continued fraction representation of numbers on the unit interval. The Gauss map

has positive Kolmogorov entropy, $h = \pi^2/6(\ln 2) \gg 0$ and it has the Bernoullian property¹⁹ and, in that sense, it is as random as that of flipping coins (or a roulette wheel).

A map, which describes the "per bounce" evolution of u (and not, merely, the "per major cycle" evolution of the u parameter) is easily found²⁰ and is known as the "Farey map": Make the substitution $u = 1/x$ in (22) and write this map in terms of the parameter $x \in [0, 1]$. This yields the following map on the unit interval (the "Farey map"²¹)

$$x \rightarrow \mathcal{F}(x) = \begin{cases} x/(1-x) & \text{if } 0 \leq x \leq 1/2 \\ (1-x)/x & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

Whereas the Gauss map has an *infiniting* of branches (cf. e.g. pictures in J. Barrow (1982) or R.M. Corkes *et al* (1990)) the Farey map has only two branches (the left, $x \leq 1/2$, and the right) and thus a natural *binary* symbolic dynamics (cf. sec.3.4) with a binary alphabet (which corresponds more directly to a symmetry-reduced binary symbolic dynamics encoding of the geodesic motion on the triangular billiard on the Poincaré disc).²² The Farey map contains the Gauss map as a sub-map, since one iteration of the Gauss map corresponds to a transition from the right branch via oscillations in the left branch back to the right branch.

As regards the evolution of the Lifshitz-Khalatnikov parameter "u", the Farey map takes into account both chaotic and "non-chaotic" segments of the one-perturbation BKL-combinatorial model for the gravitational collapse. The watch of the "Farey map" ticks one step forward (one iteration of the map) for each bounce against a wall, i.e. for each oscillation of the scale functions.

The Farey map has a *marginally stable* fixed point at the left end ($|\mathcal{F}'(0)| = 1$). This has an important influence on the instability properties of the map. Intuitively, the marginally stable point of the Farey map at $x = 1/u = 0$ corresponds to major cycles, containing an "infinite" number of oscillations (governed by the $u \rightarrow u - 1$ rule), i.e., trajectories which penetrate deeply into one of the three corner channels.

In general, a map $\mathcal{F}$ with the asymptotic expansions $\mathcal{F}(x) \approx x + \alpha x^2 + \dots$ (towards the marginally stable point at $x = 0$) and $\mathcal{F}(x) \approx 1 - b|1 - 2x|^{1/\alpha} + \dots$ (around the top point $x = 1/2$) will have an invariant measure $\mu(x)$, which has the asymptotic behavior²³

$$\mu(x) \sim x^\eta; \quad \eta = \alpha + 1 - \zeta$$

near the origin $x = 0$. The situation $\eta > 0$ is then referred to a "weak intermittency" (there is a smooth invariant measure and the Lyapunov exponent is positive, $h =$

¹⁹The Gauss map is Bernoullian in the sense that it may be extended to a two-dimensional invertible map which is isomorphic to a Bernoulli shift with an infinite alphabet.

²⁰H.H. Rugh and S.F. Rugh (1990), unpublished. See also B.K. Berger (1991).

²¹cf. e.g., M. Feigenbaum (1988) or Artuso, Aurell and Cvitanovic (1990), p.378, and references therein.

²²The Farey/Gauss map is closely related to the symbolic description of geodesic flows on so-called modular surfaces (found by Artin) on the Poincaré disc; see e.g. T. Bedford *et al* (1991), and therefore a very nice connection exists (cf. also J. Pullin (1990)) between the Farey/Gauss map encoding of the minimaxer gravitational collapse and the description of the minimaxer collapse orbit on the Poincaré disc.

²³See, e.g., Z. Kaufmann and P. Szepietulsky (1989) or P. Szepietulsky and G. Györfyi (1986).

$\lambda = \int \ln |\mathcal{F}'(x)| \mu(x) dx > 0$) while $\eta < 0$ implies "strong intermittency" (there is no normalized, smooth measure and $h = \lambda = 0$). As regards the Farey map we have $\alpha = 1, \zeta = 2$; i.e. $\eta = 1 + 1 - 2 = 0$ which is the borderline case. However, the Farey map is an example of strong intermittency. By direct inspection one may verify that $\mu(x) = 1/x$ is an invariant measure. This measure is non-normalizable, but is normalized to the δ -function measure $\delta(0)$ at the marginally stable fixpoint $x = 0$. Hence, all the measure is concentrated at the *stable* fixpoint $x = 0$ and the averaged Lyapunov exponent (the K-entropy) is zero. Although the overall Lyapunov exponent (Kolmogorov entropy) of the map is zero, all periodic orbits (which do not include $x = 0$) are unstable and have $\lambda > 0$.

This strong intermittency means that the stability properties are completely dominated by the "infinite number of corner oscillations" fixpoint at $x = 0$. However, we note that all this is within the *one-perturbation* combinatorial model (22) for the gravitational collapse, and therefore it is not valid in the regime $x = 1/u \approx 0$. Thus, there is a "cut off" towards the left end of the Farey map and the good ergodic properties are regained. A *two-perturbation* analysis has to take over and show that the trajectory then will leave the corner.²⁴

The binary "Farey tree". Self-similarity of the collapse dynamics.

The near singularity dynamics of the gravitational collapse, as regards the parameter $x = 1/u$, can be symbolically represented by the so-called "Farey tree", which is a construction associated with the Farey map. The Farey tree is a "tree of rational numbers" (the Farey numbers). In fig.6 each Farey number has been represented by its continued fraction. Originally, the tree is constructed by starting with the endpoints of the unit interval written as $0/1$ and $1/1$ and then recursively bisecting intervals by means of "Farey mediant"²⁵ $p/q \oplus p'/q' = (p+p')/(q+q')$. The Farey tree rationals can be generated by backward iterates of the number $1/2$ by the Farey map, i.e., starting at $x_0^{(0)} = 1/2$ we get the next layer $x_0^{(1)} = 1/3$ and $x_1^{(0)} = 2/3$ as the two numbers which are mapped to $1/2$ under the Farey map $\mathcal{F}$. Generally, the 2^n *n*th inverses of $1/2$ (under the Farey map) are precisely the *n*th layer of the Farey tree.

Iterating downwards the layers of the Farey tree is to follow the evolution (backwards) in the parameter u and corresponds to one bounce against one of the three walls. Compare with the displayed trees of symbols in M.H. Bugalho *et al* (1986).

A non-chaotic, segment of the type $u \rightarrow u - 1$ (a repeated bounce against two given walls) corresponds to continue along the given branch of the Farey tree. For example, drifting along the most left branch of the Farey tree corresponds to going backwards through the numbers $x = 1/2 \rightarrow 1/3 \rightarrow 1/4 \rightarrow 1/5 \rightarrow \dots$ which in the forward direction translates to the segment of evolution $\rightarrow 5 \rightarrow 4 \rightarrow 3 \rightarrow 2$ in the u parameter.

The chaotic part (an inversion $u \rightarrow 1/u$) is associated with a shift in "direction" of motion down the tree, that is when we *zig-zag* our way down the tree. For example, the $[1, 1, 1, 1] = [1]^\infty$ orbit, which only contains one oscillation in every major cycle, would here be described as an infinite zig-zag route (each zig and each zag consist of one step, interchanging without end) and the associated Farey number then converges

²⁴The avoidance of the so-called "dangerous case of small oscillations" (i.e. dangerous for the one-perturbation treatment in the BKL-combinatorial model) has been discussed in the work by Belinski *et al.*, cf. Khalatnikov *et al.* (1985) and references therein.

²⁵See p. 23 in Hardy and Wright (1938) and Cvitanovic and Myrheim (1989).

Figure 6. A continued fraction representation of the binary Farey tree (encoding the bounce dynamics of the gravitational collapse). The dotted lines continue a given branch (and correspond to repeated bouncing between the same to potential walls) while the full lines make a "zig-zag" movement down the tree (which corresponds to a transition to a third wall). The Farey tree rationals can be generated by backward iterates of the number $1/2$ by the Farey map, or, what is the same, by interpolating the rationals downwards by means of the "Farey mediant". Figure from (Vitanovic and Myrheim (1989).

to the golden mean $u = (\sqrt{5} - 1)/2$. This is the initial starting value of the parameter u which in the forward direction generates this orbit.

Each branch of the Farey tree is *similar* to the entire tree. In the gravitational collapse dynamics governed by the *classical* vacuum Einstein equations there is also no scale (the vacuum Einstein equations are scale invariant²⁶). When the evolution starts to go into the described sequence of oscillations, it does so forever and does not distinguish the situations where 10 or 10,000 major cycles have passed. According to *classical* general relativity nothing prevents this sequence of (self-similar) oscillations from going on "forever".²⁷

The gravitational collapse (3) thus "organizes itself" (is attracted) into the described (self-similar) never ending sequence of oscillations of the scale functions without any strong fine-tuning of initial-conditions (as regards the amount of matter²⁸ in the model or the initial "set up" of the gravitational degrees of freedom).

It has been stated by Belinskii *et al* that the (self-similar) oscillatory character is a generic "attractor" (locally, i.e. in the neighborhood of every space point) even in the larger space of spatially inhomogeneous gravitational collapses. (See e.g. Belinskii *et al* (1982) and Zel'dovich and Novikov (1983) §23.3). This claim is, however, a subject of substantial controversy.

2.7. Stability properties of the metric (evolved towards and away from the spacetime singularity). Does it isotropize?

It was demonstrated some time ago, E.M. Lifshitz (1946), by means of linear perturbation analysis, that the FRW cosmological solution is indeed *stable* against *local* perturbations of the metric (perturbations over small regions of the space, i.e. over regions whose linear dimensions are small compared to the FRW scale factor R) in the *forward* time direction (away from the singularity), but *unstable* in the direction towards the space time singularity!

In the "class of perturbations", studied here, which one may call *global* anisotropic $SU(2)$ homogeneous perturbations of the compact FRW solution, the study shows that in the direction towards the singularity a small anisotropy will grow up²⁹ and the gravitational collapse will go into the described BKL-sequence of oscillations, which is *unpredictable*, even if we knew the initial data with an (almost) infinite amount of precision!

²⁶There is no natural length associated with general relativity (G, c) and no natural length associated with the quantum principle ($\hbar$), individually. The union of general relativity and "the quantum" ($G, c, \hbar$) does contain a natural length, $L_P \equiv \sqrt{\hbar G/c^3}$ (The Planck length).

²⁷However, if we put an initial scale of the gravitational collapse in "by hand" (e.g., for instance, $L = c/H \sim 10^{28}$ cm which is a characteristic scale in the present universe) a restriction concerning the validity of the classical field equations themselves may reduce the number of such oscillations to be very small before Planck scales are reached and quantum fluctuations of the metric become important.

²⁸The perfect fluid matter contributions (in the case of dust or a radiation fluid) in the toy-model gravitational collapse become (in the direction towards the space-time singularity) unimportant at some point (the collapse is curvature dominated) and the model evolves into the scale-free oscillatory (chaotic) collapse dynamics.

²⁹This also happens in the well known "Taub-solution" which is an axisymmetric special case ($a = b, c \neq a, k(C)$) of the vacuum mixmaster metric. This Taub solution is integrable (Taub, 1951) and unstable!

In the direction *away from the singularity* the matter terms get increasingly important and the mixmaster cosmological model *isotropizes*³⁰ (cf., e.g., MTW §30.3-30.5, Ya.B. Zel'dovich and I.D. Novikov (1983), chapt. 22.7, and V.N. Lukash (1983)) though not fast enough to explain the remarkable degree of isotropy we see in the Universe today! An idea like, e.g., a Guth/Linde *inflationary* phase of the universe model is needed. However, while at the one hand inflation occurs in anisotropic cosmological models under a wide variety of circumstances, there are actually some difficulties in inflating the compact mixmaster model universe, if the initial anisotropy is too large, cf., e.g., sec. 6.2 in the review by K.A. Olive (1990).

3. CHAOTIC ASPECTS OF THE GRAVITATIONAL COLLAPSE (POSITIVE LYAPUNOV EXPONENTS AND ALL THAT)

Can we assign an invariant meaning to chaos in the general relativistic context? It will be clear in the following that this program of research is still in its infancy and to pursue this question will require a good deal of technical apparatus in general relativity.

A standard probe of chaos for dynamical systems with few degrees of freedom is to look at the spectrum of Lyapunov exponents, in particular the principal (maximal) Lyapunov exponent defined in the following way (J.P. Eckmann and D. Ruelle (1985)): Given a flow $f^t: M \rightarrow M$ on a manifold M and a metric (a norm) $\|\cdot\|$ on the tangent space $T_x M$, we define for $x \in M$, $\delta x \in T_x M$,

$$\lambda(x, \delta x) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|(D_x f^t) \delta x\|.$$

For any ergodic measure ρ on M this quantity for ρ -almost all $x \in M$ and almost all $\delta x \in T_x M$ defines the principal (maximal) Lyapunov exponent for the flow f^t w.r.t. the ergodic measure ρ (and is independent of x and δx).

Note, that the maximal Lyapunov exponent, and more generally the spectrum of Lyapunov exponents, extracted from non-relativistic *Hamiltonian* flows is *invariant* under non-singular *canonical* coordinate transformations which do not involve transformations of the time coordinate (see also e.g. H.-D. Meyer (1986)).

In the general relativistic context we may identify some obstacles for this construction of a "Lyapunov exponent" (e.g. for the mixmaster gravitational collapse, the orbits being three-metrics, ⁽³⁾ g_t , evolving in "time" towards the "final crunch"):

- (1) *What should we choose as a distance measure $\|\cdot\|$ on the solution space?*
- (2) *What should we choose as a time parameter?*
- (3) *Is there a natural ergodic measure on the solution space?*

At first, it is natural to try to treat the evolutionary equations (7) for the mixmaster metric (3) as a set of ordinary differential equations on *equal footing* with other

³⁰By "isotropization" we here mean that the anisotropic spacetime metric (3), coupled to perfect fluid matter, evolves into a stage of (quasi) isotropic expansion rate, that is, the metric (3) in the forward time direction evolves into a stage where the expansion is approximately uniform in all directions and is described by Hubble's law. Note, however, that the curvature of the three-dimensional space is very anisotropic and, as a rule, does not isotropize. See also Zel'dovich and Novikov, §22.7.

Figure 7. What is a *natural* distance measure $\|^{(3)}g - ^{(3)}g\|$ between two nearby space-time metrics ⁽³⁾ g and ⁽³⁾ $\tilde{g} = ^{(3)}g + \delta^{(3)}g$, where g and $\tilde{g}$ are both solutions to the vacuum Einstein equations but correspond to slightly different initial conditions, say? In the case of non-relativistic Hamiltonian dynamical systems, measures of "chaos" such as principal Lyapunov exponents are calculated by using an *Euclidean* distance measure which is naturally induced from the structure of the kinetic energy term in the non-relativistic Hamiltonian. In the general relativistic context, however, it is not obvious why one should use an Euclidean distance measure to assign a distance between two three-metrics. Do we need such a distance measure on the space of mixmaster collapses to talk about chaos?

dynamical systems governed by some set of ordinary differential equations and apply the standard probes of "chaos" available, in particular the standard methods for extracting Lyapunov exponents.

We decompose the equations (7) into the form of first order differential equations, $\dot{\vec{x}} = \vec{f}(\vec{x})$, $\vec{x} \in \mathbf{R}^6$. Write, e.g., the vacuum equations (7) as $2\alpha'' = (e^{2\beta} - e^{2\gamma})^2 - e^{4\alpha}$ (and cyclic permutations) and get for the 6-dimensional state vector

$$\vec{x} = \vec{x}(\tau) = (\alpha, \beta, \gamma, \alpha', \beta', \gamma') \quad (24)$$

the coupled *first order* equations

$$\frac{d}{d\tau} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \alpha' \\ \beta' \\ \gamma' \end{pmatrix} = \begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \frac{1}{2}(e^{2\beta} - e^{2\gamma})^2 - \frac{1}{2}e^{4\alpha} \\ \frac{1}{2}(e^{2\gamma} - e^{2\alpha})^2 - \frac{1}{2}e^{4\beta} \\ \frac{1}{2}(e^{2\alpha} - e^{2\beta})^2 - \frac{1}{2}e^{4\gamma} \end{pmatrix} \quad (25)$$

which are supplemented with the first integral constraint $I = 0$ with I given in (11) as a constraint on the initial state vector for the gravitational collapse.

Calculations of Lyapunov exponents were in all previous studies (cf., e.g., Burd et al. (1990), Hobill et al. (1991), Berger (1991), Rugh (1990a), see also Rugh and Jones

(1990)) based on the use of an Euclidean distance measure of the form,

$$\|\vec{x}(\tau) - \vec{x}^*(\tau)\| = \sqrt{\sum_{i=1,6} (x_i(\tau) - x_i^*(\tau))^2} \quad (26)$$

assigning a distance between the two state vectors $\vec{x}(\tau), \vec{x}^*(\tau) \in \mathbb{R}^6$ in the phase space.³¹

This choice of Euclidian distance measure is directly inspired from the study of ordinary (non-relativistic) Hamiltonian systems. However, in general relativity there is a priori no reason why one should use such a distance measure to give out the distance between ⁽³⁾ g and a “nearby” three-metric ⁽³⁾ $g + \delta^{(3)}g$. A distance measure like (26) is gauge dependent, i.e. *not invariant* under a change of coordinates, $g \rightarrow \tilde{g}(g)$. (Should the distance between two different “mixmaster collapses” (at some given time) be an artifact of the chosen set of coordinates for the description of the collapse?).³²

If the maximal Lyapunov exponent is extracted from the flow equations (25) for the mixmaster collapse with a Euclidian distance measure (26) on the solution space we obtain the following result (from S.E. Rough (1990a)):

In τ -time: With respect to the time parameter $\tau = \int dt/abc \sim \ln t$ introduced as a standard time variable (cf. Landau and Lifshitz (1975) §118) for the description, the approach to the singular point is *stretched* out to infinity ($\tau \rightarrow \infty$). Despite the mentioned aspect of chaotic unpredictability of the model, as described by the Farey³³ and Gauss map connected to the combinatorial model (22) for the axes, this *stretching* is “so effective” as to make the Lyapunov characteristic exponent, extracted from the $(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$ phase flow, zero in τ -time: $\lambda_\tau = 0$. (See S.E. Rough (1990a), S.E. Rough and B.J.T. Jones (1990), D. Hobill *et al* (1991)³⁴ and also A. Burd, N. Buric and R.K. Tavakol (1991)). This is not standard for a chaotic deterministic model, but it appears that the zero Lyapunov exponent is simply a consequence of the time variable chosen for the description.

In t -time: In the original synchronous time parameter $t = \int abc \, dt$ the scale functions of the metric

$$ds^2 = -dt^2 + \gamma_{ab}(t) \omega^a(x) \omega^b(x) \quad \gamma_{ab}(t) = \text{diag}(a^2(t), b^2(t), c^2(t))$$

³¹What is equivalent Lyapunov exponents were extracted from calculating the Jacobian matrix from the flow equations (25) of the state vector $(\alpha, \beta, \gamma, \alpha', \beta', \gamma')$ and integrating up along the orbit. Cf. e.g. appendix A in S.F. Rough (1990a).

³²Being somewhat acquainted with general relativity and its Hamiltonian formulation, a more natural choice may well be a distance measure which is induced from the structure of the Hamiltonian in general relativity (just like the Euclidian distance measure (26) is induced from the Hamiltonian (the kinetic term) in ordinary non-relativistic Hamiltonian dynamics), i.e. a distance measure of the type $ds^2 = \int \delta^3 x \gamma_{ij} \delta g^i \delta g^j$ where γ_{ij} is the Wheeler superspace metric on the space of three metrics. However, such a distance measure is not positive definite. (Sec 3.8).

³³We have seen that the Farey map comes closer to mimic the bounce dynamics in terms of a simple, non-invertible one-dimensional map and it has $\lambda = 0$. Moreover, the $\tau = \int dt/abc$ time between each bounce grows very fast, so it is no surprise to find $\lambda_\tau(\alpha, \beta, \gamma, \dots) = 0$. Moreover, since reliable estimates of the time intervals between bounces may be found in the literature (cf., e.g., Khalatnikov *et al* (1985) and references therein) it is possible to show this result by upper bound estimates.

³⁴Note, that Dave Hobill *et al* have extracted the entire spectrum of all six Lyapunov exponents from the phase flow - and not only the principal value (the maximal Lyapunov exponent).

exhibits an infinite sequence of oscillations and the successive major cycles condense infinitely towards the singularity which we, without loss of generality, may take to be at $t = 0$. (See also the very illustrative picture in Kip S. Thorne (1985)) of oscillating, fast growing, tidal field components of the successive oscillations as the spacetime metric approaches the singular point at $t = 0$).

In any finite time interval $t \in [T, 0]$, where T denotes an arbitrary small positive number, the model exhibits an infinite, unpredictable sequence of oscillations.

A Lyapunov characteristic exponent, although ill defined, is unbounded in t time. (A perturbation will be amplified faster than exponentially with respect to the metric t time and with respect to the distance measure (26) on approach to the singularity).

The quoted Lyapunov exponents above are for mixmaster collapses governed by the vacuum Einstein equations corresponding to the vanishing $I = 0$ of the first integral (11) or corresponding to mixmaster collapses coupled to perfect fluid matter with $p \leq \frac{2}{3}\rho$. For initial conditions which fail to satisfy the first integral constraint $I = 0$, the character of the solutions depends on the sign of the first integral I . We have qualitatively different behavior for $I < 0$, for $I > 0$ and for $I = 0$. For $I < 0$ (inclusion of negative “stiff matter” energy densities) the complicated behaviors in fig.1-3 are reflected in positive Lyapunov exponents, with respect to the τ time parameter, extracted from such numerical solutions. From the scale invariance of the Einstein equations, it follows that if one “unwittingly” put an amount $p = \rho = \rho_0 = (8\pi G)^{-1} I < 0$ of negative mass-density in the model, the maximal Lyapunov exponent roughly scales as

$$\lambda \sim \sqrt{-\rho_0}$$

This is exactly what G. Francisco and G.E.A. Matsas (1988) “unwittingly” have plotted as fig.4. in their paper.

For $I > 0$ (i.e. the inclusion of “stiff matter” with positive energy density) all three scale functions will decrease monotonically after some transient τ time and the maximal Lyapunov exponent will be zero. The reason why even V.A. Belinskii and I.M. Khalatnikov (1969) had a minor accident in obtaining qualitative accordance between numerical simulations and their theoretical model is due to the fact that they had an extremely small error in choosing the initial data. They had $I = +5.5 \times 10^{-2} > 0$, which prevents the spacetime metric from going into the described sequence of oscillations according to the *BKL* combinatorial model.

For the values of the Lyapunov exponents associated with the vacuum gravitational collapse one notes that the fact that Lyapunov exponents, as indicators of chaos, are not invariant under time reparametrizations is no surprise:

A Lyapunov exponent (which is a “per time” measure of having exponential separation of nearby trajectories in “time”) has never been invariant under transformations of the “time” coordinate!

Moreover, for a given fixed choice of time variable, transformations of the coordinates for the three-metric g_{ij} on the spacelike hypersurfaces Σ_t should be accompanied by a transformation of the distance measure $\|\cdot\|$ on the three-space. If one does not take proper account of transforming the distance measure (the metric) on the solution

space when performing a *coordinate transformation* to new variables (e.g. transformation from the ADM-variables to the Chitre-Misner variables on the Poincaré disc), then some definite value of a Lyapunov exponent (e.g. $\lambda_0 = 0$ in the ADM-variables) may be transformed into any other value by that coordinate transformation (and such measures of “chaos” which are artifacts of the coordinate transformation are not strong candidates for capturing anything interesting about inherent properties of a dynamical system.³⁵

3.1. Chaos in the “coordinates” or in the gravitational field?

The fact that Lyapunov-exponents are found to be strongly gauge dependent “touches” a deeper problem (S.E. Rugh (1990 a,b) connected with the characterization of “chaos”, or other dynamical characteristics, in theories like general relativity (and to some extent in gauge-theories) - not present in, e.g., hydrodynamical turbulence studies: How can you be sure that your measure of “chaos” is not some artifact of the “gauge” variables chosen? That is: Does the apparent “chaos” originate from the gauge-variables chosen (“chaos in the semantics”) or is it present in the “real world”?

To distinguish (in a coordinate invariant way) whether the “metrical chaos” is due to a “chaotic choice of coordinates” (we may call it “chaos in the semantics”) or reflects “real chaos and turbulence” in the gravitational field does not seem to be an easy task, as this separation between coordinate choices and gravitational fields is indeed very tricky business in general.

This question is, in some sense, the “chaos in general relativity” (“metrical chaos”) analogue of Eddingtons worries about the “spurious gravitational waves” as a propagation of “coordinate changes” with the “speed of thought”.

It is not difficult to invent (apparently) “chaotic solutions” of “space-time metrics” where the “chaotic signal” is a pure artifact of gauge (one may consider, e.g., a 2+1 dimensional study of pure “gravity” in a non-linear gauge). H.B. Nielsen and S.E. Rugh (1992 a).

The problem is rooted in the more general question (which is itself very interesting):

How gauge-invariantly may we capture any dynamics of the gravitational field (chaotic or not) in the Einsteins theory of general relativity?

³⁵Thus, when J. Pullin (1990) arrived at the conclusion that the mixer-master toy-model gravitational collapse is chaotic because a positive Lyapunov exponent “ $\lambda = 1$ ” can be associated with the geodesic motion on the Poincaré disc one should be careful to project out (separate) what are artifacts of the coordinate-transformations and what are inherent properties of the dynamical system. (Invariant under gauge choices). (Certainly the element of chaos and unpredictability of the mixer-master collapse is not (alone) due to the negatively curved interior of the Poincaré disc. Even the completely integrable Kasper metric may be mapped to the interior of the negative curved Poincaré disc (by a set of hyperbolic coordinate transformations identical to those applied to the mixer-master metric). No chaos arises, since there is no scattering in any potential boundaries. However, as an artifact of not transforming the distance measures under the coordinate transformation, we suddenly appear to have local exponential instability in all directions on the Poincaré disc - despite that the Kasper metric is completely integrable with linearly growing separation in the original ADM-variables.

It is well known that it is very difficult to extract gauge-invariant quantities *locally* from general relativity - even in the weak field limit!³⁶

While the collection of the (non-local) Wilson loop variables

$$Tr(P \exp \oint_{\gamma} g A_{\mu}^a(x) \frac{\lambda^a}{2} dx^{\mu})$$

in principle offer a tool to characterize the dynamics of non-Abelian gauge theories in a complete gauge invariant manner,³⁷ such a “projection out” of “the gauge-invariant content” is troublesome in gravity. Like in Yang-Mills theories, we may attempt to characterize the gravitational field invariantly by loops (capturing the curvature tensor components $R_{\gamma\delta}^{\alpha\beta}$ ’s, just as the loops in Yang-Mills theories capture the $F_{\mu\nu}$ ’s). But there is a problem of saying (in a diffeomorphism invariant way) *where* the loop is!

One can construct completely gauge-invariant and *global* quantities like for example

$$I(\eta) = \int d^4x \sqrt{-g} \delta(C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} - \eta) \quad (27)$$

which measures the invariant 4-volume which has the value $C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta} = \eta$. But it is very hard to capture any dynamics with such a (space-time global) signal.

One may question the very suitability (in more generic cases) of concepts like

- A Lyapunov exponent “ λ ” (a “per time” measure characterizing the “temporal chaos”: does the “distance” between “nearby” orbits grow up exponentially with time?)
- A spatial correlation length “ ξ ” (in a spatially inhomogeneous gravitational field we may have, that domains separated by distances considerably larger than $\sim \xi$ cannot “communicate” due to spatial disorder in the field configurations: “metric turbulence” indicates a lack of correlations between gravitational signals in time as well as in space.³⁸ This “spatially disordered” aspect of chaos falls away for the spatially homogeneous toy-models).

To the extent that we may split the spacetime metric ⁽⁴⁾ g into a three-metric ⁽³⁾ g evolving in a “time”³⁹ the very concept of, e.g., a Lyapunov exponent may perhaps be applicable, though *gauge-invariant generalizations* (i.e. measures of “metrical chaos” invariant under general space-time coordinate transformations or some *more restricted* class of coordinate transformations) are (obviously) better to have - if they are possible to construct?

³⁶For example, Roger Penrose has for several years attempted to extract a measure of “entropy” in the metric field itself. But it is not easy to make a *gauge-invariant* and sensible construction. So far, Roger Penrose has not succeeded. Pers. comm. with Roger Penrose (at the Niels Bohr Inst.).

³⁷Thereby loosing locality cf. also the Aharonov-Bohm effect.

³⁸Often (cf., e.g., T. Bohr (1990)) temporal and spatial chaos are somewhat connected in a relation of the type: $\xi \sim c/\lambda$ where c is the velocity of propagation (velocity of “light” in the case of propagation of disturbances of spacetime curvature in general relativity).

³⁹Note that we have been able to study our simple spatially homogeneous toy-model gravitational collapse in a (*global*) *synchronous* frame of reference. One can always choose a synchronous reference frame *locally*, but in generic situations, for strong gravitational fields, a global construction of a synchronous frame of reference easily breaks down because of the tendency of these coordinates to focus to a *caustic* (and a coordinate singularity will develop).

It may very well be that one should head for a “no g_0 ” theorem for the most ambitious task of constructing a generalization of a Lyapunov exponent (extracted from the continuous evolution equations) which is meaningful and invariant under the full class of spacetime diffeomorphisms (H.B. Nielsen and S.E. Rugh). One should therefore, at first, try to modify the intentions and construct (instability) measures which are invariant under a smaller class of diffeomorphisms (e.g. coordinate transformations on the spacelike hypersurfaces Σ_t which do not involve transformations of the time coordinate).

In the mixmaster toy-model context of a spatially homogeneous spacetime metric this question is particularly simple to address since all the physical signals are *functions of time only*.

The need for gauge invariant measures of chaos in this context of general relativity was noted and discussed in S.E. Rugh (1990a), emphasized as a major point in S.E. Rugh (1990b) and, also, emphasized in J. Pullin (1990). See also discussions in e.g. H.B. Nielsen and S.E. Rugh (1992), in M. Biesiada and S.E. Rugh (1994) and in S.E. Rugh (1994).

3.2. Chaos in ever expanding (or ever collapsing) phase spaces

In the description of the gravitational collapse involving the $(\alpha, \beta, \gamma, \dots)$ phase flows (the parametrization in, e.g., Landau and Lifshitz (1975)) and the $(\Omega, \beta_+, \beta_-, \dots)$ phase flows (the anisotropy-variables of C.W. Misner et al.), there is no “recurrence” of orbits because the phase space is ever expanding⁴⁰, and it is not straightforward to define what is meant by a concept like “ergodicity” or the “mixing property”, for example, in the outward expanding Hamiltonian billiard in the ADM-Hamiltonian description. In this sense the gravitational collapse study presents a dynamical system, which differs from most other systems treated in the theory of non-linear dynamical systems. Thus our gravitational collapse orbit is also a playing ground for investigating how to deal with chaos in expanding (or collapsing) phase spaces.

The ideas of chaos apply most naturally to time evolutions with “eternal return”. These are time evolutions of systems that come back again and again to near the same situations. In other words, if the system is in a certain state at a certain time, it will return arbitrarily near the same state at a later time. (Cf. D. Ruelle (1991), p. 86)

How do we extract measures of chaos and ergodicity in an ever expanding phase space?
 • As we have already described, one may project out (extract) lower dimensional signals from the flow which may capture (chaotic) “return properties” of the dynamics. An example is the variable “ $x = 1/u \in [0, 1]$ ” which we extracted from the phase space trajectories in sec.2.4.

• If we characterize the dynamical evolution of the gravitational collapse in terms of the infinite bounce sequence (i.e. encoding the gravitational collapse by an infinite collision

⁴⁰If we follow this metric to the final crunch singularity, the configuration space variables (a, b, c) shrink to zero volume ($abc \rightarrow 0$), while if we take the logarithmic scale factors $(\alpha, \beta, \gamma) = (\ln a, \ln b, \ln c)$ the configuration space is not bounded from below (we have $\alpha + \beta + \gamma \rightarrow -\infty$). In terms of anisotropy variables, also commonly used, $(\Omega, \beta_+, \beta_-)$ the anisotropy $\|\vec{\beta}\|$ grows to infinity and the trajectory of anisotropy $\vec{\beta}$ does never return to some given value.

sequence against the three wall boundaries) we hereby map an ever expanding phase space (in the $(\alpha, \beta, \gamma, \dots)$ variables, say) to an infinite symbolic sequence with return properties (e.g. with periodic orbits).

3.3. The many gauges and different variables describing the mixmaster gravitational collapse

The amount of work which has been put in the invention of different descriptions (different gauges) and different approaches is substantial: Cf. the study of the mixmaster gravitational collapse in the ADM variables (cf. e.g. B.K. Berger (1990)), the Misner-Chitre description (J. Pullin (1990)), the BKL-variables (D. Hobill *et al* (1991), S.E. Rugh (1990 a)), the Ellis-MacCallum-Wainwright variables (A.B. Burd, N. Buric and G.F.R. Ellis (1990)), Ashtekar variables (Ashtekar and Pullin (1989)), the description by O.I. Bogoyavlensky and S.P. Novikov (e.g. used in the approach by M. Biesiada, J. Szczyty and M. Szydowski).

The many different gauges and approaches

- BKL variables (Landau and Lifshitz)
- The ADM-Hamiltonian variables (MTW)
- The Misner-Chitre coordinates
- Ellis-MacCallum-Wainwright variables
- Ashtekar variables
- The approach by Bogoyavlensky and Novikov

Table 2. Examples of the many descriptions of the very same gravitational collapse. What are the “invariant” properties of the dynamics which can be extracted from all these descriptions ?

The “transformation theory of chaos” (which concepts of “chaos” are invariant under the transformations?) between the different gauges and approaches is not easy. As we have seen, in some gauges there is a positive (finite or infinite, depending on the gauge choice) value of a “Lyapunov exponent” - in other gauges it is zero. In many gauges the phase space is *ever expanding* and a notion like that of “ergodicity” (relying by definition on return properties of the flow) can hardly be assigned any meaning at all. In other gauges, e.g. the “Misner-Chitre” gauge, the dynamics of our spacetime metric (3) is, it appears, ergodic and, even, a K-flow! (cf., e.g., J. Pullin (1990)). This rather “messy situation” is, of course, an artifact of not having constructed gauge invariant measures of chaos (or ergodicity) in this context of general relativity.

Yet, in the mixmaster gravitational collapse, the “bounce” dynamics is there and that we must be able to capture in a gauge invariant way (i.e. that ∞ many bounces take place before the spacetime singularity is reached).

Imprints of the bounce structure of the mixmaster dynamics should be seen in various invariants which can be extracted from the mixmaster metric, e.g. in the Weyl

curvature invariant $C^2 = C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}(\tau)$ of the mixmaster metric. Thus, in order to illustrate how a “good physical signal” behaves for a typical trajectory we have set out to calculate and display (cf. Biesiada and Rugh (1994)) this Weyl curvature invariant. Premature investigations indicate that if $C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}$ is plotted with respect to the time variable τ , one observes (Biesiada and Rugh (1994)) that the bounce structure is also seen on this curvature invariant. ⁴¹

Also other (higher order) invariants could be constructed for the mixmaster gravitational collapse and one should verify whether the physical degrees of freedom of the mixmaster collapse in principle may be captured *exhaustively* in terms of the algebraic invariants constructed from the Weyl curvature tensor field.

3.4. On symbolic dynamics for the bounce structure of the gravitational collapse

Let us now consider a particular “gauge”, say, the ADM Hamiltonian description (or the “Misner-Chitre” gauge) in which the gravitational collapse is described as a “ball” exhibiting an infinite sequence of “bounces” against a potential boundary which is outward expanding⁴² (or stationary in the “Misner-Chitre” gauge). Is a “bounce” well defined? Recall, that the scattering walls - which effectively arise due to the three-curvature scalar ³ R on the space-like sections of the metric $ds^2 = -dt^2 + \gamma_{ij}(t)\omega^i(x)\omega^j(x)$ - turn into “infinitely” hard walls when sufficiently near the space-time singularity of the metric. Hence, the dynamics resembles that of an ideal billiard and it gets easier and easier to define when the orbit “bounces” against the potential wall as the metric approaches the singular point.

It may be useful to assign a *symbolic dynamics* to the dynamical system, i.e., a scheme that assigns a unique symbolic string (coding) to each orbit. The simplest qualitative way to describe an orbit is to list the order in which it hits the three boundary walls $\partial B_1, \partial B_2, \partial B_3$ using the wall labels as symbols (characterizing the orbit by its infinite collision sequence). This yields a description of the orbit in terms of ternary symbolic dynamics:

- “1” for scattering against the boundary ∂B_1 (where $c \gg a, b$)
- “2” for scattering against the boundary ∂B_2 (where $a \gg b, c$)
- “3” for scattering against the boundary ∂B_3 (where $b \gg a, c$)

As a randomly chosen reference example of a “symbolic string” (compare with Kip. S. Thorne (1985)) we may, for example, have

$$\dots\dots \underbrace{12121212}_{4 \text{ oscillations}} \underbrace{131313}_{3 \text{ oscillations}} \underbrace{232323232323}_{8 \text{ oscillations}} \underbrace{121\dots\dots}_{etc} \dots\dots \quad (28)$$

⁴¹During the workshop it was discussed (Charles W. Misner, Piotr Chrusiel *et al.*) that it is not fully investigated how the Weyl curvature tensor behaves near the generic singularity of the mixmaster collapse. For example, $C^2 = C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}(\tau)$ will not necessarily increase *monotonically* towards the big crunch singularity. The near singularity behavior of the Weyl tensor is therefore interesting in itself and deserves further examination.

⁴²Cf., e.g., MTW §30.7 or Ryan and Shepley (1975). A detailed description of the ADM-Hamiltonian description of the gravitational collapse is also given in S.E. Rugh (1990a) (which is available upon request).

The sequence of oscillations between two walls leading to one of the three corners is denoted as a “(major) cycle”. Thus, the displayed segment (28) has 4 oscillations in the first of the displayed (major) cycles (number “1”, say) 3 oscillations in the next major cycle (number “1 + 1”), 8 oscillations in cycle “1 + 2”, etc.

Binary symbolic dynamics suffices

Due to the grammar of the symbolic dynamics, which forbids the appearance of ...11..., ...22... or ...33... in a sequence (no orbit can hit the same wall twice without hitting another wall first) the three-letter alphabet (above) may be reduced to a *two-letter alphabet*. This reduction can be accomplished in many ways. One possible way of assigning binary symbolic dynamics to the dynamics is the following: Assign a “0” for scattering against the boundary ∂B_1 and a “1” for scattering against the boundary ∂B_2 . Scattering against the last of the three boundaries ∂B_3 we denote “0” if it follows a bounce against the boundary ∂B_1 and denote “1” if following a bounce against the boundary ∂B_2 . In this encoding the motion is considered as a series of choices of what boundary to go to from the present boundary (there are always two possibilities, and binary symbolic dynamics therefore suffices to describe the chosen route). Except for some arbitrariness in the beginning of a symbol sequence, this gives a one-to-one correspondence between the *pruned*⁴³ ternary dynamics and an unrestricted binary dynamics.⁴⁴

For instance, our reference-segment (28) reads

$$\dots\dots \underbrace{01010101}_{4 \text{ oscillations}} \underbrace{000000}_{3 \text{ oscillations}} \underbrace{1111111111111111}_{8 \text{ oscillations}} \underbrace{010\dots\dots}_{etc} \dots\dots \quad (29)$$

If we were to carry out a systematic exploration of system characteristics in terms of the periodic orbit structure (this includes finding periodic orbits up to some given length, quantize the system in terms of the periodic orbits, Selberg zeta functions and all that) it would be of importance to use the minimal symbolic dynamics (with no pruning). We shall, however, just use the direct symbol assignment 1, 2, 3 in order to have a fixed notation.

3.5. Chaos and the topological structure of the gravitational collapse.

A simple example illustrates that a zero Lyapunov exponent is not contradictory to an intrinsic unpredictable and chaotic behavior (from S.E. Rugh (1990 a)).

⁴³*Pruning*. If some symbolic sequences have no physical realizations, the symbolic dynamics is said to be “pruned”. In that case the symbolic alphabet must be supplemented with a set of grammatical rules (i.e. a set of pruning rules). If all possible symbol sequences correspond to physical trajectories, the symbolic dynamics is said to be “complete”. In a systematic encoding process, one will ask first if there is *at most* one orbit to a given symbolic sequence. If this is the case we have a *covering* symbolic dynamics. If there is also one orbit for any symbolic sequence, the symbolic dynamics is said to be *complete*.

⁴⁴Note, that in the three disc problem on the hyperbolic Poincaré disc we have “full symbolic dynamics” while in the three-disc problem on an Euclidian space (i.e. if the billiard had been in Euclidian space) there exist binary symbolic sequences which do not correspond to physical trajectories. In that sense, the hyperbolic three-disc dynamics is simpler to work with than the Euclidian three-disc problem. See also Giannoni and Ulino (1990).

Figure 8. The intrinsic unpredictability of the number of consecutive rotations about the two rotation centers is *independent* of the choice of time variable (whereas the spectrum of Lyapunov exponents is not).

Consider, for example the famous Lorentz attractor generated as an asymptotic attractor by the Lorentz time evolution equations (Lorentz, 1963) - just like the (BKL) oscillatory behavior is generated as an asymptotic attractor⁴⁵ of the vacuum Einstein equations for the mixmaster spacetime metric (3).

The "particle" performs an infinite number of consecutive rotations about two rotation centers, schematically shown in fig.8. It rotates, let's say, n_1 times around the left rotation center ("first major cycle") and then n_2 times around the right rotation center ("second major cycle") and return to the left again ...etc...ad infinitum.

One feature of the non-linear Lorentz system is that the number of consecutive rotations about the two rotation centers is unpredictable, since these numbers ($n_1, n_2, \dots$) depend sensitively on initial conditions. (See also Lichtenberg and Leiberman (1983), p.59-62). In that sense the behavior of the system is *intrinsically* chaotic (stochastic) and not predictable. With respect to the standard time parameter of the Lorentz evolution equations, a maximal Lyapunov characteristic exponent, extracted from the flow, mirrors this unpredictability and is greater than zero.

However, by making some kind of an "exponential stretching" of the time parameter axis (thereby making the particle moving around slower and slower with respect to the new time parameter) it is easy to make the corresponding Lyapunov exponent, extracted from the new first order autonomous differential flow, zero. The unpredictability of the number of consecutive rotations performed in each "major cycle" of course still remains!

The gravitational collapse, in the oscillatory BKL-regime on approach to the singularity, exhibits consecutive rotations around 3 "centers" (there are three "attraction centers": α and β can oscillate (γ declines), α and γ can oscillate (β declines) or β and γ can oscillate (α declines)) corresponding to oscillations in one of the three channels of the billiard game in the Hamiltonian description. (See, also, fig.9 of the triangular symmetric scattering domain in the Poincaré disc). If we, by the number of rotations around a "center", mean the number of oscillations in a period (a major cycle) where two scale functions oscillate and the third declines, the BKL-model will be intrinsically unpredictable as the Lorentz attractor described above (the number of consecutive ro-

⁴⁵The "basin of attraction" are initial conditions which are not axisymmetric (cf. the Taub-solution).

tations around now three "rotation centers" depends sensitively on initial conditions, and the "per cycle" entropy is very big: $h = \pi^2/6 \ln 2 \gg 0$). But the "particle" moves too slow with respect to the standard time parameter (constructed from the metric t time) $\tau = \int dt/abc$, as $\tau \rightarrow \infty$, to mirror this unpredictability.

The property of a dynamical system to behave "chaotically"⁴⁶ should be an intrinsic property of the system and not rely on the variables chosen for the description.

3.6. Which concepts should characterize "chaos" in the mixmaster toy-model collapse (and in more general cases) ?

There is nothing mysterious in selecting a particular "gauge" and describing physics, e.g. gravitational collapse physics, in that particular gauge. Thus, very often, it is not very useful to do the calculations in a gauge-invariant way (and in general relativity it may in fact not be possible). So one, usually, "fixes the gauge" and performs the calculations in that gauge.

However, in my opinion, it is of interest to pursue some routes in order to study how we may eventually deal more invariantly with the "chaotic" mixmaster gravitational collapse:

1. How little structure on the solution space is needed to deduce information about chaos?
2. One may argue that there is a preferred superspace metric and a preferred supertime (Wheeler's superspace) as a natural structure on the solution space
3. One may attempt a (more) invariant description (involving knowledge about extrinsic curvature invariants "Tr $\mathbf{K}$ " etc.).
4. Study our free observer and see what happens with him (tidal stretches etc.) in his local time. That is, after all, a very physical question!

I shall here attempt at preliminary sketches of how to pursue the first two routes. We have already, in sec.2.5, sketched the short, yet exciting, story of the free falling astronomer which reaches the spacetime singularity in finite proper time (the tidal stresses exerted on him by falling freely in the collapsing mixmaster spacetime metric should be properly analyzed in terms of the Riemann tensor $R_{\alpha\beta\gamma\delta}$ for the mixmaster metric and not in terms of the Riemann tensor of the Kasner metric).

We shall not pursue route no.3 here but note that in the "3+1" splitting of the spacetime into a three-metric $^{(3)}g$ evolving in a "time", a complete description of the geometry of the four manifold also needs, besides the intrinsic 3-geometry (the induced three-metric $^{(3)}g_{ij}$ on the spacelike hypersurfaces Σ_t), knowledge about how the 3-geometry is embedded in the enveloping 4-geometry, which is characterized by the extrinsic curvature K_{ij} (cf., e.g., MTW §21.5). One may attempt a more invariant description (i.e. measures of local instability of collapse orbits) by extracting the information from both the intrinsic geometry and the extrinsic curvature and its associated invariants (like its trace, "Tr $\mathbf{K}$ ").

⁴⁶I.e. the system displays a complex evolutionary pattern rather than a simple, integrable pattern (which is "just" quasi-periodic motion in the action-angle variables).

3.7. Is it possible to characterize the complexity of the gravitational collapse without use of a distance measure $\|\cdot\|$ on the solution space? (and without reference to a particular time variable?)

Since the standard measure, the Lyapunov exponent λ , is a "per time" measure, it depends, obviously, on the choice of time parameter ($\lambda = 0$ in $\tau = \int dt/abc$ time while $\lambda = " \infty "$ in metric t time). Besides, the definition of a Lyapunov exponent requires that we have at hand a metric $\|\cdot\|$ on the solution space, which measures "distances" between (neighboring) trajectories. In the chain of adding more and more structure to the solution space,

$$\left(\begin{array}{c} \text{Topological} \\ \text{Space} \end{array} \right) \subset \dots \subset \left(\begin{array}{c} \text{Manifold Structure} \\ \text{With a Metric} \end{array} \right)$$

one also hereby gradually introduces more and more arbitrariness. We need analysis to which extent one may construct measures of chaos where we do not have to rely on the (arbitrarily) chosen time-parameter " τ " or some (arbitrary) chosen distance measure " $\|\cdot\|$ " on the phase space (cf., also, H.B. Nielsen and S.E. Rugh (1992 a)).

How much "structure" on the "solution space" do we need to get information about whether a collapse dynamics is of a complex (chaotic) or regular (integrable) type, i.e. extract interesting information about "chaos"? Do we really need a manifold structure with a metric $\|\cdot\|$ on it - or can we, from knowledge of the topological structure alone, say, (on the space of solutions) tell something about chaos of interest?

For our gravitational collapse (3) - being spatially homogeneous - all physics is entirely given in terms of *functions of time* (e.g. encoded in "signals" like $C_{\partial\partial\partial}C^{\partial\partial\partial}(\tau)$ and higher order algebraic invariants) and the phase space dynamics is a low dimensional signal in "time".

It is natural to try to assign a "good symbolic dynamics"⁴⁷ from the phase space dynamics, i.e. one may think of the dynamics "described" by a string of symbols

$$\{ \dots, S_i, S_{i+1}, \dots, S_{i+n}, \dots \} \quad , \quad S_i \in \{ 1, 2, \dots, n \} \quad . \quad (30)$$

We have already noticed that there is a naturally ternary (or binary without pruning) symbolic encoding of the *bounce structure* of the gravitational collapse. (Thereby, one maps the ever expanding or collapsing phase space (without return properties of the collapse orbit), cf. sec.3.1, to symbol sequences with return properties (e.g. having periodic orbits).)

We have a *gauge-invariant* "before" and "after" - but the size of the step in "time" between two symbols in (30) is merely reflecting an arbitrary gauge choice of a "time coordinate".⁴⁸

⁴⁷This is a very difficult point. A priori, the assignment of a symbol-sequence from phase space dynamics (coding the trajectory by a discrete "alphabet") is arbitrary and accomplished by an arbitrary partitioning of the phase space into different domains (which are assigned different symbols). In many systems it is not easy to find a good partition to encode a generic orbit.

⁴⁸The continuous flow of events during the gravitational collapse is, from the point of view of "causality" (in the sense of which event is before the other) invariant under time reparametrizations

Now, let us say that we are given the set of *all possible sequences* of finite length, which may be realized by the physical system (that is usually many!). Can we then deduce, just by looking at these sequences, whether the underlying physical system is integrable or not (i.e. without having any metric $\|\cdot\|$ on the original solution space)?

Complexity measures for symbol sequences are already at hand. The *topological entropy* h_{top} of a string of symbols states how the number $N(n)$ of all the possible sequences of length n grows exponentially with n : $N(n) \approx e^{h_{top} n}$. It is 0 for an integrable system because of the quasi- or almost-periodicity⁴⁹ (setting severe restrictions on the number of possibilities for strings of length n). For a random string of symbols the topological entropy is equal to the logarithm of the number of symbols (e.g. $\log 2$ for the Bernoulli shift). For the *mixmaster gravitational collapse* the number of realized symbol sequences (characterized by its infinite bounce sequence) grows with the length n of the symbol string like $\sim 2^n$ and the topological entropy is thus $\log 2 > 0$. That is, for the encoding of the gravitational collapse with the binary (or ternary) symbolic dynamics, e.g. our segment

$$\dots \dots \underbrace{01010101}_{4 \text{ oscillations}} \underbrace{000000}_{3 \text{ oscillations}} \underbrace{111111111111}_{8 \text{ oscillations}} \underbrace{010\dots\dots}_{etc} \dots \dots$$

We have that both "0" and "1" may be realized at each place S_i of the symbol string. Every refinement (sub-partitioning) of the binary symbolic dynamics will still lead to an expression for the topological entropy which is positive.

Somewhat analogously to the concept of a Lyapunov exponent (which is, roughly, an inverse correlation length in time) one would like to extract a "correlation length" ξ_{string} from the set of all allowed strings of symbols (30) measuring, in a quantitative way, that the correlation between the symbols (as extracted from the bounce structure captured from physical signals like $C_{\partial\partial\partial}C^{\partial\partial\partial}(\tau)$) decreases with n as we go n steps along the "string" (from some i to $i+n$, averaged over i).

If all the realized string sequences (30) are *quasi-or-almost periodic*⁵⁰ (which we take to be a sign of integrability), we assign a correlation length of $\xi_{string} = \infty$. In a random sequence of symbols, on the other hand, there is no correlation between a symbol and the next, giving $\xi_{string} = 0$. If a correlation length can be defined for a string displaying "deterministic chaos" it should have a correlation length in between these two extremes.

and diagnostic tools such as a Fourier transform (an investigation in this spirit is also J. Demaret and Y. De Rop (1993)) of a signal - which can also be performed on symbol sequences - do not need a distance measure $\|\cdot\|$ on the solution space. However, it needs some sort of time variable. Note, that if we take the infinite string of symbols (30) to mean a symbolic encoding of the infinite bounce sequence of the *mixmaster gravitational collapse* we have, hereby, not escaped the introduction of a "time" variable: It is a time variable t (the label of the symbols S_i) which counts one unit for each "bounce" (a recognizable "cosmological event"). It is like "heart beats" of the *mixmaster gravitational collapse*. In terms of these "heart beats" the gravitational collapse is "infinitely old" when the "big crunch" singularity is reached. See also C.W. Misner (1969 c) and MTW, §30.7, p. 813-814.

⁴⁹Quasi-periodicity or almost-periodicity, for all realized symbol sequences, is a sign of integrability. Quasi-periodicity (multiply periodic motion) means that the Fourier transform is entirely composed of k -functions. The finite sums (trigonometric polynomials) $s(t) = \sum_{n=1}^N a_n e^{i\lambda_n t}$ (where the coefficients a_n are arbitrary complex, and the frequencies λ_n are arbitrary real) are quasi-periodic functions. According to a celebrated theorem by Harald Bohr, the class of almost periodic functions is identical to the closure $H\{s(t)\}$ of all finite $s(t)$. (The necessary and sufficient condition for an arbitrary trigonometric series, $\sum_{n=1}^{\infty} a_n e^{i\lambda_n t}$, to be the Fourier transform of an almost periodic function is that $\sum |a_n|^2$ converges, Besicovitch-Bohr).

⁵⁰"Quasi-periodicity" is captured by making a Fourier-transform on the symbol sequence.

For a given string of symbols the following limit may exist

$$C(n) = \lim_{N \rightarrow \infty} \left\{ \frac{1}{2N} \sum_{i=-N}^N S_i S_{i+n} - \left(\frac{1}{2N} \sum_{i=-N}^N S_i \right)^2 \right\}$$

and defines a correlation function $C(n)$ for the string S (i.e. along the string of symbols encoding a particular mixmaster gravitational collapse). If we can find just *one* symbol sequence for which the correlation function decays exponentially with n ,

$$C(n) \sim e^{-n/\ell}$$

it is not possible (we suspect) for the system to be integrable.

In fact, the mixmaster collapse orbit may very well (within the validity of the one-perturbation treatment of Belinskii et al.) be strongly intermittent in the “per bounce” time “ n ” and thus the correlation function will exhibit *power law decay* rather than decay exponentially along the string of symbols.⁵¹ Hence we may end up with a power law decaying correlation function, yet a positive topological entropy, for the infinite symbol strings of the mixmaster collapse orbits.

We shall not go into further details here. We remark that, starting out from our toy-model gravitational collapse (3), one is apparently naturally driven into speculations of a general sort in complexity theory. A general analysis of these problems is at the “heart of complexity theory”⁵² and is an extremely difficult subject.

3.8. Wheeler’s superspace metric as distance measure on the space of three-metrics ?

If we wish to operate with a distance measure $\|\cdot\|$ to the space of three metrics, it is natural to investigate whether we have distance measures in general relativity which are more *natural* than others. In particular one would like to construct a distance measure which is invariant under (a large class of) changes of coordinates for the description.

The original dreams by Wheeler, DeWitt and others is to consider the dynamics of the three metrics⁽³⁾ g as geodesics on some manifold called superspace equipped with the metric tensor G_{gM} which is named the supermetric. This supermetric G_{gM} induces a norm on the space of three metrics⁽³⁾ g .

$$\|\delta g_{ab}\|^2 = \int d^3x \sqrt{g} G^{gM} \delta g_{ij} \delta g_{kl} = \int d^3x \sqrt{g} G^{AB} \delta g_A \delta g_B \quad (31)$$

and one may measure the distance between two three-metrics⁽³⁾ g and⁽³⁾ $\tilde{g}$ with respect to the G_{gM} tensor. More explicitly, one read off the first natural candidate for a metric on the configuration space (the space of three metrics) from the structure of the kinetic term in the ADM-Hamiltonian

$$H = \frac{1}{2} (G_{ij}) K_i K_j \pi^K - \frac{1}{2} g^{(3)} R = \frac{1}{2} G_{AB} \pi^A \pi^B - \frac{1}{2} g^{(3)} R = 0 \quad (32)$$

⁵¹I thank Gabor Valtay, p.t. at the Niels Bohr Institute, for fruitful exchanges concerning this point

⁵²See also sec 5.4 in P. Grassberger et al (1991) for references to a number of different definitions of complexities of time sequences which are all tied to symbolic dynamics. Interesting discussions of complexity, in a wider perspective, may also be found in P. Grassberger (1986) and e.g. S. Lloyd and H. Pagels (1988).

which yields an expression like⁵³

$$G_{AB} \equiv (G_{ij}) K_i K_j = \frac{1}{2} (g_{ik} g_{jl} + g_{il} g_{jk} - 2g_{ij} g_{kl}) \quad (33)$$

This step corresponds to reading off the “usual Euclidian metric” $a^{\alpha\beta}$ from a non-relativistic Hamiltonian⁵⁴

$$H = \frac{1}{2} a^{\alpha\beta} p_{\alpha} p_{\beta} + V(q) = E \quad (34)$$

Some may stop at this point and say that G_{gM} is the superspace metric. However the dynamics of the three-metric⁽³⁾ g (the collapse orbit) is not yet a geodesic flow (cf. Biesiada and Rugh (1994)) with respect to this superspace metric.

By conformal transformation of the metric

$$\tilde{G}_{AB} = 2(E - V) G_{AB} = (-2V) G_{AB} = \tilde{R} G_{AB} = g^{(3)} R G_{AB}$$

(where the role of the potential V in the general relativistic context is played by the quantity $V = -\frac{1}{2} g^{(3)} R \equiv -\frac{1}{2} \tilde{R}$) and rescaling of the parametrization of the collapse orbit (cf. detailed discussion in Biesiada and Rugh (1994))

$$\tilde{\lambda} d\lambda = 2(E - V) d\lambda = (-2V) d\lambda = \tilde{R} d\lambda = g^{(3)} R d\lambda \quad (35)$$

one obtains (cf. also e.g. C.W. Misner (1972)) that, with respect to this new parameter $\tilde{\lambda}$ and with respect to the conformally rescaled metric $\tilde{G}_{AB}$, the three-metric⁽³⁾ g is now an affinely parametrized geodesic

$$\frac{d^2 g^A}{d\tilde{\lambda}^2} + \tilde{\Gamma}^A_{BC} \frac{dg^B}{d\tilde{\lambda}} \frac{dg^C}{d\tilde{\lambda}} = 0 \quad (36)$$

There is no “magic” involved (we have “magic without magic”): Information about the potential $V = -\frac{1}{2} g^{(3)} R$ has simply been completely encoded in the mathematical definition of the superspace-metric $\tilde{G}$ which generates the evolution of⁽³⁾ g as geodesic motion (affinely parametrized if one uses the rescaled parameter $\tilde{\lambda}$).⁵⁵

• The distance measures induced by the Wheeler superspace metric G_{gM} have the *good* property (which is not shared by artificially introduced Euclidean distance

⁵³Note that the supermetric, in this notation (cf. C.W. Misner (1972)) differs by a conformal factor ($\sqrt{g}$) from the expression by DeWitt (1967). This is allowed since the ADM action is invariant with respect to conformal transformations. I do not want to get entangled in too much detail concerning this point but refer to M. Biesiada and S.E. Rugh (1994).

⁵⁴Instead of an analogy with a free particle in special relativity, with Hamiltonian $H = \frac{1}{2} (g^{\mu\nu} p_{\mu} p_{\nu} + m^2)$ (cf. C.W. Misner (1972), p. 451) we want to point to (cf. Biesiada and Rugh (1994)) the complete one-to-one correspondence between construction of the superspace and the dynamics of a non-relativistic particle with the Hamiltonian $H = \frac{1}{2} a^{\alpha\beta} p_{\alpha} p_{\beta} + V(q)$ reduced to geodesic flow by virtue of the Maupertuis principle.

⁵⁵In a completely similar manner (cf. Biesiada and Rugh (1994)) one maps the non-geodesic motion of the configuration space variable q governed by the non-relativistic Hamiltonian (34) to affinely parametrized geodesic motion by conformal transformation $a_j \rightarrow 2(E - V(q)) a_j$, of the metric a_j , and rescaling of the time parameter $dt \rightarrow 2(E - V(q)) dt$.

measures) of being invariant under canonical coordinate transformations. Thus, if we have a change of coordinates (configuration space variables),

$$g^A \rightarrow g^{*A} = g^{*A}(g^A)$$

the distance measure $ds^2 = G_{AB} \delta g^A \delta g^B$ is invariant under such transformations, since G_{AB} transforms properly as a tensor,

$$G_{AB}^* = \frac{\partial g^C}{\partial g_A^*} \frac{\partial g^D}{\partial g_B^*} G_{CD}. \quad (37)$$

One sees this by recalling that the coordinate transformation $g^A \rightarrow g^{*A}$ induces the transformation of momenta $\pi_A \rightarrow \pi_A^* = \frac{\partial g^B}{\partial g_A^*} \pi_B$. Therefore the Hamiltonian $\mathcal{H}$ reads

$$\mathcal{H} = \frac{1}{2} (G^{AB} \pi_A \pi_B - \mathcal{R}) = \frac{1}{2} (G^{AB} \frac{\partial g^{*C}}{\partial g_A} \frac{\partial g^{*D}}{\partial g_B} \pi_C^* \pi_D^* - \mathcal{R}) = \frac{1}{2} (G^{*AB} \pi_A^* \pi_B^* - \mathcal{R})$$

justifying our formula (37).

• A bad property of the distance measure induced by the Wheeler superspace metric is that such a distance measure is *indefinite* (not positive definite). I.e., it appears that one may have situations in which two spacetime metrics (two mixmaster collapses at some given "time") are at zero distance but evolve into a distance different from zero. (This would correspond to a "Lyapunov exponent" of " ∞ " which is of course an entirely different situation from having a small finite distance which in "time" evolves into a bigger finite distance).

Note, that it is possible to construct a superspace metric G_{AB} which is *positive definite* so that two 3-geometries $^{(3)}g$ and $^{(3)}\tilde{g}$ are identical if and only if the distance between them is zero, see e.g. discussions in B.S. DeWitt (1970). However, such superspace metrics do not make the 3-geometries evolve as geodesics.

Negativity of the Ricci scalar $R < 0$ as a local instability criterium?

As a possible criterium for instability of the mixmaster gravitational collapse, M. Szydlowski and Lapeta (1990) and M. Szydlowski and M. Biesiada (1991) proposed to look at the Ricci scalar of the manifold on which the mixmaster dynamics generates a geodesic flow. (The Maupertius principle was applied to the Hamiltonian formulation of the mixmaster dynamics as given in Bogoyavlensky (1985)) and investigate if one could extract (coordinate invariant) information about instability properties of the gravitational collapse in this way.

Application of the Maupertius principle to the Bogoyavlenskii Hamiltonian formulation of the mixmaster dynamics and the associated induction of a natural distance measure on the three metrics, is (cf. M. Biesiada and S.E. Rugh (1994)) exactly to implement the dream by Wheeler (described, e.g., in C.W. Misner (1972) for the mixmaster metric) to have a superspace metric " G_{gh} " (we call it Wheelers superspace metric) with respect to which the three metrics move along geodesics - and use that metric as a natural distance measure on the space of mixmaster three-geometries. Thus, the Ricci scalar calculated from the Hamiltonian introduced by Bogoyavlenskii, cf. e.g. Szydlowski and Biesiada (1991), is exactly to calculate the Ricci scalar of the Wheelers superspace metric and the criterion $R < 0$ implies that one has *local* instability at least

in one direction (as given by the *geodesic deviation equation* for two nearby mixmaster metrics investigated with Wheelers superspace metric as a distance measure).

An obstacle to this approach (of using the conformally rescaled Wheeler superspace metric), besides the obstacle that Wheelers superspace metric is indefinite, is the accompanying introduction of a host of singularities of the superspace metric which makes the original dream of Wheeler troublesome to achieve (i.e. the three metrics $^{(3)}g$ be geodesics w.r.t. a superspace metric) - even in the restricted class of mixmaster three-geometries. Invariants calculated from the superspace metric, e.g. the Ricci scalar, inherit the singularities of the Wheeler superspace metric. In particular, the Ricci scalar quoted e.g. in M. Szydlowski and M. Biesiada (1991) has such singularities. This obstacle has also recently been emphasized by A. Burd and R. Tavakol (1993) and was previously discussed in C.W. Misner (1972).

However, as a *local* instability criterium, $R < 0$ indicates - at nonsingular points where it is defined - the *local* exponential instability in some directions of the configuration space w.r.t. the indefinite superspace metric.

Only insufficient attention, in my opinion, has been paid towards the often rather arbitrary distance measures on the space of three-metrics which are introduced in previous studies calculating Lyapunov exponents, etc.

It thus seems worthwhile to investigate if it is possible to construct and use good "gauge invariant" distance measures (for example the Wheeler superspace metric G_{gh} , with or without conformal transformation of G_{gh} , or other distance measures) on the solution space of three-metrics and if one can use such distance measures to discuss instability properties of e.g. the gravitational collapse in a way which is invariant under some (large) class of gauge transformations.

3.9. The mixmaster collapse as an interesting laboratory for testing ideas about how to apply chaos concepts in the context of general relativity

The structure of the mixmaster gravitational collapse model is so simple⁵⁶ that it is not unreasonable to investigate (cf. e.g. Contopoulos *et al* (1993)) whether the model should turn out to be integrable, after all. I.e., there could be several additional "hidden" symmetries in the governing Einstein equations of motion besides the Hamiltonian.

A model like the mixmaster model collapse is very well understood in terms of the combinatorial model by Belinskii et al (controlled by simple maps). However, if we cannot agree on how to construct indicators of chaos (from the full phase flow) in this simple example, how could we then, in principle, dream about constructing chaos indicators which can deal with more complicated situations?

Whether we can assign any invariant meaning to a concept like "metrical chaos" (or even more ambitious: "turbulence in spacetime metrics") is a theoretical problem of intrinsic interest in the mathematical study of the dynamics of the full non-linear Einstein equations.

⁵⁶In the approximation where the walls are infinitely hard we have an example of so-called "algebraic chaos". All the interesting dynamics (the chaos) lies in the transition algebra associated with the bounces against the wall boundaries.

4. IS “CHAOS” A GENERIC FEATURE OF THE EINSTEIN EQUATIONS?

The case study of our mixmaster metric (3) shows that the “generic” expectation of the self-interacting gravitational field to generate space-time “chaos and turbulence” (for strong field strength, e.g. when probing the Einstein equations near space-time singularities where invariants like $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ of the tidal field curvature tensor blow up without upper bound) is reflected even in such a *highly symmetric* model. This may be considered a very strong property:

If such a “simple” (finite degrees of freedom) deterministic system shows a chaotic, complex dynamical behavior, then one expects this to be even more the case in complicated scenarios with a larger number of interacting degrees of freedom !

In this sense the “mixmaster” toy model of a gravitational collapse should be considered as a promising starting point of subsequent investigations of more complicated situations (with less symmetry) of the classical field equations in general relativity.

(Compare, e.g., with the famous Lorenz attractor model (Lorenz, 1963) for turbulence which is a crude starting point for an investigation of some properties of hydrodynamical turbulence. Note, however, that the Lorenz attractor is a highly truncated model, where only very few modes of the Fourier expanded equations are kept in the model! The governing equations for the mixmaster, $SU(2)$ homogeneous, metric are the *exact Einstein equations* - involving few degrees of freedom because of the (a priori) made symmetry-ansatz of the metric.

4.1. Aspects of fragility of the gravitational collapse?

However, also for governing equations of motion which are modified slightly relative to the Einstein equations one would expect that “metrical chaos” is a generic feature when probing the metric in regions where the curvature gets strong and non-linear self-interactions of the involved gravitational fields get important (e.g. on approach to curvature singularities).

Let us say that we modify the Einstein-Hilbert action slightly,

$$S = \int d^4x \sqrt{-g} \{ R + \epsilon(\Phi(R, R_{\mu\nu}, R^{\mu\nu}, R_{\mu\nu\rho\sigma}, R^{\mu\nu\rho\sigma})) \} \quad (38)$$

where Φ is a general analytic function and ϵ is a small perturbation parameter. See also discussions in R.K. Tavakol (1991) and A.A. Coley and R.K. Tavakol (1992).

(Chaos in the evolution of the metric (3) should not merely be a property of the *exact* Einstein equations (i.e. for $\epsilon = 0$), but also for the modified equations, corresponding to the inclusion of higher order curvature terms (i.e. for $\epsilon \neq 0$) in the action. Such higher order curvature terms - generically arising in underlying theories of gravity like, for example, string theories - will *blow up* towards the ultraviolet (for example towards the big crunch singularity of the mixmaster metric).

Thus, when curvature gets big there is no reason why higher order curvature terms should not appear in the Lagrangian, $\mathcal{L} = \sqrt{-g} \{ R + \alpha_1 R^2 + \dots \}$. Seen in this perspective, it is puzzlesome that the chaotic mixmaster oscillations are unstable and

stop at some stage of the collapse (and proceed in a non-oscillatory manner) if higher order curvature terms, like R^2 terms, are included in the Lagrangian, according to J. Barrow and M. Sironuse-Zia (1989). See, also, J. Barrow and S. Cotsakis (1989) and a recent study by S. Cotsakis *et al* (1993).

A (premature) conclusion arises, that the mixmaster oscillatory behavior - or the very fact that the metric (3) is evolving in a complex (unpredictable) manner towards the big crunch singularity - is *not* structurally robust towards modifications of the Einstein-Hilbert action and such modifications are certainly expected when length scales get small, of the order of Planck sizes, say.

4.2. A remark on fragility and structural stability in a wider context

In a broader context R. Tavakol (1991) discusses the concept of “structural fragility” and Coley and Tavakol (1992) advocate that models which are structurally fragile are “generic” among the set of models usually employed in cosmology.

We have already mentioned that isotropization of anisotropic spacetime metrics like the compact mixmaster metric (3) occurs too slowly within the standard hot big bang model to explain the remarkable degree of isotropy in the microwave background radiation observed today. Thus, isotropy is an unstable property of physically realistic cosmological initial conditions - yet it is an observed property of the Universe. Another, famous, example is the “flatness” problem (R.H. Dicke): Why is the Universe today so near the boundary between open and closed, i.e. so nearly flat? This is a puzzle because one may show that $\Omega = \rho/\rho_c = 1$ is an *unstable equilibrium point* of the evolution of the standard hot big bang theory (i.e. it resembles the situation of a needle balancing vertically on its tip). In order for Ω to be somewhere in the allowed range today, $\Omega \sim 1/10 - 10$, this parameter had to be equal to one to an accuracy of 49 decimal places if we consider times around 10^{-35} second, say, after the big bang! This finetuning problem is, however, no problem at all relative to the completely vanishing probability for us to have a second law of thermodynamics which R. Penrose (1989) estimates as one chance in $\sim e^{10^{123}}$! (The Universe was created in a state of very low entropy compared to what it might have been).

These finetuning problems are to some extent relaxed within the inflationary Universe concept. Inflation predicts, for example, that the value of Ω today should equal one to an accuracy of about one part in 100,000, cf. A.H. Guth (1992).

In view of the variety of simplifying assumptions in the formulation of any mathematical model of a physical phenomenon in Nature and the fact that such models have a tremendous power in describing such phenomena, it appears that such a principle of “structural fragility” cannot be implemented in Nature to its extreme limit.

The extreme opposite principle, the so-called “random dynamics” principle, which concerns the structure of Natural laws themselves, has been put forward by Holger Bech Nielsen (at the Niels Bohr Institute). He contemplates (cf. e.g. H.B. Nielsen (1976, 1981)) - and has explored in many contexts (cf., e.g., C.D. Froggatt and H.B. Nielsen (1991)) - a principle of the (extreme) structural stability of Natural laws (as they appear to us) - against (almost any) modification of them in the ultraviolet.

It is a principle which postulates the structure of Natural laws (at our scales) despite a substantial lack of structure of laws at a more fundamental level and the dream.

though very ambitious here, has some analogy to, say, the phenomenon of "universality in chaos", i.e. the realization that many phenomena, like the universality of Feigenbaum's δ in a bifurcation scenario, do not hang severely on the *microscopic details* of the underlying dynamics.⁵⁷ In the context of our discussion of the mixmaster collapse it also reminds us of the "chaotic cosmology" concept (see also sec.6.3), developed by C.W. Misner, which attempts at creating our present Universe from (almost) arbitrary initial conditions.

If the universality class of fundamental models which leads to the same infrared phenomenology (the standard model of the electroweak and strong forces, say) is not too narrow it leaves (logically) the possibility of having "*chaos in the Natural laws*" (at the most fundamental "scales") - i.e. the fundamental structure may be selected randomly if it is just selected from the universality class of models restricted by the boundaries set by the "universality class"

However, such boundaries (if these could be made precise enough to be stated formally, say, by mathematical formulas) in a certain sense themselves have status as "Natural laws" (regularities) - and so on, ad infinitum. Therefore, this way, one does not circumvent the concept of some "Natural laws" and "regularities" to be implemented in our Universe.

Preliminary discussions of boundaries that such a "random dynamics" principle may come across (is it possible to define a concept of "structural stability for Natural laws"? is contained as sec.3 in H.B. Nielsen and S.E. Rugh (1993). Boundaries that such a project come across are of interest because hereby one may gradually try to isolate elements in our description of Nature which can not (so easily) be modified (and thought differently). If one contemplates - cf. e.g. S. Weinberg (1992) - that the Natural laws and parameters are the only *self-consistent* set of laws and parameters imaginable, the "random dynamics" project contributes with an analysis of what we could mean by the word "self-consistent".

4.3. Back to spacetime chaos: "Chaos" in other solutions in classical general relativity?

It would be a good idea to look for other toy models of spacetime metrics, which would exhibit chaotic behavior, e.g. in connection with solutions obtained in the (fast advancing) discipline of *numerical relativity* - if one has good "codes" available to solve the Einstein equations without too many symmetry assumptions!

If possible, it would be very nice if one could relax the symmetry properties and study the structure of the cosmological singularities of space-time metrics with only two spacelike Killing vector fields, say, instead of three. Especially, it would be wonderful if one could relax the symmetry properties and generalize the mixmaster gravitational collapse this way.

It is a reasonable expectation that the severe non-linearity present in the Einstein equations implies "chaotic" (non-integrable) solutions almost "unavoidably" in scenarios involving high gravitational field strength (e.g. near space-time singularities). This "metrical chaos" should therefore be thought of as a generic feature of the non-linear

⁵⁷ i.e. there is a huge universality class of dynamical systems, e.g. all one-dimensional maps with a quadratic maximum (cf. also Cvitanovic (1989)), which generates the same Feigenbaum δ .

Einstein equations in such strong gravitational fields - and not as a feature connected, merely, to the near-singularity behavior of some (small) subclass of metrics, i.e. the spatially homogeneous toy-models of gravitational collapses.

On the "universality class" for the intermittent behavior of the "u" parameter (discussions with V.N. Lukash)

Even in the (small) class of spatially homogeneous metrics we have, according to Peresetzki and V.N. Lukash, cf. Lukash (1983), as regards the evolution of the already mentioned "Ljshitz-Khalatnikov" parameter "u", that their evolution is chaotic (and actually given by the Farey/Gauss map) if the invariance groups/algebras - according to the standard classification of three-dimensional Lie algebras by Bianchi (1897) - (of the three-spaces) are chosen as type *VIII* and *IX* (type *IX* is actually the mixmaster space) in the case of the *empty space* equations $R_{\mu\nu} = 0$, but in fact *all the Bianchi metrics* - except for type *I* (a flat space) and *V* (where the curvature is isotropic) - in situations (a perfect fluid energy momentum tensor is assumed) where $T_{0i} \neq 0$ (but $T_{ij} = 0$), cf. Peresetzki and V.N. Lukash.

In fact, one can show that the spatially homogeneous toy-model metrics are driven into "chaos", as regards the evolution of the parameter "u", by the presence of *curvature anisotropy of spiral character* (spiral character: At least one structure constant is different from zero, $C^i_k \neq 0$, where all i, j, k are different, in the Lie-algebra formed by the three spacelike Killing vector fields): For a space of type *I* there is no curvature at all, for a metric of type *V* it is isotropic and - as a curiosity - for a space of Kantowski Sachs type, the anisotropic curvature is not of spiral character! (V.N. Lukash, private comm., NBI, and V.N. Lukash, Doctorate Thesis (1983)).

• Solutions with other "matter" sources:

What if other sources than perfect fluid matter are included in the study? It is known, e.g., that the *SU(2)* Yang Mills equations in the limit of infinite wavelength (i.e. the spatially homogeneous mode) admit "chaotic" solutions⁵⁸ (cf., e.g. S.G. Matinyan (1985) and G.K. Savvidy (1984)) with a positive metric entropy $h = \int \sum \lambda^+ d\lambda > 0$. (In this limit - the "extreme infrared" - it is also a famous result of linear stability analysis that the non-Abelian Yang-Mills configurations are *unstable* under small disturbances, cf. N.K. Nielsen and P. Olesen (1978) and S.J. Chang and N. Weiss (1979).

If the Einstein and *SU(2)* Yang Mills equations are coupled, e.g. in the context of a mixmaster toy model study of a gravitational collapse, some very complicated and chaotic behavior must arise!⁵⁹

• Spatially inhomogeneous gravitational collapses:

What is the effect of taking into account spatial degrees of freedom in the model, i.e., the inclusion of some higher order Fourier modes ("spin 2" gravitational waves) superimposed (in a gauge-invariant way?) on the *SU(2)* homogeneous (zero-mode expanded)

⁵⁸For more recent discussions of spatially inhomogeneous chaos in non-Abelian gauge theories see also e.g. M. Wehner (1992), B. Müller *et al.* (1992) and T.S. Biro *et al.* (1993). A detailed description of chaos in non-Abelian gauge theories will also be found in S.E. Rugh (1994).

⁵⁹For a study of stability properties of such coupled *SU(2)* Einstein-Yang-Mills-Higgs equations (in the context of monopole and black hole solutions) see also Straumann and Zhou (1990) and C.W. Gibbons (1990).

but anisotropic gravitational collapse? Perhaps small amplitude gravitational waves die out towards the singularity? It is, however, far from obvious that such perturbations will not do the opposite: Blow up and significantly alter the evolution of the metric on approach to the space-time singularity! (Cf. also, remarks by R. Ove (1990) in a slightly different context).

Note, that according to Belinskii, Khalatnikov and Lifshitz (1982), the simple BKL-combinatorial model for the alternation of Kasner exponents (derived for the spatially homogeneous models) remains valid locally (i.e. in the neighborhood of every space point) in the general spatially inhomogeneous gravitational collapse. See also discussion in Ya.B. Zel'dovich and I.D. Novikov (1983) §23.3.

• Chaos in colliding gravitational waves?

One may speculate whether yet another example of chaos in the classical Einstein equations occurs in the dynamics of the metric field (and extracted gauge invariant quantities) when two non-linearly interacting plane gravitational waves extending to infinity collide and generate black hole singularities - via *nonlinear focusing effects* - of the "spacetime" structure⁶⁰. The detailed nature of the evolutionary tracks of the gravitational field $\{g_{\mu\nu}(x)\}$ - when "bouncing off" parts of the spacetime manifold in the form of black hole singularities - must almost unavoidably (?) be *chaotically unpredictable!* Highly nonlinear effects (when the curvature gets strong) determines the detailed nature of the dynamics.

5. IMPLICATIONS OF "METRICAL CHAOS" ON THE QUANTUM LEVEL?

Since the classical Einstein equations are scale-independent, there is - a priori - no scale built in our toy-model study (3) of the oscillatory, chaotic collapse - when evolved according to the classical Einstein equations. However, if we - by hand - put in a length-scale (lets say "the size" of the Universe $\sim r/H \sim 10^{28} \text{ cm}$, or somewhat less) as a starting condition on the scale factors of the gravitational collapse, one notes (cf. also the displayed fig.5) that already after few major cycles of bounces some of the length scales of the gravitational collapse get very small and, in fact, very fast reach Planck scales

$$l_{Pl} = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33} \text{ cm}$$

and, correspondingly, the space-time curvature blows up

$$\sqrt{R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}} \sim l_{Pl}^{-2} = \frac{c^3}{\hbar G} \sim 10^{66} \text{ cm}^{-2}.$$

Such scales may set characteristic scales when quantum effects become important and the gravitational degrees of freedom should be quantized.

⁶⁰Cf. e.g. S. Chandrasekhar, talk at the "300 years of Gravity" celebrations at D.A.M.T.P. (1987). However, it should be realized that the self-focusing singularities most likely arise because they are colliding plane waves which extend to infinity - and such waves do not exist! (M. Demianski and D. Christodoulou, pers. comm.)

Thus, the marmaster gravitational collapse turns very fast into a quantum problem! (and is chaotic in a domain where it should be quantized)

The fact that the scale functions for the mixmaster spacetime metric change so fast can be understood from the observation that the Einstein equations for the metric (3) involve exponential functions (cf. (7) or (25)) and such exponential functions are well known to give out "large numbers" very fast!

We are lead to consider our toy-model study of a chaotic gravitational collapse as:

1. Merely a look into a "chaotic sector" of the solution space to the classical Einstein equations. One notes that the classical Einstein equations do not internally lead to contradictions at any scale (see also Landau and Lifshitz, Vol.II, §119) and considered as a *mathematical study* of the non-linear Einstein equations, "metrical chaos" is a concept (not yet properly defined) of interest in itself!
2. Presenting some evidence in favor of speculations that the process of quantization of the gravitational field - occurring at small lengthscales and correspondingly high curvatures ($\sqrt{R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}} > l_{Pl}^2$) - in generic situations rather will occur "around" field configurations which are "turbulent" (space-time chaos) than around well behaved (integrable) solutions⁶¹. In the semiclassical limit, which is built around the classical configurations, this may be of importance.

Does classical chaos obstruct the quantization of the theory? Lee Smolin (1990) makes the point that one will - very likely - only make progress on the construction of quantum general relativity to the extent that one uses *non-trivial information* about the *dynamics* of general relativity.

In perturbation theory and Monte-Carlo simulations (methods of quantization in which no special information about the dynamics of the theory is used) chaos usually does not obstruct the quantization. Ashtekar and Pullin (1989), for example, emphasize that chaotic behavior of Yang-Mills fields has not obstructed the quantization of the theory. Such questions, being far from settled, are central topics in the theory of "Quantum Chaos" and will also be discussed (and references may be found) in S.E. Rugh (1994).

Spatially homogeneous cosmologies - having a finite number of (anisotropy) degrees of freedom - are often considered a laboratory for testing ideas about the application of quantum principles to gravitational degrees of freedom. Examples could be the Robertson-Walker, the Kasner or the mixmaster spaces. Now, the Kasner or FRW solutions (being integrable) do not display any "chaos" at the classical level, while the evolution of the mixmaster metric is chaotically complicated (and in metric time the dynamical evolution condenses, as regards the oscillations of the anisotropy degrees of freedom, infinitely towards the two spacetime singularities).

Does this striking difference in the dynamical behavior at the classical level mirror itself in quantum effects in the "quantized" model cosmologies? I.e.

⁶¹Thus, our "intuition" (on gravitational collapses) based on the well known textbook example of the FRW-type "big bang"/"big crunch" collapse - with simple dynamical behavior - is a non-typical classical field configuration to have in mind when "quantizing gravity".

what is the effect of applying quantum principles to a chaotic, dynamically complicated gravitational collapse - the mixmaster, say - compared to more symmetric gravitational collapses (of the FRW- or Kasner-types, say) which are integrable all the way to the "final crunch" singularity?

In a semiclassical treatment (i.e. when the collapse has not yet reached Planck lengths) the quantum solution builds around the classical dynamics. So in the semiclassical limit it of course, for that reason, matters what the classical solution looks like!

What about the *fully quantized* regimes, when the length scales of the gravitational metric have collapsed down to Planck sizes, or less? It is hard to know what happens in the fully quantized regime of our gravitational collapse (despite that it reaches these small scales very fast) let alone that no good theory of quantum gravity is available.

5.1. The Wheeler-DeWitt quantization of the gravitational collapse

Since the pioneering works of P.A.M. Dirac in the fifties (cf., e.g., P.A.M. Dirac (1959) and references therein) and R. Arnowitt, S. Deser and C.W. Misner (1962), it is well known that a Hamiltonian formalism (ADM 3+1) may be set up for the Einstein equations and this Hamiltonian formalism is often used in a "toy-application" of quantum principles (the Wheeler-DeWitt equation) to such a model of an anisotropic gravitational collapse. See, e.g., the recent bibliographies on these topics by C. Teitelboim and J.J. Halliwell in S. Coleman *et al* (1992).

More explicitly, the Wheeler-DeWitt equation (J.A. Wheeler (1968), B.S. DeWitt (1967)) arises after appropriate translation of the coordinate and momentum variables into operators by imposing the Hamiltonian constraint⁶² as an operator constraint on the "quantum wave function Ψ " of the gravitational collapse.

For the sake of notational simplification, we put $16\pi G = 1$, $\hbar = 1$, etc., and the Wheeler-DeWitt equation (the operator form of the Hamiltonian constraint) has the form

$$\hat{H} \Psi = \left(G_{ijH} \frac{\delta^2}{\delta g_{ij} \delta g_{ij}} + g^{1/2} {}^{(3)}R - {}^{(3)}\mathcal{L}_m \right) \Psi = 0 \quad (39)$$

where $\Psi = \Psi^{(3)}(g) = \Psi(g_{ij})$ is a quantum mechanical wave-function (for our gravitational collapse), ${}^{(3)}R$ is the curvature scalar of the tree-metric in question, ${}^{(3)}\mathcal{L}_m$ denotes the three-dimensional Lagrangian density of the (non-gravitational) matter fields and

$$G_{ijH} \equiv \frac{1}{2} g^{-1/2} (g_{ik} g_{jl} + g_{il} g_{jk} - g_{ij} g_{kl})$$

is the Wheeler-DeWitt superspace metric (cf., e.g., discussion in B.S. DeWitt (1967)) on the space of the three geometries.

⁶²Such constraints always arise in field theories where the field variables have a "gauge arbitrariness" (e.g. also in the Hamiltonian formulation of Yang-Mills theory) and the vanishing of a Hamiltonian, like in equation (39), is a characteristic feature of theories which are invariant under reparametrizations of time (cf., e.g. R.M. Wald (1984), appendix E). There is, however, a crucial difference between gauge theories and *parametrized* theories like general relativity (also invariant under reparametrizations which involve time). In a parametrized theory the constraints give all the *dynamics*. See also e.g. J.B. Hartle and K.V. Kuchar (1984).

A substantial amount of uncertainty could be expressed as to whether a toy-application of quantum principles along such lines makes good sense when implemented in a context like the mixmaster gravitational collapse (3). Some points of uncertainty:

1. The "factor ordering" ambiguity in translating a classical equation like the classical Hamiltonian constraint into an operator identity (involving non-commuting quantum operators) is a severe problem. For example, one may select a factor ordering - in the case of our mixmaster toy-model gravitational collapse - that makes *any given function* Ψ (of the mixmaster three-metric variables $\Omega, \beta_{\pm}$) a solution of the Wheeler-DeWitt equation for this factor ordering! (cf. investigations by Kuchar quoted in Moncrief and Ryan (1991), p.2377).
2. It is not clear that quantum solutions of the highest-symmetry models are approximations to quantum solutions of models with less symmetry (e.g. that "quantizing" the spatially homogeneous mixmaster collapse is an approximation to some "true quantum gravity solution", which includes space-dependent modes). Cf. Kuchar and Ryan (1989).
3. If space-dependent gravitational waves are superimposed on the mixmaster metric and taken into account perturbatively in the Wheeler-DeWitt equation one will most likely end up with non-renormalizable divergencies?
4. The symbol " Ψ " (the "wave-function" of the quantized gravitational collapse including matter degrees of freedom etc.) appearing in equation (39) does not have a straightforward interpretation if it also includes the observer, for instance in an application of quantum principles to an entire model cosmology. One of the difficulties one encounters is how to allow for an experimentalist with a "free will" to perform experiments, if *everything* (including the experimentalist) is described by a big (completely deterministic) "Schrödinger equation"? The very notion of an experiment seems meaningless, since the measurement was determined in advance (at the "big bang")? ⁶³

In lack of a truly renormalizable underlying theory of quantum gravity, one may choose to ignore such obstacles and consider a toy-application of the Wheeler-DeWitt equation (39) implemented in the case of the mixmaster gravitational collapse and hope that it retains aspects of the full theory.⁶⁴

What do we obtain? We yield an operator constraint equation which roughly has a mathematical form as given below if we consider the simplest possible choice of factor ordering of the operators (we neglect non-gravitational matter fields)

$$\left\{ -\frac{\partial^2}{\partial \Omega^2} + \frac{\partial^2}{\partial \beta_+^2} + \frac{\partial^2}{\partial \beta_-^2} + e^{-6\Omega} {}^{(3)}R(\Omega, \beta_{\pm}) \right\} \Psi(\Omega, \beta_{\pm}) = 0. \quad (40)$$

It looks like a *zero energy* "Klein-Gordon-Schrödinger" wave equation. Ω, β_+, β_- are the degrees of freedom in the parameterization (17) of the metric (3) and Ω, β_+, β_- are one-to-one related to the scale functions a, b, c in (3) via

$$\begin{aligned} a &= \exp(-\Omega + \beta_+ + \sqrt{3}\beta_-) \\ b &= \exp(-\Omega + \beta_+ - \sqrt{3}\beta_-) \\ c &= \exp(-\Omega - 2\beta_+) \end{aligned}$$

⁶³See also, e.g., chapl. 15 "Quantum Mechanics for Cosmologists" in J.S. Bell (1987).

⁶⁴According to R. Graham (1993): "Given a dynamical system of Hamiltonian form, the temptation to quantize seems to be irresistible".

$\Omega = -\frac{1}{3} \ln(ab c) \propto -\log(\text{Volume})$ may function as a “time” parameter of the gravitational collapse since, as we have already noted, the three-volume $V = 16\pi^2 abc$ of the metric (3) is monotonically decreasing on approach to the space-time singularity and the variables $\beta_+ = \frac{1}{6} \ln(ab/c^2)$, $\beta_- = \frac{1}{2\sqrt{3}} \ln(a/b)$ denote the “state of anisotropy” of the gravitational collapse. The three-curvature scalar on the three-space, ${}^{(3)}R$, can be calculated from the metric (3) and has the following form

$${}^{(3)}R = -e^{2\beta_+} \left\{ e^{4\beta_+} (\cosh(4\sqrt{3}\beta_-) - 1) + \frac{1}{2} e^{-8\beta_+} - 2e^{-2\beta_-} (\cosh(2\sqrt{3}\beta_-)) \right\} \\ = -\frac{1}{2a^2 b^2 c^2} \left\{ (a^2 - (b+c)^2)(a^2 - (b-c)^2) \right\}. \quad (41)$$

The structure of ${}^{(3)}R$ gives rise to a scattering potential (in the initial stages of the collapse describing soft scattering walls which fast develop, however, into “infinitely hard” walls when approaching the space-time singularity of the metric) which makes the classical Hamiltonian dynamical problem resemble that of a billiard ball being played on a billiard table (the anisotropy plane $(\beta_+, \beta_-) \in \mathbf{R}^2$) with this potential as the boundary walls. Of course, in the isotropic case $a = b = c = R/2$ we reobtain the expression for the three-curvature scalar ${}^{(3)}R = +3/2a^2 = +6/R^2$ corresponding to the compact ($k = +1$) F/RW space.

The Wheeler-DeWitt quantized mixmaster collapse has been studied by various authors, for example C.W. Misner (1972) (see also, e.g., S.W. Hawking and J.C. Lattrell (1984), R. Graham and P. Szépfalussy (1990) and V. Moncriet and M.P. Ryan (1991), and references therein, for a toy “quantization” of the mixmaster metric).

Recently, exact solutions have been found by R. Graham if an additional supersymmetry is introduced in the study, R. Graham (1991, 1992). (See, also, P.D. Eath, S.W. Hawking and O. Obregon (1993)). These solutions describe virtual quantum wormholes, see also R. Graham (1993).

In the context of the Wheeler-DeWitt equation, applied to spatially homogeneous gravitational collapses, we could hope (preliminarily) to address the question: What is the effect of having a chaotic gravitational collapse (as opposed to an integrable solution) when quantizing it? Since in the Wheeler-DeWitt equation we are merely looking at the zero energy solution $\hat{H}\Psi = 0$, it is at first sight not obvious how we can relate to, say, an effect (cf. e.g., M.V. Berry (1981)) like that of nearest neighbour energy repulsions. Prof. R. Graham, however, pointed out that C.W. Misner already twenty years ago offered a separation of the Wheeler-DeWitt equation in the region sufficiently near the space-time singularity where the scattering potential $g^{(3)}R$ is approximately infinitely hard and where the solutions to the Wheeler-DeWitt equation are related to the spectral properties of the Laplace operator on the Poincaré disc.

Before I sketch this last point I will digress shortly into a description of the mixmaster gravitational collapse of remarkable beauty.

5.2. The Poincaré disc description of the gravitational collapse

The dynamics may be transformed⁶⁵ into that of piecewise geodesic motion in a non-compact domain with triangular symmetry (corresponding to the symmetry under the

⁶⁵Note, that the Poincaré disc description of the mixmaster collapse has gone almost unnoticed for twenty years. See, however, J. Barrow (1982). Suddenly several people looked at it again, J. Pullin (1990), Graham and Szépfalussy (1990), (S.E. Rugh (1991)).

interchange $a \leftrightarrow b \leftrightarrow c$ of the scale factors in the metric (3)) on the two dimensional Poincaré disc (the Lobachevsky space) of constant negative curvature! As is well known, geodesic motion on surfaces of constantly negative curvature is a standard laboratory for testing ideas in classical and quantum chaos, cf. e.g. Balazs and Voros (1986). However, the mixmaster gravitational collapse may very well be one of the only physically motivated⁶⁶ dynamical systems known, which realizes such geodesic motion on surfaces of constantly negative curvature! It is thus clearly an interesting and beautiful aspect of the mixmaster collapse which deserves further investigations. (The Poincaré disc description has been emphasized also by J. Pullin, recently, cf. J. Pullin (1990) and R. Graham and P. Szépfalussy (1990)).

A very brief discussion of the set of transformations which brings the mixmaster gravitational collapse into that of geodesic motion on the Poincaré disc is offered in Misner, Thorne and Wheeler (1973) §30.7. I will not drift into details here - since that is far beyond the scope of this presentation (a detailed description will be available in S.E. Rugh (1994), where the “Poincaré disc” description is worked out and discussed in exhaustive detail).

The crucial observation (which is originally due to C.W. Misner and D.M. Chitre) is that while the original Hamiltonian

$$\mathcal{H} = -p_\Omega^2 + p_+^2 + p_-^2 + g^{(3)}R(\Omega, \beta_+, \beta_-) \quad (42)$$

has outward, time dependent expanding potential boundaries in the $\Omega, \beta_\pm$ variables, a set of transformations

$$(\Omega, \beta_+, \beta_-) \mapsto (t, \xi, \phi) \mapsto (t, x, y)$$

may be devised

$$\begin{pmatrix} \Omega - \Omega_0 \\ \beta_+ \\ \beta_- \end{pmatrix} = e^t \begin{pmatrix} \cosh \xi \\ \sinh \xi \cos \phi \\ \sinh \xi \sin \phi \end{pmatrix} = e^t \frac{1}{1 - (x^2 + y^2)} \begin{pmatrix} 1 + (x^2 + y^2) \\ 2x \\ 2y \end{pmatrix} \quad (43)$$

which makes the location of the potential boundaries *time-independent* with respect to the new “ t ” time parameter when we are in a region sufficiently near the space-time singularity of the metric. (The second part of the transformation (43) is a mapping⁶⁷ of (ξ, ϕ) into coordinates (x, y) inside the Poincaré unit disc $\mathcal{D}$ in the complex plane, see below).

The transformations of the momenta p_Ω, p_+, p_- which appear in the Hamiltonian (42) are constructed to make the transformations of coordinates and momenta become *canonical* transformations, $p_i' = (\partial q^j / \partial q^i) p_j$, and the Hamiltonian is (after proper rescalings) transformed into a Hamiltonian which, in (t, ξ, ϕ) coordinates, has the form

$$\hat{\mathcal{H}} = -\dot{p}_t^2 + p_\xi^2 + \frac{p_\phi^2}{\sinh^2 \xi} + \hat{V}(t, \xi, \phi). \quad (44)$$

The scattering potential $\hat{V}(t, \xi, \phi)$ has (in the asymptotic region sufficiently near the singularity of the metric) infinitely steep potential boundaries and the location of these

⁶⁶If this system is not a “physical system” in an *empirical* sense, it is certainly a physically motivated system, being a (globally) perturbed FRW metric, evolved by the full, non-linear Einstein equations.

⁶⁷There are some discrepancies with the expressions in J. Pullin (dec. 1990). I, however, get these set of transformations.

potential boundaries (which show up to form a triangular domain $B \subset \mathcal{D}$ inside the Poincaré disc, see below) are *independent* of the new "time" variable t .⁶⁸ The location of the potential boundaries are in the (ξ, ϕ) space determined by the equations, cf. MTW (1973) §30.7,

$$2 \tanh \xi = -\frac{1}{\cos(\phi + m \frac{2\pi}{3})}, \quad m = -1, 0, 1. \quad (45)$$

Having obtained in this way the stationary (in the (ξ, ϕ) plane) scattering potential $V(\xi, \phi)$ which is zero inside the domain (we call this domain B) bounded by (45) and ∞ in the region outside this domain (an approximation which is extraordinarily good when sufficiently near the "big crunch" singularity of the mixmaster gravitational collapse)

$$\bar{V}(t, \xi, \phi) \rightarrow \bar{V}(\xi, \phi) = \begin{cases} 0 & \text{inside domain } B \\ +\infty & \text{outside domain } B \end{cases} \quad (46)$$

we have obtained a Hamiltonian (44) which, in the interior of the scattering domain B , resembles the Hamiltonian for a free particle⁶⁹ which moves along the geodesics on a curved manifold with constant negative curvature.

To see this, note, that in general the Hamiltonian for a particle that moves along the geodesics ($\delta J ds = 0$) of a manifold with the metric $g_{\mu\nu}$,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad (47)$$

has the form

$$\mathcal{H} = \frac{1}{2m} g^{\mu\nu} p_\mu p_\nu \quad (48)$$

where the metric tensor $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$, i.e. $g_{\mu\nu} g^{\nu\rho} \equiv \sum_\nu g_{\mu\nu} g^{\nu\rho} = \delta_\mu^\rho$.

Inside the (stationary) potential walls (46) we have arrived at the stationary Hamiltonian (44). As regards the *projected* motion on the two dimensional plane (ξ, ϕ) , we may therefore conclude from the form of the Hamiltonian (44) that the associated flow is a geodesic flow on a Riemannian manifold with the metric

$$d\bar{s}^2 = d\xi^2 + \sinh^2 \xi d\phi^2 \quad (49)$$

inside the (stationary) region (the "billiard table" domain B) bounded by infinitely steep potential walls (46). This metric (49) is the metric of the Poincaré disc of constant negative curvature¹ (cf. also, e.g., Balazs and Voros (1986)). Explicit computation of the Gaussian curvature K at the point (ξ, ϕ) gives

$$K = -\frac{1}{\sinh \xi} \frac{\partial^2}{\partial \xi^2} \sinh \xi = -1$$

⁶⁸Note, according to (43), that the new "time" parameter t is a somewhat "strange" new time variable.

$$t = \frac{1}{2} \ln((\Omega - \Omega_0)^2 - (\beta_+^2 + \beta_-^2)) = \frac{1}{2} \ln((\Omega - \Omega_0)^2 - \|\vec{\beta}\|^2)$$

which mixes the state of anisotropy $\vec{\beta}$ (the position of the "billiard ball") with the original Ω -time variable, $\Omega = -\frac{1}{3} \ln(abv)$, in the ADM-Hamiltonian variables.

⁶⁹Since the Hamiltonian (44) ceases to be time-dependent in the asymptotic region near the spacetime singularity, we have $p_t = -\partial \bar{H} / \partial t = 0$. Hence, p_t is a constant of motion and the bouncing of the (ξ, ϕ) variables (or the (x, y) variables (51)) within the scattering domain B will take place at *constant speed* with respect to the t time coordinate.

As is well known, by suitable transformations of the coordinates, the geodesic flow on the Riemannian manifold (49) may be represented in various (equivalent) ways. Among these equivalent models (representations) we will consider the "*Poincaré unit-disc model*" for the hyperbolic geometry. By identifying a point (ξ, ϕ) with the point $z = (x, y) = r(\cos \phi, \sin \phi) \equiv re^{i\phi} \in C$ (in the complex plane) via the coordinate transformation

$$z = x + iy = re^{i\phi} = \tanh(\xi/2) e^{i\phi}, \quad \text{i.e.} \quad (50)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \cos \phi \\ r \sin \phi \end{pmatrix} = \begin{pmatrix} \tanh \xi/2 \cos \phi \\ \tanh \xi/2 \sin \phi \end{pmatrix} \quad (51)$$

we obtain a hyperbolic flow (of constant negative curvature $K = -1$) on the unit disc

$$\mathcal{D} = \{z = x + iy \in C \mid |z| = \sqrt{x^2 + y^2} \leq 1\}. \quad (52)$$

Any point of the (ξ, ϕ) space is transformed into the unit disc (52). The boundary (i.e. points at $r = 1$) of the disc $\mathcal{D}$ corresponds to the points at infinity $\xi = \pm\infty$. By this coordinate transformation (cf. also, Balazs and Voros (1986), p. 117-118), the metric (49) is transformed into

$$d\bar{s}^2 = \frac{4(dr^2 + r^2 d\phi^2)}{(1-r^2)^2} = \frac{4(dx^2 + dy^2)}{(1-x^2-y^2)^2} \equiv \frac{4|dz|^2}{(1-|z|^2)^2} \quad (53)$$

in the interior of the unit disc.

To summarize shortly: The coordinate transformations in (43) have, in the region sufficiently near the space-time singularity, accomplished to transform the Hamiltonian dynamics (originally formulated in the $(\Omega, \beta_+, \beta_-)$ variables and their canonical conjugate momenta) into that of piecewise "geodesic motion" (free motion) which takes place on the Poincaré unit disc (with constant negative curvature) inside a scattering domain $B \subset \mathcal{D}$ (bounded by three walls) in the interior of the unit disc $\mathcal{D}$.

If one translates - via the defining coordinate transformations (51) - the equations (45) of the boundary of the scattering domain B , one finds that they describe an equilateral triangle of geodesics with the tree corners at "infinity" (at the boundary circle of the Poincaré disc). $B \subset \mathcal{D}$ is thus a zero angle triangle on the Poincaré disc with finite hyperbolic area.

Using the identity $\tanh \xi \equiv 2 \tanh(\xi/2) / (1 + \tanh^2(\xi/2))$, we express the formula for the left wall in the (x, y) coordinates. We get $4x/(1 + (x^2 + y^2)) = -1$, which is a circle $(x + 2)^2 + y^2 = 3$ with center $(x, y) = (-2, 0)$ and radius $r = \sqrt{3}$. The part of this circle which overlaps with the Poincaré disc $\mathcal{D}$ governs the equation of the left wall ∂B_1 ($2\pi/3 \leq \phi \leq 4\pi/3$). It is easily verified that ∂B_1 cuts the boundary $\partial \mathcal{D} = \{(x, y) \mid x^2 + y^2 = 1\}$ of the Poincaré disc at the points $z = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} = \{e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}\}$ and at a right angle $\varphi = \pi/2$. Hence, we conclude that the left boundary wall ∂B_1 is a geodesic curve on the Poincaré unit disc $\mathcal{D}$. The triangular symmetry of the dynamical system gives us two similar ($2\pi/3$ rotated) wall boundaries consisting of circular arcs, the radii being $\sqrt{3}$ and centres located at $z = -2 e^{\pm i2\pi/3} = 1 \pm i\sqrt{3}$. A polygon in the Poincaré disc $\mathcal{D}$ is (as in Euclidian geometry) defined as a closed, connected area whose boundary consists in parts of geodesics, called its sides. Thus, the scattering domain B is a regular *triangular polygon*, a regular 3-sided billiard (the three sides which we denote ∂B_1 , ∂B_2 and ∂B_3) on the Poincaré disc. (Cf. fig.9).

Figure 9. The triangular billiard $B \subset D$ forming the scattering domain inside the Poincaré disc ∂D . The “billiard table” is bounded by the “infinitely hard” wall boundaries $\partial B = \partial B_1 \cup \partial B_2 \cup \partial B_3$. (Each side of the 3-sided polygon, ∂B_1 , ∂B_2 , and ∂B_3 , are geodesics on the Poincaré disc, being circular arcs orthogonal to the boundary circle ∂D (i.e. cutting the Poincaré disc boundary ∂D at right angles $\varphi = \pi/2$). Hence, B is a triangular polygon, where all the interior angles between two sides at each of the three vertices C_1, C_2, C_3 are zero. The *approximations* involved in arriving at the description of the gravitational toy-model collapse are *very accurate* when sufficiently near the space-time “big crunch” singularity of the metric: The scattering potential $V(x, y)$ then (to a very high accuracy) vanishes identically in the interior of the domain B (inside the wall boundaries ∂B) and equals $+\infty$ outside B .

The periods of geodesic motion (free motion) which occur between the bounces against one of the three walls of the triangle correspond to the periods of straight line behavior (“Kasner epochs”) in fig.5. The source of the chaos of the model is to be thought of as due to the bounces in the potential boundaries ∂B in conjunction with the negative curved interior (which gives sensitive dependence on initial conditions. To the bounces are associated algebraic transition rules related to continued fraction expansions, cf. table 1. In fact, due to the symmetry properties of the domain (tiling the Poincaré disc by the action of an arithmetic group (E.B. Bogomolny *et al* (1992), J. Bolte *et al* (1992)), the chaos in the mixmaster collapse (in the region sufficiently near the singularity) is an example of arithmetic (algebraic) chaos. We thus appear to have an example of chaos of remarkable beauty.

Are the exact numerical values of the Lyapunov exponents λ an “artifact” of the considered set of transformations?

One observes, cf. J. Pullin (1990), that the constant negative curvature $K < 0$ of the interior of the Poincaré disc may be translated into a statement about a positive Lyapunov exponent. The geodesic motion takes place at constant velocity and we have for the Lyapunov exponent $\lambda = \{\text{velocity}\} \times \sqrt{-K} > 0$. (cf. Balazs and Voros (1986), p. 147). This expression scales with $\sqrt{-K}$ and the precise value of the negative curvature K of the Poincaré disc appears to be an artifact⁷⁰ of the selected set of coordinate transformations (43). However, the instability of a closed orbit γ in the triangular billiard on the Poincaré disc is given by multiplying the apparent “Lyapunov exponent” $\lambda \propto \sqrt{-K}$ with the hyperbolic length of the closed orbit. Since the length of a closed orbit $\oint_\gamma ds$ scales as $|K|^{-1/2}$ the expression for the instability exponents of the closed orbits is *invariant* under rescalings of K :

$$\{ \text{instability of closed orbit} \} \propto \sqrt{-K} \times \{ \text{length of closed orbit} \} = \sqrt{-K} \oint_\gamma ds = \sqrt{-K} \oint_\gamma \frac{2|dz|}{\sqrt{-K}(1-|z|^2)} = \oint_\gamma \frac{2|dz|}{(1-|z|^2)}.$$

Thus the instability properties of the periodic orbit structure seem to be a (slightly) more invariant characterization.

As concerns the Wheeler-DeWitt toy-quantization of our gravitational collapse in the (t, ξ, ϕ) variables, say, it will actually have a complicated form (cf. equation (74) in G.W. Misner (1972)). Being sufficiently near the singularity of the metric where the walls are approximately infinitely hard we have an operator constraint on the wave function $\Psi = \Psi^{(3)g} = \Psi(t, \xi, \psi)$ of the form

$$\left\{ e^{-t} \frac{\partial}{\partial t} (-e^{-t} \frac{\partial}{\partial t}) + \frac{1}{\sinh \xi} \frac{\partial}{\partial \xi} (\sinh \xi \frac{\partial}{\partial \xi}) + \frac{1}{\sinh^2 \xi} \frac{\partial^2}{\partial \phi^2} \right\} \Psi(t, \xi, \phi) = 0 \quad (54)$$

inside the domain $B \subset D$ with the boundary condition that Ψ vanishes at the boundary ∂B .

The substitution

$$\Psi = \Psi(t, \xi, \phi) = e^{-(1/2+\omega)t} \psi(\xi, \phi) \quad (55)$$

⁷⁰One could transform the collapse orbit to a Poincaré disc with any negative curvature $K < 0$ (One should, however, verify how the velocity of the “billiard ball” scales in that case).

yields for the ψ component

$$\left\{ \frac{1}{\sinh \xi} \frac{\partial}{\partial \xi} (\sinh \xi \frac{\partial}{\partial \xi}) + \frac{1}{\sinh^2 \xi} \frac{\partial^2}{\partial \phi^2} \right\} \psi(\xi, \phi) = (\omega^2 + \frac{1}{4}) \psi(\xi, \phi) . \quad (56)$$

The left hand side is recognized as the Laplace operator on the two-dimensional Poincaré disc of constant negative curvature with metric $ds^2 = d\xi^2 + \sinh^2 \xi d\phi^2$. We conclude that properties of solutions to the Wheeler DeWitt equation for the mixmaster collapse are one-to-one related to spectral properties of the Laplace operator on the Poincaré disc, that is the eigenvalue equation which in the more familiar Poincaré disc variables (x, y) reads

$$\Delta \psi = \frac{1}{4} (1 - x^2 - y^2)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi = (\omega^2 + \frac{1}{4}) \psi . \quad (57)$$

with the boundary condition that ψ vanishes at the boundary $\partial \mathcal{B}$ of the triangular domain.

Moreover, spectral properties of the Laplace operator are related to the Gutzwiller trace formula involving a sum over the periodic orbits inside the triangular domain $\mathcal{B} \subset \mathcal{D}$. (Cf. Gutzwiller (1990) and also Cvitanovic (1990) and references therein).

It is amusing that the spectrum of the Laplace operator exhibits “non-generic” energy level statistics in the sense that the distributions of nearest neighbor level spacings displays Poisson rather than Wigner statistics. This is due to the symmetry properties of our triangular domain (the domain tiles the disc under the action of a so-called “arithmetic group”) with a resulting exponentially large degeneracy of lengths of the periodic orbits. Note, also, that the (approximate) semiclassical Gutzwiller trace formula, cf. Gutzwiller (1971, 1990), is exact in this case, since it coincides with the exact Selberg trace formula (giving an exact relationship between the quantum spectrum and the classical periodic orbits).⁷¹

In the context of the toy-model mixmaster gravitational collapse, a contact is thus made between the disciplines of “Quantum Cosmology” and “Quantum Chaos”, each dating back twenty years to pioneering works of C.W. Misner and M.C. Gutzwiller!

6. DISCUSSION, OPEN PROBLEMS AND DREAMS (SOME FINAL REMARKS)

The toy-model sketched (the mixmaster gravitational collapse) is a very simple model. Nevertheless, it has enough degrees of freedom to behave in a dynamically complicated and non-predictable way (with positive Kolmogorov entropy) on approach to the singular space-time point (the “big crunch” point) and in this way the toy-model space-time metric captures an interesting non-linear aspect of the classical Einstein equations (probing the Einstein equations in a domain where the field strength gets strong (as captured by an invariant like $R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$, which grows without limit) and non-linearities of the Einstein equations are important).

In fact, if one were to imagine a *generic* gravitational collapse (e.g. for an entire “model-universe”) to a singular point, then it would misguide our intuition to imagine a Friedmann Robertson Walker type of collapse solution (the “standard” big crunch/big

bang). This (non generic) solution is completely integrable (and behaves in a dynamically trivial way) as an artifact of the enormous amount of symmetry imposed on that solution.

Rather one should try to imagine a sort of “metrical chaos” (chaos in the space-time metric) being developed on approach to the curvature singularity - as the gravitational field strength gets stronger and non-linear aspects of the Einstein equations are increasingly important.

6.1. How to build up concepts which characterize such “metrical chaos”?

The general problem of transferring standard indicators of chaos to the situation of general relativity (pointed out in S.E. Rugh (1990a), emphasized in S.E. Rugh (1990b) and by J. Pullin (1990)) is well illustrated by the simple spatially homogeneous mixmaster collapse.

We have noticed (S.E. Rugh (1990 a,b)) that a naive application of a “standard” Lyapunov exponent will be a highly gauge-dependent measure, even under the small subclass of gauge-transformations, which only involve (the perfectly allowed) reparametrizations of the time-coordinates. To construct a gauge invariant generalization of a “Lyapunov exponent” is a yet unsolved problem.

Perhaps, there is a “no go” theorem to be constructed towards a completely gauge invariant construction? (H.B. Nielsen and S.E. Rugh)

We have noticed that the ergodicity properties of the toy-model mixmaster collapse are interesting (and subtle): In some “gauges” the phase space is ever expanding (or ever collapsing) and there are no return properties of orbits and a concept like “ergodicity”, for example, is very ill defined. By a suitable set of coordinate transformations, however, and some very reliable approximations, the Hamiltonian can be brought into that of a geodesic flow on a two-dimensional Riemann manifold of constant negative curvature (the Poincaré disc) bounded by some stationary walls forming an equilateral triangle with three corner infinities on the Poincaré disc. In this picture a number of stochastic properties of the gravitational collapse may be stated, e.g., the property of ergodicity. It is even a K-flow!

However, the chaotic properties of a toy-model like the mixmaster gravitational collapse is of course - inherently - the same, irrespective of the choice of “gauge” (choice of description). If we get apparent “chaotic behavior” in one description (in one gauge) but absence of chaotic behavior in another description - it is because one has posed the wrong question!

I thus disagree with the conclusion arrived at by Pullin (cf. also J. Pullin (1990) about the superiority of the “Poincaré disc” gauge-choice over other gauge-choices.

It appears to me that no “gauge” is better than others. Rather, one ought to seek measures (of chaos) which are invariant under (some large class of) gauge transformations (to the extent they are possible to construct).

⁷¹For a discussion of the most recent results as concerns the Wheeler-DeWitt quantization of the mixmaster gravitational collapse, see R. Graham (1993).

6.2. Is chaos in spacetime metrics of importance for observable phenomena in Nature ?

The extreme conditions for the gravitational field referred to in this manuscript (probing the Einstein equations near space-time curvature singularities) are of course far removed from "daily life" gravity. It is hard to find observable *astrophysical* phenomena which involve very strong gravitational fields and thus probe the *non-linearities* of the Einstein equations at a really deep level. (For instance, the rather large perihelion precession in binary pulsar systems, which are indeed marvellous relativistic laboratories, do not qualify as an observational test of non-linearities of the Einstein equations at a deep level). In the description of many gravitational phenomena, even the (quasi) *linearized* equations suffice.

The relevant laboratory for probing deep non-linearities of the Einstein equations seems to be (1) violent astrophysical phenomena ⁷² (2) early moments of the Universe (could the Universe itself - and the large scale structure of the spacetime metric of the Universe - have had a violent past? Would there be observable fingerprints of this today? Cf. also next subsection 6.3)

Turning to our toy model metric (3) the Einstein equations for the metric involve exponential functions. Such exponential functions are well known to give out "large numbers" very fast! In fact, it is so that if we - by hand - put some "reasonable scale" (like $L \sim c/H \sim 10^{28}$ cm $\sim$ the characteristic size of the present Universe) for the initial length scales, $a \sim b \sim c \sim 10^{28}$ cm of the gravitational collapse, then only few (chaotic) cycles of the scale-functions a, b, c occur before they reach and pass the Planck regime, $R_{obs} R_{obs}^{3/4} \sim l_{Pl}^4$. Thus, the gravitational collapse is chaotic ("metrical chaos") in the very same region where it should be treated quantum mechanically. In that sense this gravitational collapse is very much addressing a topic on the interface of quantum physics, cosmology and chaos.

If the large scale structure of spacetime (our Universe) started out (near Planck time, say) in a chaotic, oscillatory behavior given by the mixmaster metric (3) say, would we then be able to detect signals of this behavior today? Most likely, we would observe a substantially larger anisotropy of the background microwave radiation temperature than the high degree of isotropy observed today, since the Universe dissipates away the initial anisotropies too slowly. (See also next subsection). In a Universe with an *inflationary* phase, the existence of anisotropies and inhomogeneities before the onset of inflation is not excluded, but it appears that there is not much time for spacetime *dynamics* to perform "chaotic oscillations" in the short interval from Planck times $t_{Pl} = \sqrt{(\hbar G/c^3)} \sim 10^{-43}$ seconds after the "big bang" to the onset of an inflationary phase at the G.U.T. scales $\sim 10^{14}$ GeV occurring $\sim 10^{-34}$ seconds after the "big bang" ⁷³. Moreover, inflation tends to smooth things out so pre-inflationary history (e.g. dynamics of the spacetime metric) is rendered more irrelevant.

⁷²A gravitational phenomenon like the generation of gravitational waves (which will be observable in the next century) emitted from *strong* astrophysical sources needs that we take full account of the nonlinearity of the Einstein equations, cf. e.g. D. Christodoulou (1991).

⁷³Using $R \propto T^{-1}$ (valid after inflation has occurred), it is fascinating to imagine that our current Hubble volume (size of the Universe) had a size of about ~ 10 cm at those early times ! (i.e. when the temperature in the Universe was $\sim 10^{14}$ GeV).

A remark concerning dependence of the "present" on the "past". I.

A more serious problem than whether we will be able to detect evidence of "chaotic spacetimes" in the early moments of the Universe is whether detectable signals (now) in cosmology contain information which will prove useful in probing the structure of the Natural laws at higher energies than those which can be created here on Earth. In view of the foreseeable limitations in our abilities of putting substantially more than present energies ~ 100 GeV on a single elementary particle (experiments here on Earth), one would like to "resort to" our Universe - in its very early moments - as a laboratory to probe the structure of fundamental laws at energies beyond ~ 100 GeV. In cosmology, however, we really see the big bang physics through an extremely cloudy and *little informative filter*! See also, e.g., H.B. Nielsen, S.E. Rugh and C. Surtykke (1993). Every time *thermal equilibrium phases* are reached at certain stages in the evolution of the Universe then only very little information of the physics that went on *before* that phase can reach us today. It is difficult for a signal to survive through an equilibrium phase! Basically, only *conserved* quantum numbers *survive*, like baryon and lepton numbers and energy. However, if the thermodynamic equilibrium is not reached globally, there may survive some information in the correlations or, rather, in the *spatial variations* in the densities of these conserved quantities. Gravitational signals, i.e. spacetime dynamics, may contribute to establish correlations over big distances, since gravity is a long range force.

6.3. A chaotic initial "big bang"?

Could our Universe have started out in a chaotic state? Cf. the "chaotic cosmology" concept, developed by C.W. Misner *et al* , which attempts at creating our present Universe from (almost) arbitrary irregular initial conditions, e.g. from a spacetime metric with large initial anisotropies and inhomogeneities. ⁷⁴

A remark concerning dependence of the "present" on the "past". II.
Whereas one would (in cosmology) be tempted to explain the present Universe without having to resort to considerable fine tunings of the initial data of the Universe (at Planck scales, say) a "high energy physicist" would prefer that the present depends very sensitively on the past! ⁷⁵ - in order to be able to deduce interesting *information about the past* (in particular, to get a detailed look into the Natural laws beyond ~ 100 GeV, via signals from cosmology).

The viewpoint depends on the question(s) which drives our study: Do we "want" to give a "probabilistic" explanation of the present observed Universe (i.e.: if the Universe we see today results from a larger basin of initial conditions then it is more "likely" to have it?) or do we "want" signals coming from cosmology to be a very sharp probe exploring into the structure of Natural laws at very high energies (corresponding to hot dense phases of the Universe) ?

If one contemplates, cf. e.g. S. Weinberg (1992), that the actually implemented Natural laws are the only *self-consistent* set of laws and parameters imaginable, and also consider initial conditions of the Universe as resulting from such

⁷⁴See also, e.g., discussions in B.L. Hu *et al* (1993).

⁷⁵Inflation, which only lessens the dependence of the present from the initial data of the past, is in that perspective not good news. (Even within the inflationary Universe, however, there will be an arbitrary large number of initial conditions which are inconsistent with observations today)

implemented regularities (the division line between Natural laws and initial conditions is rather arbitrary anyway) then I must admit that I have no problem with reconciling myself to the fact that *initial conditions* of the Universe may be as fine tuned as *Natural laws* appear to be (so far explored).

If the Universe had very large irregularities initially, we have - in order to obtain the present Universe with large scale smoothness - to introduce some physical mechanisms to "dissipate away" all these initial irregularities (anisotropies and inhomogeneities).

A very effective dissipative process, suggested by Zel'dovich, is that of vacuum particle creation from the changing gravitational field in anisotropic spacetimes. (Gravitational fields can create particles somewhat similar to the way magnetic fields can create electron pairs). In a test field approximation where the spacetime anisotropies are not too big, one may estimate the probability $\mathcal{P}$ of producing a pair of conformally invariant scalar particles in a world tube T of constant, co-moving cross section extending from the singularity to the present and it turns out to be proportional to the integrated Weyl-curvature invariant over the world tube (cf., e.g., J.B. Hartle (1981)),

$$\mathcal{P} \sim \int_T d^4x \sqrt{-g} C_{\alpha\beta\gamma\delta} C^{\alpha\beta\gamma\delta}.$$

If the initial Weyl tensor invariant is too large and if the Universe has managed to dissipate away all this initial anisotropy, the heat entropy generated would be very large, and the Universe would *already* have reached "heat death" which is not the case. In fact, the "small" entropy per baryon now severely limits the amount of dissipation that has taken place in the initial phases of the Universe⁷⁶

In order not to reach "heat death" too fast, the Universe has to start out in a state of low entropy⁷⁷. According to the Weyl tensor hypothesis of R. Penrose (1979), gravitational entropy should, somehow, be measured by the Weyl curvature tensor, and the constraint imposed on the Universe to start out in a state of low entropy translates, e.g., into a requirement of creating a Universe with a small Weyl curvature (whereas a final "big crunch", if the Universe is closed, may - and is expected to - have a large Weyl curvature and a large degree of disorder).

The mixmaster space-time metric appears to be a well suited toy-model for implementation of the Weyl tensor hypothesis, having a very large Weyl curvature near the space-time singularity.⁷⁸ Thus, according to the Weyl-curvature hypothesis, the chaotic collapse should be considered a (simplified) candidate for the large scale metric at a final crunch⁷⁹ rather than at the initial bang.

Turning back to our question about observational consequences of having chaos in spacetime metrics: If it is a chaotic collapse at a "big crunch" point - it will probably be too hot for us to be there and observe it!

⁷⁶In order not to over-produce entropy. See also Barrow and Matzner (1977) and Barrow (1978).

⁷⁷This fact, and the enormous amount of fine tuning and precision in the organization of the starting conditions of the Universe this implies, has been emphasized very strongly in e.g. R. Penrose (1989). Perhaps, the Guth/Linde inflationary Universe provides us, more naturally, with a "low entropy" initial condition? (see, e.g., D. Goldwirth and T. Piran (1991) and references therein).

⁷⁸M. Biesiada and S.E. Rugh, in preparation.

⁷⁹ Cf. also the title "Gravity's chaotic future" in a recent issue of Science News, Vol. 144, p. 369-384, Dec-4 (1993).

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