1. Introduction

It is well known that the sphaleron^[1–2] is the configuration of maximal energy on a non-contractible loop (NCL) in the configuration space of electroweak theory in three-dimensions. Forgács and Horváth^[3] have shown that a large variety of models admit NCLs which can be taken to signal the possible existence of saddlepoint solutions. It was also suggested that the existence of non-contractible spheres of higher dimension may also be connected with solutions of the field equations. We consider two-dimensional electroweak theory and show that the electroweak Z -string^[4] and W -strings^[5–6] are solutions of this type: each lies on a non-contractible two-sphere (NCS) in configuration space.

2. The Electroweak Z -string and W -strings

We consider static bosonic configurations in electroweak theory: there is no time dependence and we work in the temporal gauge where the time components of the gauge fields are set to zero. The energy functional is given by

$$E = \int \left[\frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{4} f_{ij} f_{ij} + (D_i \Phi)^\dagger (D_i \Phi) + \lambda (\Phi^\dagger \Phi - \frac{1}{2} \eta^2)^2 \right] d^2 x \quad (2.1)$$

where

$$F_{ij}^a = \partial_i W_j^a - \partial_j W_i^a + g \epsilon^{abc} W_i^b W_j^c, \quad (2.2)$$

$$f_{ij} = \partial_i a_j - \partial_j a_i, \quad (2.3)$$

$$D_i \Phi = \partial_i \Phi - \frac{1}{2} i g \tau^a W_i^a \Phi - \frac{1}{2} i g' a_i \Phi. \quad (2.4)$$

The Weinberg angle is given by $\tan \Theta_w = g'/g$. The semiclassical masses of the W -boson, the Z -boson and Higgs boson are, respectively,

$$M_W = \frac{1}{2} g \eta, \quad M_Z = \frac{1}{2} \alpha \eta, \quad M_H = \sqrt{2\lambda} \eta \quad (2.5)$$

where $\alpha = \sqrt{g'^2 + g^2}$.

The field equations are

$$\begin{aligned} D_j F_{ij}^a &= -\frac{1}{2} i g (\Phi^\dagger \tau^a D_i \Phi - (D_i \Phi)^\dagger \tau^a \Phi), \\ \partial_j f_{ij} &= -\frac{1}{2} i g' (\Phi^\dagger D_i \Phi - (D_i \Phi)^\dagger \Phi), \\ D_i D_i \Phi &= 2\lambda (\Phi^\dagger \Phi - \frac{1}{2} \eta^2) \Phi \end{aligned} \quad (2.6)$$

3. The NCS Passing Through the Strings

We work in the polar gauge so that $W_r = 0$ and $a_r = 0$; we are left with a global $U(2)$ gauge freedom. For finite energy configurations, we may define

$$\Phi^\infty(\varphi) = \lim_{r \rightarrow \infty} \Phi(r, \varphi) \quad (3.1)$$

satisfying

$$|\Phi^\infty| = \frac{\eta}{\sqrt{2}} . \quad (3.2)$$

We can use the global gauge freedom to fix

$$\Phi^\infty(\varphi = 0) = \frac{\eta}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \quad (3.3)$$

There remains a global $U(1)$ freedom corresponding to the unbroken gauge group; however, this is irrelevant. The vacuum configuration is now completely fixed:

$$\Phi_{vac}(\mathbf{x}) = \frac{\eta}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \quad W_{vac}(\mathbf{x}) = 0 , \quad a_{vac}(\mathbf{x}) = 0 . \quad (3.4)$$

Equations (3.1) and (3.2) imply that we can regard Φ^∞ as a map

$$\Phi^\infty : S^1 \rightarrow S^3 . \quad (3.5)$$

We parameterise a two-sphere in configuration space by (μ, ν) with $\mu \in [0, \pi]$ and $\nu \in [0, \pi]$. The boundaries of the square, $\mu = 0$ or π and $\nu = 0$ or π , give a single point in configuration space. The asymptotic Higgs field of the configurations on the two-sphere then define a family of maps,

$$\Phi^\infty(\mu, \nu) : S^1 \rightarrow S^3 , \quad (3.6)$$

which vary continuously with μ and ν . We now wish to find a single map, Ψ , where

$$\Psi : S^3 \rightarrow S^3 , \quad (3.7)$$

which is topologically equivalent to the family of maps (3.6).

AD

February 1994

DAMTP-HEP-94-13

su 9409

The Electroweak Strings and Topology in Electroweak Theory

Margaret James

*Department of Applied Mathematics and Theoretical Physics
University of Cambridge
Silver Street
Cambridge CB3 9EW
U.K.*

ABSTRACT

We show that the electroweak Z -string and the W -strings lie on non-contractible two spheres in the configuration space of electroweak theory.

for fixed δ . $p(\mu, \nu, \varphi, \delta)$ depends continuously on its arguments and the definition is consistent with φ being a spatial azimuthal angle. Note that each point p on S^3 occurs for a unique triple (μ, ν, φ) except for $p = (0, 0, 1, 0)$ where

$$p(\mu = 0, \nu, \varphi) = p(\mu, \nu = 0, \varphi) = p(\mu, \nu, \varphi = 0) = (0, 0, 1, 0) . \quad (3.15)$$

We now have

$$S^3(\mu, \nu, \varphi) = S^1(\mu) \wedge_2 (S^1(\nu) \wedge_1 S^1(\varphi)) \quad (3.16)$$

where the fixed points of $\wedge_2$ are $\mu = 0$ on $S^1(\mu)$ and $\varphi = 0$ on $S^2(\nu, \varphi)$. Hence

$$S^3(\mu, \nu, \varphi) = S^2(\mu, \nu) \wedge_3 S^1(\varphi) \quad (3.17)$$

where the fixed points of $\wedge_3$ are $\mu = 0$ or $\nu = 0$ on $S^2(\mu, \nu)$ and $\varphi = 0$ on $S^1(\varphi)$.

If we write the Higgs field

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} , \quad (3.18)$$

then we can define the related four component real vector

$$\Phi_{Re} = \begin{pmatrix} Re\Phi_1 \\ Im\Phi_1 \\ Re\Phi_2 \\ Im\Phi_2 \end{pmatrix} . \quad (3.19)$$

Asymptotically $\frac{\sqrt{2}}{\eta} \Phi_{Re}$ must lie on the unit 3-sphere (the target of the map (3.7)).

Then a two-sphere of configurations whose asymptotic Higgs fields are given by

$$\Phi_{Re}^\infty(\mu, \nu, \varphi, \delta) = \frac{\eta}{\sqrt{2}} \Psi(p(\mu, \nu, \varphi, \delta)) \quad (3.20)$$

will be non-contractible if Ψ has non-zero degree.

3.A The NCS Passing Through the Z-String

We set $\delta = 0$ and choose

$$\Psi = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (3.21)$$

where we take $T^a = -\frac{1}{2}i\tau^a$ as our basis for $L(SU(2))$ (denoting the Lie algebra of $SU(2)$) with $[T^a, T^b] = \epsilon^{abc}T^c$.

In addition we recall the usual mixing formulae which give the physical electromagnetic field and Z -field in terms of the $U(1)$ field and the third component of the $SU(2)$ field:

$$Z_i \equiv \cos \Theta_w W_i^3 - \sin \Theta_w a_i , \quad A_i \equiv \sin \Theta_w W_i^3 + \cos \Theta_w a_i . \quad (2.7)$$

The Z -string solution is given by^[4]

$$\Phi = h(r)e^{im\varphi} \begin{pmatrix} 0 \\ 1 \end{pmatrix} ,$$

$$\vec{W}^1 = 0 = \vec{W}^2 = \vec{A} , \quad \vec{Z} = -\frac{Z(r)}{r}\hat{e}_\varphi \quad (2.8)$$

The integer m is the winding number of the vortex and here we shall restrict ourselves to the case $m = 1$. On substitution of (2.8) into the field equations (2.6), they reduce to coupled ordinary differential equations:

$$h'' + \frac{h'}{r} - \left(1 - \frac{\alpha}{2}Z\right)^2 \frac{h}{r^2} - 2\lambda \left(h^2 - \frac{\eta^2}{2}\right) h = 0 ,$$

$$Z'' - \frac{Z'}{r} + \alpha \left(1 - \frac{\alpha}{2}Z\right) h^2 = 0 , \quad (2.9)$$

where primes denote differentiation with respect to r . These must be solved together with the boundary conditions $h(0) = 0 = Z(0)$, $h(\infty) = \eta/\sqrt{2}$, $Z(\infty) = 2/\alpha$.

The W -string solutions are a one-parameter family of solutions given by

$$\Phi = h \begin{pmatrix} e^{i\delta} \sin \varphi \\ \cos \varphi \end{pmatrix}$$

$$\vec{W}^1 = \sin \delta \frac{Z}{r} \hat{e}_\varphi , \quad \vec{W}^2 = \cos \delta \frac{Z}{r} \hat{e}_\varphi , \quad \vec{W}^3 = 0 = \vec{a} \quad (2.10)$$

where δ is a constant in the range $\delta \in [0, 2\pi]$ and h and Z are given by (2.9), where we replace α with g . Solutions with different δ are not actually distinct: they are related by a gauge transformation.

If we now set

$$f_3(r) = g^2 \frac{Z(r)}{2\alpha} \quad f_0(r) = g'^2 \frac{Z(r)}{2\alpha}, \quad (3.28)$$

where h and Z satisfy the field equations (2.9), then we see that configuration (3.27) is the electroweak Z -string and by our previous remarks, it lies on a non-contractible two-sphere.

3.B The NCS Passing Through the W-String

We now take Ψ in (3.20) to be the identity map (again of degree one). We now define U_δ^∞ via

$$U_\delta^\infty = \frac{\sqrt{2}}{\eta} \begin{pmatrix} \Phi_{\delta 2}^{\infty*} & \Phi_{\delta 1}^\infty \\ -\Phi_{\delta 1}^{\infty*} & \Phi_{\delta 2}^\infty \end{pmatrix}, \quad (3.29)$$

where Φ_δ^∞ is the corresponding complex Higgs isodoublet. Then the following configurations

$$\Phi_\delta = h(r) U_\delta^\infty \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$W_{\delta\varphi} = -\frac{Z(r)}{2} \partial_\varphi U_\delta^\infty (U_\delta^\infty)^{-1} \quad (3.30)$$

are defined over all space, have finite energy and satisfy the correct asymptotic boundary conditions provided that as $r \rightarrow 0$, $h \rightarrow 0$, $Z \rightarrow 0$ and as $r \rightarrow \infty$, $h \rightarrow \frac{\eta}{\sqrt{2}}$ and $Z \rightarrow 2/g$. The configuration at $\mu = \nu = \pi/2$ is the W -string configuration (2.10).

4. Discussion

We have shown that the Z -string, in both the one and two Higgs doublet models, is connected to the existence of a non-contractible two-sphere in configuration space.

We have also shown that the one-parameter family of W -string solutions is connected to a one-parameter family of non-contractible two-spheres. The energy functional on each two-sphere has a unique global minimum. The minimum is non-zero; the two-sphere does not pass through the trivial vacuum, although it could be deformed to do so.

The sphaleron lies on a NCL and has one unstable mode (for $M_H \leq 12M_W$). The Z -string and W -strings, solutions on NCSs, have (at least) two unstable modes for the physical value of the parameters $\beta = (M_H/M_Z)^2$ and Θ_w [7].

which has degree one. Note that this is compatible with our gauge choice since $\Phi_{Re}^\infty = \frac{\eta}{\sqrt{2}}(0, 0, 1, 0)$ for all (μ, ν, φ) such that $p(\mu, \nu, \varphi) = (0, 0, 1, 0)$. With our choice of map Ψ , the complex version of the Higgs isodoublet is

$$\Phi^\infty = \frac{\eta}{\sqrt{2}} \begin{pmatrix} \sin \mu \sin \nu (\cos \varphi - 1) (\cos \mu \sin \nu - i \cos \nu) \\ \sin^2 \mu \sin^2 \nu (\cos \varphi - 1) + 1 + i \sin \mu \sin \nu \sin \varphi \end{pmatrix} \quad (3.22)$$

We now define

$$F_\varphi = F_\varphi^a \frac{\tau^a}{2i} = (U^\infty)^{-1} \partial_\varphi U^\infty \quad (3.23)$$

where

$$U^\infty = \frac{\sqrt{2}}{\eta} \begin{pmatrix} \Phi_2^{\infty*} & \Phi_1^\infty \\ -\Phi_1^{\infty*} & \Phi_2^\infty \end{pmatrix}. \quad (3.24)$$

Then the following field configurations

$$\Phi = h(r) U^\infty \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$gW_\varphi = -U^\infty \left(\left(F_\varphi^1 \frac{\tau^1}{2i} + F_\varphi^2 \frac{\tau^2}{2i} \right) f_1(r) + F_\varphi^3 \frac{\tau^3}{2i} f_3(r) \right) (U^\infty)^{-1}, \quad g'a_\varphi = f_0(r) F_\varphi^3, \\ gW_r = g'a_r = 0 \quad (3.25)$$

are defined over all space, have finite energy and satisfy the correct asymptotic behaviour provided that the fields obey the boundary conditions

$$\lim_{r \rightarrow 0} h(r) = f_1(r) = f_3(r) = f_0(r) = 0, \\ \lim_{r \rightarrow \infty} h(r) = \frac{\eta}{\sqrt{2}}, \quad f_1(r) = 1, \quad f_0(r) = 1 - c, \quad f_3(r) = c \quad (3.26)$$

where c is a constant. Examining the configuration at $\mu = \pi/2$, $\nu = \pi/2$ we find

$$\Phi = h(r) \begin{pmatrix} 0 \\ e^{i\varphi} \end{pmatrix} \\ g\vec{W}^3 = -2\frac{f_3}{r} \hat{e}_\varphi, \quad \vec{W}^1 = \vec{W}^2 = 0, \quad g'\vec{a} = 2\frac{f_0}{r} \hat{e}_\varphi. \quad (3.27)$$

Figure 1.