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OBSERV C 126-1994
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PUBLICATIONS DE L'OBSERVATOIRE DE GENÈVE

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Review paper to be published in Ergodic concepts in stellar dynamics, ed. by V.G. Gurzadyan & D. Pfenniger, Springer, 1994.

Série C :

PRÉ - PUBLICATIONS

Fascicule 126 (1994)

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ignores the conservation of the local angular momentum, because this quantity is automatically conserved in the fluid differential equation model where the “elements” are infinitesimally small. But with a finite resolution no longer true, small eddies disappear within grid cells; a finite resolution acts typically as a sink of angular momentum. This can perturb significantly simulations of dissipative and turbulent rotating discs; for example the centre of a disc galaxy is then a sink of angular momentum.

Since we don’t have a proper paradigm for describing flows such as the cold ISM, we must invent new ones that reproduce an efficient mixing at small scale while conserving the global integrals at larger scales. Several attempts have been made (e.g. Schwarz 1981; Combes & Gerin 1985; Jenkins 1992; Palous et al. 1993; Binney & Gerhard 1993) along the line of describing the ISM in a Lagrangian way by a set of colliding particles with substantial mean-free paths, as in planetary rings (Brahic 1977). In these works, as well as with the “Smoothed Particle Hydrodynamics” (SPH) approach (Friedli & Benz 1992) collisions are always binary collisions conserving at least mass and momentum. This is computationally inefficient since the algorithms need to detect the nearest neighbours and to treat individual collisions. Furthermore some of these codes (Schwarz 1981, Palous et al. 1993) do not conserve angular momentum, which is physically hard to justify. In some way, these particle approaches for gas correspond to the stellar dynamical limit in the limit of zero dissipation. In the stellar dynamical case one has a well defined scale separation between the ISM gas and the gas at the stellar scale, and the interaction between the two systems can be neglected for some times.

In this work we consider whether general dissipative transformations on a set of N point mass particles can mimic complex and highly turbulent dissipative systems in a more efficient way than the previous methods. Fast dissipative schemes corresponding to the simultaneous “collisions” of N -particle that conserve exactly almost all the integrals, including angular momentum but not the energy (otherwise the flow would be conservative), are presented.

2. Mixing Transformations with Conservation Laws

We consider a small region of a physical system for a time sufficiently long during which “mixing” physical processes occur rapidly at small scale, but short enough to be able to neglect the changes due to long range interactions which are supposed to occur at scales larger than the considered region. During this time, the global integrals except energy are assumed to be strictly conserved in the subsystem, because the subsystem can be considered for a short time (“infinitesimally short” for the larger scale) as isolated. Therefore we consider the partial problem of describing consistently the global “collisions” of N isolated point mass particles.

Mixing Transformations of N Particles Conserving Almost All Classical Integrals

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1. Paradigm for Inhomogeneous Mixing Flows

The Navier-Stokes equation is often used as paradigm for modelling the gas in galaxies. Yet the interstellar medium (ISM) is far from being a smooth common flow. In fact the cold phase of the ISM, containing most of the gas mass, is *essentially* an inhomogeneous medium better described with a fractal model, and where the density can vary by 10 decades over distances much shorter than < 1 pc (Pfenniger & Combes 1994). Typically the “mean-free path” of clumps at the $10 - 50$ pc scale, such as molecular clouds, is much larger than their size, and an hydrodynamical description with particles of the size of molecular clouds can be, and has been, tried. However molecular clouds collide and dissolve because they contain internal degrees of freedom due to the smaller clumps moving inside them. For a fractal gas there is no strong reason to choose a particular scale. The resolution required to follow such an inhomogeneous flow at the 100 pc scale or larger makes the hydrodynamical approach hardly tractable with today's computers.

So for any practical smallest scale one *can* choose (not to be confused with a very small scale that might be retained in the future with very powerful computers), the differentiability of the flow required by the Navier-Stokes equation for modelling the cold ISM cannot be granted. Although the differential fluid equations are in principle inapplicable, those have nevertheless the main virtue to conserve locally some of the classical integrals at the smallest accessible scale. If the physical flow is highly turbulent and fractal at small scale it is illusory to believe that the simulated flow reproduces faithfully the detail of the physical flow. But if the physical flow mixes rapidly a part of its microscopic properties, as in statistical mechanics only the global integrals are conserved at larger scale and at least these quantities are relevant in simulations.

However the finite element Eulerian approach for describing fluids, either by grid schemes (van Albada 1985; Mulder & Liem 1986; Athanassoula 1991) or by beam schemes (e.g. Sanders & Prendergast 1974; van Albada et al. 1982)

2.3 Mass Conservation

For correcting the mass, a simple assumption is to apply a uniform linear mass correction over all the particles:

$$(4) \quad \begin{pmatrix} m_i'' \\ x_i'' \\ v_i'' \end{pmatrix} = T_1 \begin{pmatrix} A m_i' + B \\ x_i' \\ v_i' \end{pmatrix},$$

where A and B are constants. Then in order to conserve mass, A and B should satisfy

$$(5) \quad M'' \equiv \sum_i m_i'' = A \left(\sum_i m_i' \right) + B N = A M' + B N = M.$$

Clearly there are many possibilities. But in order to guarantee that this correction yields positive masses, we require $A > 0$ and $B > 0$.

For example the pair

$$(6) \quad A = \alpha \frac{M'}{M}, \quad B = (1 - \alpha) \frac{M'}{N},$$

is a possible solution when $0 \leq \alpha \leq 1$ which averages the masses when α is sufficiently small. The most mixing pair is $A = 0$, $B = M/N$ ($\alpha = 0$), and the less mixing pair is $A = M/M'$, $B = 0$ ($\alpha = 1$).
The main point of this subsection is to show that it is always possible to set up transformations conserving the total mass and positive masses.

2.4 Center of Mass and Center of Velocity Conservation

In order to conserve also the center of mass and the center of velocity, we have in general to correct T_1 . A correcting transformation is tried in the form of a constant transformation T_2 :

$$(7) \quad \begin{pmatrix} m_i'' \\ x_i'' \\ v_i'' \end{pmatrix} = T_2 \begin{pmatrix} m_i' \\ x_i' + C \\ v_i' + D \end{pmatrix},$$

where C and D are constant vectors. In order to restore the original centers of mass and velocity, we have, with $M'' \equiv \sum_i m_i'' = M$,

$$(8) \quad \begin{pmatrix} X'' \\ V'' \end{pmatrix} \equiv \sum_i \frac{m_i''}{M''} \begin{pmatrix} x_i' \\ v_i' \end{pmatrix} = \sum_i \frac{m_i''}{M''} \begin{pmatrix} x_i' \\ v_i' \end{pmatrix} + \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} X \\ V \end{pmatrix},$$

Thus we have $X'' = X$ and $V'' = V$ when

$$(9) \quad C = X - X'', \quad \text{and} \quad D = V - V''.$$

We have shown here that it is always possible to correct arbitrary transformations in order to conserve also the centers of mass and velocities.

2.1 Conservation Laws

The N particles are described in an inertial frame by their positive mass m_i , their positions $\mathbf{x}_i$, and their velocities $\mathbf{v}_i$, $i = 1 \dots N$. We require that the considered dissipative process conserves the total mass M , center of velocity $\mathbf{V}$ and center of mass $\mathbf{X}$

$$M \equiv \sum_i m_i, \quad M\mathbf{V} \equiv \sum_i m_i \mathbf{v}_i, \quad M\mathbf{X} \equiv \left(\sum_i m_i \mathbf{x}_i \right) - M\mathbf{V}t, \quad (1)$$

and the total angular momentum about its center of mass,

$$\mathbf{L} \equiv \sum_i m_i (\mathbf{x}_i - \mathbf{X}) \wedge \mathbf{v}_i, \quad (2)$$

i.e. the integrals not involving a particular form of interaction. On the other hand the total energy is not supposed to be conserved. This is motivated by the fact that in many physical cases energy is the less well conserved quantity owing to the numerous possibilities of exchanging rapidly energy by radiative processes. In contrast, momentum and angular momentum can practically only be dissipated by mass exchanges (see, e.g., Shu 1992).

2.2 Arbitrary Transformations of Coordinates

First suppose that we apply an *arbitrary* transformation T_0 to the particles coordinates at a fixed time t . We also suppose that the particle masses can be transformed, but the new masses must remain positive. Since the time is not used, we can set $t = 0$ in the following. This yields new masses ($m'_i > 0$), positions ($\mathbf{x}'_i$), and velocities ($\mathbf{v}'_i$)

$$\begin{pmatrix} m'_i \\ \mathbf{x}'_i \\ \mathbf{v}'_i \end{pmatrix} = T_0 \begin{pmatrix} m_i \\ \mathbf{x}_i \\ \mathbf{v}_i \end{pmatrix}. \quad (3)$$

In general, T_0 does not conserve the global integrals, yielding new integrals M' , $\mathbf{X}'$, $\mathbf{V}'$ and $\mathbf{L}'$. We suppose that the actual physical transformation is *mixing*, that is, except for the global invariants, the memory of the precise initial conditions are lost. So we should “correct” T_0 in order to restore a process that conserves the global integrals, except the energy. The corrections should be as uniform as possible for all particles, and should not use the detailed information about the old coordinates (the mixing process forgets the precise values of the old coordinates, but not the global integrals except the energy).

$$(13) \quad \begin{aligned} I_{xx} &= \sum_i m_i (x_i - X)^2, & I_{xy} &= \sum_i m_i (x_i - X)(y_i - Y) = I_{yx}, \\ I_{yy} &= \sum_i m_i (y_i - Y)^2, & I_{yz} &= \sum_i m_i (y_i - Y)(z_i - Z) = I_{zy}, \\ I_{zz} &= \sum_i m_i (z_i - Z)^2, & I_{zx} &= \sum_i m_i (z_i - Z)(x_i - X) = I_{xz}. \end{aligned}$$

After some manipulations, we get

$$(14) \quad S\mathbf{U} = \mathbf{L}'''' - \mathbf{L}''' = \mathbf{L} - \mathbf{L}''',$$

where S is the symmetric matrix evaluated for $I(m_i'', \mathbf{x}_i''', \mathbf{X})$,

$$(15) \quad S = \begin{pmatrix} I_{yy} + I_{zz} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{zz} + I_{xx} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{xx} + I_{yy} \end{pmatrix}.$$

If S is invertible then $\mathbf{U} = S^{-1}(\mathbf{L} - \mathbf{L}''')$. In solid body dynamics $-S$ is also called “inertia tensor” (Golstein 1950), not to be confused with I .

2.5.2 Matrix Inverse

Explicitly, if we write

$$(16) \quad S_x \equiv I_{yy} + I_{zz}, \quad S_y \equiv I_{zz} + I_{xx}, \quad S_z \equiv I_{xx} + I_{yy},$$

then the determinant $|S|$ reads

$$(17) \quad |S| = S_x S_y S_z - 2I_{xy} I_{yz} I_{zx} - S_x I_z^2 - S_y I_x^2 - S_z I_y^2,$$

and the inverse matrix S^{-1} has the form

$$(18) \quad S^{-1} = \frac{1}{|S|} \begin{pmatrix} S_y S_z - I_z^2 & I_{yz} I_{zx} + I_{xz} S_y & I_{xy} I_{yz} + I_{yz} S_y \\ I_{yz} I_{zx} + I_{xz} S_y & S_z S_x - I_x^2 & I_{zx} I_{xy} + I_{xy} S_x \\ I_{xy} I_{yz} + I_{yz} S_y & I_{zx} I_{xy} + I_{xy} S_x & S_x S_y - I_y^2 \end{pmatrix}.$$

2.5.3 Degenerate Cases

If S is not invertible, because $|S| = 0$, we have an infinite number of possible solutions to Eq. (14). In such cases, as a proxy of S^{-1} it is recommended to take the Moore-Penrose pseudo-inverse $S^\dagger$ (Moore 1920; Penrose 1955), yielding the unique solution with optimal properties in the least-squares sense.

The pseudo-inverse $A^\dagger$ of any matrix A is the unique matrix that satisfies $A^\dagger A A^\dagger = A^\dagger$, $A A^\dagger A = A$, and both $A^\dagger A$ and $A A^\dagger$ are symmetric. The derivation of the pseudo-inverse can be found symbolically (most easily by computer algebra) with the relation

$$(19) \quad A^\dagger = \lim_{\varepsilon \rightarrow 0} [(A^T A + \varepsilon^2 \text{Id})^{-1} A^T],$$

where A^T is the transpose of A and Id is the identity matrix (Frawley 1985). By a rotation and a translation the moment of inertia tensor I can be made diagonal, and then all the crossed moments I_{xy} , I_{yz} , and I_{zx} vanish.

2.5 Angular Momentum Conservation

The transformations T_1 and T_2 provide a new total angular momentum $\mathbf{L}'''$ that is generally different from the original one $\mathbf{L}$. Since constant corrections are insufficient to conserve angular momentum, we adopt the next simplest hypothesis; we suppose a transformation of the velocities only, and having the same form for all particles except for a linear dependence on the own particle position with respect to the center of mass. Although it could be possible to set up more complicated correcting transformations involving both the positions and the velocities, we don't correct the positions on the ground of simplicity. In general, position transformations produce also potential energy changes, and forces modify velocities to first order, and positions to second order.

In order to modify the total angular momentum without changing the center of velocity, the simplest velocity correction must be orthogonal to and linear with the radius vector to the center of mass. Therefore we require

$$\begin{pmatrix} m_i''' \\ \mathbf{x}_i''' \\ \mathbf{v}_i''' \end{pmatrix} = T_3 \begin{pmatrix} m_i''' \\ \mathbf{x}_i''' \\ \mathbf{v}_i''' \end{pmatrix} = \begin{pmatrix} m_i''' \\ \mathbf{x}_i''' \\ \mathbf{v}_i''' + \boldsymbol{\Omega} \wedge (\mathbf{x}_i''' - \mathbf{X}) \end{pmatrix}, \quad (10)$$

where $\boldsymbol{\Omega}$ is a constant vector to specify. This transformation is a solid rotation in velocity space that obviously conserves the mass and the center of mass, but it conserves also the center of velocity, since

$$\begin{aligned} M'''' \mathbf{V}'''' &\equiv \sum_i m_i'''' \mathbf{v}_i'''' = \sum_i m_i''' \mathbf{v}_i''' + \boldsymbol{\Omega} \wedge \underbrace{\left(\sum_i m_i''' \mathbf{x}_i''' - M \mathbf{X} \right)}_{=0 \text{ by Eqs. (8) and (9)}} \\ &= \sum_i m_i''' \mathbf{v}_i''' = M \mathbf{V}. \end{aligned} \quad (11)$$

By using the identity $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{a}) = \mathbf{a}^2 \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{a}$, then the new angular momentum is

$$\begin{aligned} \mathbf{L}'''' &\equiv \sum_i m_i''' (\mathbf{x}_i''' - \mathbf{X}) \wedge \mathbf{v}_i''' \\ &= \sum_i m_i''' (\mathbf{x}_i''' - \mathbf{X}) \wedge \mathbf{v}_i''' + \sum_i m_i''' (\mathbf{x}_i''' - \mathbf{X}) \wedge [\boldsymbol{\Omega} \wedge (\mathbf{x}_i''' - \mathbf{X})] \\ &= \mathbf{L}''' + \sum_i m_i''' (\mathbf{x}_i''' - \mathbf{X})^2 \boldsymbol{\Omega} - \sum_i m_i''' [(\mathbf{x}_i''' - \mathbf{X}) \cdot \boldsymbol{\Omega}] (\mathbf{x}_i''' - \mathbf{X}). \end{aligned} \quad (12)$$

If we require that the new angular momentum $\mathbf{L}''''$ equals the original one $\mathbf{L}$ we can solve this linear equation for $\boldsymbol{\Omega}$ in term of the coordinates $\mathbf{x}_i'''$ only. This is a familiar equation of solid body dynamics (see e.g. Goldstein 1950).

2.5.1 Moment of Inertia Tensor

First we define the moment of inertia tensor $I(m_i, \mathbf{x}_i, \mathbf{X})$ about the center of mass $\mathbf{X}$, where $\mathbf{x}_i = (x_i, y_i, z_i)$, $\mathbf{X} = (X, Y, Z)$,

where α is a constant. This transformation conserves the total mass and the centers of mass and velocities, therefore T_1 and T_2 are identity transformations. But this friction law does not conserve in general angular momentum. Indeed the angular momentum difference is then

$$L''' - L = L'' - L = -\alpha \frac{M}{m_1 m_2} (\mathbf{x}_1 - \mathbf{x}_2) \wedge (\mathbf{v}_1 - \mathbf{v}_2) .$$

Calculating explicitly the transformation T_3 by using the pseudo-inverse S^t in Eq. (23), we get, after algebraic manipulations,

$$T_3 \begin{pmatrix} m_i \\ \mathbf{x}_i \\ \mathbf{v}_i \end{pmatrix} = \begin{pmatrix} m_i \\ \mathbf{x}_i \\ \mathbf{v}_i - \alpha \frac{M}{m_j} [\mathbf{d}_{ij} \cdot (\mathbf{v}_i - \mathbf{v}_j)] \mathbf{d}_{ij} \end{pmatrix} , \quad i = 1, 2, \quad (25)$$

where $j = 3 - i$ ($j = 2$ if $i = 1$ and vice versa), and

$$\mathbf{d}_{ij} = \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|} . \quad (26)$$

This is precisely the friction law used by Brabic (1977). Only the relative velocity component parallel to the interparticle vector is reduced. This transformation conserves all the integrals but the energy. Nothing

$$C_1 \equiv \frac{m_1 m_2}{2M} [\mathbf{d}_{12} \cdot (\mathbf{v}_1 - \mathbf{v}_2)]^2 \geq 0 , \quad (27)$$

the final kinetic energy difference is then

$$\Delta E \equiv \frac{1}{2} [m_1 (v_1''^2 - v_1^2) + m_2 (v_2''^2 - v_2^2)] = -\alpha (2 - \alpha) C_1 . \quad (28)$$

So if $0 < \alpha < 2$ the kinetic energy change is negative. The kinetic energy change vanishes when either the vectors of position or velocity difference, $\mathbf{x}_1 - \mathbf{x}_2$ or $\mathbf{v}_1 - \mathbf{v}_2$, are zero, or when these vectors are orthogonal.

In order to find α for a specified energy change ΔE , we solve Eq. (28) for α , and find,

$$\alpha = 1 \pm \sqrt{1 + \frac{\Delta E}{C_1}} . \quad (29)$$

This provides the two possible transformations

$$\mathbf{v}_i''' = \mathbf{v}_i - \left\{ 1 \pm \sqrt{1 + \frac{\Delta E}{C_1}} \right\} \frac{M}{m_j} [\mathbf{d}_{12} \cdot (\mathbf{v}_1 - \mathbf{v}_2)] \mathbf{d}_{ij} . \quad (30)$$

The $(-)$ -solution is the one which is continuous with no change in velocity when $\Delta E = 0$. The $(+)$ -solution reverses the sense of the relative velocity. When the energy change is sufficiently small, $|\Delta E| \ll C_1$, by expanding the square root the $(-)$ -solution is approximately

In that case S is also diagonal and is not invertible only in the case where at least two of the three non-negative tensor components I_{xx} , I_{yy} and I_{zz} vanish (see Eq. (15)). This means that all the points are colinear, i.e. have the form $\mathbf{x}_i = \mathbf{a}t_i + \mathbf{b}$, where $\mathbf{a} \equiv \{a_x, a_y, a_z\}$ and $\mathbf{b}$ are constant vectors.

In this case, S takes the form

$$S = \frac{K}{|\mathbf{a}|^2} \begin{pmatrix} a_y^2 + a_z^2 & -a_x a_y & -a_z a_x \\ -a_x a_y & a_z^2 + a_x^2 & -a_y a_z \\ -a_z a_x & -a_y a_z & a_x^2 + a_z^2 \end{pmatrix} \quad (20)$$

where

$$K \equiv I_{xx} + I_{yy} + I_{zz} = \frac{1}{M} \sum_i \sum_{j>i} m_i m_j (t_i - t_j)^2. \quad (21)$$

If $K > 0$ then the use of Eq. (19) leads to the pseudo-inverse $S^\dagger$

$$S^\dagger = K^{-2} S. \quad (22)$$

If $K = 0$ then all the points are concomitant and $S^\dagger = 0$.

2.5.4 Case $N = 2$

In this particular case S is always not invertible since two points are always colinear. Explicitly the pseudo-inverse $S^\dagger$ is

$$S^\dagger = \frac{m_1 + m_2}{m_1 m_2} \frac{1}{(\mathbf{x}_1 - \mathbf{x}_2)^2} \times \begin{pmatrix} (y_1 - y_2)^2 + (z_1 - z_2)^2 & -(x_1 - x_2)(y_1 - y_2) & -(z_1 - z_2)(x_1 - x_2) \\ -(x_1 - x_2)(y_1 - y_2) & (z_1 - z_2)^2 + (x_1 - x_2)^2 & -(y_1 - y_2)(z_1 - z_2) \\ -(z_1 - z_2)(x_1 - x_2) & -(y_1 - y_2)(z_1 - z_2) & (x_1 - x_2)^2 + (y_1 - y_2)^2 \end{pmatrix}. \quad (23)$$

Of course, if $\mathbf{x}_1 = \mathbf{x}_2$, we have further degeneracy, and then $S^\dagger = 0$.

3. Examples: Linear Friction Laws

3.1 Binary Interactions

Often dissipative particle schemes (Brahic 1977, Schwarz 1981) use binary interactions. In this case one can consider successively pairs of particles, so we need to consider a dissipative law for $N = 2$. As a first guess a linear friction law with respect to the center of velocity is attempted

$$T_0 \begin{pmatrix} m_i \\ \mathbf{x}_i \\ \mathbf{v}_i \end{pmatrix} = \begin{pmatrix} m_i \\ \mathbf{x}_i \\ \mathbf{v}_i - \alpha(\mathbf{v}_i - \mathbf{V}) \end{pmatrix}, \quad i = 1, 2, \quad (24)$$

$$(39) \quad \begin{aligned} C_1 &\equiv \sum_{i=1}^{\frac{1}{2}} m_i v_i \cdot (v_i - V + S^{-1} L \wedge (x_i - X)) , \\ C_2 &\equiv \sum_{i=1}^{\frac{1}{2}} m_i (v_i - V + S^{-1} L \wedge (x_i - X))^2 \geq 0 . \end{aligned}$$

It is possible to show that $C_1 = C_2$ (so $C_1 \geq 0$) by expanding C_1 and C_2 and using the vector identities $\mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \wedge \mathbf{a}$ and $(\mathbf{a} \wedge \mathbf{b})^2 = \mathbf{a}^2 \mathbf{b}^2 - (\mathbf{a} \cdot \mathbf{b})^2$. Solving α as a function of ΔE , we obtain

$$(40) \quad \alpha = 1 \pm \sqrt{1 + \frac{\Delta E}{C_1}} ,$$

where the $(-)$ -solution is the one continuous with the identity at small dissipation ($\alpha \rightarrow 0$ when $\Delta E \rightarrow 0$).

The lowest energy change is negative and bounded by

$$(41) \quad \Delta E \geq -C_1 .$$

For the $(-)$ -solution, when dissipation is weak ($|\Delta E| \ll C_1$), we have small energy changes and we get

$$(42) \quad \alpha \approx -\frac{1}{2} \Delta E / C_1 .$$

For large positive energy changes ($\Delta E \gg C_1$) we get

$$(43) \quad \alpha \approx -\sqrt{\Delta E / C_1} .$$

Finally when $\alpha = 2$ we have the particular case of strong mixing and yet energy conservation, as in isothermal processes.

4. Conclusions

We have shown how to treat consistently a set of rapidly mixing particles. It is possible to devise various dissipative transformations and then correct them to conserve the mass, centers of mass and velocity, and especially the total angular momentum. Furthermore, for simple linear friction laws it is possible to specify in advance the energy dissipation. In that case the conservation of the other integrals constrains the energy loss not to exceed a maximum value.

Among all the possible corrective transformations that we have described above, only a small part may be applicable in actual physical systems. In real systems further constraints restrict the possible mixing and thus the dissipation rate, e.g., when a flow expands its internal viscosity is much smaller than when it contracts. Only studies with specific problems will allow us to choose the "right" way to model dissipation and mixing. At least the formulation here is general and consistent with the most fundamental constraints that

$$\mathbf{v}_i''' \approx \mathbf{v}_i + \frac{\Delta E}{m_i \mathbf{d}_{12} \cdot (\mathbf{v}_1 - \mathbf{v}_2)} \mathbf{d}_{ij} . \quad (31)$$

When the energy change is large and positive, $\Delta E \gg C_1$, the approximate solution reads

$$\mathbf{v}_i''' \approx \mathbf{v}_i \mp \sqrt{\frac{2\Delta E}{M} \frac{m_j}{m_i}} \mathbf{d}_{ij} . \quad (32)$$

The minimum energy change compatible with this type of dissipation is given by

$$\Delta E \geq -C_1 . \quad (33)$$

3.2 N Interacting Particles

Instead of considering particle interactions by pairs, it is much more efficient to group the interactions with larger N . This is justified if the mixing process has time to mix the properties of the N particles during a finite time-step. If a linear friction law is used for an arbitrary number N of particles, we would have

$$\begin{pmatrix} m_i'' \\ \mathbf{x}_i'' \\ \mathbf{v}_i'' \end{pmatrix} = T_0 \begin{pmatrix} m_i \\ \mathbf{x}_i \\ \mathbf{v}_i \end{pmatrix} = \begin{pmatrix} m_i \\ \mathbf{x}_i \\ \mathbf{v}_i - \alpha \mathbf{v}_i \end{pmatrix} , \quad (34)$$

where α is a constant. Then T_1 is an identity transformation and

$$\begin{pmatrix} m_i''' \\ \mathbf{x}_i''' \\ \mathbf{v}_i''' \end{pmatrix} = T_2 \begin{pmatrix} m_i \\ \mathbf{x}_i \\ \mathbf{v}_i \end{pmatrix} = \begin{pmatrix} m_i \\ \mathbf{x}_i \\ \mathbf{v}_i - \alpha(\mathbf{v}_i - \mathbf{V}) \end{pmatrix} . \quad (35)$$

The transformation T_3 is too complicated to give in closed form, but it is straightforward to evaluate numerically Eqs. (13), (15), and (18) efficiently with a computer, since the algorithm is proportional to N . Also in this linear case we do not need to evaluate $\mathbf{L}'''$ since one obtains,

$$S\Omega = \mathbf{L} - \mathbf{L}''' = \alpha \mathbf{L} , \quad \Rightarrow \quad \Omega = \alpha S^{-1} \mathbf{L} , \quad (36)$$

where S^{-1} has to be replaced by $S^\dagger$ in case of degeneracy. Finally,

$$\mathbf{v}_i''' = \mathbf{v}_i - \alpha (\mathbf{v}_i - \mathbf{V} + S^{-1} \mathbf{L} \wedge (\mathbf{x}_i - \mathbf{X})) . \quad (37)$$

The kinetic energy difference ΔE is given by

$$\Delta E = \frac{1}{2} \sum_i m_i (\mathbf{v}_i'''^2 - \mathbf{v}_i^2) = -2\alpha C_1 + \alpha^2 C_2 , \quad (38)$$

where