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Strong Infrared Effects in Quantum Gravity

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ABSTRACT: We explore "quantum cosmological gravity," or quantum general relativity with a nonzero cosmological constant. It is explained why and how QCG can be used reliably in the far infrared, despite the absence of renormalizability. We show that loop corrections to positive Λ QCG mediate powerful infrared effects for two reasons: first, the theory allows massless gravitons to self-interact via a coupling of dimension three; second, the inflationary redshift of the classical background progressively increases the population of soft gravitons. One consequence is that QCG must eventually dominate the physics of inflation with respect to any phenomenologically confirmed theory, no matter how much stronger the nongravitational couplings may seem. Another consequence is that the graviton's on-shell self-energy is negative and infrared divergent at one loop, thereby inducing a negative infrared divergence in the two-loop vacuum energy. We analyze these effects in the context of an initial patch of one Hubble volume which begins inflation at finite times in one of the homogeneous and isotropic Fock states of free QCG. Up to some tedious but probably manageable tensor algebra we show that quantum infrared effects exert an ever increasing drag on the background's expansion for as long as perturbation theory remains valid. A rough estimate of the relaxation time is easily consistent with enough inflation to solve the smoothness problem.

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1. Introduction

We refer the reader to the excellent review by Weinberg [1] for a survey of the voluminous literature on the problem of the cosmological constant. With few exceptions the search for a solution typically involves treating gravitation classically and exploiting the dynamics of some cleverly chosen matter theory.* The obvious difficulty is that any suppression mechanism must operate on very large distance scales where fundamental theory is tightly constrained by observation. Invoking new massless particles or new low dimension couplings is no good unless one can also explain why neither of those features has been otherwise detected. For this reason we shall consider the possibility that infrared processes in quantum gravity cause the effective cosmological constant to relax to zero, irrespective of the matter theory.

The problem of the cosmological constant is so notoriously enmeshed in paradox that any proposal for its resolution calls forth a plethora of questions, many of which are actually thinly veiled objections. In our case the most obvious challenges are: What do we mean by the “effective cosmological constant?” Is our definition invariant? How can a parameter in the Lagrangian change with time? How can quantum gravity possess cosmological implications at all? How can any such implications dominate those of conventional matter theories with much stronger couplings? What quantum gravitational process mediates relaxation? Why is this process powerful enough to effect complete relaxation? How can it nonetheless permit enough inflation to solve the smoothness problem? What is the evolutionary context in which we envisage relaxation? and, How can we invoke quantum gravity at all in view of the theory’s well known nonrenormalizability [5]? This introduction briefly addresses these issues, along with the essential role of infrared divergences.

* The exceptions are worth noting. Hawking [2] and Coleman [3] have proposed mechanisms involving topology change in the context of Euclidean quantum gravity. Antoniadis and Mottola [4] have proposed a mechanism in which conformal matter anomalies cause the conformal part of the metric to become dynamical.

We infer the effective cosmological constant from the expansion of spacetime the same way that observational astronomers do [6]. In the simplest case the background metric, $\hat{g}_{\mu\nu}$, is homogeneous and isotropic, and the manifold admits flat space. Under these conditions one can choose a co-moving coordinate system in which the invariant line element has the following form:

$$ds^2 \equiv \hat{g}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + R^2(t) \|d\vec{x}\|^2 \quad (1.1)$$

In these coordinates we define the effective Hubble constant:

$$H_{\text{eff}}(t) \equiv \frac{\dot{R}(t)}{R(t)} \quad (1.2)$$

In 3 + 1 dimensions the associated effective cosmological constant is:

$$\Lambda_{\text{eff}}(t) \equiv 3H_{\text{eff}}^2(t) - 8\pi G\rho(t) \quad (1.3)$$

where $\rho(t)$ is the matter or thermal energy density. For pure QCG at zero temperature $\rho(t)$ is zero. Note that $H_{\text{eff}}(t)$ and $\Lambda_{\text{eff}}(t)$ are not less invariant for having been given in co-moving coordinates. One can always make a quantity invariant by defining it in a special coordinate system, and in this case the coordinates are unique up to constant spatial rescalings that drop out in the ratio (1.2).

Our proposal is that quantum gravitational effects distort the background geometry away from the classical solution — $R_{\text{class}}(t) = \exp(Ht)$, with $H^2 \equiv \frac{1}{3}\Lambda$. This is not in conflict with the fact that the parameter Λ in the bare Lagrangian is a constant. It does require the breaking of de Sitter invariance but we shall argue later that this happens naturally through infrared divergences. Note that relaxation is consistent with the continuing expansion of spacetime. Even power law expansion — $R(t) \sim t^\alpha$ — results in a fairly rapid fall-off for the effective cosmological constant — $\Lambda_{\text{eff}}(t) \sim t^{-2}$. There does not seem to be any compelling evidence against a suitably small and time varying cosmological constant [7]. It is intriguing to speculate that some portion of the currently observed expansion may have its origin in this effect.

Although quantum effects alter the time evolution of $R(t)$ they cannot change its status as a constrained variable. This is how the effective cosmological constant can vary without embarrassing low energy phenomenology with the existence of either new light quanta or new long range forces. The relevant dynamical variables of our model are just the usual gravitons of Einstein's theory. It is their dynamics which determines $R(t)$ — classically, quantum mechanically, and for any value of Λ .

We have adopted the term, "quantum cosmological gravity," or QCG, to distinguish the $\Lambda \neq 0$ theory from conventional quantum gravity. One reason QCG can have cosmological implications is that it allows massless gravitons to self-interact via a coupling of dimension three. In flat space the physics of the far infrared is dominated by whichever massless particles self-interact via the lowest dimension coupling. This is why QCD, which possesses a dimension four self-interaction, is stronger at large distances than conventional quantum gravity, whose lowest self-interaction has dimension five.

Of course the background geometry of QCG cannot be flat, but at least for $\Lambda > 0$ the curved background *enhances* the tendency towards strong infrared effects. This is because the expansion of space induces a corresponding redshift in spatial momenta, and so increases the population of soft gravitons that drive infrared effects. It is crucial to note that all other observed zero mass particles are insensitive to the gravitational redshift by virtue of being conformally invariant in $3 + 1$ dimensions. For the same reason these particles also lack self-couplings of dimension three. It follows that QCG must eventually dominate the far infrared of an inflating universe, regardless of how much stronger other forces may seem. One way to understand this is that in conformal coordinates the dimensionless coupling constants of conformally invariant matter theories are time independent whereas the natural dimensionless coupling of QCG contains factors of $R(t)$. Even with a minuscule proportionality constant, inflation eventually makes QCG strong relative to conformally invariant matter. Competitive matter models can be imagined — for example, a massless,

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minimally coupled scalar with cubic self-interaction — but it is difficult to understand how they have escaped detection.

Fig. 1: Generation of negative graviton self-energy leads to negative vacuum energy which leads to a background geometry with reduced effective cosmological constant.

The QCG process behind relaxation is the generation of negative self-energy (see Fig. 1). This is inevitable for massless particles with a self-interaction of dimension three; modulo some tensor traces, it occurs to all orders in positive Λ QCG. Its significance is that negative self-energy induces negative vacuum energy, which contributes to reduce the effective cosmological constant. Since the couplings of QCG involve factors of $R(t)$ the induced negative vacuum energy is time dependent. If one adjusts the bare parameters so that $\Lambda_{\text{eff}}(0)$ is large and positive then the universe will inflate. As it does so, soft virtual gravitons will induce more and more negative vacuum energy, thereby slowing the expansion and tending to cut the effect off. The process ought to be strong enough to effect a complete relaxation because the induced self-energy is *infrared divergent* if one assumes that $\Lambda_{\text{eff}}(t)$ approaches any value greater than zero. Relaxation ought to permit a considerable amount of inflation because the universe must experience many e -foldings of expansion before the naturally small couplings of QCG have been enhanced enough to generate a significantly negative vacuum energy.

Although relaxation can be understood using zero temperature quantum field theory, the genesis of a physically reasonable initial condition is inherently thermal. We have sketched a hypothetical timeline in Fig. 2. The initial temperature of the universe is so large that the cosmological constant comprises a minuscule fraction of the classical stress-

energy. Since thermal fluctuations also disrupt the long range correlations needed to support the quantum infrared effect, it is just as if there were no cosmological constant at

Fig. 2: Hypothetical cosmological timeline.

all during this epoch. Eventually the temperature falls enough over some region of space that the cosmological constant comes to dominate the stress-energy density within this region and local inflation begins. It also becomes possible to begin developing long range correlations, but this process is delayed by causality. After even a few e -foldings thermal effects should be so negligible that zero temperature quantum field theory is an excellent approximation: we need only select an appropriate initial state and adjust the bare parameters of QCG to match the conditions at onset. A crucial point is that causality limits the spatial extent of the initially inflating patch to around one Hubble volume. We can therefore ignore mathematical problems associated with a de Sitter background of infinite

spatial extent; the physically relevant infrared problems of QCG derive from long periods of inflation, not from small spatial momenta at the onset of inflation [8].

The reason quantum gravity can be used reliably, in spite of its lack of renormalizability, is that the phenomenon we are studying occurs in the far infrared. There must be some consistent quantum theory of gravity, and on sufficiently large distance scales this fundamental theory should not be distinguishable from quantum general relativity plus an infinite series of counterterms to absorb ultraviolet divergences. The Lagrangian of QCG is accordingly:

$$\mathcal{L}_{QCG} = \frac{1}{16\pi G} (R - 2\Lambda) \sqrt{-g} + \delta\mathcal{L} \quad (1.4a)$$

$$\delta\mathcal{L} \equiv -\frac{\delta G}{16\pi G^2} \left(R - 2\Lambda \right) \sqrt{-g} - \frac{2\delta\Lambda}{16\pi G} \sqrt{-g} - a_1 (R - 4\Lambda)^2 \sqrt{-g} - a_2 (R_{\mu\nu} - \Lambda g_{\mu\nu}) (R^{\mu\nu} - \Lambda g^{\mu\nu}) \sqrt{-g} - a_3 (R - 4\Lambda)^3 \sqrt{-g} + \dots \quad (1.4b)$$

A few conventions must be stated before proceeding. Our metric has spacelike signature and our Riemann tensor is $R^\rho_{\sigma\mu\nu} \equiv \Gamma^\rho_{\nu\sigma\mu} - \Gamma^\rho_{\mu\sigma\nu} - (\mu \leftrightarrow \nu)$. We are also using units where $c = \hbar = 1$. Although the relaxation mechanism can be studied without recourse to perturbation theory we shall not attempt to do so here. We expand the full metric in powers of the graviton field:

$$g_{\mu\nu} \equiv \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} \quad (1.5)$$

where $\kappa^2 \equiv 16\pi G$ is the loop counting parameter and the background field $\bar{g}_{\mu\nu}$ is determined so that the expectation value of $h_{\mu\nu}$ is zero.* Amplitudes are computed as series expansions in κ using Lorentz signature perturbation theory. We do not employ the Euclidean formulation of quantum gravity, nor is there any topology change in our formalism. We assume no extra or absent spacetime dimensions, although relaxation occurs in any dimension for which gravity is dynamical. We require no special matter content, although

* Although the graviton 1-point function is not a gauge invariant, its vanishing up to a BRS transformation is a gauge invariant statement. What this means is that the background field obeys the effective action field equations of the quantum theory.

relaxation occurs for any matter theory which is naturally consistent with low energy phenomenology.

The parameters G and Λ are respectively the physical values of Newton's constant and Λ_{eff} at the onset of inflation. We take the usual inflationary mass parameter to be at or below the GUT scale:

$$M \equiv \left(\frac{\Lambda}{\kappa^2}\right)^{\frac{1}{4}} < 10^{16} \text{GeV} \quad (1.6a)$$

This implies a very small value for the natural dimensionless constant of QCG:

$$\kappa^2 \Lambda = \left(\frac{M}{M_{\text{Planck}}}\right)^4 < 10^{-12} \quad (1.6b)$$

Lest we be accused of having assumed away the problem of the cosmological constant we point out that this is the usual assumption of inflationary cosmology [9], and it can be realized naturally in some models of supergravity [10]. Note also that our initial cosmological constant can be enormously larger than the current observed bound. $G\Lambda_{\text{eff}}(\text{today}) < 10^{-120}$. There are still 108 orders of magnitude to explain! The remaining 12 orders of magnitude are in the regime where M is comparable to M_{Planck} and knowledge of the fundamental theory of quantum gravity is needed for studying inflation.

The role of the counterterms (1.4b) is to absorb ultraviolet divergences order by order in perturbation theory. The coefficient of each counterterm consists of a divergent part and a finite one. Each divergent part is fixed by demanding that a portion of some Green's function should be free of ultraviolet divergences. By virtue of the classical field equations, $R_{\mu\nu} = \Lambda g_{\mu\nu}$, the higher dimension counterterms can only contribute to the 0-point and 1-point functions through loop effects. For example, the ultraviolet divergences of the 0-point function can be removed with δG ; those of the 1-point function are removed with $\delta\Lambda$; those of the spin 0, dimension four part of the 2-point function are removed with a_1 ; those of the spin 2, dimension four part of the 2-point function are removed with a_2 ; those of the spin 0, dimension six part of the 3-point function are removed with a_3 ; and so on.

QCG is ultraviolet finite for no other reason than that we allow ourselves any sort of local counterterm to subtract off ultraviolet divergences. This sort of BPHZ renormalizability is usually devoid of physical significance because the unknown finite parts of the coefficients of counterterms contaminate the model's predictions. In QCG these finite parts derive ultimately from the fundamental theory of quantum gravity, which we do not know. It would be possible to instead determine the finite parts by requiring that QCG correctly reproduce the conditions at the onset of inflation, if only we knew what these were. For example, we can choose the initial state to make $\Lambda_{\text{eff}}(0) = \Lambda$, and a reasonable condition for $\delta\Lambda$ is that $\Lambda_{\text{eff}}(t)$ should be initially constant. A similarly plausible condition for δG is the initial constancy of Newton's Law. The problem is that the higher counterterms principally affect the physics of gravitation at very short distances, about which we know almost nothing. Indeed, were our ignorance of this regime not so great we should probably be able to infer the fundamental theory of quantum gravity.

Although the aforementioned problem is intensely frustrating to investigations of the ultraviolet it has no significance to our study of the far infrared as long as the unknown finite parts contain no unreasonably large numbers. That is, we make the usual assumption that the finite part of a dimension d counterterm is a number of order unity times κ^{4-d} . Upon expanding the associated counterterm in powers of the graviton field one finds terms which are either suppressed in the infrared because they contain many covariant derivatives or else they involve factors of the small dimensionless number, $\kappa^2 \Lambda$. As an example, consider the cubic contributions to the fourth of the counterterms in expression (1.4b):*

$$\begin{aligned} \left(R_{\mu\nu} - \Lambda g_{\mu\nu}\right)^2 \sqrt{-g} = & \dots + \frac{1}{8} \kappa^3 h_{\alpha\beta}{}^{\mu\nu} h_{\alpha\beta}{}^{\mu\nu} \sqrt{-g} + \dots \\ & + \frac{1}{2} \kappa^3 \Lambda^2 h_{\alpha\beta}{}^{\mu\nu} h_{\mu\nu}{}^{\alpha\beta} \sqrt{-g} + \dots \end{aligned} \quad (1.7)$$

* The usual rules of perturbative quantum gravity apply. The indices of $h_{\mu\nu}$ are raised and lowered using the background, $\hat{g}_{\mu\nu}$. Covariant derivatives are taken with the background compatible connection and are denoted by a semicolon. We assume that the background obeys the classical equation, $\hat{R}_{\mu\nu} = \Lambda \hat{g}_{\mu\nu}$.

The first term on the right hand side contains four derivatives and is even more heavily suppressed in the infrared than the two derivative terms of traditional quantum gravity. The second term contains no derivatives and would seem to be comparable to the dimension three coupling of QCG:

$$\frac{1}{\kappa^2} \left(R - 2\Lambda \right) \sqrt{-g} = \dots - \frac{1}{12} \kappa \Lambda \left(h^3 - 6 h h^{\alpha\beta} h_{\alpha\beta} + 8 h_{\gamma\beta} h_{\gamma}^{\beta} h_{\alpha}^{\gamma} \right) \sqrt{-g} + \dots \quad (1.8)$$

However, note that the cubic no derivative term in (1.7) contains an extra factor of the small number, $\kappa^2 \Lambda < 10^{-12}$. The argument is even stronger for higher counterterms such as the final one listed in expression (1.4b).

Our operational procedure is therefore to compute with regularized QCG, using local counterterms to subtract off ultraviolet divergences as they occur, order by order in perturbation theory. Our procedure gives reliable results because the dominant infrared effects at any fixed order reside in ultraviolet finite and nonlocal terms that are not affected by local subtractions — as long as unnaturally large numbers are avoided. These finite, non-local terms dominate the result at large distances once ultraviolet finiteness is achieved. It is this principle which explains why Bloch and Nordsieck [11] were able to resolve the infrared problem in QED more than a decade before the model's renormalizability was understood. It is also why Weinberg [12] was able to resolve the infrared problem of $\Lambda = 0$ quantum gravity — which is not even renormalizable [5] — and why his analysis will stand no matter what the fundamental theory of quantum gravity turns out to be.

Infrared divergences play a crucial role in our mechanism. We have already seen that they are responsible for the fact that QCG dominates all other known theories in the far infrared, despite the smallness of its dimensionless parameter, $\kappa^2 \Lambda < 10^{-12}$. They also give QCG a powerful push towards complete relaxation, since assuming that $\Lambda_{\text{eff}}(t)$ approaches any positive final value results in an infrared divergence, no matter how small this positive final value is. In fact we shall now argue that relaxation could be precluded without

infrared divergences, and that in this case QCG would manifest a strong gravitational force that increases with the spatial separation.

The background field is a solution to the effective action field equations. If QCG produced no infrared divergences it would make sense to expand the effective action in powers of the derivative operator:

$$\Gamma[g] = \int d^4x \sqrt{-g} \left\{ -\frac{2\Lambda}{\kappa^2} + \frac{1}{\kappa^2} R + \dots \right\} \quad (1.9)$$

The higher terms have the form of two or more curvatures sandwiched between possibly nonlocal operators. These operators may be singular when acting on covariantly constant tensors — for example, in $\Lambda = 0$ quantum gravity the operators can include powers of logarithms of the covariant d'Alembertian — but the curvature terms upon which they act must have the property that they vanish on de Sitter space. This is the same convention we employed in organizing the local counterterms of expression (1.4b). Failure to obey it implies either a trivial renormalization of Λ or else an infrared divergence, which conflicts with the stated assumption of this discussion. The resulting field equations are satisfied by de Sitter space because at most one of the higher curvatures will be lost in the variation. It follows that there can be no relaxation unless the expansion we have just discussed breaks down, which implies the existence of infrared divergences.

Another way of seeing the same thing is that infrared divergences break de Sitter invariance. The background $\hat{g}_{\mu\nu}$ is defined to make the expectation value of $h_{\mu\nu}$ vanish. If there were a de Sitter invariant vacuum then this expectation value could only be some number times the de Sitter metric. But any such number could be absorbed into the counterterm $\delta\Lambda$, at which point we would have established that the background remains de Sitter for all time. Infrared divergences are what endows the expectation value of $h_{\mu\nu}$ with the crucial time dependence that precludes us from cancelling it with $\delta\Lambda$ at more than one instant.*

* We should mention that there are two other, much less significant, sources of de Sitter

To see the point about strong gravity consider the force needed to keep a test mass, m_1 , at a fixed invariant distance from a much larger source mass, m_2 . If m_1 lies on a sphere of surface area $4\pi r^2$, whose center is at m_2 , then the classical result for this force is:

$$F = m_1 \frac{\partial}{\partial r} \sqrt{1 - \frac{1}{3}\Lambda r^2 + 2N(r)} \quad (1.10a)$$

$$N(r) = -\frac{Gm_2}{r} \quad (1.10b)$$

The term under the radical is just $-g_{00}$ evaluated at m_1 in a static coordinate system centered on m_2 . We therefore recognize the inverse radical that comes from taking the derivative as the gravitational redshift factor. Neglecting this effect we see that G scales the inverse-square force law proposed by Newton. The cosmological constant scales a very different sort of force which grows linearly once the redshift is ignored. One must apply this force in order to counteract the fact that the distance between freely falling observers expands exponentially for $\Lambda > 0$ and contracts sinusoidally for $\Lambda < 0$. The accuracy with which no such expansion or contraction is observed constrains the dimensionless number $G\Lambda$ to be less than 10^{-120} [6]. The problem of the cosmological constant is finding a good theoretical reason for why this number should be so small.

Let us see how QCG corrections could modify this observable if there were no infrared divergences. The final, renormalized result can only depend upon m_1 , m_2 , r , G and Λ . If m_1 is very small we can neglect any dependence upon it. Further, it turns out that the worldline of a spatially localized source such as m_2 does not count towards closing up loops. The classical result (1.10) is in fact obtained by summing over all diagrams (see noninvariance. The first is a mathematical problem associated with zero modes. It turns out that even in free QCG on a de Sitter background there can be no normalizable, de Sitter invariant state [13]. (The problem was first noticed for massless, minimally coupled scalars by Allen and Folacci [14].) The second source of de Sitter noninvariance is physical and derives from the implausibility of any causal cosmological process resulting in a patch of de Sitter background of infinite spatial extent [9]. Since this affects all the sufficiently long wavelength modes it is the more significant effect, but it is still minuscule compared with the de Sitter noninvariance induced by infrared divergences.

Fig. 3) where any number of gravitons are emitted from m_2 and then combine into a single line that reaches m_1 without forming closed graviton loops. The multiple emissions just give the series expansion of the square root in powers of the Newtonian potential, $N(r)$.

Fig. 3: Some classical contributions to the force law between a heavy source m_2 and a light source m_1 .

If the same principle applies to loops then we can assume the force is given generally by an expression of the form (1.10a), but with a loop corrected Newtonian potential that is linear in m_2 . Each loop brings an additional factor of G , so we see that loop corrections to the classical result (1.10b) will give an asymptotic series in the dimensionless parameters, $G r^{-2}$ and $G \Lambda^2 r^2$.

In the $\Lambda = 0$ we can only get powers of $G r^{-2}$, which is why traditional quantum gravity is insignificant at large distances. Positive powers of $G \Lambda^2 r^2$ give an increasing potential. One might think that the second dimensionless expansion parameter should really be $G \Lambda$, but each vertex with the crucial dimension three self-interaction (1.8) contributes a factor of $\kappa \Lambda$. The infrared dominant contribution from one-loop QCG comes from two such vertices, which gives $G \Lambda^2$. Since the loop integral only diverges logarithmically for $\Lambda = 0$, we know that the remaining dimensional factor must be r^2 , *provided there are no infrared divergences for $\Lambda \neq 0$* . If there are infrared divergences then the required two powers of length will tend to come from the infrared cutoff.

The rest of the argument is simple. The universally attractive nature of gravitation means that the $G \Lambda^2 r^2$ terms must enter so as to increase the force at each order, thus precluding the possibility of offsetting terms as in the expansion, $1 - x + x^2 - \dots = (1 + x)^{-1}$. If there were no infrared divergences QCG would therefore give a gravitational force law that grows with increasing spatial separation, and in fact grows faster than any power of r . Of course no such growth is allowed, which is one way to see that infrared divergences must be present. Their appearance adds a dimensionful parameter in the form of the infrared cutoff time. The net effect is to change the relevant sense of "infrared" from large spatial separations at fixed time to large temporal separations at fixed distance. This is why we get a time dependent relaxation of the effective cosmological constant rather than gravitational confinement.

It remains to describe the work of succeeding sections. Section 2 is devoted to a study of the nearest flat space analog to QCG: the theory of a massless scalar with cubic self interaction. Although the analogy is not perfect, many useful things can be learned from it. Because both models allow massless particles to self-interact via a coupling of dimension three, they possess physical infrared divergences that cannot be eliminated by the Lee-Nauenberg technique [15] of restricting to transition amplitudes between degenerate

ensembles of states. The structure of perturbation theory is also so similar between the two theories that only very slight generalizations of the scalar self-energy calculation give the crucial result for the one-loop graviton self-energy. We present this in Section 3 and then go on to show how the infrared divergences of QCG infect invariant observables such as loop corrections to the force (1.10).

Section 4 describes a calculation of the amputated graviton 1-point function which implies that relaxation happens for at least as long as perturbation theory is reliable and the back reaction can be neglected. Our result for the effective cosmological constant has the following form:

$$\Lambda_{\text{eff}}(t) = \Lambda \left\{ 1 - G^2 \Lambda^2 \left[\left(\frac{16\pi}{3} \right)^2 \left(\frac{1}{6} a_0 + \frac{1}{2} c_0 \right) (H t)^2 + O(H t) \right] + O(G^3 \Lambda^3) \right\} \quad (1.11)$$

The pure numbers a_0 and c_0 result from some two-loop tensor algebra which we have not completed but which is probably within the grasp of the most recent symbolic manipulation programs. Of course perturbation theory breaks down when the (two) loop correction becomes comparable to the tree result, so our present methods can give little information about the critical period during which most inflation ends. However, the obvious estimate for the number of inflationary e-foldings is just the Hubble constant multiplied by the time until this critical period is reached:

$$N_{\text{TOT}} \approx \frac{1}{G \Lambda} > 10^{12} \quad (1.12)$$

This is easily consistent with enough inflation to solve the smoothness problem. Section 5 discusses possible implications of our work to particle physics and cosmology. In this section we also describe the problems that have so far prevented us from obtaining a result for negative Λ , and we indicate how the relaxation mechanism might be further studied by attempting to solve for the exponent of a power law that, at least asymptotically, may provide a stable background geometry. Finally, we explain why understanding relaxation via a renormalization group analysis is likely to be difficult.

This is perhaps the most appropriate point to comment on our original published analysis of relaxation in QCG [16]. The easy way to study operators in quantum field theory is through their “in”-“out” matrix elements. It is considerably harder to make the formalism yield expectation values, but a set of Feynman rules for accomplishing this was worked out long ago by Schwinger [17] and has been more recently studied by Jordan [18].* The distinction is without a difference for the most commonly studied problems in which the vacuum is stable. Of course relaxation occurs in QCG precisely because the theory lacks a stable vacuum, so one must properly study relaxation using a suitably modified version of Schwinger’s formalism which we shall describe in Section 2.

It is very important to understand that while the modified Schwinger expectation values must be used to gain quantitative control of relaxation, “in”-“out” amplitudes still give a qualitative understanding of the phenomenon. In other words, although asymptotic quantum field theory is not the appropriate formalism — as we noted in our previous work [16] — QCG does signal its tendency to relax through the appearance of infrared divergences in its “in”-“out” amplitudes (see Section 3). Our modified Schwinger expectation values are infrared finite because there is no contribution from loops of virtual quanta that persist longer than the latest observation time. What happens instead is that expectation values of the same operators whose “in”-“out” matrix elements harbor infrared divergences grow as they are observed at later and later times. Of course this is a simple and necessary consequence of the “in”-“out” infrared divergences. We know that something like relaxation has to occur since otherwise the QCG vacuum would be stable and the “in”-“out” amplitudes would be infrared finite. What one cannot necessarily guess

* Another way is to use the Heisenberg field equations to determine the field at any time in terms of the initial value operators, and then to take its expectation value in the presence of a suitable Heisenberg state. We have recently been informed that this method is being applied to QCG by A. D. Dolgov, M. B. Einhorn and V. I. Zakharov [19]. The reflection of this technique in the Schrödinger picture can also be used but it has so far been applied only to free scalar fields on a de Sitter background [20]. We comment in this regard that the interesting case is the massless, minimally coupled scalar with a cubic self-interaction, and that one must go to two loops to see relaxation.

right this way is the quantitative relation between the rate at which “in”-“out” matrix elements diverge and the rate at which our modified Schwinger expectation values of the same operators grow. “In”-“out” matrix elements typically diverge as powers of the scale factor $R(t)$, whereas we shall see that our modified Schwinger expectation values typically give percentage corrections to classical results which grow as powers of $\ln[R(t)]$.

“In”-“out” amplitudes are so much simpler to compute than the modified Schwinger expectation values, that we first attempted to approximate the time evolution of the latter using “in”-“out” amplitudes with a sharp cutoff on the duration of the cosmological interaction term. The idea was that such a calculation would be self-consistent if it showed the effective cosmological constant falling suddenly to near zero at the cutoff time. By adjusting the cutoff time to make this happen we could obtain an estimate for the relaxation time. Although our result was not self-consistent in that it failed to show the effective cosmological constant falling suddenly to zero, we did encounter the power law divergences characteristic of the “in”-“out” formalism. This led us to estimate the relaxation time as when GA — the natural dimensionless parameter of QCG — is enhanced by infrared effects to order one. In other words, our previous estimate for the relaxation time T came from assuming $GA R(T) \approx 1$ [16]. In fact our modified Schwinger expectation values show that the correct estimate is $GA \ln[R(T)] \approx 1$, which was used above in (1.12). Once the necessity of using our modified Schwinger expectation values became apparent we decided to re-work our previous “in”-“out” analysis around them.*

2. The Scalar Analog

In 1972 Veneziano [21] was able to show that there is no S-matrix for ϕ^3 theory in the limit of zero physical mass. The model is useful for us as a simple theoretical environment.

* We acknowledge J. Polchinski for urging us to use the Schwinger formalism. Of course he bears no responsibility for the length of time it has taken to convert our earlier analysis to this formalism.

free of tensor indices and exotic backgrounds, in which we can study the dynamics of massless particles with a self-interaction of dimension three. We shall first compute the one-loop self-energy, both to illustrate the complete irrelevance of ultraviolet divergences in the far infrared and to observe the negative divergence that occurs on shell. It is also shown that the negative sign persists to all orders, a result which applies as well to QCG up to some tensor traces. We will then explain why infrared divergences are real in the scalar model in a sense that they are not for QED, QCD or traditional quantum gravity. We next translate the momentum space analysis of the scalar self-energy to position space. The result is a formula that is simply generalized to QCG in the next section. We close by stretching the scalar model slightly so as to simulate the mechanism of relaxation.

The work of this section is done in 3+1 dimensional Minkowski space. The Lagrangian is:

$$\mathcal{L} = -\frac{1}{2}\phi_{,\mu}\phi^{,\mu} - \frac{1}{6}\lambda\phi^3 - \frac{1}{24}g\phi^4 + \delta\mathcal{L} \quad (2.1a)$$

$$\delta\mathcal{L} = -k\phi - \frac{1}{2}\delta m^2\phi^2 - \frac{1}{2}\delta Z\phi_{,\mu}\phi^{,\mu} - \frac{1}{6}\delta\lambda\phi^3 - \frac{1}{24}\delta g\phi^4 \quad (2.1b)$$

Three points deserve mention. First, the linear counterterm in (2.1b) is not a traditional feature of scalar field theory. We have included it to simulate the cosmological counterterm of QCG — the second of the terms in expression (1.4b). This is one of the ways in which we are stretching the scalar model. Second, although $\widehat{\phi} = 0$ is not stable it is a classical solution and we can organize perturbation theory around it. Classical stability is one of the features of QCG [22] that is not reflected in the scalar analog. Finally, we have included a quartic coupling in the classical Lagrangian to make the energy bounded below. This means that the decay of the $\widehat{\phi} = 0$ vacuum does eventually result in a stable vacuum, so the theory can be said to exist in some nonperturbative sense. However, we shall never actually use the quartic interaction. One point of the ultraviolet analysis is that we could as easily have included even *higher* interactions which spoil renormalizability. This entails adding more counterterms but it does not change the infrared limit.

To evaluate the scalar self-energy at one loop we need to regulate and renormalize.

Since the infrared limit is insensitive to the ultraviolet cutoff we may as well use dimensional regularization. With this method the one-loop contribution from the quartic vertex (Fig. 4a) is identically zero so the full one-loop result derives from cubic vertices (Fig. 4b).*

Fig. 4: Contributions to the scalar self-energy at order g (Fig. 4a) and λ^2 (Fig. 4b).*

Including the counterterms gives:

$$\begin{aligned} -i\Sigma(p^2) &= -i\delta m^2 - i\delta Z p^2 + \frac{(-i\lambda)^2}{2\mu D-4} \int \frac{d^D k}{(2\pi)^D} \frac{-i}{k^2 - i\epsilon} \frac{-i}{(p-k)^2 - i\epsilon} + O(\lambda^4, g^2) \quad (2.2a) \\ &= -i\delta m^2 - i\delta Z p^2 + \frac{1}{2}\lambda^2 \frac{\Gamma(2-\frac{D}{2})}{2D\pi^{D/2}} \frac{\Gamma^2(\frac{D}{2}-1)}{\Gamma(D-2)} \left(\frac{p^2 - i\epsilon}{\mu^2} \right)^{D/2-2} + O(\lambda^4, g^2) \quad (2.2b) \end{aligned}$$

Taking the limit $D \rightarrow 4$ we find:

$$\Sigma(p^2) = \delta m^2 + \delta Z p^2 - \frac{\lambda^2}{32\pi^2} \left\{ \frac{2}{4-D} + \ln\left(\frac{\mu^2}{p^2 - i\epsilon}\right) + \text{constant} \right\} + O(\lambda^4, g^2) \quad (2.3)$$

Ultraviolet finiteness implies the following forms for the two quadratic counterterms:

$$\delta m^2 = \frac{\lambda^2}{32\pi^2} \left\{ \frac{2}{4-D} + \alpha_1 + \text{constant} \right\} + O(\lambda^4, g^2) \quad (2.4a)$$

$$\delta Z = \frac{\lambda^2}{32\pi^2} \frac{\alpha_2}{\mu^2} + O(\lambda^4, g^2) \quad (2.4b)$$

where α_1 and α_2 are arbitrary constants of order one. The final result is therefore:

$$\Sigma(p^2) = \frac{\lambda^2}{32\pi^2} \left\{ -\ln\left(\frac{\mu^2}{p^2 - i\epsilon}\right) + \alpha_1 + \alpha_2 \frac{p^2}{\mu^2} \right\} + O(\lambda^4, g^2) \quad (2.5)$$

* The contribution from Fig. 4a would in any case be an infrared finite constant, independent of p^2 , and hence completely absorbable into the counterterm δm^2 .

Note that the logarithm drives $\Sigma(p^2)$ to negative infinity as p^2 approaches zero, no matter what values we assign to α_1 and α_2 . Note as well that we need not fret over the strength of g versus λ because the order g^2 contribution that occurs at two loops (Fig. 5) must be proportional to p^2 — and hence completely irrelevant in the infrared limit. If we had added higher interactions to (2.1a) their contributions would be even less relevant in the infrared.

Fig. 5: Contributions to the scalar self-energy at order g^2 .

The negative sign of the on-shell self-energy is a generic, all-orders effect for massless particles with self-interactions of dimension three. The physical reason is obvious from the shape of the scalar potential (Fig. 6): quantum fluctuations lower the self-energy because they explore the decreasing potential. The mathematics is scarcely less transparent. First note that the dominant contribution at any loop order comes from diagrams which involve only the 3-point coupling. Now count the factors of -1 and i . At ℓ loops such a diagram acquires 2ℓ factors of $-i$ from the vertex, $3\ell - 1$ factors of $-i$ from the propagators, and ℓ factors of $+i$ from the Wick rotations:

$$\begin{aligned} -i\Sigma_\ell(p) &= \left(\text{positive Euclidean integral} \right) \cdot (-i\lambda)^{2\ell} \cdot (-i)^{3\ell-1} \cdot (+i)^\ell \\ &= \left(\text{positive Euclidean integral} \right) \cdot (+i) \cdot \lambda^{2\ell} \end{aligned} \quad (2.6)$$

It follows that $\Sigma_\ell(p^2)$ is necessarily negative as p^2 approaches zero, to all orders. Note that subtracting off the negative ultraviolet divergences cannot change this because of the infrared divergences. We know that the integrals to the right of (2.6) grow without bound as p^2 approaches zero, so they eventually become positive even after the subtraction. We saw this explicitly in expression (2.5).

Fig. 6: Vertical axis: The scalar potential $V(\phi) = \frac{1}{2}\lambda\phi^3 + \frac{1}{24}g\phi^4$ for $\lambda, g > 0$. Horizontal axis: The scalar field ϕ .

We stress that the infrared problems of the $\hat{\phi} = 0$ vacuum are real in this model. The theorem of Lee and Nauenberg [15] is not applicable here. What they proved is that transition rates between ensembles of degenerate states are free of infrared divergences *provided the relevant self-energies are infrared finite*. Theories in which the massless particles have interactions of canonical field dimension four or higher (QED, QCD and $\Lambda = 0$ gravity) automatically obey this proviso so the theorem is sometimes mistakenly construed as meaning that the appearance of an infrared divergence has no physical significance beyond our inability to distinguish degenerate states. This is not what Lee and Nauenberg said

and Veneziano's work [21] shows that it is not even true in the scalar model.

Fig. 7: Off-shell infrared divergent contribution to the scalar 4-point function at two loops.

What goes wrong is obvious from the form (2.5) of the renormalized one-loop self-energy. In a normal theory we would renormalize the self-energy to vanish on mass shell so that insertions of it on internal loop lines would give no problems when the attached propagators became singular. The infrared divergence precludes any such renormalization. There is simply no choice of the constants α_1 and α_2 that can cancel the logarithm at $p^2 = 0$. Of course the tree order self-energy is finite, so the procedure of Lee and Nauenberg does give infrared finite transition amplitudes at one loop. The unavoidable breakdown occurs at two loops, an example of which is given by the 4-point box diagram displayed in Fig. 7. The top side of the box contains an insertion of the renormalized self-energy so the total integral is:

$$(-i\lambda)^4 \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 - i\epsilon} \left[\frac{-i}{i\Sigma(k^2)} \right] \frac{-i}{k^2 - i\epsilon} \times \frac{-i}{(k+p_1)^2 - i\epsilon} \frac{-i}{(k+p_1+p_2)^2 - i\epsilon} \frac{-i}{(k-p_4)^2 - i\epsilon} \quad (2.7)$$

Because the self-energy fails to vanish at $k^2 = 0$ — in fact it diverges logarithmically at this point — this amplitude is not even infrared finite off mass shell. The same thing happens in all two-loop Green's functions.

The appearance of an infrared divergence means that something is unphysical about the question being asked. In theories where the lowest self-interaction has dimension four the on-shell self-energy is infrared finite and it is only on-shell scattering amplitudes that experience infrared divergences. There is nothing wrong with the perturbative vacuum in this case, the problem is rather that one cannot exclude stray soft quanta with infinite precision. The Lee-Nauenberg procedure of studying transitions between degenerate ensembles is the correct solution to this problem. However, we have just seen that the situation is very different for theories where the lowest self-interaction has dimension three. In this case the on-shell self-energy is infrared divergent and the problem afflicts even off-shell Green's functions. The fact that the on-shell self-energy is infinitely *negative* signals an instability in the perturbative vacuum. The stable vacuum of our scalar model (2.1) is based on the classical background, $\hat{\phi} = -(3\lambda/g)$. There are no states for which the background remains at $\hat{\phi} = 0$ over an asymptotically long period of time. Pretending otherwise is what gives infrared divergences, just as pretending that we can measure arbitrarily soft quanta leads to infrared divergences in QED, QCD and conventional quantum gravity.

Having reached this insight we might reformulate perturbation theory around the background $\hat{\phi} = -(3\lambda/g)$ and go on using asymptotic scattering to study the steady state phenomenology of the resulting infrared finite, massive scalar field theory. Were we for some reason concerned with the model (2.1) in its own right this would be a reasonable course of action. However, in its guise as a paradigm for QCG the most interesting feature of the scalar model is not its steady state behavior but rather the transient effects associated with the decay of the background in a prepared state for which $\hat{\phi}$ and its first time derivative are initially zero. This is what most closely resembles the QCG relaxation of $\Lambda_{\text{eff}}(t)$ in an initially inflating universe.

A crucial distinction between the scalar model and QCG is that the decay of the scalar background is essentially classical once quantum processes nudge it away from zero whereas

there is no classical analog for the relaxation of $\Lambda_{\text{eff}}(t)$. This has two important implications for us. First, only the initial stages of the scalar decay are of use in understanding QCG. Second, we want to study *perturbative* contributions to the scalar decay rather than the tunnelling amplitude. This is not because the former are stronger than the latter — in fact the reverse is true — but rather because the perturbative effect provides the better analog to QCG.

It should be obvious at this stage that asymptotic quantum field theory is not the appropriate tool for this problem. Asymptotic quantum field theory gives matrix elements between states of free vacuum in the far past and future; we wish instead to follow a short period of the evolution from a state which is initially prepared to have $\hat{\phi} = \hat{\phi} = 0$. This means first that we must translate from momentum space to position space — which would be necessary in any case for a theory such as QCG whose classical background is not constant. Second, we want a formalism for computing expectation values rather than matrix elements. Finally, we need to be able to specify the state at finite times rather than asymptotically.

The first step — computing the usual in-out matrix elements in position space — is easy because massless propagators are so simple, even in QCG. For the massless scalar we have:

$$i\Delta(x; x') = \frac{1}{4\pi^2} \frac{1}{(x - x')^2 + i\epsilon} \quad (2.8a)$$

$$= \frac{1}{4\pi^2} \text{Pr} \left[\frac{1}{(x - x')^2} \right] - \frac{i}{4\pi} \delta((x - x')^2) \quad (2.8b)$$

(The symbol, “Pr,” denotes the principal value.) Although the $i\epsilon$ in (2.8a) resolves the same ambiguity as its momentum space cousin it is *not* the same term. Keeping the $i\epsilon$ of momentum space nonzero regulates the infrared without affecting the ultraviolet, whereas keeping the $i\epsilon$ of position space nonzero regulates the ultraviolet without affecting the infrared. It should also be noted that expression (2.8b) is only valid as a distributional

identity when multiplied by functions which are nonsingular at coincidence (i.e. at $x^\mu = x'^\mu$). Another factor of $i\Delta(x; x')$ does not meet this criterion; the correct procedure for handling squares of propagators is to use expression (2.8a) to derive a new distributional identity in the limit that ϵ goes to zero.

$$-i\Sigma(x; x') = \text{---}\times\text{---} + \text{---}\bigcirc\text{---} + \text{---}\bigcirc\text{---} + \dots$$

Fig. 8: The diagrammatic expansion of $\Sigma(x; x')$.

A good check of the position space formalism is the on-shell self-energy. It can be extracted from the quadratic part of the effective action:

$$\Gamma^{(2)}[\bar{\phi}] = \frac{1}{2} \int d^4x d^4x' \bar{\phi}(x) [\partial^2 \delta^4(x - x') - \Sigma(x; x')] \bar{\phi}(x') \quad (2.9)$$

where $\Sigma(x; x')$ stands for the 1PI 2-point function. The perturbative expansion of $\Sigma(x; x')$ gives (see Fig. 8):

$$\begin{aligned} \Sigma(x; x') = & \delta m^2 \delta^4(x - x') + \frac{1}{2} g i\Delta(x; x') \delta^4(x - x') - \frac{i}{2} \lambda^2 [i\Delta(x; x')]^2 - \frac{i}{6} g^2 [i\Delta(x; x')]^2 \\ & + \frac{i}{2} \lambda^4 \int d^4x d^4x' [i\Delta(x; y)] [i\Delta(x; y')] [i\Delta(y; y')] [i\Delta(y; x')] \\ & + \dots \end{aligned} \quad (2.10)$$

The equations of motion coming from the quadratic effective action (2.9) are:

$$\partial^2 \bar{\phi}(x) - \int d^4x' \Sigma(x; x') \bar{\phi}(x') = 0 \quad (2.11)$$

and are solved perturbatively:

$$\bar{\phi} = \phi_0 + \delta\bar{\phi} + \dots, \quad \Sigma(x; x') = \Sigma_1(x; x') + \dots \quad (2.12a)$$

The linearized equations are:

$$\partial^2 \phi_0 = 0 \quad (2.12b)$$

while the first order ones give:

$$\partial^2 \delta \phi(x) = \int d^4 x' \Sigma_1(x; x') \phi_0(x') \quad (2.12c)$$

The physical state wave functions of the linearized theory are:

$$\Phi(t, \vec{x}; \vec{k}) = \frac{1}{\sqrt{2k}} \exp[-ik t + i\vec{k} \cdot \vec{x}] \quad (2.13a)$$

$$\int d^3 x \Phi^*(t, \vec{x}; \vec{k}) i \overleftrightarrow{W}_0 \Phi(t, \vec{x}; \vec{k}) = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') \quad (2.13b)$$

(Here $k \equiv \|\vec{k}\|$, and the Wronskian operator for this theory is $\overleftrightarrow{W}_\mu \equiv \overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu$.) The on-shell self-energy Σ_1 is defined as follows:

$$\Sigma_1 \Phi(x; \vec{k}) \equiv \int d^4 x' \Sigma_1(x; x') \Phi(x'; \vec{k}) \quad (2.14)$$

and is spacetime independent since the theory is spacetime translation invariant. If we only possessed space translation invariance, as is the case in QCG, we would have obtained a time and frequency dependent "self-energy" $\Sigma_1(t; k)$.

The formalism we have just described gives the following result at one loop:

$$\Sigma_1 = \exp[ikt - i\vec{k} \cdot \vec{x}] \int d^4 x' \left(-\frac{i}{2} \lambda^2 [\Delta(x; x')]^2 \right) \exp[-ikt' + i\vec{k} \cdot \vec{x}'] \quad (2.15)$$

Substituting (2.8a) results in a straightforward integral:

$$\begin{aligned} \Sigma_1 &= \frac{-i\lambda^2}{2} \exp(ikt) \int_{-\infty}^{+\infty} dt' \exp(ikt') \int d^3 x \exp(-i\vec{k} \cdot \vec{x}) \left[\frac{1}{4\pi^2} \frac{1}{\|\vec{x}\|^2 - (t' - t)^2 + i\epsilon} \right]^2 \\ &= \frac{-i\lambda^2}{32\pi^2} \int_{-\infty}^{+\infty} dt' \exp[-ik(t' - t)] \int d^3 x \frac{\exp(-i\vec{k} \cdot \vec{x})}{(\|\vec{x}\|^2 - (t' - t)^2 + i\epsilon)^2} \\ &= -\frac{i\lambda^2}{32\pi^2} \int_{-\infty}^{+\infty} dt' \exp[-ik(t' - t)] \frac{\exp(-ik|t' - t|)}{(-i\pi^2) \frac{1}{|t' - t| - i\epsilon}} \\ &= -\frac{\lambda^2}{32\pi^2} \left\{ \int_{-\infty}^{t'} dt' \frac{1}{t - t' - i\epsilon} + \int_t^{+\infty} dt' \frac{\exp[-2ik(t' - t)]}{t' - t - i\epsilon} \right\} \end{aligned} \quad (2.16)$$

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Both the ultraviolet divergence at $t' = t$ (when the regularization parameter ϵ goes to zero) and the infrared divergence at $t' = -\infty$ are manifest in (2.16). Of course it is dangerous to compare unregulated expressions — or results obtained using different methods — but the sign certainly agrees with (2.2b). If we keep the energy of the right hand wave function off shell in order to regulate the infrared, then the term which diverges logarithmically as the mass shell is approached agrees with (2.5).

So much for the position space formalism. A procedure for computing expectation values in the presence of the "in" vacuum was given by Schwinger [17] and elaborated by Jordan [18]. The method uses "+" fields to evolve forward to an arbitrarily chosen time greater than or equal to the last observation; "-" fields then evolve backwards to the "in" vacuum. By turning the interaction off before $t = 0$ we can modify this formalism to make it give expectation values in the presence of a state which is formally in the free $\hat{\phi} = 0$ vacuum at $t = 0$. This is because the asymptotic "in" vacuum is evolved without change to $t = 0$ if there are no interactions before that time. This choice of initial state guarantees that both $\hat{\phi}$ and its first time derivative are zero at $t = 0$; we use the linear counterterm, $-\kappa\phi$, to keep them zero for as long as possible. The most natural way of accomplishing this is to enforce the condition that the background's *second* initial time derivative also vanishes. In fact ultraviolet divergences complicate the process slightly because one cannot take the unregulated limit after renormalization until it has been decided that t is either large or small compared with the ultraviolet regularization scale. Of course we want t to be large, but if this is assumed then the resulting expression will contain spurious divergences when one attempts to evaluate it at $t = 0$. We therefore use the linear counterterm to make the background's second time zero at a small positive time which is nonetheless large compared with the ultraviolet regularization scale.

The Feynman rules are straightforward. One can study the expectation value of any combination of "+" and "-" fields; by convention we shall study Green's functions of

the “+” fields. Vertices can be either “+” or “−”. The “+” vertices are identical to the vertices of the usual “in”-“out” formalism except that the interaction is absent for times before $t = 0$: the “−” vertices are complex conjugated. There are no mixed vertices but propagators can link fields of either sign. If we propagate between points x (to the left) and x' (to the right) the four possibilities are:

$$+ \quad + \quad \Rightarrow \quad i\Delta(x; x') = \frac{1}{4\pi^2} \frac{1}{(x - x')^2 + i\epsilon} \quad (2.17a)$$

$$- \quad + \quad \Rightarrow \quad -i\Delta^+(x; x') = \frac{1}{4\pi^2} \frac{1}{(x - x')^2 + i\text{sgn}(x^0 - x'^0)\epsilon} \quad (2.17b)$$

$$+ \quad - \quad \Rightarrow \quad i\Delta^-(x; x') = \frac{1}{4\pi^2} \frac{1}{(x - x')^2 - i\text{sgn}(x^0 - x'^0)\epsilon} \quad (2.17c)$$

$$- \quad - \quad \Rightarrow \quad -i\Delta^*(x; x') = \frac{1}{4\pi^2} \frac{1}{(x - x')^2 - i\epsilon} \quad (2.17d)$$

Of course one can always eliminate (2.17b) in favor of (2.17c) by merely reversing the arguments. Note also that (2.17b) and (2.17c) are harmonic.

An important property of this formalism is causality — observables can only be affected by interactions within the past lightcone of some observation. Note also that Hermitian fields give real Green’s functions. When the interaction is turned off before $t = 0$ the formalism has another important property: no infrared divergences can possibly occur as long as all observables are evaluated at finite times. If for some reason the background did not change then we should recover infrared divergences for asymptotically late observation times. However, what really happens is that the background moves away from $\hat{\phi} = 0$ and the resulting induced mass suppresses any potential infrared problems.

The simplest and most revealing object of study is the background, i.e., the scalar 1-point function:

$$\hat{\phi}(t) \equiv \langle 0 | \phi(t, \vec{x}) | 0 \rangle \quad (2.18)$$

Note that we are taking advantage of manifest spatial translation invariance to recognize that $\hat{\phi}$ depends only on time. The generic diagrams — making no distinction between “+”

and “−” vertices — that contribute to $\hat{\phi}$ are shown through two-loop order in Fig. 9. Since they all possess an external propagator it is simplest to compute the amputated 1-point function:

$$J(t) \equiv -\square \langle 0 | \phi(t, \vec{x}) | 0 \rangle \quad (2.19a)$$

and then obtain $\hat{\phi}$ by a trivial integration:

$$\hat{\phi}(t) = \int_0^t dt' (t - t') J(t') \quad (2.19b)$$

Fig. 9: Generic diagrams which contribute to the scalar background.

At one loop, and for $t > 0$, only the first two diagrams of Fig. 9 contribute:

$$J(t) = -k - \frac{1}{2}\lambda i\Delta(x; x) + O(\lambda^3, \lambda g) \quad (2.20)$$

With the understanding that the linear counterterm is:

$$k = -\frac{1}{2}\lambda i\Delta(x; x) + O(\lambda^3, \lambda g) \quad (2.21)$$

Fig. 10: Generic contribution to the scalar background at order λ^3 (Fig. 10a), $\lambda\delta m^2$ (Fig. 10b) λg (Fig. 10c), λg (Fig. 10d).

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like t^{-3} — possibly times logarithms — and hence to $\hat{\phi}(t)$ like t^{-1} .

(11a)

(11b)

Fig. 11: Vertex permutations of Figs. 10a-10b with all "+" internal vertices (Fig. 11a), ultraviolet finite "+ + -" vertex permutation of Fig. 10a (Fig. 11b), ultraviolet finite "- + +" vertex permutation of Fig. 10a (Fig. 11c), vertex permutations of Figs. 10a-10b with "-" internal vertices (Fig. 11d) of Fig. 10a (Fig. 11e), vertex permutations of Figs. 10a-10b with "-" internal vertices (Fig. 11f)

with the mass insertions necessary to renormalize the internal loop. Fig. 11a gives the

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contribution from diagrams which are composed entirely of “+” vertices:

$$\begin{aligned}
J_{11a}(t) &= \frac{1}{4} \lambda^3 \int_0^\infty dy^0 \int d^3 y \int_0^\infty dz^0 \int d^3 z \, i\Delta(x; y) \left[i\Delta(y; z) \right]^2 i\Delta(z; x) \\
&\quad - \frac{1}{2} \lambda \int_0^\infty dy^0 \int d^3 y \, i\Delta(x; y) \left[\text{at order } \lambda^2 \right] i\Delta(y; x) \quad (2.22a) \\
&= \frac{\lambda^3}{210 \pi^8} \int_0^\infty dy^0 \int d^3 y \int_0^\infty dz^0 \int d^3 z \, \frac{1}{(x-y)^2 + i\epsilon} \left[\frac{1}{(y-z)^2 + i\epsilon} \right]^2 \frac{1}{(z-x)^2 + i\epsilon} \\
&\quad - \frac{\lambda}{25 \pi^4} \int_0^\infty dy^0 \int d^3 y \, \frac{1}{(x-y)^2 + i\epsilon} \left[\text{at order } \lambda^2 \right] \frac{1}{(y-x)^2 + i\epsilon} \quad (2.22b)
\end{aligned}$$

We shall impose ultraviolet regularization by the simple device of leaving the parameter ϵ nonzero. To extract the ultraviolet divergence of the interior loop we make use of the following identity:

$$\frac{1}{(x^2 \pm i\epsilon)^2} = -\frac{1}{4} \square \left\{ \frac{\ln(x^2 \pm i\epsilon)}{x^2 \pm i\epsilon} \right\} \pm \frac{3i\epsilon}{(x^2 \pm i\epsilon)^3} \mp \frac{2i\epsilon \ln(x^2 \pm i\epsilon)}{(x^2 \pm i\epsilon)^3} \quad (2.23a)$$

On sufficiently smooth test functions the right hand side can be recast to give the following distributional identity:

$$\frac{1}{(x^2 \pm i\epsilon)^2} \longrightarrow -\frac{1}{4} \square \left\{ \frac{\ln(x^2 \pm i\epsilon)}{x^2 \pm i\epsilon} \right\} \pm i\pi^2 \left[-\frac{3}{2} + \ln(\epsilon) \right] \delta^4(x) \quad (2.23b)$$

(The distributional limit of the final term in (2.23a) follows from the identity: $\int d^4 x \ln(x^2 + i)/(x^2 + i)^3 = 0$.) It is simplest to choose δm^2 to absorb the two local terms:

$$\delta m^2 = \frac{\lambda^2}{25 \pi^2} \left\{ \frac{3}{2} - \ln(\mu^2 \epsilon) \right\} + O(g, \lambda^4) \quad (2.24)$$

With this choice we can express the result for Fig. 11a as:

$$\begin{aligned}
J_{11a}(t) &= \frac{-\lambda^3}{212 \pi^8} \int_0^\infty dy^0 \int d^3 y \, \frac{1}{(x-y)^2 + i\epsilon} \\
&\quad \times \int_0^\infty dz^0 \int d^3 z \, \square_z \left\{ \frac{\ln[\mu^2(y-z)^2 + i\mu^2 \epsilon]}{(y-z)^2 + i\epsilon} \right\} \frac{1}{(z-x)^2 + i\epsilon} \quad (2.25)
\end{aligned}$$

Manipulations of this sort are at the heart of the new technique known as, “differential regularization” [23].*

* Perhaps because they work in Euclidean space the method’s inventors do not view ϵ as the parameter of regularization. They also make no distinction between regularization and renormalization. For example, they would render the coefficient of the delta function in (2.23b) as an arbitrary, finite constant, and they would regard the right hand side of (2.23b) as defining the left.

The next step is partial integration. The volume term is simple because it gives a delta

function:

$$\square_z \left\{ \frac{1}{(z-x)^2 + i\epsilon} \right\} = \frac{-8i\epsilon}{(z-x)^2 + i\epsilon} \longrightarrow 4\pi^2 i \delta^4(z-x) \quad (2.26)$$

The surface terms at infinity vanish but those at $z^0 = 0$ give nonvanishing contributions.

The total is:

$$\begin{aligned}
J_{11a}(t) &= \frac{-\lambda^3}{212 \pi^8} \int_0^\infty dy^0 \int d^3 y \, \frac{1}{(x-y)^2 + i\epsilon} \left(4\pi^2 i \frac{\ln[\mu^2(x-y)^2 + i\mu^2 \epsilon]}{(x-y)^2 + i\epsilon} \right. \\
&\quad \left. + \int_{z^0=0} d^3 z \, \frac{\partial}{\partial z^0} \left\{ \frac{\ln[\mu^2(y-z)^2 + i\mu^2 \epsilon]}{(y-z)^2 + i\epsilon} \right\} \frac{1}{(z-x)^2 + i\epsilon} \right. \\
&\quad \left. - \int_{z^0=0} d^3 z \, \frac{\ln[\mu^2(y-z)^2 + i\mu^2 \epsilon]}{(y-z)^2 + i\epsilon} \frac{\partial}{\partial z^0} \left\{ \frac{1}{(z-x)^2 + i\epsilon} \right\} \right) \quad (2.27)
\end{aligned}$$

The further reduction of the volume term involves another identity like that of (2.23):

$$\begin{aligned}
\frac{\ln(x^2 \pm i\epsilon)}{(x^2 \pm i\epsilon)^2} &= -\frac{1}{8} \square \left\{ \frac{\ln^2(x^2 \pm i\epsilon)}{x^2 \pm i\epsilon} + 2 \frac{\ln(x^2 \pm i\epsilon)}{x^2 \pm i\epsilon} \right\} \\
&\quad \mp \frac{i\epsilon \ln^2(x^2 \pm i\epsilon)}{(x^2 \pm i\epsilon)^3} \pm \frac{i\epsilon \ln(x^2 \pm i\epsilon)}{(x^2 \pm i\epsilon)^3} \pm \frac{2i\epsilon}{(x^2 \pm i\epsilon)^3} \quad (2.28a) \\
&\longrightarrow -\frac{1}{8} \square \left\{ \frac{\ln^2(x^2 \pm i\epsilon)}{x^2 \pm i\epsilon} + 2 \frac{\ln(x^2 \pm i\epsilon)}{x^2 \pm i\epsilon} \right\} \\
&\quad \pm \frac{1}{2} \pi^2 i \left[\ln^2(\epsilon) - \ln(\epsilon) - 2 \right] \delta^4(x) \quad (2.28b)
\end{aligned}$$

The two surface terms are reduced using another identity yet:

$$\begin{aligned}
\frac{\ln(x^2 \pm i\epsilon)}{x^2 \pm i\epsilon} &\longrightarrow \frac{1}{8} \square \left[\ln^2(x^2 \pm i\epsilon) - 2 \ln(x^2 \pm i\epsilon) \right] \mp \frac{i\epsilon \ln(x^2 \pm i\epsilon)}{(x^2 \pm i\epsilon)^2} \pm \frac{2i\epsilon}{(x^2 \pm i\epsilon)^2} \quad (2.29a) \\
&\longrightarrow \frac{1}{8} \square \left[\ln^2(x^2 \pm i\epsilon) - 2 \ln(x^2 \pm i\epsilon) \right] \quad (2.29b)
\end{aligned}$$

(The final two terms in (2.29a) disappear as ϵ goes to zero because they reside upon functions that are not sufficiently singular at $x^\mu = 0$ to give a delta function.) The partial integrations on y^μ that these identities facilitate gives rise to infrared divergent surface terms at infinity which are precisely cancelled by similar terms from the three other diagrams. The reason for this is that the various vertex permutations must destructively

interfere when either y^μ or z^μ is outside the past lightcone of x^μ . After an uninteresting series of further manipulations we reach the following forms for the volume and surface terms of (2.27):

$$\begin{aligned} & -\frac{i\lambda^3}{210\pi^6} \int_0^\infty dy^0 \int d^3y \frac{\ln[\mu^2(x-y)^2 + i\mu^2\epsilon]}{[(x-y)^2 + i\epsilon]^2} \\ & = -\frac{i\lambda^3}{211\pi^5} \frac{\partial}{\partial\epsilon} \int_0^\infty d\tau \tau \frac{\partial}{\partial\tau} \left\{ \frac{1}{6} \ln^3[-\mu^2\tau^2 + \mu^2r^2 + i\mu^2\epsilon] + \frac{1}{2} \ln^2[-\mu^2\tau^2 + \mu^2r^2 + i\mu^2\epsilon] \right\} \\ & \quad + \frac{\lambda^3}{211\pi^4} [\ln^2(\mu^2\epsilon) - \ln(\mu^2\epsilon) - 2] + (\text{spurious surface terms}) \quad (2.30a) \end{aligned}$$

$$\begin{aligned} & \frac{\lambda^3}{212\pi^8} \int_0^\infty dy^0 \int d^3y \frac{1}{(x-y)^2 + i\epsilon} \\ & \quad \times \int_{z^0=0} d^3z \left(-\frac{\partial}{\partial z^0} \left\{ \frac{\partial}{\partial\tau} \left\{ \frac{\ln[\mu^2(y-z)^2 + i\mu^2\epsilon]}{(y-z)^2 + i\epsilon} \right\} \frac{1}{(z-x)^2 + i\epsilon} \right. \right. \\ & \quad \left. \left. + \frac{\ln[\mu^2(y-z)^2 + i\mu^2\epsilon]}{(y-z)^2 + i\epsilon} \frac{\partial}{\partial z^0} \left\{ \frac{1}{(z-x)^2 + i\epsilon} \right\} \right\} \right) \\ & = -\frac{i\lambda^3}{211\pi^5} \frac{\partial}{\partial\epsilon} \int_0^\infty d\tau \tau \frac{\partial}{\partial\tau} \left\{ \frac{1}{6} \ln^3[-\mu^2\tau^2 + \mu^2r^2 + i\mu^2\epsilon] + \frac{1}{2} \ln^2[-\mu^2\tau^2 + \mu^2r^2 + i\mu^2\epsilon] \right\} \\ & \quad - \frac{\lambda^3}{212\pi^6} \int_0^\infty \frac{y dy}{-\tau^2 + y^2 + i\epsilon} \int_0^\infty \frac{z dz}{-\tau^2 + z^2 + i\epsilon} \left\{ \ln^2[\mu^2(y+z)^2] - 2 \ln[\mu^2(y+z)^2] \right\} \\ & \quad - \frac{\lambda^3}{213\pi^6} \int_0^\infty \frac{dy}{-\tau^2 + y^2 + i\epsilon} \left\{ \frac{y}{-\tau^2 + y^2 + i\epsilon} \right\} \int_0^\infty \frac{dz}{-\tau^2 + z^2 + i\epsilon} \left\{ \frac{z}{-\tau^2 + z^2 + i\epsilon} \right\} (y+z)^2 \\ & \quad \times \left\{ \ln^2[\mu^2(y+z)^2] - 4 \ln[\mu^2(y+z)^2] + 4 \right\} + (\text{spurious surface terms}) \quad (2.30b) \end{aligned}$$

where we have not bothered to write out the aforementioned, spurious surface terms.

Reducing the vertex permutation diagrams is very similar. The chief difference is that the propagators $i\Delta^+$ and $i\Delta^-$ are harmonic so their square is free of ultraviolet divergences. The internal loops of Figs. 11b and 11c accordingly require no subtraction from mass insertion diagrams as did Fig. 11a. The distributional identities analogous to (2.26), (2.23b) and (2.29b) are respectively:

$$\square \left\{ \frac{1}{x^2 \pm i\text{sgn}(x^0)\epsilon} \right\} \longrightarrow 0 \quad (2.31a)$$

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$$\frac{1}{[x^2 \pm i\text{sgn}(x^0)\epsilon]^2} \longrightarrow -\frac{1}{4} \square \left\{ \frac{\ln|x^2 \pm i\text{sgn}(x^0)\epsilon|}{x^2 \pm i\text{sgn}(x^0)\epsilon} \right\} \quad (2.31b)$$

$$\frac{\ln|x^2 \pm i\text{sgn}(x^0)\epsilon|}{x^2 \pm i\text{sgn}(x^0)\epsilon} \longrightarrow \frac{1}{6} \square \left\{ \ln^2|x^2 \pm i\text{sgn}(x^0)\epsilon| - 2 \ln|x^2 \pm i\text{sgn}(x^0)\epsilon| \right\} \quad (2.31c)$$

For Fig. 11b we use (2.31b) and (2.31a) to partially integrate on z^μ , we then employ (2.31c) to partially integrate on y^μ .

$$\begin{aligned} J_{11b}(t) & = -\frac{1}{4} \lambda^3 \int_0^\infty dy^0 \int d^3y \int_0^\infty dz^0 \int d^3z \, i\Delta^-(x;y) \left[i\Delta^-(y;z) \right]^2 - i\Delta^+(z;x) \quad (2.32a) \\ & = -\frac{\lambda^3}{212\pi^8} \int_0^\infty dy^0 \int d^3y \frac{1}{(x-y)^2 + i\epsilon} \\ & \quad \times \int_{z^0=0} d^3z \left(-\frac{\partial}{\partial z^0} \left\{ \frac{\partial}{\partial\tau} \left\{ \frac{\ln[\mu^2(y-z)^2 + i\text{sgn}(y^0-z^0)\mu^2\epsilon]}{(y-z)^2 + i\text{sgn}(y^0-z^0)\epsilon} \right\} \frac{1}{(z-x)^2 - i\epsilon} \right. \right. \\ & \quad \left. \left. + \frac{\ln[\mu^2(y-z)^2 + i\text{sgn}(y^0-z^0)\mu^2\epsilon]}{(y-z)^2 + i\text{sgn}(y^0-z^0)\epsilon} \frac{\partial}{\partial z^0} \left\{ \frac{1}{(z-x)^2 - i\epsilon} \right\} \right\} \right) \quad (2.32b) \\ & = \frac{i\lambda^3}{211\pi^5} \frac{\partial}{\partial\epsilon} \int_0^\infty d\tau \tau \frac{\partial}{\partial\tau} \left\{ \frac{1}{6} \ln^3[-\mu^2\tau^2 + \mu^2r^2 - i\mu^2\epsilon] + \frac{1}{2} \ln^2[-\mu^2\tau^2 + \mu^2r^2 - i\mu^2\epsilon] \right\} \\ & \quad + \frac{\lambda^3}{212\pi^6} \int_0^\infty \frac{y dy}{-\tau^2 + y^2 + i\epsilon} \int_0^\infty \frac{z dz}{-\tau^2 + z^2 - i\epsilon} \left\{ \ln^2[\mu^2(y+z)^2] - 2 \ln[\mu^2(y+z)^2] \right\} \\ & \quad + \frac{\lambda^3}{213\pi^6} \int_0^\infty \frac{dy}{-\tau^2 + y^2 + i\epsilon} \left\{ \frac{y}{-\tau^2 + y^2 + i\epsilon} \right\} \int_0^\infty \frac{dz}{-\tau^2 + z^2 - i\epsilon} \left\{ \frac{z}{-\tau^2 + z^2 - i\epsilon} \right\} (y+z)^2 \\ & \quad \times \left\{ \ln^2[\mu^2(y+z)^2] - 4 \ln[\mu^2(y+z)^2] + 4 \right\} + (\text{spurious surface terms}) \quad (2.32c) \end{aligned}$$

Of course Fig. 11c gives the same result as Fig. 11b:

$$\begin{aligned} J_{11c}(t) & = -\frac{1}{4} \lambda^3 \int_0^\infty dy^0 \int d^3y \int_0^\infty dz^0 \int d^3z \, i\Delta^-(x;y) \left[-i\Delta^+(y;z) \right]^2 i\Delta^-(z;x) \\ & = J_{11b}(t) \quad (2.33) \end{aligned}$$

because they differ only by interchange of the dummy variables y^μ and z^μ . The reduction of Fig. 11d is similar to that of the other three:

$$\begin{aligned} J_{11d}(t) & = \frac{1}{4} \lambda^3 \int_0^\infty dy^0 \int d^3y \int_0^\infty dz^0 \int d^3z \, i\Delta^-(x;y) \left[-i\Delta^+(y;z) \right]^2 - i\Delta^+(z;x) \\ & \quad - \frac{1}{2} \lambda^3 \int_0^\infty dy^0 \int d^3y \, i\Delta^-(x;y) \left[\frac{i\lambda^2}{25\pi^2} \left\{ \frac{3}{2} - \ln(\mu^2\epsilon) \right\} \right] - i\Delta^+(y;x) \quad (2.34a) \end{aligned}$$

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$$\begin{aligned}
&= -\frac{\lambda^3}{2^{12}\pi^8} \int_0^\infty dy^0 \int_0^\infty dz^3 y \frac{1}{(x-y)^2 - i\text{sgn}(x^0 - y^0)\epsilon} \\
&\quad \times \int_0^\infty dz^0 \int_0^\infty dz^3 \square_z \left\{ \frac{\ln[\mu^2(y-z)^2 - i\mu^2\epsilon]}{(y-z)^2 - i\epsilon} \right\} \frac{1}{(z-x)^2 + i\text{sgn}(z^0 - x^0)\epsilon} \quad (2.34b) \\
&= -\frac{\lambda^3}{2^{12}\pi^8} \int_0^\infty dy^0 \int_0^\infty dz^3 y \frac{1}{(x-y)^2 - i\text{sgn}(x^0 - y^0)\epsilon} \\
&\quad \times \int_{z^0=0}^\infty dz^3 \left\{ \frac{\partial}{\partial z^0} \left\{ \frac{\ln[\mu^2(y-z)^2 - i\mu^2\epsilon]}{(y-z)^2 - i\epsilon} \right\} \frac{1}{(z-x)^2 - i\epsilon} \right. \\
&\quad \left. - \frac{\ln[\mu^2(y-z)^2 - i\mu^2\epsilon]}{(y-z)^2 - i\epsilon} \frac{\partial}{\partial z^0} \left\{ \frac{1}{(z-x)^2 - i\epsilon} \right\} \right\} \quad (2.34c) \\
&= -\frac{\lambda^3}{2^{12}\pi^8} \int_0^\infty \frac{y dy}{-t^2 + y^2 - i\epsilon} \int_0^\infty \frac{z dz}{-t^2 + z^2 - i\epsilon} \left\{ \ln^2[\mu^2(y+z)^2] - 2 \ln[\mu^2(y+z)^2] \right\} \\
&\quad - \frac{\lambda^3}{2^{13}\pi^8} \int_0^\infty \frac{dy}{dt} \left\{ \frac{\partial}{\partial t} \left\{ \frac{y}{-t^2 + y^2 - i\epsilon} \right\} \int_0^\infty \frac{dz}{dt} \left\{ \frac{\partial}{\partial t} \left\{ \frac{z}{-t^2 + z^2 - i\epsilon} \right\} (y+z)^2 \right. \right. \right. \\
&\quad \left. \left. \left. \times \left\{ \ln^2[\mu^2(y+z)^2] - 4 \ln[\mu^2(y+z)^2] + 4 \right\} + (\text{spurious surface terms}) \right\} \right\} \quad (2.34d)
\end{aligned}$$

It is worth remarking that diagrams which mix “+” and “−” vertices do not produce primitive ultraviolet divergences. This is as it should be since there are no counterterms which mix “+” and “−” fields.

The spurious surface terms cancel in combining the four figures. Using the distributional identities:

$$\ln[-\mu^2 t^2 + \mu^2 r^2 \pm i\epsilon] \longrightarrow \ln[\mu^2(t^2 - r^2)] \pm i\pi \theta(t - r) \quad (2.35a)$$

$$\frac{y}{-t^2 + y^2 + i\epsilon} - \frac{y}{-t^2 + y^2 - i\epsilon} \longrightarrow -i\pi \delta(t - y) \quad (2.35b)$$

a few simple manipulations allow us to reach the following form:

$$\begin{aligned}
J_{11}(t) &= \frac{\lambda^3}{2^{10}\pi^4} \frac{\partial}{\partial t} \int_0^t dt' \left\{ -\ln^2[\mu^2(t'^2 - r^2)] - 2 \ln[\mu^2(t'^2 - r^2)] + \frac{1}{3}\pi^2 \right\} \\
&\quad + \frac{\lambda^3}{2^{12}\pi^4} \int_0^\infty dy \delta(t - y) \int_0^\infty dz \delta(t - z) \left\{ \ln^2[\mu^2(y+z)^2] - 2 \ln[\mu^2(y+z)^2] \right\} \\
&\quad + \frac{\lambda^3}{2^{13}\pi^4} \int_0^\infty dy \delta'(t - y) \int_0^\infty dz \delta'(t - z) (y+z)^2 \\
&\quad \times \left\{ \ln^2[\mu^2(y+z)^2] - 4 \ln[\mu^2(y+z)^2] + 4 \right\}
\end{aligned}$$

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$$\begin{aligned}
&\quad + \frac{\lambda^3}{2^{11}\pi^4} \left\{ \ln^2(\mu^2\epsilon) - \ln(\mu^2\epsilon) - 2 \right\} \quad (2.36a) \\
&= \frac{\lambda^3}{2^8\pi^4} \left\{ -\frac{1}{8} \ln^2(4\mu^2 t^2) - \frac{1}{2} \ln(4\mu^2 t^2) \right. \\
&\quad \left. - \ln(2) + \frac{1}{2} - \frac{1}{6}\pi^2 + \frac{1}{8} \ln^2(\mu^2\epsilon) - \frac{1}{8} \ln(\mu^2\epsilon) \right\} \quad (2.36b)
\end{aligned}$$

The obvious renormalization condition to impose is:

$$k = -\frac{1}{2}\lambda i \Delta(x, x) + \frac{\lambda^3}{2^8\pi^4} \left\{ \frac{1}{8} \ln^2(\mu^2\epsilon) - \frac{1}{8} \ln(\mu^2\epsilon) - \ln(2) + \frac{1}{2} - \frac{1}{6}\pi^2 \right\} + O(\lambda^5, \lambda g) \quad (2.37)$$

Recall that the apparent divergences at $t = 0$ are artifacts of the tacit assumption behind our various distributional identities that the observation time, t , is very much larger than the ultraviolet regularization scale, $\sqrt{\epsilon}$.

To recover the background $\widehat{\phi}(t)$ we must perform the integration (2.19b). The spurious divergence at $t = 0$ is integrable so we obtain a perfectly finite result:

$$\widehat{\phi}(t) = \frac{-\lambda^3}{2^{10}\pi^4} t^2 \left\{ \ln^2(2\mu t) - \ln(2\mu t) + \frac{1}{2} \right\} + O(\lambda^5, \lambda g) \quad (2.38)$$

As noted previously, the contributions of order λg — cf. Fig. 10d — fall off as t^{-1} , times perhaps as many as two factors of $\ln(\mu t)$. The leading infrared effect at any loop derives from diagrams which are composed solely of 3-point vertices. The asymptotic behavior of these contributions is simple to evaluate. From our argument about the self energy the sign must be negative. Standard diagram topology gives $2\ell - 1$ factors of λ at ℓ loops; reconciling the dimensions with factors of t and allowing one logarithm per loop integrations gives the following asymptotic form at all loops:

$$J^\ell(t) \sim -\lambda^{2\ell-1} t^{2\ell-4} \ln^\ell(\mu t) \quad (2.39a)$$

$$\widehat{\phi}^\ell(t) \sim -\lambda^{2\ell-1} t^{2\ell-2} \ln^\ell(\mu t) \quad (2.39b)$$

Note that even though the on-shell self-energy is negative for either sign of λ — cf. equation (2.6) — the source of the background couples through an extra 3-point vertex. The

background therefore goes like an odd power of λ and will become negative or positive as λ is positive or negative respectively. This is obvious from the shape of the classical potential but we wish to emphasize that it follows also from the simple diagram topology which this scalar model shares with QCG.

Of course the true behavior of the background is monotonic decay to the stable vacuum at $\tilde{\phi} = -(3\lambda/g)$, followed by damped oscillations around this value as the kinetic energy of the background is converted into radiation. The last period is a nonperturbative regime which cannot be probed using the methods so far developed. Perturbation theory can break down either through the background becoming of order λ/g , in which case the 4-point vertex becomes dominant, or through the higher loop corrections of (2.39) exceeding the two-loop result. This last occurs for $t \approx t_{\text{crit}}$ where:

$$\lambda^2 t_{\text{crit}}^2 \ln(\mu t) = 1 \quad (2.40)$$

Unless the dimensionless number λ/μ is unreasonably large or small, t_{crit} will be λ^{-1} to within a few orders of magnitude. Since $\tilde{\phi}(t_{\text{crit}}) \sim -\lambda \ln(\mu/\lambda)$ the background will enter the period of damped oscillations before t_{crit} if $g \ln(\mu/\lambda) \gg 1$.

Although our modified "in-in" formalism is free of infrared divergences the reader should not be deceived into discounting the significance of infrared divergences in the "in-out" formalism. The latter unquestionably signal a decay of the background, the gross evolution of which can be inferred without reference to the new formalism. Were the "in-out" formalism free of infrared divergences there would be no decay. To see this note that any constant shift in the vacuum can be absorbed using the linear counterterm, $-k\phi$. We can understand this process by rather inappropriately including the linear counterterm, which has loop strength, along with the classical scalar potential as in Fig. 12. The linear term is chosen to create a perturbatively small barrier to the field's tendency to roll down the potential well. If the free vacuum is translationally invariant in spacetime then

any infrared finite contribution to the 1-point function is necessarily constant, and hence completely absorbable into the linear counterterm. It is only the infrared divergent contributions which break translation invariance by virtue of the infrared regularization. These contributions are not constant and so cannot be absorbed into the linear counterterm. Further, if the vacuum did not decay they would become arbitrarily large as the infrared regularization was removed. This fact ensures that the background field receives however strong a push is needed to start it rolling down the potential well.

Fig. 12: Vertical axis: The scalar potential $V(\phi) = \kappa\phi + \frac{1}{2}\lambda\phi^3 + \frac{1}{4!}g\phi^4$ for $\kappa < 0$ and $\lambda, g > 0$. Horizontal axis: The scalar field ϕ .

Although the fact of decay, along with its basic evolution, follows from asymptotic

quantum field theory, the details can only be observed using the modified "in-in" formalism we have developed. The point of following the decay of the scalar background in detail was to illustrate this method, much of which generalizes to QCG. It is therefore worthwhile concluding with a summary of what has been learned.

- (1) First and foremost, the modified "in-in" formalism is *simple* to use for massless theories. This is because keeping the parameter ϵ nonzero regulates the ultraviolet while interference between "+" and "-" vertices regulates the infrared. Distributional identities such as (2.23), (2.26), (2.28), (2.29) and (2.31) allow us to reduce loop integrals by successive partial integrations.
- (2) Though using ϵ as an ultraviolet regulator can sometimes require counterterms which are not Hermitian — cf. equation (2.21) — there does not seem to be anything wrong with this. Further, the problem does not occur for logarithmic divergences. As a check on the method's validity we computed $J_{11}(t)$ in momentum space using a combination of dimensional regularization for the internal loop and a momentum cutoff for the external one. The result has the same double logarithm term as (2.36b).
- (3) The formalism can be implemented directly in Minkowski signature position space. It is not necessary or even particularly desirable to work in Euclidean position space, and momentum space calculations are actually *harder*.
- (4) The formalism is causal. That is, all contributing interaction vertices must lie on or inside the past lightcone of some operator which is being observed.
- (5) The formalism gives real results for the expectation values of Hermitian operators. In particular the background, defined as the expectation value of the field, is real.
- (6) Higher dimension couplings are irrelevant in the far infrared so long as perturbation theory is valid.
- (7) Although ultraviolet divergences are completely irrelevant to the far infrared the *finite*

parts of low dimensional counterterms can have significance. The scalar model illustrates this with the scale-setting parameter μ which entered through k and δm^2 . Then too, the finite part of k is used to enforce the condition that $J(0) = 0$.

3. The Infrared Divergences of QCG

The structure of perturbative quantum gravity on a de Sitter background has been studied thoroughly in [8, 13, 24]. There it was shown that a natural set of coordinates is given by:

$$ds^2 = -dt^2 + \exp(2Ht) \|d\vec{x}\|^2 = -dt^2 + \exp(2Ht)(dx^2 + dy^2 + dz^2) \quad (3.1)$$

Although these "open" coordinates cover only a portion of the full de Sitter manifold they provide a completely valid arena for physics because no causal evolution off the submanifold is possible. Indeed, if one wishes to imagine inflation as a phase in the development of a previously open Friedmann-Robertson-Walker universe there is no alternative to using such an open coordinate system. Since the entire open coordinate patch is conformal to flat space, it is convenient to work in the "open-conformal" set of coordinates. The conformal time u is defined by:

$$u \equiv \frac{1}{H} \exp(-Ht) \quad (3.2)$$

A peculiarity of this system is that while the spatial coordinate, $\vec{x}$, can take any value in 3-dimensional Euclidean space the time coordinate, u , runs only from zero to infinity. It is also inverted with respect to physical time; that is, the far future is obtained by letting u approach zero while the far past is probed by taking u to $+\infty$.

The perturbation theory of the QCG Lagrangian (1.4) derives from the expansion (1.5) of the full metric $g_{\mu\nu}$ in terms of the graviton field $h_{\mu\nu}$. The big advantage of conformal coordinates is that the background $\bar{g}_{\mu\nu}$ is simply related to the Lorentz metric:

$$\bar{g}_{\mu\nu} = \frac{1}{(Hu)^2} \eta_{\mu\nu} \equiv \Omega^2 \eta_{\mu\nu} \quad (3.3)$$

Interactions in QCG are most easily described in terms of the pseudo-graviton field $\psi_{\mu\nu}$:

$$\psi_{\mu\nu} \equiv \Omega^{-2} h_{\mu\nu} \quad (3.4)$$

However, it is a slightly different rescaling that gives the simplest formulation of the free theory:

$$\chi_{\mu\nu} \equiv \Omega^{-1} h_{\mu\nu} \quad (3.5)$$

The indices on $h_{\mu\nu}$ are raised and lowered with $\hat{g}_{\mu\nu}$ but those of both $\psi_{\mu\nu}$ and $\chi_{\mu\nu}$ are raised and lowered with the Minkowski metric. It is of course completely trivial to convert the mode solutions, Green's functions or propagators from one of these fields to any other.

The simplest gauge fixing functional seems to be $-\frac{1}{2}\eta^{\mu\nu}F_\mu F_\nu$ where:

$$F_\mu \equiv \chi_{\mu,\rho} - \frac{1}{2}\chi_{\rho,\mu} + \chi_\mu(\ln\Omega)_\rho + \frac{1}{2}\chi_\rho(\ln\Omega)_\mu \quad (3.6)$$

After some partial integrations the quadratic gauge fixed action can be written as:

$$\mathcal{L}_{\text{GF}}^{(2)} = \frac{1}{2}\chi^{\mu\nu}\mathcal{D}_{\mu\nu}{}^{\rho\sigma}\chi_{\rho\sigma} \quad (3.7a)$$

with the kinetic operator equal to:

$$\begin{aligned} \mathcal{D}_{\mu\nu}{}^{\rho\sigma} = & \left[\frac{1}{2}\bar{\partial}_\mu{}^{(\rho}\bar{\partial}_\nu{}^{\sigma)} - \frac{1}{4}\eta_{\mu\nu}\eta^{\rho\sigma} - \frac{1}{2}\delta_\mu^0\delta_\nu^0\delta_0^\rho\delta_0^\sigma \right] \left(\partial^2 + \frac{2}{u^2} \right) \\ & + \delta_\mu^0\bar{\partial}_\nu{}^{(\rho}\delta_0^{\sigma)}\partial^2 + \delta_\mu^0\delta_\nu^0\delta_0^\rho\delta_0^\sigma\partial^2 \end{aligned} \quad (3.7b)$$

The pseudo-graviton propagator is given by:

$$\begin{aligned} i\left[\rho\sigma\Delta^{\alpha\beta}\right](x,x') = & \frac{H^2}{8\pi^2}\left\{ \frac{2u'u}{(x-x')^2 + i\epsilon} \left[2\delta_\rho^{(\alpha}\delta_\sigma^{\beta)} - \eta_{\rho\sigma}\eta^{\alpha\beta} \right] \right. \\ & \left. - \ln\left(H^2(x-x')^2 + i\epsilon\right) \left[2\bar{\partial}_\rho^{(\alpha}\bar{\partial}_\sigma^{\beta)} - 2\bar{\eta}_{\rho\sigma}\bar{\eta}^{\alpha\beta} \right] \right\} \end{aligned} \quad (3.8)$$

Parenthesized indices are symmetrized and a bar above a Lorentz metric or a Kronecker delta symbol means that the zero (i.e., u) component is projected out:

$$\bar{\eta}_{\mu\nu} \equiv \eta_{\mu\nu} + \delta_\mu^0\delta_\nu^0 \quad (3.9)$$

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From the quadratic part of the ghost action we obtain the following ghost propagator:

$$i\left[\rho\Delta^\alpha\right](x,x') = \frac{H^2}{8\pi^2}\left\{ \frac{2u'u}{(x-x')^2 + i\epsilon}\delta_\rho^\alpha - \ln\left(H^2(x-x')^2 + i\epsilon\right)\bar{\partial}_\rho^\alpha \right\} \quad (3.10)$$

The propagators (3.8) and (3.10) can be decomposed into their real and imaginary by using the expressions:

$$\frac{1}{(x-x')^2 + i\epsilon} = \text{Pr}\left[\frac{1}{(x-x')^2}\right] - i\pi\delta\left((x-x')^2\right). \quad (3.11a)$$

$$\ln\left(H^2(x-x')^2 + i\epsilon\right) = \ln\left(H^2(x-x')^2\right) + i\pi\theta\left(-H^2(x-x')^2\right) \quad (3.11b)$$

The self interactions of the pseudo-graviton field are obtained by expanding the interaction Lagrangian:

$$\begin{aligned} \mathcal{L}_{\text{int}} = & \sqrt{-g}\bar{\partial}^{\alpha\beta}\bar{\partial}^{\rho\sigma}\bar{\partial}^{\mu\nu}\left[\frac{1}{2}\psi_{\alpha\rho\mu}\psi_{\nu\sigma\beta} - \frac{1}{2}\psi_{\alpha\beta\sigma}\psi_{\sigma\mu\nu} + \frac{1}{4}\psi_{\alpha\beta\sigma}\psi_{\mu\nu\sigma} - \frac{1}{4}\psi_{\alpha\rho\mu}\psi_{\beta\sigma\nu}\right]\Omega^2 \\ & - \frac{1}{2}\sqrt{-g}\bar{\partial}^{\rho\sigma}\bar{\partial}^{\mu\nu}\psi_{\rho\sigma\mu}\psi_\nu{}^\alpha(\Omega^2)_\alpha \end{aligned} \quad (3.12)$$

where we define $\bar{g}_{\mu\nu} \equiv \eta_{\mu\nu} + \kappa\psi_{\mu\nu}$. If ω_μ is the ghost field and $\bar{\omega}^\mu$ the antighost field, their interactions are obtained from:

$$\begin{aligned} \mathcal{L}_{\text{ghost,int}} = & -\kappa\Omega^2\bar{\omega}^{\mu,\nu}\left[\psi_{\mu\rho}\omega^\rho{}_\nu + \psi_{\nu\rho}\omega^\rho{}_\mu + \psi_{\mu\nu,\rho}\omega^\rho + 2\psi_{\mu\nu}\omega^\rho(\ln\Omega)_{,\rho}\right] \\ & + \kappa\left(\Omega^2\bar{\omega}^\rho\right)_\rho\left[\psi_{\mu\nu}\omega^{\mu,\nu} + \frac{1}{2}\psi^\mu{}_{\mu,\nu}\omega^\nu + \psi^\mu{}_\mu\omega^\nu(\ln\Omega)_\nu\right] \end{aligned} \quad (3.13)$$

Note that almost all of the interaction vertices are just those of flat space times a factor of Ω^2 .

We now turn our attention to the examination of the infrared behaviour of QCG.

The relevant infrared problems of the theory occur for large time separations. Thus, as a general rule, to isolate the dominant Feynman diagrams of a certain order in perturbation theory, we must determine the time dependence of each graph. For instance, each vertex carries factors of $\Omega = (Hu)^{-1}$ which is singular as $u \rightarrow 0$. To a fixed order in κ we wish to have as many factors of Ω as possible. As a result, graphs with the maximum possible

number of vertices will typically dominate in the infrared. Such diagrams will be generally built out of 3-point vertices:

Fig. 13: (a) Tree order process which can become infrared finite with a certain *ad-hoc* restriction on the allowed external states. (b) Infrared divergent process for any choice of external states.

It is sufficient to stay at tree order in perturbation theory to get a good taste of the infrared problems of QCG [8]. Even the simple process of two gravitons going into one via a 3-point vertex is infrared divergent (see Fig. 13a). Although it is possible to choose the graviton wave functions so that the amplitude becomes finite, no such choice exists so that every tree amplitude is infrared finite. Indeed, it is easy to exhibit a tree diagram which is infrared divergent independent of the definition of the vacuum (see Fig. 13b). The key feature of this diagram is the presence of a 3-point vertex with only propagators and no external states. Now the part of this diagram involving the imaginary part of the three propagators is infrared divergent. But this imaginary part is the same for any choice of the ground state and we conclude that the infrared infinity cannot be transformed into something finite. Such infinities signal, as early as at tree order, the instability of the background [8].

Another instability signal is furnished by the QCG on-shell self-energy. In analogy with the scalar model of Section 2, it can be obtained from the quadratic part of the

pseudo-graviton effective action:

$$\Gamma_{\bar{\psi}}^{(2)}[\bar{\psi}] = \frac{1}{2} \int d^4x d^4x' \bar{\psi}^{\mu\nu}(x) \left\{ \left[\Omega D_{\mu\nu}{}^{\rho\sigma} \Omega \right] \delta^4(x-x') - \left[{}_{\mu\nu}^{\Sigma\rho\sigma} \right](x,x') \right\} \bar{\psi}_{\rho\sigma}(x') \quad (3.14)$$

where $\left[{}_{\mu\nu}^{\Sigma\rho\sigma} \right](x,x')$ is the 1PI 2-point function and has the usual perturbative diagrammatic expansion. The linearized physical state wave functions are given by [24]:

$$\Psi_{\alpha\beta}(u, \vec{x}; \vec{k}, \lambda) = \frac{\Omega^{-1}}{\sqrt{2k}} \left(1 + \frac{i}{ku} \right) \exp \left[ik(u - \frac{i}{k}) + i\vec{k} \cdot \vec{x} \right] \epsilon_{\alpha\beta}(\vec{k}, \lambda) \quad , \quad k \equiv \|\vec{k}\| \quad (3.15)$$

and have as their inner product:

$$- \int d^3x \Psi^{\alpha\beta}(u, \vec{x}; \vec{k}, \lambda) i \left[{}_{\alpha\beta}^{\dagger\gamma} \nabla_0^{\mu\nu} \right] \Psi_{\mu\nu}(u, \vec{x}; \vec{k}', \lambda') = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') \delta_{\lambda\lambda'} \quad (3.16)$$

where the QCG Wronskian operator is:

$${}_{\alpha\beta}^{\dagger\gamma} \nabla_\mu^{\rho\sigma} = \left[\frac{1}{2} \delta_\alpha^{(\rho} \delta_\beta^{\sigma')} - \frac{1}{4} \eta_{\alpha\beta} \eta^{\rho\sigma} \right] \Omega \vec{\partial}_\mu^\gamma \Omega \quad (3.17)$$

The physical polarization tensors $\epsilon_{\mu\nu}(\vec{k}, \lambda)$ have vanishing zero components, are transverse and traceless:

$$\epsilon_{\mu 0}(\vec{k}, \lambda) = k_\mu \epsilon_{\mu\nu}(\vec{k}, \lambda) = \epsilon_{\mu\mu}(\vec{k}, \lambda) = 0 \quad (3.18a)$$

and satisfy:

$$\epsilon_{\mu\nu}^\bullet(\vec{k}, \lambda) \epsilon^{\mu\nu}(\vec{k}, \lambda) = 2 \delta_{\lambda\lambda'} \quad , \quad \sum_\lambda \epsilon_{\mu\nu}(\vec{k}, \lambda) \epsilon^{\rho\sigma}(\vec{k}, \lambda) = 2 \Pi_\mu^{(\rho} \Pi_\nu^{\sigma')} - \Pi_{\mu\nu} \Pi^{\rho\sigma} \quad (3.18b)$$

where:

$$\Pi_{\mu\nu} \equiv \bar{\eta}_{\mu\nu} - \frac{\vec{k}_\mu \vec{k}_\nu}{\|\vec{k}\|^2} \quad (3.18c)$$

Since there is only spatial translational invariance in the theory, there is a time and frequency dependent one-loop on-shell self-energy $\Sigma_1(u; k)$:

$$\Sigma_1(u; k) \Psi_{\mu\nu}(u, \vec{x}; \vec{k}, \lambda) = \int_0^{+\infty} du' \int d^3x' \left[{}_{\mu\nu}^{\Sigma_1^{\rho\sigma}} \right](u, \vec{x}; u', \vec{x}') \Psi_{\rho\sigma}(u', \vec{x}'; \vec{k}, \lambda) \quad (3.19)$$

By multiplying both sides of (3.19) by $\epsilon^{\mu\nu}(\vec{k}, \lambda)$, summing over all polarizations λ , and, utilizing (3.15), (3.18b) as well as the space translation invariance we obtain the following expression for the on-shell self-energy:

$$\Sigma_1(u, k) = \frac{\exp(-iku)}{4(1-iku)} \int_0^{+\infty} du' (1-iku') \exp(iku') \int d^3x \mathcal{A}(u, u'; \vec{x}) \exp(-i\vec{k} \cdot \vec{x}) \quad (3.20a)$$

where:

$$\mathcal{A}(u, u'; \vec{x}) \equiv \left[\mu\nu \Sigma_1^{\rho\sigma} \right] (u, \vec{x}; u', \vec{0}) \left(2\Pi_{(\rho}^{\mu} \Pi_{\sigma)}^{\nu} - \Pi^{\mu\nu} \Pi_{\rho\sigma} \right) \quad (3.20b)$$

Note that the observation event is $(u, \vec{x})$ while the interaction one is $(u', \vec{x}')$.

Fig. 14: One-loop contributions to the graviton self-energy.

There are three contributions to the one-loop 1PI 2-point function $\left[\mu\nu \Sigma_1^{\rho\sigma} \right] (x; x')$. The one coming from the 4-point vertex (Fig. 14a) is proportional to the coincidence limit of the graviton propagator and contains no infrared divergences — there is nothing that can cause large time separations. There are, as expected, ultraviolet divergences present and they are subtracted away as usual. Note that it is impossible to have a ghost loop since there is no quartic vertex with two gravitons and a ghost-antighost pair. This is not the case, however, when the 3-point vertices are involved: there is a graviton loop contribution (Fig. 14b) and a ghost loop contribution (Fig. 14c).

The explicit calculation of the last two diagrams is straightforward but extremely tedious. A brief description of it must start by mentioning the different structures one may get in the course of the computation. The generic form of the basic ingredients is:

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$\frac{1}{x^2} + \ln x^2$ for the pseudo-graviton and ghost propagators, $\partial\Omega \partial\psi \psi + \partial\psi \partial\psi \psi$ for the pseudo-graviton 3-point vertex with the understanding that the external of the three pseudo-gravitons is a physical one, and, an analogous expression for the ghost vertex where the only pseudo-graviton is again given by (3.15).

It is worth noting the absence of terms of the form $\ln^2 x^2$ in evaluating $\left[\mu\nu \Sigma_1^{\rho\sigma} \right] (x; x')$ — the latter being basically the square of a propagator. Because the action of a derivative on $\ln x^2$ turns it into a power, to obtain a $\ln^2 x^2$ term we must have both propagators undifferentiated. In turn this can occur from vertex products with two undifferentiated fields at each point. In the case of Fig. 14b the only such product is: $(\delta_\mu^0 \partial\psi \psi) (\delta_\mu^0 \partial\psi \psi)$ — where we used that: $\partial\Omega \sim \delta_\mu^0$. If even one δ_μ^0 contracts on an external pseudo-graviton we get zero due to (3.18a). Furthermore the same result is obtained when a δ_μ^0 contracts to an internal pseudo-graviton: the $\ln x^2$ piece of its propagator does not couple to time components (see (3.8)). The only remaining possibility is for δ_μ^0 to contract to another derivative. But then this contraction — which by inspection can only involve a derivative acting on an internal pseudo-graviton — gives a time derivative which changes the logarithm into a power. A similar argument can be made for the ghost loop (Fig. 14c).

We should now display the general types of spatial integrals that can appear in (3.20a). We start from the fundamental integral:

$$I(k, \Delta u) \equiv \int d^3x \frac{\exp(-i\vec{k} \cdot \vec{x})}{\left(\|\vec{x}\|^2 - \Delta u^2 + i\epsilon \right)} = \frac{2\pi^2}{k} \exp(-ik|\Delta u|) \quad (3.21a)$$

By taking derivatives we can increase the power of the denominator:

$$\begin{aligned} I_n(k, \Delta u) &\equiv \int d^3x \frac{\exp(-i\vec{k} \cdot \vec{x})}{\left(\|\vec{x}\|^2 - \Delta u^2 + i\epsilon \right)^{n+1}} \\ &= \frac{1}{2^n n!} \left(\frac{1}{\Delta u - i\epsilon} \frac{\partial}{\partial |\Delta u|} \right)^n I(k, \Delta u) \end{aligned} \quad (3.21b)$$

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and bring powers of $i\vec{k} \cdot \vec{x}$ in the numerator:

$$\begin{aligned} \Gamma_n^m(k, \Delta u) &\equiv \int d^3x \ (i\vec{k} \cdot \vec{x})^m \frac{\exp(-i\vec{k} \cdot \vec{x})}{(\|\vec{x}\|^2 - \Delta u^2 + i\epsilon)^{n+1}} \\ &= (-k)^m \left(\frac{\partial}{\partial \vec{k}} \right)^m \Gamma_n(k, \Delta u) \end{aligned} \quad (3.21c)$$

From the second fundamental integral:

$$\begin{aligned} \mathcal{J}(k, \Delta u) &\equiv \int d^3x \ \exp(-i\vec{k} \cdot \vec{x}) \frac{\ln \left[\frac{H^2(\|\vec{x}\|^2 - \Delta u^2 + i\epsilon)}{(\|\vec{x}\|^2 - \Delta u^2 + i\epsilon)} \right]}{\left(\|\vec{x}\|^2 - \Delta u^2 + i\epsilon \right)} \\ &= 2\pi^2 \left\{ \frac{\exp(-ik|\Delta u|)}{k} \left[\ln(H^2 u^2) - \ln(ku) - \gamma - \frac{i}{2}\pi \right] \right. \\ &\quad \left. - \frac{\exp(-ik|\Delta u|)}{k} \text{Ei}(-2ik|\Delta u|) \right\} \end{aligned} \quad (3.22a)$$

we obtain its necessary generalizations:

$$\begin{aligned} \mathcal{J}_n(k, \Delta u) &\equiv \int d^3x \ \exp(-i\vec{k} \cdot \vec{x}) \frac{\ln \left[\frac{H^2(\|\vec{x}\|^2 - \Delta u^2 + i\epsilon)}{(\|\vec{x}\|^2 - \Delta u^2 + i\epsilon)} \right]}{\left(\|\vec{x}\|^2 - \Delta u^2 + i\epsilon \right)^{n+1}} \\ &= \frac{1}{2^n n!} \left(\frac{1}{\Delta u} \frac{\partial}{\partial |\Delta u|} \right)^n \left\{ \mathcal{J}(k, \Delta u) + \sum_{l=1}^n \frac{1}{l} l(k, \Delta u) \right\} \end{aligned} \quad (3.22b)$$

$$\begin{aligned} \mathcal{J}_n^m(k, \Delta u) &\equiv \int d^3x \ (i\vec{k} \cdot \vec{x})^m \exp(-i\vec{k} \cdot \vec{x}) \frac{\ln \left[\frac{H^2(\|\vec{x}\|^2 - \Delta u^2 + i\epsilon)}{(\|\vec{x}\|^2 - \Delta u^2 + i\epsilon)} \right]}{\left(\|\vec{x}\|^2 - \Delta u^2 + i\epsilon \right)^{n+1}} \\ &= (-k)^m \left(\frac{\partial}{\partial \vec{k}} \right)^m \mathcal{J}_n(k, \Delta u) \end{aligned} \quad (3.22c)$$

As argued above — and verified by the actual calculation — there is no need for a generating spatial integral involving $\ln^2 x^2$.

It becomes appropriate at this stage to provide the basic steps involved in reducing a contribution to $\Sigma(u, k)$ to the aforementioned integrals (3.21)-(3.22). As an example we

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shall select the part of the diagram (12b) where at the observation point $(u, \vec{x})$ the vertex is:

$$i\kappa \psi^{\alpha\beta} \epsilon_{\alpha\beta\mu} \psi^{\mu\nu} \Omega_\nu \Omega \quad (3.23a)$$

while at the interaction point $(u', \vec{x}')$ it is:

$$-\frac{i}{2} \kappa \psi^{\alpha'\beta'} \epsilon_{\alpha'\beta'\mu'} \psi^{\mu'\nu'} \Omega'^2 \quad (3.23b)$$

In terms of the pseudo-graviton propagator (3.8) the resulting 1PI 2-point function takes the form:

$$\begin{aligned} \left[{}^{\alpha\beta} S_1^{\alpha'\beta'} \right](x; x') &= -\frac{i\kappa^2}{2H^4 u^3 u'^2} \left\{ i \left[{}^{\alpha\beta} \Delta^{\alpha'\beta'} \right](x; x') i \left[{}^{\mu 0} \Delta_{\mu'} \right](x; x') \right. \\ &\quad \left. + i \left[{}^{\alpha\beta} \Delta_{\mu'} \right](x; x') i \left[{}^{\mu 0} \Delta^{\alpha'\beta'} \right](x; x') \right\} \end{aligned} \quad (3.24)$$

Because the index pairs $(\alpha\beta)$ and $(\alpha'\beta')$ are contracted into physical polarization tensors — and the latter satisfy (3.18) — the second term in (3.24) vanishes. From the remaining one we get:

$$\begin{aligned} \left[{}^{\alpha\beta} S_1^{\alpha'\beta'} \right](x; x') &= \frac{i\kappa^2}{16\pi^4} \eta^{\alpha\alpha'} \eta^{\beta\beta'} \left\{ \overleftarrow{\partial}_u \frac{2}{u' u [(x - x')^2 + i\epsilon]^2} \overrightarrow{\partial}_{u'} \right. \\ &\quad + \overleftarrow{\partial}_u \frac{4(x - x')^{\mu'}}{u [(x - x')^2 + i\epsilon]^3} \overrightarrow{\partial}_{\mu'} \\ &\quad - \overleftarrow{\partial}_u \frac{\ln[H^2(x - x')^2 + i\epsilon]}{u'^2 u^2 [(x - x')^2 + i\epsilon]} \overrightarrow{\partial}_{u'} \\ &\quad \left. - \overleftarrow{\partial}_u \frac{(x - x')^{\mu'} \ln[H^2(x - x')^2 + i\epsilon]}{u' u^2 [(x - x')^2 + i\epsilon]^2} \overrightarrow{\partial}_{\mu'} \right\} \end{aligned} \quad (3.25)$$

According to (3.19) the first step in arriving at the self-energy is to multiply (3.25) on the right with $\Psi_{\alpha'\beta'}(u', \vec{x}'; \vec{k}, \lambda)$ and on the left with $\epsilon_{\alpha\beta}(\vec{k}, \lambda)$, and then to sum over all polarizations λ using (3.18). The second step is to act the right oriented derivative operator on $\Psi(u', \vec{x}'; \vec{k}, \lambda)$ where we have defined:

$$\Psi_{\alpha\beta}(u', \vec{x}'; \vec{k}, \lambda) \equiv \Psi(u', \vec{x}'; \vec{k}, \lambda) \ \epsilon_{\alpha\beta}(\vec{k}, \lambda) \quad (3.26)$$

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The result of these two steps is:

$$\begin{aligned}
\sum_{\lambda} \epsilon_{\alpha\beta}(\bar{k}, \lambda) \left[{}^{\alpha\beta} S_1^{\alpha\beta} \right] (u, \bar{x}, u', \bar{x}') &= \frac{i\kappa^2}{16\pi^4} \\
&\left\{ \bar{\partial}_u \frac{2}{u'u[(x-x')^2 + i\epsilon]^2} \frac{ik}{(1 + \frac{1}{\kappa u})} \Psi(u', \bar{x}'; \bar{k}, \lambda) \right. \\
&+ \bar{\partial}_u \frac{4(u-u')}{u[(x-x')^2 + i\epsilon]^3} \frac{ik}{(1 + \frac{1}{\kappa u})} \Psi(u', \bar{x}'; \bar{k}, \lambda) \\
&+ \bar{\partial}_u \frac{4i\bar{k} \cdot (\bar{x} - \bar{x}')}{u[(x-x')^2 + i\epsilon]^3} \Psi(u', \bar{x}'; \bar{k}, \lambda) \\
&- \bar{\partial}_u \frac{\ln[H^2(x-x')^2 + i\epsilon]}{u^2 u'^2 [(x-x')^2 + i\epsilon]} \frac{ik}{(1 + \frac{1}{\kappa u})} \Psi(u', \bar{x}'; \bar{k}, \lambda) \\
&- \bar{\partial}_u \frac{2(u-u') \ln[H^2(x-x')^2 + i\epsilon]}{u'u^2 [(x-x')^2 + i\epsilon]^2} \frac{ik}{(1 + \frac{1}{\kappa u})} \Psi(u', \bar{x}'; \bar{k}, \lambda) \\
&\left. - \bar{\partial}_u \frac{2i\bar{k} \cdot (\bar{x} - \bar{x}') \ln[H^2(x-x')^2 + i\epsilon]}{u'u^2 [(x-x')^2 + i\epsilon]^2} \Psi(u', \bar{x}'; \bar{k}, \lambda) \right\} \quad (3.27)
\end{aligned}$$

The action of the left oriented derivative is understood to be — by integration by parts — as right oriented action with reversed sign. We reach the desired form after multiplying on the left with $\Psi^{-1}(u, \bar{x}; \bar{k}, \lambda)$ and integrating over the interaction spacetime position as (3.19) instructs — and then effecting the change of variables $\bar{x} - \bar{x}' = \bar{x}'$:

$$\begin{aligned}
S_1(u; k) &= \frac{i\kappa^2}{16\pi^4} \frac{\exp(-iku)}{(1 - iku)} \bar{\partial}_u \\
&\left\{ -\frac{2\kappa^2}{u} \int_0^{+\infty} du' \exp(iku') I_1(k, \Delta u) \right. \\
&- \frac{4\kappa^2}{u} \int_0^{+\infty} du' \exp(iku') u'(u-u') I_2(k, \Delta u) \\
&- \frac{1}{u} \int_0^{+\infty} du' \exp(iku') (1 - iku') I_2^1(k, \Delta u) \\
&+ \frac{\kappa^2}{u^2} \int_0^{+\infty} du' \exp(iku') \frac{1}{u'} J_1(k, \Delta u) \\
&+ \frac{2\kappa^2}{u^2} \int_0^{+\infty} du' \exp(iku') (u-u') J_1(k, \Delta u) \\
&\left. + \frac{2}{u^2} \int_0^{+\infty} du' \exp(iku') \frac{(1 - iku')}{u'} J_1^1(k, \Delta u) \right\} \quad (3.28)
\end{aligned}$$

By using *Mathematica* 2.1 and *FeynCalc* 1.0 we have been able to carry out the requisite

tensor algebra and reduce the overall answer in a satisfactory fashion. Since all spatial integrations give a factor of $|\Delta u|$ we must distinguish the two possibilities. The answer will be given as a function $\mathcal{P}_1(u; u', k)$ from which we arrive at the self-energy by integrating over the interaction time:

$$\Sigma_1(u; k) = \frac{\kappa^2}{8\pi} \int_0^{+\infty} du' \mathcal{P}_1(u; u'; k) \quad (3.29)$$

When $u' < u$ the function $\mathcal{P}_1(u; u'; k)$ equals:

$$\begin{aligned}
&\left\{ -\frac{360u^5 - 2800u^4 u' + 6376u^3 u'^2 - 3940u^2 u'^3 + 1220u u'^4 - 120u^5}{15u^3 u'^2 (-u + u')^5 \|1 - iku\|^2} \right. \\
&+ \frac{-380u^7 - 610u^6 u' + 1178u^5 u'^2 - 29942u^4 u'^3 + 27308u^3 u'^4}{15u^3 u'^2 (-u + u')^5 \|1 - iku\|^2} k^2 \\
&+ \frac{(-11250u^2 u'^5 + 1670u u'^6 + 120u'^7) k^2}{15u^3 u'^2 (-u + u')^5 \|1 - iku\|^2} \\
&+ \frac{(910u^4 - 4920u^3 u' + 9572u^2 u'^2 - 4960u u'^3 + 750u'^4) k^4}{15u u' (-u + u')^3 \|1 - iku\|^2} \\
&+ i \frac{(400u^5 - 2600u^4 u' + 7952u^3 u'^2 - 4280u^2 u'^3 + 1240u u'^4 - 120u^5) k}{15u^3 u'^2 (-u + u')^4 \|1 - iku\|^2} \\
&+ i \frac{(220u^6 + 1540u^5 u' - 13132u^4 u'^2 + 29960u^3 u'^3 - 23812u^2 u'^4) k^3}{15u^2 u'^2 (-u + u')^4 \|1 - iku\|^2} \\
&+ i \frac{(8140u u'^5 - 930u'^6) k^3}{15u^2 u'^2 (-u + u')^4 \|1 - iku\|^2} \\
&+ i \frac{(-64u^2 + 32u u' - 30u'^2) k^5}{3(-u + u')^2 \|1 - iku\|^2} \exp[-2ik(u-u')] \\
&+ \frac{32(u-u')(1 + k^2 u u') k^2}{u^2 u'^2 \|1 - iku\|^2} \\
&\left\{ \ln[H^2(u-u')^2] - \ln[k(u-u')] - \gamma - \frac{i}{2}\pi \right\} \exp[-2ik(u-u')] \\
&- i \frac{16(1 - ku + ku' + k^2 u u')(1 + ku - ku' + k^2 u u') k}{u^2 u'^2 \|1 - iku\|^2} \\
&\left\{ \ln[H^2(u-u')^2] - \ln[k(u-u')] - \gamma - \frac{i}{2}\pi \right\} \exp[-2ik(u-u')] \\
&+ \frac{16k^2}{u^3 \|1 - iku\|^2} \text{Ei}[-2ik(u-u')] \\
&+ i \frac{16(1 + k^2 u^2 + k^2 u'^2) k}{u^2 u'^2 \|1 - iku\|^2} \text{Ei}[-2ik(u-u')] \quad (3.30a)
\end{aligned}$$

Let us isolate the dominant terms as we keep the observation time u fixed and make the interaction occur in the far future ($u' \rightarrow 0$):

$$\begin{aligned} & \left\{ \left(\frac{112}{3} + \frac{76}{3} k^2 u^2 \right) + i \left(\frac{80}{3} k u + \frac{44}{3} k^3 u^3 \right) \right\} \frac{\exp(-2ik u)}{u^3 u'^2 \|1 - ik u\|^2} \\ & + (32k^2 u^2 - i 16k u) \left\{ \ln \left(\frac{H^2 u}{k} \right) - \gamma - \frac{i}{2} \pi \right\} \frac{\exp(-2ik u)}{u^3 u'^2 \|1 - ik u\|^2} \\ & + i \frac{16k u}{u^3 u'^2 \|1 - ik u\|^2} \text{Ei}(-2ik u) \end{aligned} \quad (3.30b)$$

It is apparent that the self-energy has infrared divergences both in its real and imaginary parts. These divergences are oscillatory in nature and they provide yet another strong signal of the instability of the de Sitter background. We should not be concerned with the lack of a definite sign in these divergences since the region $u' < u$ cannot contribute to the background decay. Such a decay involves — as in the scalar analog — the calculation of a real expectation value using the modified “in-in” formalism. All contributions to it have to be causal and this is not the case when the interaction occurs after the observation. Although the above infrared infinities show the breakdown of the theory, they do not drive any relaxation mechanism.

When $u < u'$ we get the following $\mathcal{P}_1(u, u', k)$:

$$\begin{aligned} & \frac{560u^5 - 2800u^4 u' + 6376u^3 u'^2 - 3940u^2 u'^3 + 1220u u'^4 - 120u'^5}{15u^3 u'^2 (-u + u')^5 \|1 - ik u\|^2} \\ & + \frac{(60u^7 + 1650u^6 u' - 9756u^5 u'^2 + 23358u^4 u'^3 - 22756u^3 u'^4) k^2}{15u^3 u'^2 (-u + u')^5 \|1 - ik u\|^2} \\ & + \frac{(10490u^2 u'^5 - 1630u u'^6 - 120u'^7) k^2}{15u^3 u'^2 (-u + u')^5 \|1 - ik u\|^2} \\ & + \frac{(1350u^4 - 5920u^3 u' + 8928u^2 u'^2 - 5680u u'^3 + 1350u'^4) k^4}{15u u' (-u + u')^3 \|1 - ik u\|^2} \\ & + \frac{(-48u + 8u') k}{u^3 u'^2 \|1 - ik u\|^2} + i \frac{180 k^5}{\|1 - ik u\|^2} \\ & + i \frac{(-220u^6 + 40u^5 u' + 2812u^4 u'^2 - 7468u^3 u'^3 + 9252u^2 u'^4) k^3}{15u^2 u'^2 (-u + u')^4 \|1 - ik u\|^2} \\ & - i \frac{(3400u u'^5 + 1230u'^6) k^3}{15u^2 u'^2 (-u + u')^4 \|1 - ik u\|^2} \end{aligned}$$

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$$\begin{aligned} & - \frac{16k^2}{u^3 \|1 - ik u\|^2} \left\{ \ln[H^2(u' - u)^2] - \ln[k(u' - u)] - \gamma - \frac{i}{2} \pi \right\} \\ & - i \frac{16(1 + k^2 u^2 + k^2 u'^2) k}{u^2 u'^2 \|1 - ik u\|^2} \left\{ \ln[H^2(u' - u)^2] - \ln[k(u' - u)] - \gamma - \frac{i}{2} \pi \right\} \\ & + \frac{32(-u + u')(1 + k^2 u u')}{u^2 u'^2 \|1 - ik u\|^2} \exp[2ik(u - u')] \text{Ei}[2ik(u - u')] \\ & + i \frac{16(1 - ku + ku' + k^2 u u')(1 + ku - ku' + k^2 u u') k}{u^2 u'^2 \|1 - ik u\|^2} \\ & \exp[2ik(u - u')] \text{Ei}[2ik(u - u')] \end{aligned} \quad (3.31a)$$

In this case the interaction occurs before the observation and we have the proper causal order to contribute to an expectation value calculated with the modified “in-in” formalism. Indeed we expect that part of (3.31a) will participate in the evaluation of the two-loop graviton 1-point function (see Fig. 1). Therefore the sign of the dominant infrared terms — as our observation time increases with respect to the interaction time — becomes important. As $u \rightarrow 0$ we find:

$$- \frac{8}{u^3} \left\{ \frac{1}{u^2} + k^2 - i \frac{k}{u'} \right\} - \frac{16k^2}{u^3} \left\{ \ln \left(\frac{H^2 u'}{k} \right) - \gamma - \frac{i}{2} \pi \right\} \quad (3.31b)$$

The leading contributions to the self-energy are *negative definite* and *growing* with the observation time. This strong infrared effect is accountable for the background decay. It has the proper causal structure and the capability to generate in time arbitrarily large amounts of negative self-energy. It is the generation in time of such amounts — something explicitly verified here — which is responsible for the change in the background geometry and the associated reduction in time of the effective cosmological constant.

The presence of a divergent imaginary part in (3.31b) — as in the previous case — is further evidence of the graviton instability. For instance it implies that the tree order process of one graviton going into two is allowed (see Fig. 13a) — something which can also be seen directly [8]. Finally we should note the need to renormalize the self-energy from the ultraviolet divergences that are present in both cases when $u' \rightarrow u$. This can be

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achieved by using the covariant counterterms (1.4) as well as the BRS counterterms of the gauge fixed theory.

If the appearance of infrared divergences in the on-shell self-energy of QCG is to have physical significance it must either be independent of the choice of gauge, or else the phenomenon must be understood — in our fixed gauge — as forming part of a larger effect from which all gauge dependence cancels. In fact both conditions seem to apply. We will see in Section 4 that the infrared problems of the one-loop self-energy imply a growth for the two-loop 1-point function which produces a secular term in the effective cosmological constant. Even if the self-energy's infrared divergences were to change with the gauge, $\Lambda_{\text{eff}}(t)$ ought still to be gauge independent by virtue of having been defined in a unique coordinate system. However, there is every reason for believing that the infrared divergences of the on-shell self-energy of QCG are gauge independent in their own right. This might seem surprising in view of the known gauge dependence of the Yang-Mills self-energy [25], and of conventional quantum gravity in flat space perturbation theory [26]. The key distinction between these quantities and the infrared divergent part of the on-shell QCG self-energy is locality. On shell quantities must either be gauge independent or else they must be absorbable into local counterterms. The on-shell self-energies of both Yang-Mills and conventional quantum gravity are just numbers times the appropriate tensor functions of the 3-momentum $\vec{k}$. Any such term can be absorbed into a local counterterm, and we typically use this freedom to make the self-energies of these theories vanish on shell. The self-energy of QCG has such terms too, and they are presumably as gauge dependent as those of conventional quantum gravity. However, infrared divergent contributions depend in a complicated, non-local way upon $\vec{k}$, even after the tensor factors have all been extracted. This dependence cannot be absorbed into a local counterterm. Since it is on shell it must therefore be independent of the choice of gauge.

So far we have shown that S-matrix elements and the on-shell self-energy contain

infrared infinities in QCG. Although this is perfectly adequate to demonstrate that these infinities have physical consequences, the field theoretic aesthete may insist on a proof using a gauge invariant quasi-localized object. To do so we can examine the infrared behaviour of a physically meaningful observable such as the force that must be exerted to hold a test mass m_1 stationary with respect to a large mass m_2 . Quantum corrections to the classical force law, given by expression (1.10), must be free of infrared infinities. To infer such corrections one must have a suitable QCG observable. Finding such an operator and analyzing it properly is an undertaking of sufficient subtlety that the answer to the analogous problem in QCD is worth mentioning. The relevant observable for the latter theory is the Wilson loop:

$$W_{\text{QCD}}(t, r) \equiv \text{Tr} \left(P \exp \left[ig \oint dx^\mu A_\mu(x) \right] \right) \quad (3.32)$$

where P is the path-ordering symbol, A_μ is the matrix-valued, Yang-Mills connection operator and the integration proceeds along a rectangular path of spatial extent r and temporal extent t . The timelike segments of this path can be thought of as describing the interaction between the color field and two oppositely charged, point particles. The force exerted by the one particle upon the other is given by the following limit:

$$F_{\text{QCD}}(r) = \lim_{t \rightarrow \infty} \frac{1}{it} \frac{\partial}{\partial t} \text{Im} \ln \left(W_{\text{QCD}}(t, r) \right) \quad (3.33)$$

The expectation value is evaluated in pure QCD. Note that this definition is free of anomalous dimensions and that the differentiation with respect to r eliminates the self energy induced by each particle's interaction with its own color field. Furthermore, the Wilson loop is gauge invariant and of finite spatial and temporal extent. Thus, it is protected from contamination by the infrared divergences that plague the on-shell S-matrix of QCD.

It is not simple to construct an acceptable gravitational analog to the Wilson loop. The most natural candidate might seem to be the amplitude for propagating around a

suitably shaped, closed loop:

$$\exp \left[-im \oint d\tau \sqrt{-g_{\mu\nu}(\chi(\tau)) \dot{\chi}^\mu(\tau) \dot{\chi}^\nu(\tau)} \right] \quad (3.34)$$

The first problem with this observable is that it fails to account for a crucial difference between gravity and QCD. Just as the timelike segments of the Wilson loop describe the propagation of oppositely charged particles so the timelike segments of this amplitude represent two future-directed particles of opposite mass. There is a strong physical prejudice against this sort of thing.

The second problem concerns the use of point particles. Arnowitt, Deser and Misner showed long ago that for an uncharged point particle any finite bare mass is renormalized to zero by nonperturbative classical effects [27]. What we need, therefore, is two forward-directed sources which are dispersed enough to make the gravitational self-interaction of each source small, at least classically.

We shall take each source to be a continuous distribution of particles of infinitesimal mass executing very slow orbits around a central point. Any one of these particles contributes the following sort of factor to the full amplitude:

$$\exp \left[-i\delta m \int_{-t/2}^{t/2} d\tau \sqrt{-g_{\mu\nu}(\chi(\tau)) \dot{\chi}^\mu(\tau) \dot{\chi}^\nu(\tau)} \right] \quad (3.35)$$

The path $\chi^\mu(\tau)$ executes a prescribed motion with respect to the center of the appropriate source. Just as in expression (1.10) there is a heavy source and a light one. We shall assume that the particles which comprise each source move slowly with respect to one another and are sufficiently dispersed that their self interactions can be ignored. In this case the masses m_1 and m_2 are just the sums of the individual particle masses δm in each source.

In order to avoid complications arising from the fluctuating quantum metric we shall specify the relation between the two centers with respect to the geometry of the classical

limit — which we define to include the large source. It is natural to take the center $C^\mu(\tau)$ of the large source to be a timelike geodesic. Neglecting self-interactions this reduces the classical contribution of the entire first source to $\exp(-im_2 t)$, where t is the geodesic time. The effect we are studying comes from maintaining the center of the small source, $c^\mu(\tau)$, static with respect to $C^\mu(\tau)$.

The construction which gives $c^\mu(\tau)$ is tedious but straightforward. One first picks any spacelike vector U^μ perpendicular to the geodesic $C^\mu(\tau)$ at the point $\tau = 0$.

$$g_{\alpha\beta}(\dot{C}(\tau)) \dot{C}^\alpha(\tau) U^\beta(\tau) = 0 \quad (3.36)$$

By parallel transporting U^μ along $C^\mu(\tau)$ one can then define a family of spacelike vectors $U^\mu(\tau)$ which are perpendicular to $C^\mu(\tau)$:

$$P^{\mu\alpha}(\tau, \sigma) + \Gamma^\mu_{\alpha\beta}(P(\tau, \sigma)) P^{\alpha\beta}(\tau, \sigma) P^\beta(\tau, \sigma) = 0 \quad (3.37a)$$

$$P^\mu(\tau, 0) = C^\mu(\tau) \quad (3.37b)$$

$$P^{\mu\mu}(\tau, 0) = U^\mu(\tau) \quad (3.37c)$$

(A dot denotes differentiation with respect to τ ; a prime denotes differentiation with respect to σ .) The center of the small source is:

$$c^\mu(\tau) \equiv P^\mu(\tau, \sigma(r)) \quad (3.38)$$

where $\sigma(r)$ is the geodesic parameter at which P intersects the sphere of surface area $4\pi r^2$ that is centered on the heavy mass.

Let us call the observable constructed in this way $W_{\text{QCG}}(t, r)$. It gives the QCG force law in analogy to expression (3.33):

$$F_{\text{QCG}}(r) = \lim_{t \rightarrow \infty} \frac{1}{it} \frac{\partial}{\partial r} \text{Im} \ln \langle W_{\text{GR}}(t, r) \rangle \quad (3.39)$$

The expectation value is taken in the theory governed by Lagrangian (1.4), in the appropriate vacuum and with the appropriate subtractions to remove ultraviolet divergences. A

well defined observable such as this should be free of infrared divergences provided that the vacuum is stable.

The hard way to evaluate $F_{QCG}(\tau)$ at the classical level is to sum the effects of tree order graviton exchange. The easy way is to merely substitute the appropriate classical metric into expression (3.35) and then use (3.39). The most convenient coordinate system is one in which the two sources are fixed. For points outside the mass distribution of the large source the line element is:

$$ds^2 = -\left(1 - \frac{1}{3}\lambda r^2 + 2N(r)\right)dt^2 + \frac{dr^2}{1 - \frac{1}{3}\lambda r^2 + 2N(r)} + r^2 d\Omega^2 \quad (3.40a)$$

$$N(r) = -\frac{Gm_2}{r} \quad (3.40b)$$

Relativists refer to this as the de Sitter-Schwarzschild solution and we stress that it is an exact classical solution to the local Einstein equations. In this static coordinate system the two source worldlines are:

$$C^\mu(\tau) = \delta_0^\mu \tau \quad (3.41a)$$

$$r^\mu(\tau) = \delta_0^\mu \tau + (1 - \delta_0^\mu) x^\mu \quad (3.41b)$$

If the particles orbiting the light source are moving very slowly then only the g_{00} component of the metric contributes. Assuming their distribution is small with respect to r the resulting classical force law is the same as (1.10):

$$F_{\text{class}}(\tau) = m_1 \frac{\partial}{\partial \tau} \sqrt{1 - H^2 \tau^2 + 2N(\tau)} \quad (3.42)$$

The point of this exercise is that we now have the machinery for computing quantum corrections to the force law.

The most convenient coordinate system for calculating such quantum corrections is not the static system we used to obtain the classical result, but the "open-conformal" system in which the Feynman rules of de Sitter space take a particularly simple form. We must

first obtain the worldlines of the two masses in "open-conformal" coordinates. To do so we appropriately transform the worldlines (3.41) obtained in static coordinates. The timelike geodesic of the heavy mass goes from $(t, \vec{0})$, where t is the geodesic time, to $(u, \vec{0})$, where u is the conformal geodesic time. Similarly, the worldline of the light mass goes from $(t, r\vec{r})$ to $[1 - H^2 r^2]^{-\frac{1}{2}}(u, uHr\vec{r})$.

Fig. 15: Some one-loop contributions to the force law between a heavy source m_2 and a light source m_1 .

To compute the aforementioned quantum corrections we perform a perturbative expansion of the contribution from the heavy source geodesic:

$$\begin{aligned} & \exp \left[-im_2 \int d\tau \sqrt{-g_{\mu\nu}(\chi_{m_2}(\tau)) \dot{\chi}_{m_2}^\mu(\tau) \dot{\chi}_{m_2}^\nu(\tau)} \right] \\ &= \exp \left\{ -im_2 \int d\tau \Omega(\tau) \left[1 - \frac{\kappa}{2} \psi_{00}(\tau, \vec{0}) + \frac{\kappa^2}{8} \psi_{00}^2(\tau, \vec{0}) - O(\kappa^3) \right] \right\} \end{aligned}$$

$$= \exp(-im_2 t) \left[1 + \frac{i\kappa m_2}{2} \int d\tau \Omega(\tau) \psi_{00}(\tau, \vec{0}) - \frac{i\kappa^2 m_2}{8} \int d\tau \Omega(\tau) \psi_{00}^2(\tau, \vec{0}) - \frac{\kappa^2 m_2^2}{8} \int d\tau \Omega(\tau) \psi_{00}(\tau, \vec{0}) \int d\tau' \Omega(\tau') \psi_{00}(\tau', \vec{0}) + O(\kappa^3) \right] \quad (3.43a)$$

the light source worldline:

$$\begin{aligned} \exp \left[-im_1 \int d\tau \sqrt{-g_{\mu\nu}(\chi_{m_1}(\tau))} \dot{\chi}_{m_1}^\mu(\tau) \dot{\chi}_{m_1}^\nu(\tau) \right] = \\ \exp(-im_1 t) \left[1 + \frac{i\kappa m_1}{2(1-H^2 r^2)} \int d\tau \Omega(\tau) \Upsilon(\chi_{m_1}(\tau)) - \frac{i\kappa^2 m_1}{8(1-H^2 r^2)^2} \int d\tau \Omega(\tau) \Upsilon^2(\chi_{m_1}(\tau)) - \frac{\kappa^2 m_1^2}{8(1-H^2 r^2)^2} \int d\tau \Omega(\tau) \Upsilon(\chi_{m_1}(\tau)) \int d\tau' \Omega(\tau') \Upsilon(\chi_{m_1}(\tau')) + O(\kappa^3) \right] \quad (3.43b) \end{aligned}$$

where:

$$\Upsilon(\chi_{m_1}(\tau)) \equiv \psi_{00}(\chi_{m_1}(\tau)) + 2Hr\psi_{0r}(\chi_{m_1}(\tau)) + H^2 r^2 \psi_{rr}(\chi_{m_1}(\tau)) \quad (3.43c)$$

and the gauge fixed action of the theory.

It is apparent that the dependence of all connected tree diagrams on κ and the masses is of the form $\kappa^2 m_1 m_2$ times some function $f_0(\kappa^2 m_1, \kappa^2 m_2)$. The summation of all such diagrams, examples of which are provided in Fig. 3, gives the classical result expressed by (1.10).

We shall be interested in the one-loop corrections to the force law. The general dependence of a connected one-loop diagram on κ and the masses is $\kappa^4 m_1 m_2$ times some function $f_1(\kappa^2 m_1, \kappa^2 m_2)$. Typical examples of such graphs can be seen in Fig. 15. Suppose we restrict ourselves to the class of diagrams such that $f_1(\kappa^2 m_1, \kappa^2 m_2) = 1$. Since the loop integrations cannot alter the mass dependence of a graph, if the restricted class of diagrams contains infrared divergences so will the complete set of one-loop diagrams. Thus, it is perfectly adequate to examine all connected one-loop graphs proportional to $\kappa^4 m_1 m_2$ (see Fig. 16 for the complete list). Notice that the diagrams (f) and (g) of Fig. 16 are cancelled exactly by the appropriate counterterm to ensure the vanishing of the one-loop

tadpole. Furthermore, there is no ghost loop possibility in Fig. 16e because there is no vertex quadratic in the graviton and ghost fields.

Fig. 16: Complete set of one-loop connected exchanges between the heavy source m_2 and the light source m_1 proportional to $\kappa^4 m_1 m_2$.

The remaining five diagrams (a)-(e) of Fig. 16 contain infrared infinities. Consider, for instance, one of the many terms contributing to the first diagram of Fig. 17:

$$4 \int d^4 x \int d^4 y \exp(-im_2 t) \frac{i\kappa m_2}{2} \int d\tau \Omega(\tau) \exp(-im_1 t) \frac{i\kappa m_1}{2(1-H^2 r^2)} \int d\tau' \Omega(\tau') H^2 r^2 \left[\bar{u}_{00} \Delta_{\alpha_1 \beta_1} \right](\tau, \vec{0}; x) i \left[\alpha_2 \beta_2 \Delta_{\mu_2 \nu_2} \right](x; y) i \left[\alpha_3 \beta_3 \gamma_3 \Delta_{\mu_3 \nu_3, \lambda_3} \right](x; y) i \left[\mu_1 \nu_1 \Delta_{rr} \right](y; \chi_{m_1}(\tau'))$$

$$\frac{\kappa}{H^2 u_z^2} V_{\alpha_1 \beta_1 \alpha_2 \beta_2 \alpha_3 \beta_3 \gamma_3} \frac{\kappa}{H^2 u_y^2} V_{\mu_1 \nu_1 \mu_2 \nu_2 \mu_3 \nu_3 \lambda_3} \quad (3.44)$$

where we have written the cubic piece of (3.12) as:

$$\begin{aligned} \mathcal{L}_{\text{int}}^{(3)} = & \frac{\kappa}{H^2} \left(\frac{1}{u^3} \psi_{\alpha_1 \beta_1} \psi_{\alpha_2 \beta_2} \psi_{\alpha_3 \beta_3 \gamma_3} V^{\alpha_1 \beta_1 \alpha_2 \beta_2 \alpha_3 \beta_3 \gamma_3} \right. \\ & \left. + \frac{1}{u^2} \psi_{\alpha_1 \beta_1} \psi_{\alpha_2 \beta_2 \gamma_2} \psi_{\alpha_3 \beta_3 \gamma_3} V^{\alpha_1 \beta_1 \alpha_2 \beta_2 \gamma_2 \alpha_3 \beta_3 \gamma_3} \right) \quad (3.45) \end{aligned}$$

To determine the superficial infrared degree of divergence we merely count the powers of u associated with each vertex integration. Keeping in mind that the first term of each graviton propagator (3.8) can add factors of u in the numerator while derivatives acting on it can subtract such factors, we can easily conclude that the leading small u behaviour in each of the two vertices is du/u^2 . We expect (3.44) to diverge quadratically with the conformal time cutoff.

Fig. 17: Two of the five one-loop contributions to the gravitational force law.

Another diagram we may wish to consider is that of Fig. 17b. A particular contribution is equal to:

$$\begin{aligned} & 2 \int d^4 x \exp(-im_2 t) \frac{(-i\kappa^2 m_2)}{8} \int d\tau \Omega(\tau) \exp(-im_1 t) \frac{i\kappa m_1}{2(1-H^2 r^2)} \int d\tau' \Omega(\tau') 2Hr \\ & i [00\Delta_{\alpha_1 \beta_1}] (\tau, \vec{0}; x) i [00\Delta_{\alpha_2 \beta_2 \gamma_2}] (\tau, \vec{0}; x) i [\alpha_3 \beta_3 \gamma_3 \Delta_{\tau\tau'}] (x; \backslash m_1(\tau')) \\ & \frac{\kappa}{H^2 u_z^2} V_{\alpha_1 \beta_1 \alpha_2 \beta_2 \gamma_2 \alpha_3 \beta_3 \gamma_3} \quad (3.46) \end{aligned}$$

There is only one vertex being integrated and its leading behaviour is du/u^2 making this contribution subdominant with respect to (3.44).

It is clearly beyond the scope of this paper to explicitly calculate the one-loop correction to the gravitational force law (1.10) coming from the five diagrams (a)-(e) of Fig. 16. The

leading infrared correction should be of the form:

$$N(r) = -\frac{Gm^2}{r} \left(1 + b \, G\Lambda \exp(2HT) + \dots \right) \quad (3.47)$$

where T is a real time infrared cutoff. One expects, due to the nature of gravitation, the infrared singular corrections to be attractive and, thus, the numerical coefficient b to be positive.

The infrared infinities of QCG have a physical interpretation similar to that of the scalar field infinities. In the massless ϕ^3 scalar analog they signal the physical instability of the theory. This instability is eliminated because the background changes. In the correct background the scalar field acquires a mass and the infrared problem is cut off. Indeed as long as real infrared infinities are present the background will keep changing until it reaches a configuration which leads to infrared finiteness. Similarly, the severe infrared divergences of QCG imply the need for the theory to change its background geometry until it solves its infrared nightmare. The perturbative demonstration of this change and its physical consequences form the subject of the next section.

4. The Relaxation Mechanism

The purpose of this section is to explain our formula (1.11) for the effective cosmological constant. We begin by inferring the general form of the amputated 1-point function in a homogeneous and isotropic state: we next derive the effective Hubble constant associated with this general form. The selection of a plausible initial condition for QCG at the onset of inflation permits us to give a simple set of Feynman rules for a modified “in”-“in” formalism which is very similar to that of the scalar analog model considered in section 2. The parallel between the two theories is very close. In each case we must chose the counterterms to completely absorb any one-loop contribution to the amputated 1-point function. The first nontrivial contributions come in each model from infrared processes in

two-loop diagrams which have the same topology as those of Fig. 11. A crucial distinction between the two theories arises due to their different background geometries: the dominant infrared effect of flat space comes from interaction vertices near the initial value surface whereas the causal structure of de Sitter space is such that the dominant infrared effect of QCG must come from interaction vertices at late times. Recognition of this fact, and exploitation of the Feynman rules of QCG, allows us to infer (1.11) even though we do not obtain in this work the numerical coefficients α_0 and α_0 . We also explain what one should expect at higher loops, and how to treat small perturbations in the initial conditions.

We begin by reminding the reader of the relation between the full metric operator, $g_{\mu\nu}$, and the pseudo-graviton field, $\psi_{\mu\nu}$:

$$g_{\mu\nu} \equiv \Omega^2 \tilde{g}_{\mu\nu} \equiv \Omega^2 (\eta_{\mu\nu} + \kappa \psi_{\mu\nu}) \quad , \quad \Omega \equiv (H u)^{-1} \quad (4.1)$$

The pseudo-graviton kinetic operator is [13]:

$$D_{\mu\nu}{}^{\rho\sigma} \equiv \left[\frac{1}{2} \bar{\delta}_{\mu}^{\rho} \bar{\delta}_{\nu}^{\sigma} - \frac{1}{4} \eta_{\mu\nu} \eta^{\rho\sigma} - \frac{1}{2} \delta_{\mu}^0 \delta_{\nu}^0 \delta^{\rho} \delta^{\sigma} \right] D_A + \delta_{(\mu}^0 \delta_{\nu)}^0 D_B + \delta_{\mu}^0 \delta_{\nu}^0 \delta^{\rho} \delta^{\sigma} D_C \quad (4.2)$$

where $D_A \equiv \Omega(\partial^2 + \frac{2}{u^2})\Omega$ is the kinetic operator for a massless, minimally coupled scalar and $D_B = D_C \equiv \Omega \partial^2 \Omega$ is the kinetic operator for a conformally coupled scalar. Recall as well that a barred quantity has its natural zero components nulled, for example:

$$\bar{\delta}_{\nu}^{\mu} \equiv \delta_{\nu}^{\mu} - \delta_0^{\mu} \delta_{\nu}^0 \quad (4.3a)$$

$$\bar{\eta}_{\mu\nu} \equiv \eta_{\mu\nu} + \delta_{\mu}^0 \delta_{\nu}^0 \quad (4.3b)$$

Let us assume a normalized homogeneous and isotropic initial state, $|0\rangle$, for which the expectation value of both the pseudo-graviton field and its conformal time derivative vanish at $u = H^{-1}$. (This is required by the condition that the background be initially de Sitter with $\Lambda_{\text{eff}}(t)|_{t=0} = \Lambda$.) We want to follow the evolution of the expectation value of the metric and to infer from this the effective cosmological constant. Since we are

perturbing around a de Sitter background we need only compute the expectation value of the pseudo-graviton field, $\psi_{\mu\nu}(u, \vec{x})$. From the manifest invariance of the theory and the state under spatial rotations and translations we know the amputated 1-point function must have the following form:

$$\begin{aligned} \kappa \widehat{\psi}_{\mu\nu}(u) &\equiv \left\langle 0 \left| \kappa \psi_{\mu\nu}(u, \vec{x}) \right| 0 \right\rangle \\ &= A(u) \bar{\eta}_{\mu\nu} + C(u) \delta_{\mu}^0 \delta_{\nu}^0 \end{aligned} \quad (4.4a) \quad (4.4b)$$

Note that we cannot assume $C(u) = -A(u)$, even though the background is proportional to $\eta_{\mu\nu}$. Note as well that the assumption of de Sitter invariance would imply not only that $C(u) = -A(u)$ but that they are both constant. Enforcing the renormalization condition $\Lambda_{\text{eff}}(0^+) = 0$ would further imply that this constant is zero.

To find the effective Hubble constant we first compare the quantum-corrected invariant element in conformal coordinates with the standard form of co-moving coordinates:

$$\begin{aligned} \bar{g}_{\mu\nu} dx^{\mu} dx^{\nu} &= \Omega^2 \left\{ - \left[1 - C(u) \right] du^2 + \left[1 + A(u) \right] \|d\vec{x}\|^2 \right\} \\ &= -dt^2 + R^2(t) \|d\vec{x}\|^2 \end{aligned} \quad (4.5a) \quad (4.5b)$$

This allows us to identify the scale factor and the quantum-corrected coordinate transformation which relates u to t :

$$R(t) = \Omega \sqrt{1 + A(u)} \quad (4.6a)$$

$$dt = -\Omega \sqrt{1 - C(u)} du \quad (4.6b)$$

We then apply the standard definition (1.2) of the effective Hubble constant:

$$H_{\text{eff}}(t) = \frac{d}{dt} \ln[R(t)] \quad (4.7a)$$

$$= \frac{d}{dt} \frac{d}{du} \left\{ \ln \Omega + \frac{1}{2} \ln \left[1 + A(u) \right] \right\} \quad (4.7b)$$

to get:

$$H_{\text{eff}}(u) = \frac{H}{\sqrt{1 - C(u)}} \left\{ 1 - \frac{1}{2} u \frac{d}{du} A(u) \right\} \quad (4.7c)$$

Note that this is a *nonperturbative* formula; it requires only the definition (1.2) of H_{eff} and the general form (4.4) of the 1-point function.

Just as with the scalar analog model it turns out to be simpler to compute the *amputated* 1-point function and then attach the retarded propagator. (The external propagator is retarded — as it was for the scalar model — because we are computing expectation values rather than matrix elements.) The most general form of the amputated 1-point function is:

$$D_{\mu\nu}^{\rho\sigma} \kappa_{\rho}^{\mu} \hat{\nu}^{\sigma}(u) = a(u) \bar{\eta}_{\mu\nu} + c(u) \delta_{\mu}^0 \delta_{\nu}^0 \quad (4.8)$$

Substitution of (4.4) reveals that $A(u)$ and $C(u)$ obey the following differential equations:

$$-\frac{1}{4} D_A \left[A(u) - C(u) \right] = \frac{1}{4} \Omega^2 \left\{ \left(\frac{d}{du} \right)^2 - \frac{2}{u} \frac{d}{du} \right\} \left[A(u) - C(u) \right] = a(u) \quad (4.9a)$$

$$D_C C(u) = \Omega^2 \left\{ - \left(\frac{d}{du} \right)^2 + \frac{2}{u} \frac{d}{du} - \frac{2}{u^2} \right\} C(u) = 3 a(u) + c(u) \quad (4.9b)$$

The solution can be given in terms of the retarded Green's functions for D_A and D_C :

$$A(u) = -4 G_A^{\text{ret}} \left[a \right] (u) + G_C^{\text{ret}} \left[3a + c \right] (u) \quad (4.10a)$$

$$C(u) = G_C^{\text{ret}} \left[3a + c \right] (u) \quad (4.10b)$$

Since inflation commences at $u = H^{-1}$ the appropriate boundary conditions are that the field and its first u derivative vanish at this point. It is simple to enforce these conditions on the inverses of D_A and D_C for any power of u :

$$G_A^{\text{ret}} \left[u^{-4} (Hu)^{\epsilon} \right] (u) = \frac{H^2}{3\epsilon(1 - \frac{1}{3}\epsilon)} \left\{ (Hu)^{\epsilon} - (1 - \frac{1}{3}\epsilon) - \frac{1}{3}\epsilon (Hu)^3 \right\} \quad (4.11a)$$

$$G_C^{\text{ret}} \left[u^{-4} (Hu)^{\epsilon} \right] (u) = \frac{H^2}{2(1 - \epsilon)(1 - \frac{1}{2}\epsilon)} \left\{ -(Hu)^{\epsilon} + 2(1 - \frac{1}{2}\epsilon)Hu - (1 - \epsilon)(Hu)^2 \right\} \quad (4.11b)$$

We will shortly see that the strongest infrared effects obtainable from perturbation theory result in the functions $a(u)$ and $c(u)$ behaving for small u as u^{-4} times powers of $\ln(Hu)$.

The contributions such terms make to $A(u)$ and $C(u)$ can be obtained by differentiation with respect to ϵ : $\ln(Hu) = \frac{d}{d\epsilon} (Hu)^{\epsilon} |_{\epsilon=0}$. For example, with zero, one and two logarithms the inverse of D_A gives:

$$G_A^{\text{ret}} \left[u^{-4} 1 \right] (u) = H^2 \left\{ \frac{1}{3} \ln(Hu) + \frac{1}{9} - \frac{1}{9} (Hu)^3 \right\} \quad (4.12a)$$

$$G_A^{\text{ret}} \left[u^{-4} \ln(Hu) \right] (u) = H^2 \left\{ \frac{1}{6} \ln^2(Hu) + \frac{1}{9} \ln(Hu) + \frac{1}{27} - \frac{1}{27} (Hu)^3 \right\} \quad (4.12b)$$

$$G_A^{\text{ret}} \left[u^{-4} \ln^2(Hu) \right] (u) = H^2 \left\{ \frac{1}{6} \ln^3(Hu) + \frac{1}{9} \ln^2(Hu) + \frac{2}{27} \ln(Hu) + \frac{2}{81} - \frac{2}{81} (Hu)^3 \right\} \quad (4.12c)$$

The same exercise for D_C gives:

$$G_C^{\text{ret}} \left[u^{-4} 1 \right] (u) = H^2 \left\{ -\frac{1}{2} + Hu - \frac{1}{2} (Hu)^2 \right\} \quad (4.13a)$$

$$G_C^{\text{ret}} \left[u^{-4} \ln(Hu) \right] (u) = H^2 \left\{ -\frac{1}{2} \ln(Hu) - \frac{3}{4} + Hu - \frac{1}{4} (Hu)^2 \right\} \quad (4.13b)$$

$$G_C^{\text{ret}} \left[u^{-4} \ln^2(Hu) \right] (u) = H^2 \left\{ -\frac{1}{2} \ln^2(Hu) - \frac{3}{2} \ln(Hu) - \frac{7}{4} + 2Hu - \frac{1}{4} (Hu)^2 \right\} \quad (4.13c)$$

To obtain (1.11) suppose — as we will shortly argue — that the coefficients $a(u)$ and $c(u)$ have the form:

$$a(u) = \frac{\kappa^4 H^2}{u^4} \left\{ a_0 \ln^2(Hu) + (\text{subdominant}) \right\} + O(\kappa^6 H^4) \quad (4.14a)$$

$$c(u) = \frac{\kappa^4 H^2}{u^4} \left\{ c_0 \ln^2(Hu) + (\text{subdominant}) \right\} + O(\kappa^6 H^4) \quad (4.14b)$$

where the “subdominant” terms diverge less strongly as u approaches zero. Substituting (4.12) and (4.13) into (4.10) gives:

$$A(u) = \kappa^4 H^4 \left\{ -\frac{4}{9} a_0 \ln^3(Hu) + (\text{subdominant}) \right\} + O(\kappa^6 H^6) \quad (4.15a)$$

$$C(u) = \kappa^4 H^4 \left\{ -\frac{1}{2} (3a_0 + c_0) \ln^2(Hu) + (\text{subdominant}) \right\} + O(\kappa^6 H^6) \quad (4.15b)$$

Inserting these results into (4.7) we obtain the following expression for the effective Hubble constant:

$$H_{\text{eff}}(u) = H \left\{ 1 - \kappa^4 H^4 \left[\frac{1}{12} a_0 + \frac{1}{4} c_0 \right] \ln^2(Hu) + (\text{subdominant}) \right\} + O(\kappa^6 H^6) \quad (4.16)$$

Using the definition (1.3) of Λ_{eff} and making a few simple substitutions results in (1.11).

So much for the form of $\kappa\hat{\psi}_{\mu\nu}(u)$ and its relation to Λ_{eff} , we turn now to the issue of initial conditions. We are not interested in perfect, mathematical de Sitter space, but rather the sort of inflationary cosmology that might be reached by causal evolution from the Big Bang. The initial temperature of such a universe must be so much greater than $(\Lambda/\kappa^2)^{\frac{1}{4}}$ that thermal contributions to the stress-energy tensor far exceed those due to the cosmological constant. There is no possibility for strong infrared effects during this period because thermal fluctuations would disrupt the long range correlations needed to sustain them. Under these conditions it is just as if there were no cosmological constant: the universe is radiation dominated and its background geometry evolves classically. The universe cools as it expands, and eventually the temperature drops below $(\Lambda/\kappa^2)^{\frac{1}{4}}$. At this point the cosmological constant begins to dominate the stress-energy tensor and inflation commences. After a few e-foldings the temperature will have redshifted to so near zero, and any initial inhomogeneities or anisotropies will have been smoothed to such an extent, that it is an excellent approximation to use zero temperature QCG in a homogeneous and isotropic state. It is at this time that the infrared effect we are studying begins to build up.

It is crucial to note that the onset of inflation is very unlikely to be simultaneous throughout space. This is a trivial consequence of statistics and causality. The Big Bang is not likely to have been perfectly homogeneous — and homogeneity would in any case be spoiled by thermal fluctuations. There is no reason to suppose that thermal equilibrium can be maintained across a region much larger than the effective Hubble radius. Since the effective Hubble constant at the moment of onset is just H , we are interested in following inflation that commences on a patch of radius H^{-1} . This region need not be the first to experience inflation, nor is the larger universe beyond it of any interest to us. Interior points rapidly lose causal contact with the boundary and with everything beyond it: to late

time observers throughout most of the region's volume, the inflating patch is the universe.

The initial condition we have described has two important consequences. The first is that we can entirely ignore problems associated with trying to formulate even free QCG on an initial value surface of infinite spatial extent [8,13]. However fascinating this problem may be to the mathematically inclined, it is irrelevant to physics because its initial condition has zero probability of having occurred. The second consequence is that we can be sloppy about spatial boundary conditions and the associated zero modes because physics in the interior is very largely insensitive to them. We shall accordingly simplify the mode sums by imposing periodic boundary conditions. To be more explicit, we assume each of the three spatial coordinates lies within the range:

$$-\frac{1}{2H} < r_i \leq \frac{1}{2H} \quad (4.17)$$

with the endpoints identified. This means we are specializing to the manifold $T^3 \times R$, which does admit the de Sitter background even though topological obstructions prevent it from possessing the full de Sitter group of isometries.

Of course the Lagrangian of QCG is the same on any manifold, so expressions (3.12) and (3.13) for the interaction vertices are unchanged. However, to define the initial state and to find the associated propagator we must diagonalize free QCG on $T^3 \times R$. The allowed spatial momenta are:

$$\vec{k} = 2\pi H \vec{n} \quad (4.18)$$

for any triplet of integers $\vec{n} = (n_1, n_2, n_3)$. The free field mode expansion in our gauge is given by [8]:

$$\begin{aligned} \psi_{\mu\nu}(u, \vec{x}) = & H^3 \sum_{\lambda} \sum_{\vec{k} \neq 0} \left\{ \Psi(u, \vec{x}; \vec{k}, \lambda) a_{\mu\nu}(\vec{k}, \lambda) + \Psi^*(u, \vec{x}; \vec{k}, \lambda) a_{\mu\nu}^\dagger(\vec{k}, \lambda) \right\} \\ & + \sum_{\lambda \in A} \left\{ 1 q_{\mu\nu}(\lambda) + \frac{H^2}{3} (H^3 u^3 - 1) p_{\mu\nu}(\lambda) \right\} \\ & + \sum_{\lambda \in B \cup C} \left\{ H u (2 - H u) q_{\mu\nu}(\lambda) + H^2 u (-1 + H u) p_{\mu\nu}(\lambda) \right\} \end{aligned} \quad (4.19)$$

We class the ten modes as “A”, “B” or “C” depending upon which part of the kinetic operator (4.2) fails to annihilate their tensor structure [8, 13, 24]. The six A polarizations are purely spatial. The u dependence of the A zero modes is given in (4.19); the u and $\vec{x}$ dependence of the A nonzero modes is:

$$\Psi\left(u, \vec{x}, \vec{k}, \lambda\right) = \frac{H^u}{\sqrt{2k}} \left(1 + \frac{i}{k u}\right) \exp\left[ik\left(u - \frac{1}{H}\right) + i\vec{k} \cdot \vec{x}\right], \quad \forall \lambda \in A, \vec{k} \neq 0 \quad (4.20a)$$

The A operator coefficients give the following polarization sums:

$$\sum_{\lambda, \lambda' \in A} \left[q_{\mu\nu}(\lambda), p_{\rho\sigma}(\lambda') \right] = i \left[{}^{\mu\nu}T_{\rho\sigma}^A \right] \quad (4.20b)$$

$$\sum_{\lambda, \lambda' \in A} \left[a_{\mu\nu}(\vec{k}, \lambda), a_{\rho\sigma}^\dagger(\vec{k}', \lambda') \right] = H^{-3} \delta_{\vec{k}, \vec{k}'} \left[{}^{\mu\nu}T_{\rho\sigma}^A \right], \quad \vec{k}, \vec{k}' \neq 0 \quad (4.20c)$$

where the A tensor coefficient is:

$$\left[{}^{\mu\nu}T_{\rho\sigma}^A \right] \equiv 2 \left(\bar{\eta}_{\mu(\rho} \bar{\eta}_{\sigma)\nu} - \bar{\eta}_{\mu\nu} \bar{\eta}_{\rho\sigma} \right) \quad (4.20d)$$

The three B polarizations have mixed space and time tensor indices. Their zero mode functions are given in (4.19) and their nonzero mode functions are:

$$\Psi\left(u, \vec{x}, \vec{k}, \lambda\right) = \frac{H^u}{\sqrt{2k}} \exp\left[ik\left(u - \frac{1}{H}\right) + i\vec{k} \cdot \vec{x}\right], \quad \forall \lambda \in B, \vec{k} \neq 0 \quad (4.21a)$$

The B operator coefficients have the following polarization sums:

$$\sum_{\lambda, \lambda' \in B} \left[q_{\mu\nu}(\lambda), p_{\rho\sigma}(\lambda') \right] = i \left[{}^{\mu\nu}T_{\rho\sigma}^B \right] \quad (4.21b)$$

$$\sum_{\lambda, \lambda' \in B} \left[a_{\mu\nu}(\vec{k}, \lambda), a_{\rho\sigma}^\dagger(\vec{k}', \lambda') \right] = H^{-3} \delta_{\vec{k}, \vec{k}'} \left[{}^{\mu\nu}T_{\rho\sigma}^B \right], \quad \vec{k}, \vec{k}' \neq 0 \quad (4.21c)$$

where the B tensor coefficient is:

$$\left[{}^{\mu\nu}T_{\rho\sigma}^B \right] \equiv 4 \delta_{(\mu}^0 \bar{\eta}_{\nu)(\rho} \eta_{\sigma)0} \quad (4.21d)$$

The single C polarization is proportional to the Euclidean metric and in 3 + 1 dimensions it has the same spacetime dependence as the B modes:

$$\Psi\left(u, \vec{x}, \vec{k}, \lambda\right) = \frac{H^u}{\sqrt{2k}} \exp\left[ik\left(u - \frac{1}{H}\right) + i\vec{k} \cdot \vec{x}\right], \quad \forall \lambda \in C, \vec{k} \neq 0 \quad (4.22a)$$

The C polarization sums are:

$$\left[q_{\mu\nu}(C), p_{\rho\sigma}(C) \right] = i \left[{}^{\mu\nu}T_{\rho\sigma}^C \right] \quad (4.22b)$$

$$\left[a_{\mu\nu}(\vec{k}, C), a_{\rho\sigma}^\dagger(\vec{k}', C) \right] = H^{-3} \delta_{\vec{k}, \vec{k}'} \left[{}^{\mu\nu}T_{\rho\sigma}^C \right], \quad \vec{k}, \vec{k}' \neq 0 \quad (4.22c)$$

where the C tensor coefficient is:

$$\left[{}^{\mu\nu}T_{\rho\sigma}^C \right] \equiv \left(\delta_\mu^0 \delta_\nu^0 + \bar{\eta}_{\mu\nu} \right) \left(\delta_\rho^0 \delta_\sigma^0 + \bar{\eta}_{\rho\sigma} \right) \quad (4.22d)$$

We remark in passing that this mode analysis is for the gauge fixed theory. The physical degrees of freedom for nonzero $\vec{k}$ are the two transverse-traceless A polarizations. The three (positive norm) longitudinal A polarizations and the single (positive norm) C polarization combine with the three (negative norm) B polarizations and the (negative norm) trace A polarization to decouple from the theory [24]. The physical zero mode structure is complicated by the four linearization stability constraints of perturbation theory on $T^3 \times R$.^{*} In fact the A trace zero mode is a non-dynamical function of the other variables. The other three constraints correspond to compact residual symmetry groups and so are best realized by simply using invariant states. We shall ignore these subtleties because they are irrelevant to the relaxation effect we shall find. It should also be noted that our use of $T^3 \times R$ was itself a simplification; if one wishes to quibble over zero modes and unenforced linearization constraints then we can always revert to an open patch with fixed boundary conditions on which there are no zero modes and no linearization constraints.

A glance at the mode expansion (4.19) suggests a plausible initial state. The nonzero modes are nearly conventional oscillators. If we recall that conformal time u is reversed with respect to physical time then we recognize the $a_{\mu\nu}$'s as annihilation operators -- at least in the infinite past (i.e., at $u = \infty$) -- and the $a_{\mu\nu}^\dagger$'s as creation operators. The zero

^{*} The usual de Sitter manifold $-S^3 \times R-$ has ten such constraints but topological obstructions prevent the realization of six of them on this manifold.

modes are like free particles so they do not possess normalizable eigenstates; the best we can do is make them minimum uncertainty Gaussians. Our initial state is accordingly:

$$a(\vec{k}, \lambda) |0\rangle = 0 \quad , \quad \forall \vec{k} \neq 0 \quad (4.23a)$$

$$p_{\mu\nu}(\lambda) |0\rangle = i \frac{q_{\mu\nu}(\lambda)}{\Delta^2_{\varphi}(\lambda)} |0\rangle \quad (4.23b)$$

The spreads — $\Delta\varphi = (H/\sqrt{3})$ for the A modes and $\Delta\varphi = (H/\sqrt{2})$ for the B-C modes — were chosen to minimize the dispersion in the infinite future (i.e., at $u = 0$). Note that, modulo our use of a noninvariant gauge, (4.23a) is a de Sitter invariant condition. The physical breaking of de Sitter invariance comes, in free QCG, from the zero modes. Although we have seen that de Sitter invariance must be broken in order for relaxation to occur, the zero modes do not in fact play any role in relaxation. The far more significant breaking of de Sitter invariance comes from the strong infrared effects of interacting QCG.

With the initial state described above we can find the propagator using the standard methods of free field theory:

$$i \left[\mu\nu \Delta_{\rho\sigma} \right] (x, x') = \left\langle 0 \left| \bar{T} \left\{ \psi_{\mu\nu}(x) \psi_{\rho\sigma}(x') \right\} \right| 0 \right\rangle \quad (4.24a)$$

$$= i \Delta_A(x, x') \left[\mu\nu T_{\rho\sigma}^A \right] + i \Delta_B(x, x') \left\{ \left[\mu\nu T_{\rho\sigma}^B \right] + \left[\mu\nu T_{\rho\sigma}^C \right] \right\} \quad (4.24b)$$

where:

$$i \Delta_A(x, x') \equiv H^3 \sum_{\vec{k} \neq 0} \frac{H^2 u u'}{2k} \left[1 + \frac{1 + ik|\Delta u|}{k^2 u u'} \right] \exp \left[-ik|\Delta u| + i\vec{k} \cdot (\vec{x}' - \vec{x}) - \epsilon k \right] \quad (4.24c)$$

$$+ \frac{1}{6} H^2 \left[2 - H^3 u^3 - H^3 u^3 + H^6 u^3 u^3 \right] + \frac{1}{6} H^2 \left[H^3 u^3 - H^3 u^3 \right]$$

and:

$$i \Delta_B(x, x') \equiv H^3 \sum_{\vec{k} \neq 0} \frac{H^2 u u'}{2k} \exp \left[-ik|\Delta u| + i\vec{k} \cdot (\vec{x}' - \vec{x}) - \epsilon k \right] \\ + \frac{1}{4} H^4 u^2 u'^2 \left[8 - 6H u - 6H u' + 5H^2 u u' \right] + \frac{1}{2} H^4 u^2 u'^2 \left[H u' - H u \right] \quad (4.24d)$$

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The symbol $\bar{T}$ denotes anti-time-ordering (to overcome the temporal inversion of conformal coordinates), and $\Delta u \equiv u' - u$. Since the ghost kinetic operator necessarily involves the same scalar operators as that of the pseudo-graviton field we can write down the ghost propagator immediately:

$$i \left[\mu\nu \Delta_\nu \right] (x, x') \equiv \left\langle 0 \left| \bar{T} \left\{ \omega_\mu(x) \bar{\omega}_\nu(x') \right\} \right| 0 \right\rangle \quad (4.25a)$$

$$= i \Delta_A(x, x') \bar{\eta}_{\mu\nu} + i \Delta_B(x, x') \delta_\mu^0 \delta_\nu^0 \quad (4.25b)$$

The mode expansions given above are too complicated to be of much use but they can be simplified by exploiting two facts about the calculation. First, we are only interested in conformal times from the onset of inflation at $u = H^{-1}$ to the infinite future at $u = 0$. Second, we are evaluating the expectation value of the amputated pseudo-graviton field rather than its "in"-"out" amplitude. Recall from our discussion of the modified "in"-"in" formalism introduced in section 2 that no interaction can contribute to the amputated 1-point function outside the past lightcone of the observed field. This means that we need only obtain tractable approximations for $i \Delta_A(x, x')$ and $i \Delta_B(x, x')$ which are valid when x^μ and $x^{\mu'}$ are causally related for small $H u$ and $H u'$. One of the many advantages of conformal coordinates is that lightcones are the same as those of flat space. Hence the requirement that x^μ and $x^{\mu'}$ are causally related means:

$$(x - x')^2 \equiv -\Delta u^2 + \Delta x^2 < 0 \quad (4.26a)$$

$$\Delta u \equiv u' - u \quad , \quad \Delta x \equiv \|\vec{x}' - \vec{x}\|^2 \quad (4.26b)$$

Since we also know that $H u$ and $H u'$ are small we can conclude that $H \Delta x$ is small as well.

The obvious approximation under these circumstances is to represent the mode sums as integrals. The integrand for $i \Delta_B(x, x')$ is sufficiently regular that the integration can be extended to include the zero modes:

$$i \Delta_B(x, x') \approx \int \frac{d^3 k}{(2\pi)^3} \frac{H^2 u u'}{2k} \exp \left[-ik|\Delta u| + i\vec{k} \cdot (\vec{x}' - \vec{x}) - \epsilon k \right] \quad (4.27a)$$

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$$\begin{aligned}
&= \frac{H^2 u u'}{4\pi^2 \Delta x} \int_0^\infty dk \sin(k \Delta x) \exp[-ik|\Delta u| - \epsilon k] \\
&= \frac{1}{4\pi^2} \frac{H^2 u u'}{\Delta x^2 - \Delta u^2 + i\epsilon} \quad (4.27c)
\end{aligned}$$

For $i\Delta A(x; x')$ we must cut the integral off at $k \sim H$ and include the zero modes separately.

The result is [8]:

$$\begin{aligned}
i\Delta A(x; x') &\approx \frac{H^2 u u'}{4\pi^2 \Delta x} \int_H^\infty dk \sin(k \Delta x) \left[1 + \frac{1 + ik|\Delta u|}{k^2 u u'} \right] \exp[-ik|\Delta u| - \epsilon k] \\
&\quad + \frac{1}{6} H^2 \left[2 - H^3 u^3 - H^3 u'^3 + H^6 u^3 u'^3 \right] + \frac{1}{6} H^2 \left[H^3 u^3 - H^3 u'^3 \right] \quad (4.28a) \\
&= \frac{H^2 u u'}{8\pi^2 \Delta x} \left\{ \frac{\exp(iH \Delta x)}{\Delta x - |\Delta u| + i\epsilon} + \frac{\exp(-iH \Delta x)}{\Delta x + |\Delta u| - i\epsilon} \right\} \exp[-iH(|\Delta u| - i\epsilon)] \\
&\quad + \frac{H^2 \sin(H \Delta x)}{4\pi^2} \frac{H \Delta x}{H \Delta x} \exp[-iH(|\Delta u| - i\epsilon)] \\
&\quad - \frac{H^2}{8\pi^2} \left\{ \text{Ei} \left[iH(\Delta x - |\Delta u| + i\epsilon) \right] - \text{Ei} \left[-iH(\Delta x + |\Delta u| - i\epsilon) \right] \right\} \\
&\quad + \frac{1}{6} H^2 \left[2 - H^3 u^3 - H^3 u'^3 + H^6 u^3 u'^3 \right] + \frac{1}{6} H^2 \left[H^3 u^3 - H^3 u'^3 \right] \quad (4.28b) \\
&= \frac{1}{4\pi^2} \frac{H^2 u u'}{\Delta x^2 - \Delta u^2 + i\epsilon} - \frac{H^2}{8\pi^2} \ln \left[H^2 (\Delta x^2 - \Delta u^2 + i\epsilon) \right] \\
&\quad + \frac{H^2}{4\pi^2} \left\{ 1 - \gamma + \frac{4}{3} \pi^2 \right\} + H^2 \times O \left(H u, H u', H \Delta x \right) \quad (4.28c)
\end{aligned}$$

where γ is Euler's constant. It is important to recognize that these expressions are only valid inside the lightcone and for small extensions beyond it. In particular, the apparent logarithmic growth of $i\Delta A(x; x')$ outside the lightcone is a fiction of the asymptotic expansion which was used in passing from (4.28b) to (4.28c). In fact there are no very large spatial coordinate separations in our manifold; and, if there were, the integral in (4.28a) would actually fall off as $(\Delta x)^{-2}$ far outside the lightcone.

In the modified "in"-"in" formalism there are four kinds of propagators. They all have the same tensor structure and their scalar coefficient functions are obtained from $i\Delta A(x; x')$ and $i\Delta B(x; x')$ by varying the $i\epsilon$ structure precisely as in (2.17a-d). Note that by keeping ϵ nonzero we again obtain an ultraviolet regulator, although now there are quadratic and

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quartic divergences in addition to the logarithmic ones. Because of the temporal inversion, it is actually the "+" vertices which are conjugated and the "-" ones which are those of the original theory. (This subtlety is not relevant to our two-loop computation which involves an even number of vertices.) As in section 2, the formalism's causality and the finite onset time combine to prevent the appearance of infrared divergences. What happens instead is that later observation times show ever larger corrections to the expectation values of operators whose "in"-"out" matrix elements are infrared divergent.

Renormalization is accomplished by subtracting primary divergences using local counterterms. We use the cosmological counterterm to remove primary divergences in the annulated 1-point function and to enforce the condition that the effective Hubble constant be initially time independent: *

$$\frac{d}{dt} H_{\text{eff}}(t) \Big|_{t=0^+} = 0 \quad (4.29a)$$

Since both the coefficients $A(u)$ and $C(u)$, and their first derivatives, vanish at onset we can express the condition for $\delta\lambda$ in terms of the initial values of the coefficients $a(u)$ and $c(u)$:

$$\left(\frac{d}{dt} \right)^2 A(u) \Big|_{u=H^{-1}} = a(H^{-1}) - c(H^{-1}) = 0 \quad (4.29b)$$

If desired we can enforce the vanishing of the other linearly independent combination of $a(H^{-1})$ and $c(H^{-1})$ by using time dependent changes in the coordinate system.** We use δG to remove primary divergences of dimension two in the 2-point function. Since the imaginary part of our propagator gives the correct classical response to a point mass [13] we know that Newton's Law is true at onset. We could use the finite part of δG to enforce the initial constancy of Newton's law, but it turns out that the finite part of δG is no more obeyed by [0].

* The condition that $H_{\text{eff}}(0) = H$ is enforced by the choice of initial state, and is indeed obeyed by [0].

** These time dependent coordinate transformations do not affect $H_{\text{eff}}(t)$, so it is irrelevant whether or not we use them to absorb the initial values of $a(u)$ and $c(u)$. Of course these transformations can be used to impose an independent condition at each value of u . A nice example would be $c(u) = -a(u)$.

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relevant to the leading infrared contributions to the amputated 1-point function than was the finite part of δm^2 in the scalar analog model of section 2. Of course the finite parts of the higher dimension counterterms are even less relevant.

We turn now to the crucial issue of where in the manifold one should expect interaction vertices to make the strongest infrared contributions to the amputated 1-point function. To first approximation the effects of distant vertices add coherently, so the strength of infrared contributions simply tracks the invariant volume of spacetime within the past lightcone of the observation point. Let us therefore suppose an observer at co-moving time t and let us compute the invariant volume of his past lightcone from co-moving times t_1 to t_2 . In flat space the light cones are straight lines and the result is:

$$V_{\text{flat}}(t_1, t_2; t) = \int_{t_1}^{t_2} dt' \frac{4\pi}{3} (t - t')^3 \quad (4.30a)$$

$$= \frac{4\pi}{3} t^3 \left\{ (t_2 - t_1) - \frac{3}{2} \frac{t_2^2 - t_1^2}{t} + \frac{t_2^3 - t_1^3}{t^2} - \frac{1}{4} \frac{t_2^4 - t_1^4}{t^3} \right\} \quad (4.30b)$$

Not only is the largest contribution from the earliest interval, but the contribution from any fixed interval diverges as the observer evolves into the infinite future. The analogous result for a de Sitter background is rather different:

$$V_{\text{dS}}(t_1, t_2; t) = \int_{t_1}^{t_2} dt' e^{3Ht'} \frac{4\pi}{3} \left\{ H^{-1} e^{-Ht'} - e^{-Ht} \right\}^3 \quad (4.31a)$$

$$= \frac{4\pi}{3} H^{-3} \left\{ (t_2 - t_1) - \frac{3}{H} \left[e^{-H(t-t_2)} - e^{-H(t-t_1)} \right] \right. \\ \left. + \frac{3}{2H} \left[e^{-2H(t-t_2)} - e^{-2H(t-t_1)} \right] - \frac{1}{3H} \left[e^{-3H(t-t_2)} - e^{-3H(t-t_1)} \right] \right\} \quad (4.31b)$$

$$\xrightarrow{t \rightarrow \infty} \frac{4\pi}{3} H^{-3} (t_2 - t_1) \quad (4.31c)$$

Although the largest contribution still comes from the earliest interval, the contribution from any fixed interval approaches a finite constant as the observer evolves into the infinite future. Of course our spatial manifold is finite, but causality would in any case permit only a finite segment of it to contribute. Thus we reach the important conclusion that one

should not expect interactions near the time of onset to make a strong contribution in QCG as they did in the scalar analog model.

That there can be an infrared effect at all in QCG derives from the fact that space expands even as the light cone constricts. The full volume of the past lightcone still diverges as the observer evolves into the infinite future:

$$V_{\text{dS}}(0, t; t) = \frac{4\pi}{3} H^{-3} t \left\{ 1 - \frac{11}{6} \frac{1}{Ht} + \frac{e^{-Ht}}{3} \frac{1}{Ht} - \frac{3}{2} \frac{e^{-2Ht}}{Ht} + \frac{1}{4} \frac{e^{-3Ht}}{Ht} \right\} \quad (4.32)$$

Less there be concern about mixing between infrared and ultraviolet regimes we note that linear growth occurs even if t_2 is cut off at some fraction $0 < f < 1$ of the observer time:

$$V_{\text{dS}}(0, ft; t) = \frac{4\pi}{3} H^{-3} ft + O\left(H^{-4} e^{-(1-f)Ht}\right) \quad (4.33)$$

We therefore conclude that one must look for infrared effects in QCG from late time interactions which contribute to even later observations.

It remains to translate this insight into the conformal coordinate system in which the Feynman rules have such a simple expression. For this purpose it is useful to consider the invariant volume of the past lightcone from $t_1 = f_1 t$ to $t_2 = f_2 t$ for fixed fractions $0 < f_1 < f_2 < 1$:

$$V_{\text{dS}}(f_1 t, f_2 t; t) = \frac{4\pi}{3} H^{-3} (f_2 - f_1) t + O\left(H^{-4} e^{-(1-f_2)Ht}\right) \quad (4.34)$$

Though the initial and final co-moving times grow linearly with t , the associated conformal times approach one another — and zero — as the observer evolves into the infinite future:

$$u_i = H^{-1} e^{-H f_i t} = H^{-1} (H u)^{f_i}, \quad i = 1, 2 \quad (4.35)$$

We stress that the invariant volume of the region bounded by u_1 and u_2 actually diverges as u approaches zero, and that its separation in co-moving time from the observation point grows as well.* Conformal coordinates disguise this portion of the infrared to make it seem

* Note also that one obtains good ultraviolet separation while still having u_2 be linear in u . For example, if the conformal cutoff time is $u_2 = su$ then the associated co-moving cutoff is $t_2 = t - H^{-1} \ln(s)$.

ultraviolet. Since any portion of the past light cone which is separated from $u = 0$ must have a finite invariant volume we conclude that *the significant infrared effects in QCG come from regions of the conformal coordinate manifold whose lower boundaries shrink down on the observer as he approaches the infinite future at $u = 0$.*

We are finally ready to justify our ansatz (4.14) for the coefficient functions $a(u)$ and $c(u)$ of the amputated 1-point function. Let us begin by extracting factors of κ and H . The n -point vertex contributes a factor of $\kappa^{n-2} H^{-2}$. From our expansions (4.27) and (4.28) we see that each propagator is H^2 times functions which depend only very weakly on H . If an ℓ -loop contribution consists entirely of 3-point interactions there will be $V = 2\ell - 1$ vertices and $P = 3\ell - 2$ propagators. Since we include an extra factor of κ in the definition (4.4), the ℓ -loop contributions to $a(u)$ and $c(u)$ can be written as:

$$\kappa^{V+1} H^{2P-2V} = (\kappa H)^{2\ell} H^{-2} \quad (4.36)$$

times the integral of a function which depends only weakly on H — and not at all on κ — over the $2\ell - 2$ free vertices. To see that this result is correct even if higher point vertices are involved, note that diagrams involving the higher point vertices at the same loop order can be obtained by contracting propagators.** Suppose we make such a contraction between an m -point and an n -point vertex. Before the contraction the structure contributed a factor of:

$$\kappa^{m-2} H^{-2} \times H^2 \times \kappa^{n-2} H^{-2} = \kappa^{m+n-4} H^{-2} \quad (4.37)$$

Afterwards we are left with a single vertex having $m+n-2$ legs, so the factor is unchanged. What *does* change is the number of vertex integrations. We shall see that this is responsible for the irrelevance of diagrams which include higher interaction vertices.

We remind the reader that κ has the dimensions of length, while H and the pseudo-
** Indeed, one way to view the role of the higher interaction vertices is as filling out certain portions of the moduli space of Schwinger parameters which are needed for decoupling of the primitive diagram.

graviton field have the dimensions of inverse length. It follows that the coefficient functions $a(u)$ and $c(u)$ have the same dimensions as the pseudo-graviton kinetic operator: namely, inverse length squared. At ℓ loops these functions can be written as $(\kappa H)^{2\ell} H^{-2}$ times the integral of a function which depends only weakly on H . This integral has dimension of length⁻⁴ and it can depend only upon the observation time u , the ultraviolet regularization parameter ϵ — which has dimensions of length squared — and the Hubble constant H . The only strong source of dependence upon H is through the upper limit — H^{-1} — of the conformal time integrations. Since we have seen that infrared effects derive mostly from the lower limits, and since they must be ultraviolet finite, it follows that the leading infrared effects at any order go like u^{-4} times dimensionless functions of u . These dimensionless functions can only be logarithms, and there can be at most one for every vertex integration. At ℓ loops there are at most $2\ell - 2$ free vertices, so we expect the ℓ -loop coefficient functions to behave as:

$$a_\ell(u), c_\ell(u) \sim (\kappa H)^{2\ell} H^{-2} u^{-4} \left\{ \# \ln^{2\ell-2}(Hu) + (\text{subdominant}) \right\} \quad (4.38)$$

Since the inclusion of higher point vertices at the same loop order results in fewer free vertices, the leading term at any order is composed entirely of 3-point vertices.

Even on the basis of the general analysis just given it is clear that one-loop effects cannot make significant corrections to $H_{\text{eff}}(t)$. From (4.38) we see that the strongest possible contributions to the amputated 1-point function would be proportional to u^{-4} with no logarithms. Applying (4.12a) and (4.13a), and substituting into (4.7c) reveals that the resulting corrections to $H_{\text{eff}}(t)$ approach a constant in the far future. Since this constant must be a pure number times $(\kappa H)^2 \sim 10^{-12}$, any one-loop effect has to be irrelevant.

In fact there are good reasons for believing one-loop effects are absent altogether. The two diagrams that contribute are displayed in Fig. 18. Because the result is the coincidence

limit of (pseudo-graviton and ghost) propagators, the symmetries of the theory dictate that it can be absorbed into a local counterterm plus a time dependent redefinition of the coordinate system. In fact the highest inverse power of u that the vertices can contribute is u^{-2} , vertices with u^{-3} and u^{-4} carry tensor factors which only survive when contracted into a part of the propagator which is proportional to u or u^2 respectively. The extra dimensionality is made up by factors of the regularization parameter, ϵ . We must therefore obtain quadratically divergent terms of the form, $u^{-2}\epsilon^{-1}$, and quadratically divergent contributions of the form, ϵ^{-2} . These terms must be zero in an invariant regularization but they may survive with our method. In any case, they can certainly be absorbed into time dependent coordinate redefinitions.

Fig. 18: One-loop contributions to the graviton background.

There may be a very small physical effect at one loop arising from the spread of the zero modes. Since any such effect must involve non-negative powers of u , its impact on $H_{\text{eff}}(t)$ would vanish in the asymptotic future. Physically this corresponds to the inflationary redshift which renders the dynamics of any finite number of low modes ultimately invisible to a local observer. The powerful infrared effects of QCG derive from the fact that inflation is redshifting a constantly increasing number of modes down to low physical momenta.

The generic two-loop graphs are depicted in Fig. 19. Although we have not evaluated them as we did for their scalar cousins in section 2, the problem seems to be no harder than the tensor algebra of the on-shell self-energy which was described in section 3. We shall

confine ourselves here to arguing that the double log terms anticipated in (4.14) should arise, barring *miraculous* cancellations in the tensor algebra. The first diagram of Fig. 19 has only one free vertex so it can give at most a single logarithm. Nonetheless, it is worth exploiting the simple topology of this first diagram in order to understand mathematically why there are any logarithms at all. Of course the *physical* reason for logarithms is the coherent superposition of interactions over the ever larger invariant volume of the observer's past light cone as he evolves into the future.

Fig. 19: Two-loop contributions to the graviton background.

If the observation is at x^μ and the interaction is at x'^μ then diagrams with the topology of the first graph in Fig. 19 make contributions having the qualitative form:

$$\kappa \int_u^{H^{-1}} du' \int d^3x' V_4(x) \left[i\Delta(x, x') \right]^3 V_3(x') \quad (4.39)$$

Here $V_n(y)$ denotes the n -point vertex operator — it contains derivatives — at the point y^μ , and we have ignored all tensor indices. The obvious reduction strategy is to decompose both the vertex operator and the propagator into sums of terms and then to multiply the five sums into one another. Each product in the resulting 5-fold do-loop is tractable with existing symbolic manipulation programs.

Within the region of interest each propagator can be written as the sum of the following two terms:

$$\frac{H^2}{4\pi^2} \frac{uu'}{(x-x')^2 \pm i\epsilon} \left[2\delta^{(a)}_{\delta^{(j)}} - \eta_{\alpha\sigma} \eta^{\alpha\beta} \right] \quad (4.40a)$$

$$-\frac{H^2}{8\pi^2} \ln \left[(x-x')^2 + \pm i\epsilon \right] \left[2\bar{\partial}_\rho \frac{(\alpha\bar{\alpha})^\beta}{\partial_\sigma} - 2\bar{\eta}_\rho \bar{\eta}^{\alpha\beta} \right] \quad (4.40b)$$

The 3-point vertex operators descend from sixteen distinct cubic interactions, two of which are:

$$\frac{1}{2} \kappa H^{-2} u^{-2} \psi^{\rho\sigma} \psi_{\rho\sigma\mu} \psi^{\mu\nu} \quad (4.41a)$$

$$2\kappa H^{-2} u^{-3} \psi \psi_{,\mu} \psi^{\mu 0} \quad (4.41b)$$

We call the twelve distinct terms of the form (4.41a) “conventional” because they come from flat space perturbation theory with the replacement $h_{\mu\nu} \rightarrow \psi_{\mu\nu}$ and multiplication by $\Omega^2 = (Hu)^{-2}$. We call the four distinct terms of the form (4.41b) “new.” Although the new interactions may seem to be more singular at $u = 0$ than the conventional ones, this is an illusion. All new interactions contain an undifferentiated pseudo-graviton field with a free zero index, $\psi^{\mu 0}$. Such a field gives zero when contracted into the purely spatial tensor of (4.40b), so it can only couple through (4.40a). Since no derivative acts on $\psi^{\mu 0}$ the undifferentiated u in (4.40a) cancels the apparent extra factor of u^{-1} .

In reducing any product of the 5-fold sum one first performs the tensor contractions, then acts the various derivatives, and finally computes the spatial integrations. Since the dependence upon spatial coordinates is through the Lorentz interval, $(x-x')^2 = -\Delta u^2 + \Delta x^2$, the spatial integrations must result in simple functions of $\Delta u = u' - u$, times whatever powers of u and u' remain after acting the derivatives. For example, suppose each propagator contributes the term (4.40a) — with different tensor indices of course — and that the interaction vertices are all conventional. If the derivatives act on internal lines — as depicted in Fig. 20a — then one of the many terms which arises is where the free u and u' on two of the propagators are stripped away. Suppressing the $i\epsilon$, this gives:

$$\kappa^4 H^2 \frac{1}{u} \int_u^{H^{-1}} du' \frac{1}{u'} \int d^3x' \left[\frac{1}{(x-x')^2} \right]^3 \quad (4.42a)$$

$$= \# \kappa^4 H^2 \frac{1}{u} \int_u^{H^{-1}} du' \frac{1}{u'} \frac{1}{(u' - u)^3} \quad (4.42b)$$

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$$= \# \kappa^4 H^2 \frac{1}{u} \int_u^{H^{-1}} du' \left\{ -\frac{1}{u^3} \frac{1}{u'} + \frac{1}{u^3} \frac{1}{u' - u} - \frac{1}{u^2} \frac{1}{(u' - u)^2} + \frac{1}{u} \frac{1}{(u' - u)^3} \right\} \quad (4.42c)$$

The last three terms inside the brackets correspond to primitive ultraviolet divergences which can be absorbed into a combination of the 1-point counterterm and a time dependent coordinate transformation. * The first term gives the single logarithms which were the point of this exercise.

Fig. 20: In these two-loop contributions to the graviton amputated 1-point function, the arrows indicate the action of the vertex derivatives.

We should make two comments before considering the other two-loop graphs. First, note the essential role played by free factors of u in the vertices and the propagators. Were it not for these terms the result would have the form:

$$\int_u^{H^{-1}} du' f(u' - u) = \mathcal{F}(H^{-1} - u) - \mathcal{F}(0) \quad (4.43)$$

where $\mathcal{F}(u) = f(u)$. This is essentially constant near $u = 0$, whereas we see from (4.7c), (4.10) and (4.11) that the amputated 1-point function must diverge as rapidly as u^{-4} just * The second term in (4.42c) is logarithmically divergent while the fourth is quadratically divergent. The third term would give a linear divergence which presumably vanishes when proper attention is paid to regularization.

to induce a constant shift in the asymptotic value of $H_{\text{eff}}(t)$. Any weaker u dependence is lost in the inflationary redshift.

Of course the free u is originated in conformal factors, and these fail to cancel because QCG is not conformally invariant. All such factors *do* cancel in conformally invariant models, up to regularization induced anomalies. This is why QCG eventually dominates conformally invariant theories such as QCD, in spite of the fact that the QCD coupling constant is at least twelve orders of magnitude larger than the natural dimensionless constant of QCG, $(\kappa H)^2$. Massive theories are irrelevant for a different reason. Although these models are not conformally invariant, their propagators oscillate inside the past lightcone and this prevents distant interactions from adding coherently.

Our second comment concerns the curious way in which decomposing by partial fractions in (4.42c) distills ultraviolet and infrared effects. These effects originally became entangled when we performed partial integrations to simplify the invariant Lagrangian of QCG [13]. When a background field is not constant one can always write it as the derivative of something and then partially integrate. The following manipulations are suggestive of how this works:

$$\psi^3 \frac{1}{u^4} = -\frac{1}{6} \psi^3 \partial^2 \left(\frac{1}{u^2} \right) \quad (4.44a)$$

$$= -\frac{1}{6} \partial^\mu \left[\psi^3 \partial_\mu \left(\frac{1}{u^2} \right) \right] + \frac{1}{2} \psi^2 \psi^{\mu\mu} \partial_\mu \left(\frac{1}{u^2} \right) \quad (4.44b)$$

$$= \partial^\mu \left[-\frac{1}{6} \psi^3 \partial_\mu \left(\frac{1}{u^2} \right) + \frac{1}{2} \psi^2 \psi_{,\mu} \frac{1}{u^2} \right] - \left(\psi \psi^{\mu\mu} \psi_{,\mu} + \frac{1}{2} \psi^2 \psi^{\mu\mu} \right) \frac{1}{u^2} \quad (4.44c)$$

The partial integration replaces two factors of u^{-1} with two derivatives, at which point we can combine with the original dimension five interactions.

Of course the physical effect of the dimension three self-interaction is still present, but it is mixed up with that of the higher dimension coupling. The dimension three effects are less ultraviolet divergent and more significant in the infrared; the dimension five effects

are more ultraviolet divergent and less significant in the infrared. The way we separate them is through decomposition of the combined result by partial fractions. Consider, for instance:

$$\frac{1}{u'^2} \frac{1}{u' - u} = \frac{1}{u^2} \left[-\frac{u}{u'^2} - \frac{1}{u'} + \frac{1}{u' - u} \right] \quad (4.45)$$

The first term is ultraviolet finite as u' approaches u , but it cannot contribute infrared logarithms. These come from the second term while the third term contains a logarithmic ultraviolet divergence and is irrelevant to the infrared. Other examples are:

$$\frac{1}{u'} \frac{1}{u' - u} = \frac{1}{u} \left[-\frac{1}{u'} + \frac{1}{u' - u} \right] \quad (4.46a)$$

$$\frac{1}{u'} \frac{1}{(u' - u)^2} = \frac{1}{u^2} \left[\frac{1}{u'} - \frac{1}{u' - u} + \frac{u}{(u' - u)^2} \right] \quad (4.46b)$$

$$\frac{1}{u'} \frac{1}{(u' - u)^3} = \frac{1}{u^3} \left[-\frac{1}{u'} + \frac{1}{u' - u} - \frac{u}{(u' - u)^2} + \frac{u^2}{(u' - u)^3} \right] \quad (4.46c)$$

Notice that the partial fractions decomposition gives prefactors of u^{-1} . This is operationally how parts of the amputated 1-point function contrive to acquire an overall factor of u^{-4} even though the interaction vertex of the observer can only contribute an overall factor of u^{-2} .

The physics of our infrared effect descends ultimately from the dimension three interaction:

$$\frac{\kappa}{H^2} \frac{1}{u^4} \left\{ -\frac{1}{4} \psi^3 + \frac{3}{2} \psi \psi^{\alpha\beta} \psi_{\alpha\beta} - 2 \psi^\alpha_\beta \psi^\beta_\gamma \psi^\gamma_\alpha \right\} \quad (4.47)$$

We have seen that this interaction is heavily disguised by partial integrations. Had we foregone the partial integrations the interactions would have been more complicated, but the infrared effect would have been more clearly displayed. Consider, for example, a contribution with the topology of the first term in Fig. 19 but constructed using interaction vertices with no free derivatives. Of course there would still be many terms. It is trivial to evaluate the one where each of the three propagators contributes the imaginary part of

(4.40b):

$$\kappa^4 H^2 \frac{1}{u^4} \int_u^{H^{-1}} \frac{du'}{(u')^4} \int \frac{du''}{u''} \int \frac{d^3 x'}{x'} \theta \left[(x' - x)^2 \right] \quad (4.48a)$$

$$\begin{aligned} &= \kappa^4 H^2 \frac{1}{u^4} \int_u^{H^{-1}} \frac{du'}{(u')^4} \int \frac{du''}{u''} \frac{4\pi}{3} (u' - u)^3 \\ &= \frac{4\pi}{3} \kappa^4 H^2 \frac{1}{u^4} \left\{ -\ln(Hu) + 3(Hu - 1) - \frac{3}{2}(H^2 u^2 - 1) + \frac{1}{3}(H^3 u^3 - 1) \right\} \quad (4.48c) \end{aligned}$$

The parallel with (4.32) could not be clearer: QCG manifests strong infrared effects through the *coherent* and *causal* superposition of vacuum fluctuations from throughout the *expanding past lightcone*.

The strongest possible effect at any fixed order comes from diagrams constructed entirely from 3-point vertices because this maximizes the number of integrations. At two loops we can potentially get double logarithms from the second, third and fourth graphs of Fig. 19, each of which has the following qualitative form:

$$\kappa \int_u^{H^{-1}} du' \int d^3 x' \int_u^{H^{-1}} du'' \int d^3 x'' V_3(x') V_3(x'') \left[\Delta(x', x'') \right]^2 V_3(x'') i\Delta(x'', x) \quad (4.49)$$

These graphs are more complicated to evaluate because they harbor a subdivergence — from when the internal loop contracts — and because the spatial integrations can result in more complicated functions of the two independent differences, $u' - u$ and $u'' - u'$. We confine ourselves to noting two facts about the prospects for double logarithms. First, they are certainly present if one retains only the original, dimension three vertices. For example, the term where each of the four propagators contributes the imaginary part of (4.40b) gives:

$$\begin{aligned} &\frac{\kappa^4 H^2}{u^4} \int_u^{H^{-1}} \frac{du'}{u'^4} (u' - u)^3 \int_u^{H^{-1}} \frac{du''}{u''^4} (u'' - u')^3 = \\ &\frac{\kappa^4 H^2}{u^4} \left\{ \frac{1}{2} \ln^2(Hu) + \left[\frac{11}{3} + 9Hu + \frac{9}{2}H^2 u^2 + \frac{1}{3}H^3 u^3 \right] \ln(Hu) + \frac{97}{12} - \frac{27}{4}H^2 u^2 - \frac{4}{3}H^3 u^3 \right\} \end{aligned} \quad (4.50)$$

Note that this term is completely free of ultraviolet divergences.

The second point we wish to note is that there are many dimensionally correct terms which can result from the spatial integrations and which give double infrared logarithms.

Consider, for example:

$$\begin{aligned} &\frac{\kappa^4 H^2}{u^2} \int_u^{H^{-1}} \frac{du'}{u'} \frac{1}{u'} \int_u^{H^{-1}} \frac{du''}{u''} \frac{1}{u''} \frac{1}{(u'' - u)} \\ &= \frac{\kappa^4 H^2}{u^4} \int_u^{H^{-1}} \frac{du'}{u'} \left[-\frac{1}{u'} + \frac{1}{u' - u} \right] \int_u^{H^{-1}} \frac{du''}{u''} \left[-\frac{1}{u''} + \frac{1}{u'' - u} \right] \quad (4.51a) \\ &= \frac{\kappa^4 H^2}{u^4} \int_u^{H^{-1}} \frac{du'}{u'} \left[-\frac{1}{u'} + \frac{1}{u' - u} \right] \left\{ \ln(Hu') + \ln(1 - Hu) - \ln[H(u' - u)] \right\} \\ &= \frac{\kappa^4 H^2}{u^4} \left\{ -\frac{1}{2} \ln^2(Hu') - \ln(Hu') \ln(1 - Hu) + \ln(Hu') \ln[H(u' - u)] \right. \\ &\quad \left. + \ln(1 - Hu) \ln[H(u' - u)] - \frac{1}{2} \ln^2[H(u' - u)] \right\} \Big|_u^{H^{-1}} \quad (4.51b) \end{aligned}$$

A potentially obtainable term that shows the subdivergence is:

$$\begin{aligned} &\frac{\kappa^4 H^2}{u^2} \int_u^{H^{-1}} \frac{du'}{u'} \frac{1}{u'} \frac{1}{u'} \int_u^{H^{-1}} \frac{du''}{u''} \frac{1}{u''} \frac{1}{u'' - u'} \\ &= \frac{\kappa^4 H^2}{u^4} \int_u^{H^{-1}} \frac{du'}{u'} \left[-\frac{u}{u^2} - \frac{1}{u'} + \frac{1}{u' - u} \right] \int_u^{H^{-1}} \frac{du''}{u''} \left[-\frac{1}{u''} + \frac{1}{u'' - u'} \right] \quad (4.52a) \\ &= \frac{\kappa^4 H^2}{u^4} \int_u^{H^{-1}} \frac{du'}{u'} \left[-\frac{u}{u^2} - \frac{1}{u'} + \frac{1}{u' - u} \right] \left\{ -\ln(Hu'') + \ln[H(u'' - u')] \right\} \Big|_u^{H^{-1}} \quad (4.52b) \end{aligned}$$

Although this integral cannot be evaluated exactly, the dominant infrared contribution is easy to isolate:

$$\frac{\kappa^4 H^2}{u^4} \int_u^{H^{-1}} \frac{du'}{u'} \frac{1}{u'} \ln(Hu') = \frac{\kappa^4 H^2}{2u^4} \ln^2(Hu) \quad (4.53)$$

and again reveals double logarithms.

It remains to discuss the effect of including gravitons in the initial state. In this case we cannot rigorously define the effective Hubble constant using co-moving coordinates but it is simple to see both that the amputated 1-point function becomes homogeneous in the far future, and that it experiences qualitatively the same infrared growth as for the pure vacuum case. The key is understanding how reduction works for our modified “in”-“in” formalism.

The two physical graviton polarizations of free QCG are transverse, traceless, and purely spatial. We obtain the associated wave functions by multiplying a polarization tensor by the Λ mode function (4.20a):

$$\Psi_{\mu\nu}(u, \vec{x}; \vec{k}, \lambda) = \frac{H u}{\sqrt{2k}} \left(1 + \frac{i}{ku}\right) \exp\left[ik(u - \frac{i}{H}) + i\vec{k} \cdot \vec{x}\right] \epsilon_{\mu\nu}(\vec{k}, \lambda), \quad k \equiv \|\vec{k}\| \quad (4.54a)$$

$$\epsilon_{\mu 0}(\vec{k}, \lambda) = k_i \epsilon_{i\mu}(\vec{k}, \lambda) = \epsilon_{\mu\mu}(\vec{k}, \lambda) = 0 \quad (4.54b)$$

$$\epsilon'_{\mu\nu}(\vec{k}, \lambda) \epsilon^{\mu\nu}(\vec{k}, \lambda') = 2 \delta_{\lambda\lambda'} \quad (4.54c)$$

These wavefunctions obey the free equations of motion:

$$D_{\mu\nu}{}^{\rho\sigma} \Psi_{\rho\sigma}(u, \vec{x}; \vec{k}, \lambda) = 0 \quad (4.55)$$

Since the pseudo-graviton kinetic operator (4.2) is of Stürm-Liouville type we can obtain a conserved inner product from its Wronskian:

$${}^{\mu\nu} \overleftrightarrow{W}_\alpha{}^{\rho\sigma} = \left[\frac{1}{2} \delta_\mu{}^\rho \delta_\nu{}^\sigma - \frac{1}{4} \eta_{\mu\nu} \eta^{\rho\sigma} \right] \Omega \left(\overleftrightarrow{\partial}_\alpha - \overleftrightarrow{\partial}_\alpha \right) \Omega \quad (4.56)$$

With the stated conventions for the wavefunctions we have the following orthogonality relation on T^3 :

$$-i \int_{T^3} d^3x \Psi^*{}^{\mu\nu}(u, \vec{x}; \vec{k}, \lambda) \left[{}^{\mu\nu} \overleftrightarrow{W}_0{}^{\rho\sigma} \right] \Psi_{\rho\sigma}(u, \vec{x}; \vec{k}', \lambda') = H^{-3} \delta_{\vec{k}\vec{k}'} \delta_{\lambda\lambda'} \quad (4.57)$$

The free field expansion (4.19) is exact at onset because interactions are absent before $u = H^{-1}$. We can therefore employ the onset fields to extract the operators which create and annihilate single gravitons:

$$A_\pm^\dagger(\vec{k}, \lambda) = iH^{\frac{3}{2}} \int_{T^3} d^3x \Psi^*{}^{\mu\nu}(u, \vec{x}; \vec{k}, \lambda) \left[{}^{\mu\nu} \overleftrightarrow{W}_0{}^{\rho\sigma} \right] \psi_{\rho\sigma}^\pm(u, \vec{x}) \Big|_{u=H^{-1}} \quad (4.58a)$$

$$A_\pm(\vec{k}, \lambda) = -iH^{\frac{3}{2}} \int_{T^3} d^3x \Psi^*{}^{\mu\nu}(u, \vec{x}; \vec{k}, \lambda) \left[{}^{\mu\nu} \overleftrightarrow{W}_0{}^{\rho\sigma} \right] \psi_{\rho\sigma}^\pm(u, \vec{x}) \Big|_{u=H^{-1}} \quad (4.58b)$$

Note that the normalized operators $A_\pm(\vec{k}, \lambda)$ contain a factor of $H^{-\frac{3}{2}}$ relative to the operators $a_{\mu\nu}(\vec{k}, \lambda)$ which appear in (4.19). Note as well that we can use either “+”

fields — which generate the forward evolution from onset to the turn around point — or “−” fields — which generate backwards evolution from turn around to the time of onset. The distinction is without a difference for isolated operators but it is useful inside an anti-time-ordered product because onset “+” fields stand to the extreme right while onset “−” fields are on the extreme left.*

Let us denote by the symbol $\mathcal{M}$ the portion of $T^3 \times R$ that covers the time from onset to turn around — which can be at any conformal time smaller than that of the latest observation. The key identities of “in”-“out” reduction are:

$$A_+^\dagger(\vec{k}, \lambda) = A_-^\dagger(\vec{k}, \lambda) - iH^{\frac{3}{2}} \int_{\mathcal{M}} d^4x \Psi^*{}_{\mu\nu}(u, \vec{x}; \vec{k}, \lambda) D^{\mu\nu\rho\sigma} \left[\psi_{\rho\sigma}^+(u, \vec{x}) - \psi_{\rho\sigma}^-(u, \vec{x}) \right] \quad (4.59a)$$

$$A_-(\vec{k}, \lambda) = A_+(\vec{k}, \lambda) - iH^{\frac{3}{2}} \int_{\mathcal{M}} d^4x \Psi^*{}_{\mu\nu}(u, \vec{x}; \vec{k}, \lambda) D^{\mu\nu\rho\sigma} \left[\psi_{\rho\sigma}^+(u, \vec{x}) - \psi_{\rho\sigma}^-(u, \vec{x}) \right] \quad (4.59b)$$

One uses them the same way as in the reduction of “in”-“out” amplitudes. Suppose, for example, that the state $|\Phi\rangle$ does not contain the $(\vec{k}, \lambda)$ graviton:

$$A_\pm(\vec{k}, \lambda) |\Phi\rangle = 0 \quad (4.60a)$$

and let us consider reducing the state formed by adding this graviton to $|\Phi\rangle$:

$$|\Phi + (\vec{k}, \lambda)\rangle \equiv A_\pm^\dagger(\vec{k}, \lambda) |\Phi\rangle \quad (4.60b)$$

Application of the two identities (4.59) and a trivial commutation gives the following relation:

$$\begin{aligned} \langle \Phi + (\vec{k}, \lambda) | \kappa \psi_{\mu\nu}^\pm(u, \vec{x}) | \Phi + (\vec{k}, \lambda) \rangle &= \langle \Phi | \overleftrightarrow{T} \left\{ A_-(\vec{k}, \lambda) \kappa \psi_{\mu\nu}^\pm(u, \vec{x}) A_+^\dagger(\vec{k}, \lambda) \right\} | \Phi \rangle \quad (4.61a) \\ &= \langle \Phi | \kappa \psi_{\mu\nu}^\pm(u, \vec{x}) | \Phi \rangle \end{aligned}$$

* Our ordering convention is according to the progress from the initial ket to the final bra. Hence “+” fields stand to the right of “−” fields. The “+” fields are distributed from right to left according to decreasing values of conformal time; that is, “+” fields with smaller conformal times stand to the left. The “−” fields are distributed from right to left according to increasing values of conformal time; that is, “−” fields with larger conformal times stand to the left.

$$\begin{aligned}
& -H^3 \int_{\mathcal{M}} d^4 x' \Psi_{\alpha\beta}^*(u', \vec{x}', \vec{k}, \lambda) D'^{\alpha\beta\delta} \int_{\mathcal{M}} d^4 x'' \Psi_{\theta\theta}(u'', \vec{x}'', \vec{k}, \lambda) D''^{\theta\theta\zeta\zeta} \\
& \times \langle \Phi | \overline{T} \left\{ \left[\psi_{\gamma\delta}^+(u', \vec{x}') - \psi_{\gamma\delta}^-(u', \vec{x}') \right] \kappa \psi_{\mu\nu}^+(u, \vec{x}) \left[\psi_{\zeta\zeta}^+(u'', \vec{x}'') - \psi_{\zeta\zeta}^-(u'', \vec{x}'') \right] \right\} | \Phi \rangle
\end{aligned} \quad (4.61b)$$

Successive applications of this formula can be used to extract any number of initial gravitons, just as in conventional reduction.

The first term in (4.61b) is the usual forward scattering contribution. In an asymptotic scattering process it is typically zero because none of the outgoing particles is likely to have precisely the same momentum and helicity as any of the incoming particles. There is no such restriction for our modified “in”-“in” formalism: in our case the outgoing particles are identical to the incoming ones. This means that the final term certainly contributes.

To better understand the second term in (4.61b) let us consider the case of a single external graviton and let us amputate the observed field:

$$\begin{aligned}
D_{\mu\nu}^{\rho\sigma} \langle (\vec{k}, \lambda) | \kappa \psi_{\rho\sigma}^+(u, \vec{x}) | (\vec{k}, \lambda) \rangle &= D_{\mu\nu}^{\rho\sigma} \langle 0 | \kappa \psi_{\rho\sigma}^+(u, \vec{x}) | 0 \rangle \\
&- H^3 D_{\mu\nu}^{\rho\sigma} \int_{\mathcal{M}} d^4 x' \Psi_{\alpha\beta}^*(u', \vec{x}', \vec{k}, \lambda) D'^{\alpha\beta\gamma\delta} \int_{\mathcal{M}} d^4 x'' \Psi_{\theta\theta}(u'', \vec{x}'', \vec{k}, \lambda) D''^{\theta\theta\zeta\zeta} \\
&\times \langle 0 | \overline{T} \left\{ \left[\psi_{\gamma\delta}^+(u', \vec{x}') - \psi_{\gamma\delta}^-(u', \vec{x}') \right] \kappa \psi_{\rho\sigma}^+(u, \vec{x}) \left[\psi_{\zeta\zeta}^+(u'', \vec{x}'') - \psi_{\zeta\zeta}^-(u'', \vec{x}'') \right] \right\} | 0 \rangle
\end{aligned} \quad (4.62)$$

The first term just gives the vacuum effect we have already discussed. Note the subtractions in the two bracketed parts of the second term. This term can only contribute when interaction vertices link the bracketed fields with the observed field at $(u, \vec{x})$. In this case the bracketed fields behave very much as a new kind of interaction vertex, and one only gets a nonzero contribution when the points $(u', \vec{x}')$ and $(u'', \vec{x}'')$ lie on or within the past lightcone of the observer at $(u, \vec{x})$.*

* One might worry that the two fields in each bracket add outside the observer’s lightcone, rather than interfere, because their relative minus sign just cancels the relative minus sign between “+” and “−” vertices. This is not correct. There is a *third* relative minus sign derived from the fact that $\psi_{\rho\sigma}^+$ is amputated using $D_{\mu\nu}^{\rho\sigma}$ whereas $\psi_{\rho\sigma}^-$ is amputated using $-D_{\mu\nu}^{\rho\sigma}$.

Fig. 21: One-loop contributions to the graviton background (top line) from a single initial graviton (bottom lines).

The generic one-loop diagrams from the second term in (4.62) are displayed in Fig. 21. Of these, diagrams (d) and (e) give one-loop contributions of the same order as the two-loop vacuum effect with the following qualitative structure:

$$\begin{aligned}
& \kappa H^3 \int_u^{H^{-1}} du' \int d^3 x' \Psi^*(u', \vec{x}', \vec{k}, \lambda) \int_u^{H^{-1}} \int d^3 x'' \Psi(u'', \vec{x}'', \vec{k}, \lambda) \\
& \times V_3(x) i\Delta(x; x') V_3(x') i\Delta(x'; x'') V_3(x'') i\Delta(x'', x)
\end{aligned} \quad (4.63)$$

Just as with the vacuum term, the most significant infrared contributions come from when u' and u'' approach the observation time. In this region causality requires the three spatial coordinates to be close as well, and we see from (4.54a) that the product of the two wave functions simplifies:

$$\Psi_{\alpha\beta}^*(u', \vec{x}', \vec{k}, \lambda) \cdot \Psi_{\theta\theta}(u'', \vec{x}'', \vec{k}, \lambda) \longrightarrow H^2 k^{-3} \epsilon_{\alpha\beta}(\vec{k}, \lambda) \epsilon_{\theta\theta}(\vec{k}, \lambda) \quad (4.64)$$

Since all momenta are proportional to H , the dominant infrared contribution has the same coupling constant and u dependence as that of the vacuum:

$$\frac{\kappa^4 H^2}{u^4} \left\{ \ln^2(Hu) + (\text{subdominant}) \right\} \quad (4.65)$$

The only difference is that the constant tensor factors of the single graviton contribution can depend upon $\vec{k}$. Thus we recover homogeneity but not necessarily isotropy.

A point worth noting is that although there can be any number of spectator gravitons at a given order, no more than two initial gravitons can become involved with the observed

field at order κ^4 . The topology of the two graviton contribution is that of a tree diagram with five external lines, four for the two gravitons of the initial state and one for the observed field (see Fig. 22). The dominant infrared contributions of this type will be ones composed of three 3-point vertices, two of which are free (diagrams (d) and (e) of Fig. 22). It might be very much simpler to compute this tree diagram than either the pure vacuum contribution at two loops or the single graviton contribution at one loop. Note as well that these are tree diagrams so they should be ultraviolet finite.

Fig. 22: Tree contributions to the graviton background (top line) from two initial gravitons (bottom lines).

Our analysis of initial gravitons leads to the following understanding of the physics of relaxation. The first term in (4.62) gives contributions in which the initial graviton is a spectator. The relaxation effect in this case derives purely from the fact that when self-interactions of dimension three are available, vacuum fluctuations always act to lower the ground state energy. The effect becomes significant because it acts coherently over the ever larger invariant volume covered by the observer's past lightcone.

The second term in (4.62) describes contributions in which an initial graviton becomes involved with the observation. Of course the dominant infrared effect comes long after the initial graviton's physical wavelength has been redshifted beyond the observer's causal horizon. He is not detecting the graviton *per se*, only the contribution its negative one-loop self-energy makes to the local stress-energy. This is why he sees a homogeneous effect. We stress that this is a completely causal process: just because a particle's physical wavelength

redshifts beyond the Hubble radius does not mean that its contribution to the stress-energy disappears.

Both of the relaxation mechanisms described above could have been inferred — and were [8] — from the infrared problems of the “in”-“out” formalism. What could not be guessed in this way is the time dependence; accurate estimates require our modified “in”-“in” formalism. Based on a detailed consideration of this formalism we have given strong physical and mathematical reasons to support the time dependence posited in relation (1.11).

5. Discussion

Our present study of perturbative QCG with a positive cosmological constant leads to five significant conclusions:

- (1) *The theory can be used reliably in the far infrared*, in spite of the fact that it is not perturbatively renormalizable. Since we know that there is a finite quantum theory of gravitation our formalism is very similar to that of the modified Fermi theory which results when one integrates out the electroweak gauge fields and then expands the resulting action in powers of derivatives of the Fermi fields. The counterterms needed to absorb ultraviolet divergences are present order by order in perturbation theory, even though the model is not renormalizable. Of course the counterterms have finite coefficients as well, but the higher ones are irrelevant to the physics of long times and distances.
- (2) *The theory exhibits very strong infrared effects* because it allows massless gravitons to self-interact through a coupling of dimension three, and because the inflationary redshift continually increases the population of soft gravitons. We stress that this is not simply intuitive reasoning based on experience from flat space, we saw it emerge

in sections 3 and 4 from a detailed analysis of perturbative QCG on a de Sitter background.

- (3) *QCG dominates the physics of an inflating universe with respect to any matter theory whose phenomenological viability does not require fine tuning.* Most familiar matter fields are either massive — and so do not generate effects which can add coherently over long times — or else they are massless but conformally invariant — and so do not feel the inflationary redshift. A massless, minimally coupled scalar with a cubic self-interaction would be competitive with QCG — and would enhance, not diminish relaxation — but it is not natural to assume that the scalar mass is small.

- (4) *Infrared effects are so strong in QCG that the conventional S-matrix is not even finite at tree order [8].* The mathematics behind this result is that in our gauge, and with our choice of field variables, the interaction vertices diverge quadratically as the conformal time goes to zero, while the propagators and the external wave functions approach nonzero constants. The physics is just that all momenta redshift to zero at late times, so gravitational waves interact coherently over larger and larger volumes. The infrared problems of the S-matrix strongly suggest a relaxation of the effective cosmological constant, although accurate estimates of the time dependence must be made using the modified “in”-“in” formalism described in section 2 for scalars and in section 4 for QCG.

- (5) *The time derivative of the effective cosmological constant is negative in perturbation theory, modulo some tensor algebra.* The effect for initially empty space occurs through the causal and coherent superposition of vacuum fluctuations in the past lightcone of a local observer. These fluctuations lower the vacuum energy — and hence slow the rate of expansion — because they act through a cubic self-interaction. The effect becomes large because the invariant volume of a local observer’s past lightcone grows without bound as he evolves to later and later times. If the initial state contains gravitons

then it experiences both the vacuum effect and a comparably strong enhancement of relaxation coming from the graviton’s tendency to develop a negative self-energy.

The fact that our mechanism operates through the coherent superposition of distant but causally related interactions is one of its most desirable features. The time lag required for such an effect to build up is what permits an extensive period of inflation. If local effects were instead responsible for screening the cosmological constant then — unless one resorts to fine tuning — inflation is bound to be extinguished very rapidly.

A related point is that there are no gravity shields. It would be very difficult to imagine stopping the sea of soft gravitons that we believe is responsible for screening Λ . They have been redshifted to such an extent that it is almost impossible for them to interact locally, but of course we are still affected by their contribution to the stress-energy. Nor should one be surprised to see no spatial variation. The effect is homogeneous for the same reason that the rest of the universe is: it has been blown up from very tiny scales.

We believe that QCG has a plethora of implications for inflationary cosmology:

- (1) Inflation occurs for no other reason than that the bare cosmological constant is positive and not unreasonably small. In particular, inflation does not require the fine tuning of scalar potentials, nor does it even require the existence of any light fundamental scalars.

- (2) The visible universe we inhabit grew out of a patch of about one Hubble radius which began inflation when its temperature dropped below $(\Lambda/G)^{1/4}$. Before this time there were no strong infrared effects because thermal fluctuations screened long range correlations.

- (3) Relaxation occurs in QCG through the causal and coherent superposition of interactions throughout a steadily increasing volume of spacetime. A substantial period of inflation is permitted because the infrared effect must accumulate from a large vol-

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time of spacetime before it can enhance the naturally small dimensionless couplings of gravity. For GUT scale inflation we estimate the relaxation time to be about 10^{12} e-foldings, which is comfortably more than the sixty necessary to explain the smoothness problem.

(4) Since matter is irrelevant to the phenomenon of relaxation the predictions of QCG should be unique.

(5) Perturbative QCG breaks down when relaxation becomes significant. For most interesting questions — the relaxing temperature, the magnitude and spectrum of late time density perturbations, the response to low energy phase transitions — predictions from QCG may be as difficult to obtain as those of low energy QCD.

(6) It is conceivable that our universe is still experiencing a residual inflationary expansion. This is one of the few interesting questions for which perturbative methods might be able to make a definite prediction.

(7) It is possible that couplings other than $\Lambda_{\text{eff}}(t)$ are affected by relaxation. For example, one should be able to define an effective Newton's constant using a variant of our force observable, and then check how it is affected by relaxation.

(8) Since we are no longer fine tuning the cosmological constant to be zero in the present epoch, we must expect that there was probably a period of inflation or deflation, followed by relaxation, after every sufficiently rapid cosmological phase transition. Some of these phase transitions — for example, the breaking of the electroweak and QCD chiral symmetries — occurred quite late in the evolution of our universe. The response of QCG to such late phase transitions is likely to have had a crucial impact on the content of the current universe.

Phase transitions occur when whatever is the appropriate free energy of the system under study develops a new global minimum. Although it is possible to imagine entrop-

ically driven transitions in which the vacuum energy of the new phase exceeds that of the old one, the obvious assumption — and the one realized in both the electroweak and QCD phase transitions — is that the new vacuum energy lies below the old one. We must therefore expect that any phase transitions subsequent to the first period of inflation and relaxation induce a negative effective cosmological constant and a corresponding period of deflation.

What happens in this case is a fascinating question to which we do not know the answer. Just as with the case of a positive cosmological constant, there is an uninteresting mathematical obstacle and a vitally important physical one. The mathematical problem is how to formulate QCG on anti-de Sitter space. The peculiar thing about this manifold is that its casual structure allows information to flow in from spatial infinity, and out to it, in a finite amount of time. Some boundary condition must be identified before the theory is even self-contained. The standard choice is the “reflective” boundary condition [28] in which every source is shackled to an anti-source located at the antipodal point.

We do not feel that reflective boundary conditions are very physical, but the point is scarcely worth arguing because there is zero probability for the simultaneous appearance of an infinite, correlated patch of anti-de Sitter vacuum. The physical problem is that any anti-de Sitter patch must have finite size. Therefore what goes on inside it is very strongly influenced by the outside universe. It was the lack of such outside influence which allowed us to be relatively careless about formulating QCG at the onset of inflation. Now we must be careful, and the trouble is that we do not understand the physics of the situation very well.

An intriguing possibility is that deflation is self-regulating because it tends to dissipate the infrared effect which is responsible for screening the GUT scale bare cosmological constant. Another possibly relevant point is that the scalar analog model shows relaxation for either sign of its cubic coupling constant, even though the vacuum energy is negative

definite. As the sign of the cubic coupling changes, so too does the sign of the induced 1-point function. The close correspondence between the two theories suggests that the same thing may occur in QCG.

In fact our understanding of QCG breaks down long before we face the prospect of $\Lambda_{\text{eff}}(t) < 0$. An obvious first thought is that control might be gained over the non-perturbative region by extending into it using a renormalization group analysis. The procedure would be to follow the relaxation of $\Lambda_{\text{eff}}(t)$ for a short period Δt during which perturbation theory is valid. One would then reset the coupling constant Λ to $\Lambda_{\text{eff}}(\Delta t)$ and repeat the process for another short period. Since the cosmological constant is driven to ever smaller values, one might expect that perturbation theory in QCG must improve just as it does for QCD when we use the renormalization group to sum the leading logarithms of the deep ultraviolet.

The renormalization group has shown itself to be such a powerful technique in the ultraviolet that we should not like to argue generally against its application to this infrared problem. However, the specific scheme just outlined does not work because it fails to capture the essential nonlocality of relaxation. Inflation is slowed in our model by the causal and coherent superposition of interactions through the ever growing invariant volume of the past lightcone. A universe which contains this sea of virtual gravitons is not the same as an empty universe with a smaller cosmological constant. In fact no adjustment of parameters in the local theory should be generally able to mimic the virtual infrared gravitons because they do not represent an even approximately local effect. We do expect at very late times, and over small spatial and temporal separations, that QCG should resemble classical gravitation with a very small cosmological constant. However, we do not expect this at all when relaxation first becomes significant, and there may be important residual effects at large spatial and temporal separations even at late times. The analogy seems not be with QCD in the ultraviolet but rather with the physics of confinement.

Unless the previous reasoning is incorrect it is going to be very difficult to gain quantitative control over the regime in which relaxation becomes significant. During this period of time the scale factor may suffer violent fluctuations as the momentum of inflation is thermalized. However, it is conceivable that the universe then enters a “quiet phase” in which the scale factor is characterized by a slow residual expansion. A simple ansatz would be the arbitrary power law: $R(t) \sim t^p$. The possibility of slow residual expansion is exciting for two reasons: first, the residual expansion rate may be observable; and second, we may be able to use perturbation theory to predict it. One can certainly express the propagator as a mode sum for any value of α . The procedure would then be to check whether α can be chosen so that at asymptotically late times the 1-point function induced by quantum corrections cancels the 1-point function derived from the fact that the background metric does not obey the classical Einstein equations.

Though we have largely defined the apparatus of perturbative QCG on $T^3 \times R$ we have barely begun to exploit the formalism, and mastery of the crucial nonperturbative regime seems to be a problem comparable in difficulty to that of making low energy predictions using QCD. Nevertheless, it is often simpler to reach a goal when one knows roughly which way to head. That perturbative quantum gravity manifests enormously strong infrared effects when the cosmological constant is positive seems beyond doubt. We believe that these effects provide the physical basis for gradually slowing the expansion of an inflating universe. The implications of this proposal for inflationary cosmology and for low energy phenomenology are numerous and profound. Perhaps the most intriguing aspect of our work is that it has the potential for bringing quantum gravity for the first time within the realm of making testable predictions.

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