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KUCP-64
January 1994

Representation Blocks of Conformal Fields for the $N=4$ $SU(2)_k$ Superconformal Algebras

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ABSTRACT

The representation theories of the $SU(2)_k$ -extended $N=4$ superconformal algebras (SCAs) with *arbitrary* level k are developed being based on their Feigin-Fuchs representations found recently by the present author. A basic unit of the representation blocks consisting of eight "boson-like" and eight "fermion-like" conformal fields is found to describe arbitrary representations of the $N=4$ $SU(2)_k$ SCAs, including *unitary* and *nonunitary* representations. The transformation properties of the fundamental sets of the conformal fields under the $N=4$ $SU(2)_k$ superconformal symmetries are given. Then, the whole sets of the charge-screening operators of the $N=4$ $SU(2)_k$ SCAs are identified out of the sixteen conformal fields in the basic unit of the representation blocks. The conditions for the *eligible* charge-screening operators are analyzed in terms of the continuous parameters which enter in our vertex-operator forms for the fundamental conformal fields of the representation blocks.

* Work supported in part by the Grant-in-Aid for Scientific Research on Priority Area "Infinite Analysis" (#05230029) from the Ministry of Education, Science and Culture.

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1. Introduction

The representation theories of the $N=4$ $SU(2)_k$ -extended superconformal algebras (SCAs) have been studied extensively by now¹⁻⁸. In particular, Eguchi and Taormina^{2,3} gave a comprehensive survey of the unitary representations, whereas the superfield formalism was investigated systematically by Matsuda and Uematsu^{5,6}. Also, the conjecture of the Kac determinant formulae of the $N=4$ $SU(2)_k$ SCAs has been made by Kent and Riggs⁷. The expressions of the $N=4$ Kac formulae were obtained by a mixture of analytic arguments and computational experiment. Their rigorous analytic proof was left for future studies.

Recently the present author has found the Feigin-Fuchs representations of the $N=4$ $SU(2)_k$ SCAs expressed in terms of the four pairs of boson and fermion fields⁸. The representations incorporate the bosonic representations of the non-abelian $SU(2)_k$ Kac-Moody algebras *a la* Nemeshtansky⁹, where we have $k = \hat{k} + 1$. The study of the chiral and nonchiral vertex operators in the discovered Feigin-Fuchs representations allowed us⁸ to accomplish the analytic proof of the $N=4$ Kac determinant formulae. In fact, in ref.8, we correctly identified the necessary and sufficient number of the charge-screening operators of the $N=4$ $SU(2)_k$ SCAs, thereby proving analytically the conjectured expressions⁷ of the $N=4$ Kac formulae by the use of the *complex contour method* of Kato and Matsuda^{10,11,12}. The full analytic proof requires a rather elaborate analysis, and the details of the study will be presented in a future publication.

In the present paper we report on the representation blocks of the conformal fields, which describe the whole representations of the $N=4$ $SU(2)_k$ SCAs. Within the framework of the discovered Feigin-Fuchs representations of the $N=4$ $SU(2)_k$ SCAs, we have found a basic unit of the representation blocks which is composed

of eight “boson-like” and eight “fermion-like” conformal fields in the form of vertex operators. By varying the continuous parameters which enter in the vertex operator forms of the fundamental conformal fields in the representation blocks, we can explore the arbitrary representations of the $N=4$ $SU(2)_k$ SCAs including *unitary* and *nonunitary* ones. The transformation properties of the representation blocks under the $N=4$ $SU(2)_k$ superconformal symmetries are given. Then, the whole sets of the charge-screening operators of the $N=4$ $SU(2)_k$ SCAs are identified out of the sixteen conformal fields by investigating the conditions for the *eligible* charge-screening operators in terms of the continuous parameters entering in the vertex operator forms.

The immediate application of our results is to confirm that the charge-screening operators we have obtained in ref.8 exhaust the whole sets of them for the $N=4$ $SU(2)_k$ SCAs. As an important consequence of this observation, we have found⁸ that the expressions of the conjectured $N = 4$ Kac formulae⁷ exhaust all the necessary factors of the determinant zeroes to be included in the formulae.

2. The $N=4$ $SU(2)_k$ superconformal algebras

Now, the structure of the SCAs is essentially specified by N and for each N there is a range of possible modings for the algebraic generators parametrizing distinct algebras with this basic supersymmetry structure.¹³ To get around the complexity of the moding assignments till the last moment, we generate the $N=4$

$SU(2)_k$ SCAs by operators $L(z), T^i(z), G^a(z), \bar{G}_a(z)$ and a c -number central charge $c = 6k$ in the form of operator product expansions (OPEs):

$$\begin{aligned} L(z)L(w) &\sim \frac{3k}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w}, \\ T^i(z)T^j(w) &\sim \frac{\frac{1}{2}k\eta^{ij}}{(z-w)^2} + \frac{i\epsilon^{ijk}\eta_k T^l(w)}{z-w}, \quad L(z)T^i(w) \sim \frac{T^i(w)}{(z-w)^2} + \frac{\partial_w T^i(w)}{z-w}, \\ T^i(z)G^a(w) &\sim -\frac{\frac{1}{2}(\sigma^i)^a{}_b G^b(w)}{z-w}, \quad T^i(z)\bar{G}_a(w) \sim \frac{\frac{1}{2}\bar{G}_b(w)(\sigma^i)^b{}_a}{z-w}, \\ L(z)G^a(w) &\sim \frac{\frac{3}{2}G^a(w)}{(z-w)^2} + \frac{\partial_w G^a(w)}{z-w}, \quad L(z)\bar{G}_a(w) \sim \frac{\frac{3}{2}\bar{G}_a(w)}{(z-w)^2} + \frac{\partial_w \bar{G}_a(w)}{z-w}, \\ G^a(z)G^b(w) &\sim 0, \quad \bar{G}_a(z)\bar{G}_b(w) \sim 0 \\ G^a(z)\bar{G}_b(w) &\sim \frac{4k\delta^a{}_b}{(z-w)^3} - \frac{4(\sigma^i)^a{}_b \eta_{ij} T^j(w)}{(z-w)^2} - \frac{2(\sigma^i)^a{}_b \eta_{ij} \partial_w T^j(w)}{z-w} + \frac{2\delta^a{}_b L(w)}{z-w} \end{aligned} \quad (1)$$

The superscripts and subscripts $i=0,\pm$ denote $SU(2)$ triplets in the diagonal basis, whereas the superscripts (subscripts) $a=1,2$ label $SU(2)$ doublet (antidoublet) representations. The group tensors ϵ^{ijk} , $\eta^{ij} = \eta_{ij}$ and the Pauli matrices $(\sigma^i)^a{}_b$ are defined in the diagonal basis. The symmetric tensor $\eta^{ij} = \eta^{ji}$ is: $\eta^{+-} = \eta^{00} = 1, \epsilon^{tc}$, otherwise zero, whereas the antisymmetric tensor ϵ^{ijk} is: $\epsilon^{+-0} = -i, \epsilon^{tc}$, otherwise zero. The Pauli matrices are: $\sigma^\pm = (\sigma^1 \pm \sigma^2)/\sqrt{2}$, $\sigma^0 = \sigma^3$. For later convenience the fermionic operators are hereby re-labeled according to the canonical notation of $SU(2)$ spins, thus the index a runs over $\pm$ corresponding to the increase(+) or decrease(−) of their third components by a half unit when the operators are ap-

plied to conformal states. We henceforth have $G^- \equiv G^1$, $G^+ \equiv G^2$, $\bar{G}_+ \equiv \bar{G}_1$ and $\bar{G}_- \equiv \bar{G}_2$. The symmetric delta function $\delta^a_b = \delta^b_a$ has the standard meaning with the $SU(2)$ doublet or antidoublet labels, therefore we have $\delta^-_+ = \delta^1_1 = 1$, $\delta^+_- = \delta^2_2 = 1$, $\delta^+_+ = \delta^2_1 = 0$, $\delta^-_- = \delta^1_2 = 0$.

3. The Feigin-Fuchs representations

Our Feigin-Fuchs (Coulomb gas) theories of the $N=4$ $SU(2)_k$ SCAs are represented in terms of four real bosons $\varphi_\alpha(z)$ ($\alpha=1,2,3,4$), and four real fermions forming⁵ a pair of complex fermion doublet $\gamma^a(z)$ and antidoublet $\bar{\gamma}_a(z)$ ($a=1,2$ or $\pm$). The four real bosons and the pairs of complex fermions satisfy the following OPEs :

$$\varphi(z)\partial\varphi(w) \sim \frac{1}{z-w}, \quad \gamma^a(z)\bar{\gamma}_b(w) \sim \frac{1}{z-w}\delta^a_b \quad (2)$$

We first consider the currents J^i ($i=0,\pm$) forming the $SU(2)_k$ Kac-Moody subalgebras with *arbitrary* level $\hat{k}$. Their Feigin-Fuchs forms are given in terms of the first three bosons by⁹

$$\begin{aligned} J^0(z) &= i\sqrt{\frac{\hat{k}}{2}}\partial\varphi_3 \\ J^\pm(z) &= i\sqrt{\frac{1}{2}}\left(\sqrt{\frac{\hat{k}+2}{2}}\partial\varphi_1 \pm i\sqrt{\frac{\hat{k}}{2}}\partial\varphi_2\right)e^{\pm i\sqrt{\frac{\hat{k}}{2}}(\varphi_3 - i\varphi_2)} : \end{aligned} \quad (3)$$

The corresponding contribution of the energy-momentum tensor is then given due to the Sugawara construction as

$$\frac{1}{\hat{k}+2}\sum_{i,j=0,\pm} : J^i(z)\eta_{ij}J^j(z) : = -\frac{1}{2}\sum_{\alpha=1}^3 : (\partial\varphi_\alpha)^2 : + i\frac{\tau}{2}\partial^2\varphi_1 \quad (4)$$

where $\tau \equiv \sqrt{2/(\hat{k}+2)}$.

Now, the total energy-momentum tensor $L(z)$ is given by adding the contributions from the fermion doublets and the fourth boson to eq.(4) as:

$$L(z) = -\frac{1}{2}\sum_{\alpha=1}^4 : (\partial\varphi_\alpha)^2 : + i\frac{\tau}{2}\partial^2\varphi_1 - i\kappa\partial^2\varphi_4 + \frac{1}{2} : (\partial\bar{\gamma} \cdot \gamma - \bar{\gamma} \cdot \partial\gamma) : \quad (5)$$

with the parameter value of $\kappa \equiv i(\hat{k}+1)\tau/2$, where one should note the i factor in front.

The $N=4$ supercurrents $G^a(z)$ and $\bar{G}_a(z)$ in our Feigin-Fuchs representations are:

$$\begin{aligned} G^a(z) &= i\gamma^a\partial\varphi_4 - 2\kappa\partial\gamma^a - i\tau J^i\eta_{ij}(\sigma^j\gamma)^a + i\tau : (\bar{\gamma} \cdot \gamma)\gamma^a : \\ \bar{G}_a(z) &= i\bar{\gamma}_a\partial\varphi_4 - 2\kappa\partial\bar{\gamma}_a + i\tau J^i\eta_{ij}(\bar{\gamma}\sigma^j)_a - i\tau : (\bar{\gamma} \cdot \gamma)\bar{\gamma}_a : \end{aligned} \quad (6)$$

We finally define the total $SU(2)_k$ Kac-Moody currents, $T^i(z)$, with level $\hat{k} \equiv \hat{k}+1$ by adding the fermionic contribution as

$$T^i(z) = J^i(z) + \frac{1}{2} : \bar{\gamma}\sigma^i\gamma(z) : \quad (7)$$

We note that our Feigin-Fuchs representations just presented in fact can be confirmed to satisfy the $N=4$ $SU(2)_k$ SCAs, eq.(1), for the parameter values being summarized as follows:

$$\mu \equiv i\tau = i\sqrt{\frac{2}{\hat{k}+2}}, \quad \kappa = i(\hat{k}+1)\frac{\tau}{2}, \quad k = \hat{k}+1 \quad (8)$$

with *arbitrary* level $\hat{k}$.

4. Vertex operators and conformal weights

The vertex operators representing basic primary fields of the $N=4$ $SU(2)_k$ SCAs are given by

$$V(t, j, j_0, z) = : e^{it\varphi_4(z)} : V_{j,j_0}(z) \quad (9)$$

where the vertex operator $V_{j,j_0}(z)$ is defined as

$$V_{j,j_0}(z) = : e^{-ij\tau\varphi_1} e^{ij_0\sqrt{\frac{2}{k}}(\varphi_3 - i\varphi_2)} : \quad (10)$$

which gives the representations labeled by $SU(2)$ -spin (j, j_0) of the $SU(2)_k$ Kac-Moody algebra and satisfy the following OPEs:

$$J^0(z)V_{j,j_0}(w) \sim \frac{j_0}{z-w}V_{j,j_0}(w), \quad \sqrt{2}J^\pm(z)V_{j,j_0}(w) \sim \frac{-j \pm j_0}{z-w}V_{j,j_0 \pm 1}(w) \quad (11)$$

In our Coulomb gas representations a primary state with conformal dimension h and $SU(2)$ -spin (j, j_0) is defined as

$$|h, j_0\rangle \sim V(t, j, j_0, z=0)|0\rangle \quad (12)$$

where $|0\rangle$ is the ground state and the conformal dimension h is given by

$$h = h(t, j) \equiv \frac{t^2}{2} + \kappa t + \frac{\tau^2}{2}j(j+1) = \frac{1}{2}(t+\kappa)^2 + \frac{\tau^2}{2}\left(j + \frac{1}{2}\right)^2 + \frac{\hat{k}}{4} \quad (13)$$

As discussed extensively in ref.14, the vertex operator $V(-t-2\kappa, j, j_0, z)$ conjugate to $V(t, j, j_0, z)$ has the same conformal weight $h(-t-2\kappa, j) = h(t, j)$ as its conjugate, thus plays crucial roles in investigating the Kac determinant formulae in the t parameter space. The reference 14 should be referred to for the details in this context.

Quite similarly, the vertex operator $V_{-j-1, j_0}(z)$ conjugate to $V_{j, j_0}(z)$ has the same conformal weight Δ_j as $V_{j, j_0}(z)$ does, with the definition of

$$\Delta_j \equiv \frac{\tau^2}{2}j(j+1) = \frac{j(j+1)}{\hat{k}+2} = \Delta_{-j-1} \quad (14)$$

The role of the conjugate operator has to be taken into proper account whenever the conjugate expressions obtained by its usage provide valid results. Thus in our Coulomb gas representation, for a given conformal weight $h = h(t, j) = h(t, -j-1)$, there are, in general terms, two ways of defining a primary state with conformal weight h and *highest weight* $SU(2)$ -spin j_0 :

$$|h = h(t, j), j_0 = j\rangle \equiv V(t, j, j_0 = j, z=0)|0\rangle \quad (15)$$

$$|h = h(t, -j-1), j_0 = -j-1\rangle \equiv V(t, -j-1, j_0 = -j-1, z=0)|0\rangle \quad (16)$$

For simplicity we present the equations here for the Neveu-Schwarz case, but our analysis is of course general and the results hold for other modings as well, which will be explicit later.

At this point, following ref.7, let us introduce the “charge” operator $\hat{C}$ whose action is defined by

$$\begin{aligned} \hat{C}|h, j_0\rangle &= 0, \quad [\hat{C}, G_a(z)] = G_a(z), \quad [\hat{C}, \tilde{G}_a(z)] = -\tilde{G}_a(z), \\ [\hat{C}, L(z)] &= 0, \quad [\hat{C}, T^i(z)] = 0 \end{aligned} \quad (17)$$

The eigenvalues $\hat{C}=\pm 1$ actually correspond to nothing but the up and down of the third component of $SU(2)_{global}$ spin.⁵

5. Representation blocks of conformal fields

Next we proceed to the construction of the *representation blocks* of conformal fields. As illustrated by the rhombic dodecahedron in fig.1, a basic unit of the representation blocks has the top and bottom points and the three layers in between. The fundamental set of the conformal fields forming the representation blocks can be identified by successively taking the OPEs of each entry in the representation blocks with the $N = 4 \text{ SU}(2)_k$ supercurrents $G^a(z)$, $\bar{G}_a(z)$ ($a = \pm$).

One starts with the “boson-like” fundamental conformal field sitting at the bottom point of the rhombic dodecahedron, which is given by the basic vertex operator form $V(t, j, j_0, z)$. By taking the OPEs of $V(t, j, j_0, z)$ with the supercurrents:

$$G^\mp(z)V(t, j, j_0, w) \sim \frac{1}{z-w}V^\mp(t, j, j_0, w), \quad (18)$$

$$\bar{G}_\pm(z)V(t, j, j_0, w) \sim \frac{1}{z-w}\bar{V}_\pm(t, j, j_0, w),$$

the four “fermion-like” *chiral* conformal fields, $V^a(t, j, j_0, z)$ and $\bar{V}_a(t, j, j_0, z)$ ($a = \pm$), sitting at the four corners of the first layer are found to be given by

$$V^\mp(t, j, j_0, z) = (t \mp i\tau j_0)\gamma^\mp V(t, j, j_0, z) + i\tau(j \pm j_0)\gamma^\pm V(t, j, j_0 \mp 1, z), \quad (19)$$

$$\bar{V}_\pm(t, j, j_0, z) = (t \pm i\tau j_0)\bar{\gamma}_\pm V(t, j, j_0, z) - i\tau(j \mp j_0)\bar{\gamma}_\mp V(t, j, j_0 \pm 1, z).$$

The fundamental six “boson-like” conformal fields, $X^\pm(t, j, j_0, z)$, $Y(t, j, j_0, z)$, $\bar{Y}(t, j, j_0, z)$, $W(\pm)(t, j, j_0, z)$, sitting singly at the four corners of and doubly in the center of the middle layer of the rhombic dodecahedron are constructed by the

OPE relations:

$$G^\mp(z)V^\pm(t, j, j_0, w) \sim \pm \frac{1}{z-w}Y(t, j, j_0, w), \quad (20)$$

$$\bar{G}_\pm(z)\bar{V}_\mp(t, j, j_0, w) \sim \pm \frac{1}{z-w}\bar{Y}(t, j, j_0, w),$$

$$G^\mp(z)V^\mp(t, j, j_0, w) \sim 0, \quad \bar{G}_\pm(z)\bar{V}_\pm(t, j, j_0, w) \sim 0 \quad (21)$$

$$G^\mp(z)\bar{V}_\mp(t, j, j_0, w) \sim \frac{2(j \pm j_0)}{(z-w)^2}V(t, j, j_0 \mp 1, w) - \frac{1}{z-w}X^\mp(t, j, j_0, w), \quad (22)$$

$$\bar{G}_\pm(z)V^\pm(t, j, j_0, w) \sim -\frac{2(j \mp j_0)}{(z-w)^2}V(t, j, j_0 \pm 1, w) + \frac{1}{z-w}X^\pm(t, j, j_0, w).$$

and

$$G^\mp(z)\bar{V}_\pm(t, j, j_0, w) \sim \frac{2(h \mp j_0)}{(z-w)^2}V(t, j, j_0, w) + \frac{1}{z-w}[\partial V(t, j, j_0, w) \pm W^\mp(t, j, j_0, w)],$$

$$\begin{aligned} \bar{G}_\pm(z)V^\mp(t, j, j_0, w) &\sim \frac{2(h \pm j_0)}{(z-w)^2}V(t, j, j_0, w) \\ &+ \frac{1}{z-w}[\partial V(t, j, j_0, w) \mp W^\mp(t, j, j_0, w)], \end{aligned} \quad (23)$$

where

$$Y(t, j, j_0, z) = (t + i\tau j)\left(t - i\tau(j+1)\right)\gamma^-\gamma^+V(t, j, j_0, z), \quad (24)$$

$$\bar{Y}(t, j, j_0, z) = (t + i\tau j)\left(t - i\tau(j+1)\right)\bar{\gamma}_+\bar{\gamma}_-V(t, j, j_0, z),$$

$$X^\pm(t, j, j_0, z) = i\tau(j \mp j_0) : (i\partial\varphi_4 \pm i\tau J^0)V(t, j, j_0 \pm 1, z) :$$

$$+ i\tau(t \pm i\tau j_0) : \sqrt{2}J^\pm V(t, j, j_0, z) :$$

$$\begin{aligned} & \mp i\tau(j \mp j_0)(t \pm i\tau j_0) : i\sqrt{\frac{2}{k}}(\partial\varphi_3 - i\partial\varphi_2)V(t, j, j_0 \pm 1, z) : \\ & - i\tau(j \mp j_0)(t + \mu \pm i\tau j_0)(\tilde{\gamma}_\mp \gamma^\pm - \tilde{\gamma}_\pm \gamma^\mp)V(t, j, j_0 \pm 1, z) \\ & + (t \pm i\tau j_0)(t + \mu \pm i\tau j_0)\tilde{\gamma}_\pm \gamma^\pm V(t, j, j_0, z) \\ & + \tau^2(j \mp j_0)(j \mp j_0 - 1)\tilde{\gamma}_\mp \gamma^\mp V(t, j, j_0 \pm 2, z) , \end{aligned} \quad (25)$$

and

$$\begin{aligned} W^{(\pm)}(t, j, j_0, z) = & \left[i\tau j_0 \partial\varphi_4 - \mu t \left(i\sqrt{\frac{k}{2}} \partial\varphi_3 \right) - \tau^2 j \sqrt{\frac{k}{2}} \partial\varphi_2 + i\tau j_0 \partial\varphi_1 \right. \\ & \left. - \tau^2(j(j+1) - j_0^2) i\sqrt{\frac{2}{k}}(\partial\varphi_3 - i\partial\varphi_2) \right. \\ & \left. \pm (t^2 + \tau^2 j_0^2) \tilde{\gamma}_\mp \gamma^\pm \pm \left(\tau^2 j(j+1) - \tau^2 j_0^2 - \mu t \right) \tilde{\gamma}_\pm \gamma^\mp \right] V(t, j, j_0, z) : \\ & - i\tau(j + j_0)(t + i\tau j_0) \tilde{\gamma}_+ \gamma^+ V(t, j, j_0 - 1, z) \\ & + i\tau(j - j_0)(t - i\tau j_0) \tilde{\gamma}_- \gamma^- V(t, j, j_0 + 1, z) . \end{aligned} \quad (26)$$

The another four “fermion-like” conformal fields, $U^a(t, j, j_0, z)$ and $\bar{U}_a(t, j, j_0, z)$

($a = \pm$), sitting at the four corners of the upper layer show up in the OPE relations as

$$\begin{aligned} \bar{G}_\pm(z)Y(t, j, j_0, w) \sim & \pm \frac{2}{(z-w)^2} [(h \pm j_0 + 1)V^\pm(t, j, j_0, w) \\ & + (j \mp j_0)V^\mp(t, j, j_0 \pm 1, w)] \pm \frac{1}{z-w} U^\pm(t, j, j_0, w) , \end{aligned} \quad (27)$$

$$\begin{aligned} G^\mp(z)\bar{Y}(t, j, j_0, w) \sim & \pm \frac{2}{(z-w)^2} [(h \mp j_0 + 1)\bar{V}_\mp(t, j, j_0, w) \\ & - (j \pm j_0)\bar{V}_\pm(t, j, j_0 \mp 1, w)] \pm \frac{1}{z-w} \bar{U}_\mp(t, j, j_0, w) , \end{aligned}$$

where

$$\begin{aligned} G^\mp(z)X^\pm(t, j, j_0, w) \sim & \frac{2}{(z-w)^2} (h \mp j_0)V^\pm(t, j, j_0, w) \\ & + \frac{1}{z-w} [2\partial V^\pm(t, j, j_0, w) - U^\pm(t, j, j_0, w)] , \end{aligned}$$

$$\begin{aligned} \bar{G}_\mp(z)X^\pm(t, j, j_0, w) \sim & -\frac{2}{(z-w)^2} (h \mp j_0)\bar{V}_\pm(t, j, j_0, w) \\ & - \frac{1}{z-w} [2\partial \bar{V}_\pm(t, j, j_0, w) - \bar{U}_\pm(t, j, j_0, w)] , \end{aligned} \quad (28)$$

and

$$\begin{aligned} G^\mp(z)W^{(\pm)}(t, j, j_0, w) \sim & \pm \frac{1}{(z-w)^2} [2(j \pm j_0)V^\pm(t, j, j_0 \mp 1, w) \\ & + V^\mp(t, j, j_0, w)] \mp \frac{1}{z-w} [\partial V^\mp(t, j, j_0, w) - U^\mp(t, j, j_0, w)] , \end{aligned}$$

$$\begin{aligned} \bar{G}_\pm(z)W^{(\pm)}(t, j, j_0, w) \sim & \pm \frac{1}{(z-w)^2} [2(j \mp j_0)\bar{V}_\mp(t, j, j_0 \pm 1, w) \\ & - \bar{V}_\pm(t, j, j_0, w)] \pm \frac{1}{z-w} [\partial \bar{V}_\pm(t, j, j_0, w) - \bar{U}_\pm(t, j, j_0, w)] , \end{aligned} \quad (29)$$

$$\begin{aligned}
U^\mp(t, j, j_0, z) = & (t + i\tau j) \left(t - i\tau(j+1) \right) \\
& : \left\{ \gamma^\mp(i\partial\varphi_4 \mp i\tau J^0) - i\tau\gamma^\pm\sqrt{2}J^\mp + \mu\partial\gamma^\mp \right\} V(t, j, j_0, z) \\
& \mp i\tau(j \pm j_0)\gamma^\pm i\sqrt{\frac{2}{k}}(\partial\varphi_3 - i\partial\varphi_2)V(t, j, j_0 \mp 1, z) \\
& + (t + \mu \mp i\tau j_0)\bar{\gamma}_\mp\gamma^\pm V(t, j, j_0, z) \\
& + i\tau(j \pm j_0)\bar{\gamma}_\pm\gamma^\mp V(t, j, j_0 \mp 1, z) : ,
\end{aligned} \tag{30}$$

$$\begin{aligned}
\bar{U}_\pm(t, j, j_0, z) = & (t + i\tau j) \left(t - i\tau(j+1) \right) \\
& : \left\{ \bar{\gamma}_\pm(i\partial\varphi_4 \pm i\tau J^0) + i\tau\bar{\gamma}_\mp\sqrt{2}J^\pm + \mu\partial\bar{\gamma}_\pm \right\} V(t, j, j_0, z) \\
& \mp i\tau(j \mp j_0)\bar{\gamma}_\mp i\sqrt{\frac{2}{k}}(\partial\varphi_3 - i\partial\varphi_2)V(t, j, j_0 \pm 1, z) \\
& + (t + \mu \pm i\tau j_0)\gamma^\pm\bar{\gamma}_\mp\bar{\gamma}_\pm V(t, j, j_0, z) \\
& - i\tau(j \mp j_0)\gamma^\mp\bar{\gamma}_\pm\bar{\gamma}_\mp V(t, j, j_0 \pm 1, z) : .
\end{aligned}$$

Let us also note the following OPE relations which are already closed without generating the new entries, $U^a(t, j, j_0, z)$ and $\bar{U}_a(t, j, j_0, z)$ ($a = \pm$) :

$$G^\mp(z)Y(t, j, j_0, w) \sim 0, \quad \bar{G}_\pm(z)\bar{Y}(t, j, j_0, w) \sim 0, \tag{31}$$

$$\begin{aligned}
G^\pm(z)X^\pm(t, j, j_0, w) & \sim \frac{2(j \mp j_0)}{(z-w)^2}V^\pm(t, j, j_0 \pm 1, w), \\
\bar{G}_\pm(z)X^\pm(t, j, j_0, w) & \sim \frac{2(j \mp j_0)}{(z-w)^2}\bar{V}_\pm(t, j, j_0 \pm 1, w),
\end{aligned} \tag{32}$$

$$\begin{aligned}
G^\pm(z)W^{(\pm)}(t, j, j_0, w) & \sim \pm \frac{1}{(z-w)^2} \left[2(h \pm j_0) + 1 \right] V^\pm(t, j, j_0, w) \\
& \pm \frac{1}{z-w} \partial V^\pm(t, j, j_0, w),
\end{aligned} \tag{33}$$

$$\begin{aligned}
\bar{G}_\mp(z)W^{(\pm)}(t, j, j_0, w) & \sim \mp \frac{1}{(z-w)^2} \left[2(h \mp j_0) + 1 \right] \bar{V}_\mp(t, j, j_0, w) \\
& \mp \frac{1}{z-w} \partial \bar{V}_\mp(t, j, j_0, w).
\end{aligned}$$

Finally we proceed to the construction of the last fundamental conformal field, $U(t, j, j_0, z)$, sitting at the *top* corner of the representation block . Taking the OPEs of the four “fermion-like” conformal fields, $U^a(t, j, j_0, z)$ and $\bar{U}_a(t, j, j_0, z)$, in the upper layer with the supercurrents again, we generate $U(t, j, j_0, z)$ as

$$\begin{aligned}
\bar{G}_\pm(z)U^\mp(t, j, j_0, w) & \sim \frac{2}{(z-w)^2} \left[(h \pm j_0 + 1) \left\{ \partial V(t, j, j_0, w) \pm W^{(\pm)}(t, j, j_0, w) \right\} \right. \\
& \quad \left. + (j \mp j_0)X^\mp(t, j, j_0 \pm 1, w) \right] \\
& \quad + \frac{1}{z-w} \left[\partial W^{(+)}(t, j, j_0, w) - \partial W^{(-)}(t, j, j_0, w) + U(t, j, j_0, w) \right], \\
G^\mp(z)\bar{U}_\pm(t, j, j_0, w) & \sim \frac{2}{(z-w)^2} \left[(h \mp j_0 + 1) \left\{ \partial V(t, j, j_0, w) \mp W^{(\pm)}(t, j, j_0, w) \right\} \right. \\
& \quad \left. + (j \pm j_0)X^\pm(t, j, j_0 \mp 1, w) \right] \\
& \quad - \frac{1}{z-w} \left[\partial W^{(+)}(t, j, j_0, w) - \partial W^{(-)}(t, j, j_0, w) - U(t, j, j_0, w) \right].
\end{aligned} \tag{34}$$

where the *celebrated* final entry $U(t, j, j_0, z)$ in the representation block is given by the extremely elaborate expression as follows:

$U(t, j, j_0, z) = (t + i\tau j)(t - i\tau(j+1))$
The other combinations of the OPE operations of the supercurrents on $U^a(t, j, j_0, z)$ and $\bar{U}_a(t, j, j_0, z)$ do not produce $U(t, j, j_0, z)$, but are given by

$$\begin{aligned}
& : \left[(t - i\tau j)(t + i\tau(j+1))\bar{\gamma}_+\gamma^-\bar{\gamma}_-\gamma^+V(t, j, j_0, z) \right. \\
& \quad + (1 - \mu t)(\partial\bar{\gamma}_+ \cdot \gamma^- + \partial\bar{\gamma}_- \cdot \gamma^+ - \bar{\gamma}_+ \cdot \partial\gamma^- - \bar{\gamma}_- \cdot \partial\gamma^+)V(t, j, j_0, z) \\
& \quad + \tau^2(j + j_0)\partial(\bar{\gamma}_+ \cdot \gamma^+)V(t, j, j_0 - 1, z) + \tau^2(j - j_0)\partial(\bar{\gamma}_- \cdot \gamma^-)V(t, j, j_0 + 1, z) \\
& \quad + i\tau\mu j_0\partial(\bar{\gamma}_+ \cdot \gamma^- - \bar{\gamma}_- \cdot \gamma^+)V(t, j, j_0, z) \\
& \quad + \left(-\sum_{\alpha=1}^4 (\partial\varphi_\alpha)^2 + i\tau\partial^2\varphi_1 + i\mu\partial^2\varphi_4 \right)V(t, j, j_0, z) \\
& \quad + \tau^2 j_0 i\sqrt{\frac{2}{k}}(\partial^2\varphi_3 - i\partial^2\varphi_2)V(t, j, j_0, z) \\
& \quad + i\sqrt{\frac{2}{k}}(\partial\varphi_3 - i\partial\varphi_2)\left(\tau^2(j + j_0)\sqrt{2}J^+V(t, j, j_0 - 1, z) \right. \\
& \quad \quad \left. - \tau^2(j - j_0)\sqrt{2}J^-V(t, j, j_0 + 1, z) \right) \\
& \quad - \left(\tau^2j(j+1) - \tau^2j_0^2 \right) \left(i\sqrt{\frac{2}{k}}(\partial\varphi_3 - i\partial\varphi_2) \right)^2 V(t, j, j_0, z) \\
& \quad + (\bar{\gamma}_+\gamma^- - \bar{\gamma}_-\gamma^+) \left\{ -\frac{1}{2}(t - i\tau j_0)(i\partial\varphi_4 + i\tau J^0)V(t, j, j_0, z) \right. \\
& \quad \quad \left. + \frac{1}{2}(t + i\tau j_0)(i\partial\varphi_4 - i\tau J^0)V(t, j, j_0, z) \right. \\
& \quad \quad \left. - \frac{1}{2}\tau^2(j + j_0)\sqrt{2}J^+V(t, j, j_0 - 1, z) + \frac{1}{2}\tau^2(j - j_0)\sqrt{2}J^-V(t, j, j_0 + 1, z) \right. \\
& \quad \quad \left. + \left(\tau^2j(j+1) - \tau^2j_0^2 \right) i\sqrt{\frac{2}{k}}(\partial\varphi_3 - i\partial\varphi_2)V(t, j, j_0, z) \right\} \\
& \quad - i\tau(t - i\tau j_0)\sqrt{2}J^+\bar{\gamma}_-\gamma^-V(t, j, j_0, z) - i\tau(t + i\tau j_0)\sqrt{2}J^-\bar{\gamma}_+\gamma^+V(t, j, j_0, z) \\
& \quad - i\tau(j - j_0)(i\partial\varphi_4 - i\tau J^0)\bar{\gamma}_-\gamma^-V(t, j, j_0 + 1, z) \\
& \quad - i\tau(j + j_0)(i\partial\varphi_4 + i\tau J^0)\bar{\gamma}_+\gamma^+V(t, j, j_0 - 1, z) \\
& \quad + i\tau(j - j_0)(t - i\tau j_0)i\sqrt{\frac{2}{k}}(\partial\varphi_3 - i\partial\varphi_2)\bar{\gamma}_-\gamma^-V(t, j, j_0 + 1, z) \\
& \quad \left. - i\tau(j + j_0)(t + i\tau j_0)i\sqrt{\frac{2}{k}}(\partial\varphi_3 - i\partial\varphi_2)\bar{\gamma}_+\gamma^+V(t, j, j_0 - 1, z) \right] : . \tag{35}
\end{aligned}$$

Let us also note here that further operations of the supercurrents $G^a(z)$, $\bar{G}_a(z)$ ($a = \pm$) on $U(t, j, j_0, z)$ do not generate any new entry, but only produce the

$$\begin{aligned}
\bar{G}_\mp(z)U^\mp(t, j, j_0, w) & \sim -\frac{2}{(z-w)^2} \left[(h \mp j_0 + 1)X^\mp(t, j, j_0, w) \right. \\
& \quad \left. + (j \pm j_0) \left\{ \partial V(t, j, j_0 \mp 1, w) \pm W^{(\pm)}(t, j, j_0 \mp 1, w) \right\} \right] \\
\bar{G}^\mp(z)\bar{U}_\mp(t, j, j_0, w) & \sim \frac{2}{(z-w)^2} \left[(h \mp j_0 + 1)X^\mp(t, j, j_0, w) \right. \\
& \quad \left. + (j \pm j_0) \left\{ \partial V(t, j, j_0 \mp 1, w) \pm W^{(\mp)}(t, j, j_0 \mp 1, w) \right\} \right] \tag{36}
\end{aligned}$$

$$\begin{aligned}
G^\mp(z)U^\mp(t, j, j_0, w) & \sim \mp \frac{2}{(z-w)^2} (j \pm j_0)Y(t, j, j_0 \mp 1, w) \\
\bar{G}_\mp(z)\bar{U}_\mp(t, j, j_0, w) & \sim \pm \frac{2}{(z-w)^2} (j \pm j_0)\bar{Y}(t, j, j_0 \mp 1, w) \tag{37}
\end{aligned}$$

$$\begin{aligned}
G^\pm(z)U^\mp(t, j, j_0, w) & \sim \mp \frac{2}{(z-w)^2} (h \pm j_0)Y(t, j, j_0, w) \mp \partial_w \left(\frac{2}{z-w} Y(t, j, j_0, w) \right) \\
\bar{G}_\mp(z)\bar{U}^\mp(t, j, j_0, w) & \sim \pm \frac{2}{(z-w)^2} (h \pm j_0)\bar{Y}(t, j, j_0, w) \pm \partial_w \left(\frac{2}{z-w} \bar{Y}(t, j, j_0, w) \right) \tag{38}
\end{aligned}$$

following closed OPE relations:

$$\begin{aligned}
G^\mp(z)U(t, j, j_0, w) &\sim \frac{2}{(z-w)^3} \left[(h \mp j_0 + 1)V^\mp(t, j, j_0, w) \right. \\
&\quad + (j \pm j_0)V^\pm(t, j, j_0 \mp 1, w) \left. + \frac{2}{(z-w)^2} \left[\left(h \mp j_0 + \frac{3}{2} \right) U^\mp(t, j, j_0, z) \right. \right. \\
&\quad \left. \left. + (j \pm j_0)U^\pm(t, j, j_0 \mp 1, w) \right] + \frac{1}{z-w} \partial U^\mp(t, j, j_0, w) \right],
\end{aligned}$$

$$\begin{aligned}
\bar{G}_\pm(z)U(t, j, j_0, w) &\sim \frac{2}{(z-w)^3} \left[(h \pm j_0 + 1)\bar{V}_\pm(t, j, j_0, w) \right. \\
&\quad - (j \mp j_0)\bar{V}_\mp(t, j, j_0 \pm 1, w) \left. + \frac{2}{(z-w)^2} \left[\left(h \pm j_0 + \frac{3}{2} \right) \bar{U}_\pm(t, j, j_0, z) \right. \right. \\
&\quad \left. \left. - (j \mp j_0)\bar{U}_\mp(t, j, j_0 \pm 1, w) \right] + \frac{1}{z-w} \partial \bar{U}_\pm(t, j, j_0, w) \right].
\end{aligned} \tag{39}$$

We have thus completed the assignment of the fundamental sets of the eight “fermion-like” and eight “boson-like” conformal fields to the fourteen *nondegenerate* corners and the one *doubly degenerate* center of the rhombic dodecahedron forming the representation blocks.

Now let us mention the transformation properties under $L(z)$ of each entry of the fundamental conformal fields in the representation blocks. The three “boson-like” fields, $V(t, j, j_0, z)$, $Y(t, j, j_0, z)$ and $\bar{Y}(t, j, j_0, z)$, and the four “fermion-like” fields, $V^a(t, j, j_0, z)$ and $\bar{V}_a(t, j, j_0, z)$ ($a = \pm$), transform as *primary* fields, i.e.

$$L(z)V(t, j, j_0, w) \sim \frac{1}{z-w} \partial V(t, j, j_0, w) + \frac{1}{(z-w)^2} h V(t, j, j_0, w)$$

$$L(z)V^a(t, j, j_0, w) \sim \frac{1}{z-w} \partial V^a(t, j, j_0, z) + \frac{1}{(z-w)^2} \left(h + \frac{1}{2} \right) V^a(t, j, j_0, w)$$

$$L(z)\bar{V}_a(t, j, j_0, w) \sim \frac{1}{z-w} \partial \bar{V}_a(t, j, j_0, z) + \frac{1}{(z-w)^2} \left(h + \frac{1}{2} \right) \bar{V}_a(t, j, j_0, w) \tag{40}$$

$$L(z)Y(t, j, j_0, w) \sim \frac{1}{z-w} \partial Y(t, j, j_0, w) + \frac{1}{(z-w)^2} (h+1) Y(t, j, j_0, w)$$

$$L(z)\bar{Y}(t, j, j_0, w) \sim \frac{1}{z-w} \partial \bar{Y}(t, j, j_0, w) + \frac{1}{(z-w)^2} (h+1) \bar{Y}(t, j, j_0, w).$$

The conformal fields $W^{(\pm)}(t, j, j_0, z)$ satisfy the OPEs as

$$\begin{aligned}
L(z)W^{(\pm)}(t, j, j_0, w) &\sim \frac{1}{z-w} \partial W^{(\pm)}(t, j, j_0, w) \\
&\quad + \frac{1}{(z-w)^2} (h+1) W^{(\pm)}(t, j, j_0, w) - \frac{2}{(z-w)^3} j_0 V(t, j, j_0, w).
\end{aligned} \tag{41}$$

Thus we find that the difference of the two fields,

$$\begin{aligned}
W^{(+)}(t, j, j_0, z) - W^{(-)}(t, j, j_0, z) = \\
(t + i\tau j) \left(t - i\tau(j+1) \right) \left(\bar{\gamma}_+ \gamma_- + \bar{\gamma}_- \gamma_+ \right) V(t, j, j_0, w),
\end{aligned} \tag{42}$$

being the fourth “boson-like” *primary* combination, satisfy the primary field condition with conformal weight $h+1$.

The other “boson-like” conformal fields, $X^\pm(t, j, j_0, z)$ and $U(t, j, j_0, z)$, satisfy

$$\begin{aligned}
L(z)X^\pm(t, j, j_0, w) &\sim \frac{1}{z-w} \partial X^\pm(t, j, j_0, w) + \frac{1}{(z-w)^2} (h+1) X^\pm(t, j, j_0, w) \\
&\quad - \frac{2}{(z-w)^3} (j \mp j_0) V(t, j, j_0 \pm 1, w)
\end{aligned} \tag{43}$$

The OPE operation of $L(z)$ on the four “fermion-like” fields, $U^a(t, j, j_0, z)$ and

$\bar{U}_a(t, j, j_0, z)$, being assigned to the upper layer gives

$$\begin{aligned}
L(z)U^\mp(t, j, j_0, w) &\sim \frac{1}{z-w}\partial U^\mp(t, j, j_0, w) + \frac{1}{(z-w)^2}\left(h + \frac{3}{2}\right)U^\mp(t, j, j_0, w) \\
&+ \frac{2}{(z-w)^3}\left[(h \mp j_0 + 1)V^\mp(t, j, j_0, w) + (j \pm j_0)V^\pm(t, j, j_0 \mp 1, w)\right] \\
L(z)\bar{U}_\pm(t, j, j_0, w) &\sim \frac{1}{z-w}\partial \bar{U}_\pm(t, j, j_0, w) + \frac{1}{(z-w)^2}\left(h + \frac{3}{2}\right)\bar{U}^\pm(t, j, j_0, w) \\
&+ \frac{2}{(z-w)^3}\left[(h \pm j_0 + 1)\bar{V}^\pm(t, j, j_0, w) + (j \mp j_0)\bar{V}^\mp(t, j, j_0 \pm 1, w)\right]
\end{aligned} \tag{44}$$

Finally, the OPE relation of $L(z)$ with the top entry, $U(t, j, j_0, z)$, in the rhombic dodecahedron of the representation block is given by

$$\begin{aligned}
L(z)U(t, j, j_0, w) &\sim \frac{1}{z-w}\partial U(t, j, j_0, w) + \frac{1}{(z-w)^2}(h+2)U(t, j, j_0, w) \\
&+ \frac{2}{(z-w)^3}\left[(j+j_0)\bar{\gamma}_+\gamma^+V(t, j, j_0-1, w) + (j-j_0)\bar{\gamma}_-\gamma^-V(t, j, j_0+1, w) \right. \\
&\quad \left. - j_0(\bar{\gamma}_+\gamma^- - \bar{\gamma}_-\gamma^+)V(t, j, j_0, w)\right]
\end{aligned} \tag{45}$$

6. Identification of Charge-Screening Operators

The results in the previous section allow us to identify the whole sets of the charge-screening operators of the $N = 4$ $SU(2)_k$ SCAs. In the following we shall explain how the whole sets of the screening operators with conformal dimension one can be constructed.

The basic requirements for an *eligible* charge-screening operator $\mathcal{S}(z)$ are that it transforms as a primary field with conformal dimension one and that it stays invariant under the transformations of the $N = 4$ $SU(2)_k$ SCAs. Operationally

these requirements are satisfied if and only if the following OPE relations

$$\mathcal{O}(z)\mathcal{S}(w) \sim \partial_w \left[\frac{1}{z-w} Q(w) \right] \quad \text{or} \quad \mathcal{O}(z)\mathcal{S}(w) \sim 0 \tag{46}$$

should be valid for any generator $\mathcal{O}(z)$ of the $N = 4$ $SU(2)_k$ SCAs with $Q(z)$ being a certain nonzero operator or just being zero. In particular, the following primary field OPE relation should hold

$$L(z)\mathcal{S}(w) \sim \partial_w \left[\frac{1}{z-w} \mathcal{S}(w) \right] \tag{47}$$

for any *eligible* charge-screening operator $\mathcal{S}(z)$.

The OPE relations we have obtained in the previous section allow us to study unambiguously whether any of the sixteen conformal fields assigned to the rhombic dodecahedron could be an *eligible* candidate for the charge-screening operators of the $N = 4$ $SU(2)_k$ SCAs.

6.1. Nonchiral screening operator

As an illustration we shall start with giving the *nonchiral* charge-screening operator $\mathcal{N}(z)$ which can be constructed out of the *celebrated* top entry $U(t, j, j_0, z)$ in the representation blocks. In order for $U(t, j, j_0, z)$ to be a screening operator it has to be primary. Therefore, from eq.(45) we obtain the parameter conditions for $U(t, j, j_0, z)$: $h(t, j) + 2 = 1$, $j + j_0 = 0$, $j - j_0, j_0 = 0$. Thus $j = j_0 = 0$ and $h = h(t, j = 0) = -1$, whose last equation determines t : $t = \mu$ or $t = -2\kappa - \mu$. The former solution is excluded because $U(t = \mu, j = 0, j_0 = 0, z)$ trivially vanishes, while the second solution could correspond to a possible candidate for a charge-screening operator. In fact, $U(t = -2\kappa - \mu, j = 0, j_0 = 0, z)$ is an *eligible* charge-screening operator, which we call the *nonchiral screening operator* $\mathcal{N}(z)$ ⁸ and is

given by

$$\begin{aligned} \mathcal{N}(z) &\equiv U(t = -2\kappa - \mu, j = 0, j_0 = 0, z) \\ &= 2(\kappa + \mu)(2\kappa + \mu) : \left[(i\partial\varphi_4)^2 + i\mu\partial^2\varphi_4 + \kappa(2\kappa + \mu)(\bar{\gamma} \cdot \gamma)^2 \right. \\ &\quad \left. - (\partial\bar{\gamma} \cdot \gamma - \bar{\gamma} \cdot \partial\gamma) + \tau^2 j^i \eta_{ij} j^j - 2(\bar{\gamma}\sigma^i\gamma)\eta_{ij} j^j \right] V(t = -2\kappa - \mu, j = 0, j_0 = 0, z) : \end{aligned} \quad (48)$$

Its OPE's with the $N=4$ generators either vanish or turn into total derivatives.¹²

To save space, we only present here the nontrivial relations which we obtain from eqs.(39) and (45):

$$\begin{aligned} L(z)\mathcal{N}(w) &\sim \partial_w \left[\frac{1}{z-w} \mathcal{N}(w) \right] \\ G^a(z)\mathcal{N}(w) &\sim \partial_w \left[\frac{1}{z-w} U^a(t = -2\kappa - \mu, j = 0, j_0 = 0, w) \right] \end{aligned} \quad (49)$$

The similar OPE relation holds for $\bar{G}_a(z)\mathcal{N}(w)$ as well.

The nonchiral screening operator $\mathcal{N}(z)$ basically generates those null states with the same isospin and “charge” quantum numbers as the primary highest weight state. Only the mass level n is raised relative to the primary ground state.

6.2. $SU(2)_\kappa$ Kac-Moody screening operator

Next we shall consider the possibility of getting an *eligible* charge-screening operator from $X^\pm(t, j, j_0, z)$. In order for $X^\pm(t, j, j_0, z)$ to be primary, we have to have $h(t, j) + 1 = 1$, $j \mp j_0 = 0$ from eq.(43). Also, in eqs.(28) and (32) we have to satisfy the relations $h(t, j) \mp j_0 = 1$, $j \mp j_0 = 0$, and $(t + i\tau j)(t - i\tau(j + 1)) = 0$, so that $U^a(t, j, j_0, z) \equiv 0$, $\bar{U}_a(t, j, j_0, z) \equiv 0$ and consequently the OPE relations $G^a(z)X^\pm(t, j, j_0, w)$ and $\bar{G}_a(z)X^\pm(t, j, j_0, w)$ either turn into total derivatives or vanish. The nontrivial solution exists only for the choice $(t - i\tau(j + 1)) = 0$ with the solution $t = 0, j = -1, j_0 = \mp 1$. The other choice $(t + i\tau j) = 0$ gives the trivially

vanishing solution $X^\pm(t = 0, j = 0, j_0 = 0, z) \equiv 0$ as a result of the constraints $t = 0, j = j_0 = 0$, henceforth is excluded.

We thus find another “charge” preserving screening operators to be given by⁸

$$\begin{aligned} \mathcal{J}_\pm(z) &\equiv \mp X^\pm(t = 0, j = -1, j_0 = \mp 1, z) = \tau^2 : j^\pm(z) V_{j=-1, j_0=\mp 1}(z) : \\ &= \tau^2 : i \left(\frac{1}{\tau} \partial\varphi_1 \pm i \sqrt{\frac{\hat{k}}{2}} \partial\varphi_2 \right) e^{i\tau\varphi_1} : \end{aligned} \quad (50)$$

whose OPE relations with the $N=4$ generators either are trivially zero or become nontrivial total derivatives. Only some of the nontrivial OPEs are presented here:

$$\begin{aligned} J^\mp(z)\mathcal{J}_\pm(w) &\sim \partial_w \left[\frac{1}{z-w} V_{-1, \mp 1}(w) \right], \\ G^a(z)\mathcal{J}_\pm(w) &\sim \partial_w \left[\frac{1}{z-w} (\sigma^\pm \gamma)^a(w) V_{-1, \mp 1}(w) \right] \end{aligned} \quad (51)$$

with the similar OPE relations for $\bar{G}_a(z)\mathcal{J}_\pm(w)$ being valid. We remark here that we have in effect a single $SU(2)_\kappa$ Kac-Moody screening operator $\mathcal{J}(z)$:

$$\mathcal{J}(z) \equiv \mathcal{J}_+(z) \simeq -\mathcal{J}_-(z) \quad (52)$$

since the sum $\mathcal{J}_+(z) + \mathcal{J}_-(z)$ is just a total derivative.

By the use of the similar consideration for $W^\pm(t, j, j_0, z)$ we also obtain another *eligible* charge-screening operator $\mathcal{W}(z)$, which is actually equivalent to $\mathcal{J}_\pm(z)$ as follows:

$$\begin{aligned} \mathcal{W}(z) &\equiv W^\pm(t = 0, j = -1, j_0 = 0, z) = \tau^2 : \sqrt{\frac{\hat{k}}{2}} \partial\varphi_2 e^{i\tau\varphi_1} : \\ &= \pm \mathcal{J}_\mp(z) \mp \partial V(t = 0, j = -1, j_0 = 0, z) \simeq -\mathcal{J}(z) \end{aligned} \quad (53)$$

As for $\mathcal{W}(z)$ we have the following OPEs in the form of total derivatives:

$$\begin{aligned} G^a(z)\mathcal{W}(z) &\sim \mp \partial_w \left[\frac{1}{z-w} V^a(t=0, j=-1, j_0=0, w) \right], \\ \bar{G}_a(z)\mathcal{W}(z) &\sim \pm \partial_w \left[\frac{1}{z-w} \bar{V}_a(t=0, j=-1, j_0=0, w) \right], \end{aligned} \quad (54)$$

We remark that the conformal fields $Y(t, j, j_0, z)$ and $\bar{Y}(t, j, j_0, z)$ do not give rise to any eligible charge-screening operators, which can be confirmed from eqs. (40), (27) and (31). The only allowed solution $t=0, j=-1, j_0=0$ for the parameters being constrained by the eligible conditions for charge-screening operators gives the trivially vanishing results such as $Y(t=0, j=-1, j_0=0, z) \equiv 0$ and $\bar{Y}(t=0, j=-1, j_0=0, z) \equiv 0$. Similarly, the conformal field $V(t, j, j_0, z)$ cannot satisfy the conditions eq.(46) for an *eligible* screening operator except in the form of the trivial solution $V(t=0, j=0, j_0=0, z) = 1$.

6.3. Chiral screening operators

Next we shall proceed to constructing “fermion-like” charge-screening operators. The eligible candidates for them can be obtained by studying the “fermion-like” conformal fields $V^a(t, j, j_0, z)$, $\bar{V}_a(t, j, j_0, z)$, $U^a(t, j, j_0, z)$ and $\bar{U}_a(t, j, j_0, z)$.

First we consider the conformal fields $V^\mp(t, j, j_0, z)$. In this case there is some judicious investigation involved, since the primary condition eq.(40) for an *eligible* charge-screening operator just gives the constraint on the conformal dimension such as $h(t, j) = 1/2$ and nothing else. From eqs.(20)~(26) we get additional constraints for the parameters t, j, j_0 such that the equalities $Y(t, j, j_0, z) = 0, j \pm j_0 = 1, X^\mp(t, j, j_0, z) = -2\partial V(t, j, j_0 \mp 1, z), h(t, j) \pm j_0 = 1, W^\mp(t, j, j_0, z) = \mp \partial V(t, j, j_0, z)$ should hold. The first equality provides the constraint $(t + i\tau j)(t -$

$i\tau(j+1)) = 0$. Remarkably enough, these constraints have nontrivial solutions: $j = 1/2, j_0 = \pm 1/2, t = -i\tau j = -i\tau/2, h(t = -i\tau j, j) = j = 1/2$ due to the following identities

$$X^\mp\left(t = -i\frac{\tau}{2}, j = \frac{1}{2}, j_0 = \pm\frac{1}{2}, z\right) = -2\partial V\left(t = -i\frac{\tau}{2}, j = \frac{1}{2}, j_0 = \mp\frac{1}{2}, z\right), \quad (55)$$

$$W^\mp\left(t = -i\frac{\tau}{2}, j = \frac{1}{2}, j_0 = \pm\frac{1}{2}, z\right) = \mp \partial V\left(t = -i\frac{\tau}{2}, j = \frac{1}{2}, j_0 = \pm\frac{1}{2}, z\right).$$

We thus find the *eligible chiral* screening operators $\mathcal{V}^{(\mp)}(z)$ carrying the “charge” of $\hat{C}=1$ to be given in terms of the chiral vertex operators, eq.(19), as follows:

$$\mathcal{V}^{(\mp)}(z) \equiv V^\mp\left(t = -i\frac{\tau}{2}, j = \frac{1}{2}, j_0 = \pm\frac{1}{2}, z\right) = -\mathcal{V}^{(\pm)}(z) \equiv \mp \mathcal{C}(z), \quad (56)$$

By making a similar investigation for $\bar{V}_\pm(t, j, j_0, z)$, we get the *anti-chiral* screening operators with the “charge” of $\hat{C}=-1$ as

$$\bar{\mathcal{V}}^{(\pm)}(z) \equiv \bar{V}_\pm\left(t = -i\frac{\tau}{2}, j = \frac{1}{2}, j_0 = \mp\frac{1}{2}, z\right) = \bar{\mathcal{V}}^{(\mp)}(z) \equiv \bar{\mathcal{C}}(z), \quad (57)$$

One should note that the “charge” conjugate expressions of the above chiral vertex operators identically vanish:

$$\bar{V}_\pm\left(t = -i\frac{\tau}{2}, j = \frac{1}{2}, j_0 = \pm\frac{1}{2}, z\right) \equiv 0, \quad V^\mp\left(t = -i\frac{\tau}{2}, j = \frac{1}{2}, j_0 = \mp\frac{1}{2}, z\right) \equiv 0. \quad (58)$$

Here again the OPEs of $\mathcal{C}(z)$ and $\bar{\mathcal{C}}(z)$ with the $N=4$ generators either vanish or are expressed as total derivatives. In particular, $\mathcal{C}(w)$ is annihilated by $G^a(z)$

while $\bar{\mathcal{C}}(w)$ by $\bar{G}_a(z)$, that is,

$$G^a(z)\mathcal{C}(w) \sim 0, \quad \bar{G}_a(z)\bar{\mathcal{C}}(w) \sim 0. \quad (59)$$

We present here some of the nontrivial relations taking the form of total derivatives:

$$\bar{G}_{\pm}(z)\mathcal{C}(w) \sim \mp 2\partial_w \left[\frac{1}{z-w} V\left(t = -i\frac{\tau}{2}, j = \frac{1}{2}, j_0 = \pm\frac{1}{2}, w\right) \right], \quad (60)$$

whose “charge” conjugate OPE relations for $G^{\mp}(z)\bar{\mathcal{C}}(w)$ also hold in a similar form.

We finally remark that the “fermion-like” conformal fields $U^a(t, j, j_0, z)$ and $\bar{U}_a(t, j, j_0, z)$ do not give rise to any eligible charge-screening operators. The only allowed solutions $t = -i\tau j = i\tau/2, j = -1/2, j_0 = \pm 1/2, h(t = -i\tau j, j) = j = -1/2$ for $U^{\mp}(t, j, j_0, z)$ to be eligible screening operators satisfy the constraint $(t + i\tau j)(t - i\tau(j + 1)) = 0$, which in turn kills in a trivial manner not only $U(t, j, j_0, z) \equiv 0$ and $Y(t, j, j_0, z) \equiv 0$, but also $U^{\mp}(t, j, j_0, z) \equiv 0$ itself. Similarly this is the case for $\bar{U}_a(t, j, j_0, z)$.

To summarize, those *eligible* charge-screening operators that we have obtained in the above are conformally invariant vertex operators.

7. Comments and conclusions

We have some comments in order. In the two illustrations 2-(a) and 2-(b) of fig.2 are shown the steps of the OPE operations for the fundamental conformal field $V(t, j, j_0, z)$ sitting at the bottom of the rhombic dodecahedron to finally reach the top point, thus generating the *celebrated* $U(t, j, j_0, z)$ field. First of all, the corresponding OPE operations in figs.2-(a) and 2-(b) are orthogonal to each other. Therefore, the two illustrations are at right angles to each other as a whole.

Also, the OPE operations in the upper rectangle in one illustration are twisted at right angles from those in the lower rectangle in the same illustration. That is, the OPE operations on the conformal fields $W^{(\pm)}(t, j, j_0, z)$ to let them climb one floor up from the middle layer generating the “fermion-like” fields, $U^a(t, j, j_0, z)$ and $\bar{U}_a(t, j, j_0, z)$, are orthogonal to those OPE operations which have generated $W^{(\pm)}(t, j, j_0, z)$ themselves from $V(t, j, j_0, z)$ sitting at the bottom point.

In spite of the complicated structure of the sixteen conformal fields assigned to the fourteen edge corners and the one degenerate center of the rhombic dodecahedron, they possess a neat substructure representing the underlying subalgebras of the $N=4$ $SU(2)_k$ SCAs, which is the $N=2$ $U(1)$ SCAs.

We first note that the two subsets of the operators, $\{L(z), G^{\pm}(z), \bar{G}_{-}(z), T^0(z)\}$ and $\{L(z), \bar{G}^{+}(z), G_{-}(z), T^0(z)\}$, in the $N=4$ $SU(2)_k$ SCAs respectively form the $N=2$ $U(1)$ SCAs. Consequently, the two correspondingly chosen subsets of the “boson-like” and “fermion-like” conformal fields, $\{V(t, j, j_0, z), V^{+}(t, j, j_0, z), \bar{V}_{-}(t, j, j_0, z), W^{(+)}(t, j, j_0, z)\}$ and $\{V(t, j, j_0, z), \bar{V}_{+}(t, j, j_0, z), V^{-}(t, j, j_0, z), W^{(-)}(t, j, j_0, z)\}$, respectively stand^{11,12} for the complete representations of the $N=2$ $U(1)$ SCAs which are given by the above mentioned two subsets of the generators. In the generic notation of ref.12 each set of these conformal fields forms a *closed* primary multiplet $\{F(z), G^{\pm}(z), H(z)\}$ representing the $N=2$ $U(1)$ SCAs. The closed property can be confirmed from eq.(33), where no new entries are generated by the OPE operations, that is, the OPE relations are closed among the primary multiplets under the corresponding $N=2$ $U(1)$ SCAs. In fig.2 this is shown by the orthogonality of the OPE operations in the lower rectangles to those in the upper ones. As already stated in the above, the supercharge generators of the $N=2$ $U(1)$ SCAs in the lower rectangles of fig.2(a) and fig.2(b) respectively do

not generate any new conformal field in the directions parallel to the lower rectangles, when they are applied to the conformal fields $W^{(\pm)}(t, j, j_0, z)$. One set of the supercharge generators forming the $N = 2$ $U(1)$ subalgebra has to switch over to another set of the supercharge generators in the orthogonal rectangle in order to generate new entries by those OPE operations on $W^{(\pm)}(t, j, j_0, z)$.

Now, in conclusion, we have the necessary and sufficient tools of accomplishing the complete study of the representation theory of the $N=4$ $SU(2)_k$ SCAs. For example, the fundamental unitary representations of the $N=4$ $SU(2)_k$ SCAs constructed in ref.2 can be studied systematically within our framework. Also, the rigorous analytic proof of the $N=4$ Kac determinant formulae can be accomplished by the use of the whole sets of the charge-screening operators. The preliminary results are already reported in ref.8.

We also mention here that the extension of our approach⁸ to the $N=4$ $SU(2) \times SU(2)$ extended SCAs has been attempted in ref.15. However, the construction of the representation blocks of the $N=4$ $SU(2) \times SU(2)$ SCAs as in the context of the present paper has not been achieved yet. It should be worth studying for making further systematic investigations of the representation theories of the $N=4$ SCAs.

These topics and other applications will be treated in our future publications.

ACKNOWLEDGEMENTS

We thank T. Eguchi and M. Kato at University of Tokyo and T. Uematsu at Kyoto University for useful discussions. We also express our sincere thanks to Theory Division at CERN and Theory Group at SLAC for the kind hospitality during my visit. Most of my elaborate calculations in the present paper were performed and completed in a pleasant and encouraging atmosphere during my stay at Geneva and at Stanford in 1993.

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Figure Captions

Fig. 1 :The basic unit of the representation blocks consisting of the sixteen conformal fields. The eight “boson-like” and eight “fermion-like” conformal fields (represented by circles and triangles respectively) are assigned to the fourteen edge corners and the one doubly degenerate center of the *rhombic dodecahedron* which are located at the top and bottom points and on the three layers in between.

Fig. 2 :The steps of the OPE operations for V to reach the top point of U are illustrated. The upper rectangles are at right angles to those in the lower ones. Each set of conformal fields, $\{V, V^+, \bar{V}_-, W^{(+)}\}$ in (a) or $\{V, \bar{V}_+, V^-, W^{(-)}\}$ in (b), forms a *closed* primary multiplet representing the corresponding $N = 2$ $U(1)$ subalgebras of the $N = 4$ $SU(2)$ SCAs.

Fig. 1

Fig. 2