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1. Introduction

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SW 9410**The crossover from first to second-order finite-size scaling:
a numerical study**

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Abstract: We consider a particular case of the two dimensional Blume-Emery-Griffiths model to study the finite-size scaling for a field driven first-order phase transition with two coexisting phases not related by a symmetry. For low temperatures we verify the asymptotic (large volume) predictions of the rigorous theory of Borgs and Kotecký, including the predictions concerning the so-called equal-weight versus equal-height controversy. Near the critical temperature we show that all data fit onto a unique curve, even when the correlation length ξ becomes comparable to or larger than the size of the system, provided the linear dimension L of the system is rescaled by ξ .

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First-order phase transitions play an important role in physics. Examples range from well known transitions in solid state physics (like ordinary melting) to phase transitions in the early universe. For many models describing such a transition, exact solutions are not available, and systematic expansions such as “weak coupling” or “high temperature series” are of limited value. In this situation, the numerical simulation of physically interesting models has gained more and more popularity. With the increasing speed of computers, the statistical errors of Monte Carlo simulations have been drastically reduced, but a systematic limitation of this method is the finite size of the simulated system. A clear understanding of finite-size effects is therefore more and more important in order to interpret the resulting numerical data.

The theory of finite-size scaling (FSS) for second-order phase transitions has a rich and longstanding literature, starting with the pioneering work of Fisher [1] and Fisher and Barber [2]. In the last ten years, the theory of FSS has been extended to first-order transitions. For systems with two coexisting phases, three main approaches can be distinguished in the literature: the phenomenological renormalisation group approach to FSS of Berker/Fisher [3] and Blöte/Nightingale [4], the thermodynamic fluctuation theory for the order parameter distribution $P_V(s)$ of Binder and co-workers [5,6,7] and the transfer matrix method of Privman and Fisher [8], see Refs. [9] and [10] for reviews.

Recently, the theory of FSS near first order transitions has been put on a rigorous basis [11,12] within the context of the Pirogov-Sinai theory [13] which applies whenever it is possible to describe the configurations of the system in terms of energetically unfavourable contours. Typical models to which these methods can be applied are field-driven transitions in low temperature spin systems with discrete spins, temperature-driven

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transitions for Potts models, lattice field theories with double well potential, Higgs/confinement transitions in large N lattice gauge theories, and certain 2d continuum field theories, such as $P(\phi)_2$ field theories in the broken phase and supersymmetric Wess-Zumino models.

In the context of the field-driven transitions with two coexisting phases considered in this paper, the main result of Ref. [11] can be summarized by a formula of the form

$$Z(V, h) = \left[e^{-\beta J_+(\beta, h) V} + e^{-\beta J_-(\beta, h) V} \right] (1 + O(V e^{-L/L_0})) \quad (1.1)$$

for the partition function in a cubic volume $V = L^d$ with periodic boundary conditions [14]. Here L_0 is a constant of the order of the infinite volume correlation length, and $f_m(\beta, h)$ is some sort of metastable free energy for the phase m . It is equal to the free energy $f(\beta, h)$ if m is stable, strictly larger than $f(\beta, h)$ if m is unstable, and may be chosen as a smooth function of h . The finite size scaling of the partition function and its derivatives is obtained by expanding $f_m(\beta, h)$ around the transition point h_* .

In a recent paper [16], we have analysed a controversy stemming from the work of Binder and co-workers: within the phenomenological fluctuation theory for the order parameter distribution $P_V(s)$, different formulae for the FSS of the order parameter are obtained assuming different ansätze [6, 7] for the relative normalisation of the two Gaussians. If the normalisation is chosen in such a way that both Gaussians have *equal weight* at the infinite volume transition point h_* , the resulting formula for the magnetisation in a cube with periodic boundary conditions leads to the prediction [6] that the shift of the susceptibility maximum with respect to h_* is proportional to L^{-d} , while a normalisation with *equal peak height* at h_* leads to the prediction [7] that this shift is proportional to L^{-d} .

To some extent, this controversy had already been resolved in Ref. [11], where it had been shown that the predictions of the *equal weight* prescription agree with those following from equation (1.1) above. Unfortunately,

equation (1.1) is only *proven* for very low temperatures (for Ising type systems in $d = 2$, this restricts the temperature to about half T_c), while most numerical simulations are performed near T_c . It therefore seemed desirable to test the predictions following from (1.1) in an actual numerical simulation in order to definitely settle the controversy between [6] and [7].

This was done in [16]. We considered a spin model with spins $s_x \in \{-1, 0, 1\}$ and Hamiltonian

$$H = \frac{1}{2} \sum_{\langle xy \rangle} |\sigma(s_x) - \sigma(s_y)|^2 - h \sum_x s_x, \quad (1.2a)$$

where the first sum goes over the set of dL^d nearest neighbours in V and

$$\sigma(s_x) = \begin{cases} -1 & \text{if } s_x = -1 \\ +1 & \text{if } s_x = 0, +1. \end{cases} \quad (1.2b)$$

It is defined in such a way that boundaries between -1 and 0 or $+1$ are energetically unfavourable, while boundaries between 0 and $+1$ do not cost energy. As a consequence, the low temperature phases of the model consist of a phase “minus” corresponding to spin configurations where most spins s_x are equal to -1 with small islands of 0 or $+1$, and a phase “plus” corresponding to spin configurations which are a mixture of 0 and $+1$ with small islands of -1 . The phase minus is stable as long as h is smaller than a certain critical value $h_* = h_*(\beta)$, while the plus phase is stable for $h > h_*$. Because in the “plus” phase spins can be flipped from 0 to $+1$ without creating any energetically unfavourable boundaries we expect the magnetic susceptibility of this phase to be larger than that of the “minus” phase. A large susceptibility difference between the phases is desirable because it leads to a clear difference between the predictions of the equal-weight and equal-height theories of first order finite size effects.

This model is a special case of the Blume-Emery-Griffiths (BEG) model [17], which is the most general model with a three-valued spin and nearest neighbour interactions. The general BEG model has a complicated phase

structure, with up to three coexisting phases. Here we have chosen the coupling constants in such a way that the model has two coexisting phases at the phase transition point h_t , with a large susceptibility difference.

In [16], we simulated the model (1.2) in $d = 2$ and determined the shift of the susceptibility maximum with respect to h_t . For $\beta = 0.5$, and lattices with linear dimensions L from 4 to 30, we found overwhelming evidence for the equal weight prescription, as to be expected from the rigorous results of Ref. [11]. We summarize the main results of Ref. [16], as well as certain extensions (in particular to $\beta = 0.45$ and $\beta = 0.47$) and refinements in Section 3.

In [16], we did not analyse, however, other finite-size scaling predictions following from equation (1.1), such as the volume dependence of the susceptibility-maximum, or the volume dependence of the susceptibility at the transition point h_t . In view of the results of Billioire and co-workers [18], who found that the corresponding asymptotic scaling form for two dimensional Potts models was only reached for systems of linear dimensions L as big as 5 to 10 times the correlation length ξ of the infinite system, it seems highly desirable to check more finite-size scaling predictions for the model considered above. This will be done in Section 4.

As we will see, the asymptotic FSS-form obtained from equation (1.1) will again only be reached for fairly large volumes. While these volumes are easily accessible for $\beta = 0.5$ and $\beta = 0.47$ (the corresponding correlation length ξ is 2.19238... and 4.35091...), it is not accessible for temperatures near T_c (we did do a third simulation for $\beta = .45$, corresponding to $\xi = 13.5096...$). For these temperatures, however, a scaling ansatz using the length scale ξ to rescale all lengths in consideration, should be reasonable. In fact, we will see in Section 5, that this ansatz allows to fit all data onto a unique curve which describes the crossover between first and second order phase transitions.

2. The Model and its Relation to the standard Ising Model

In the next three sections, we will analyse the FSS for the model (1.2). We recall that the low temperature phases of the model consist of a phase “minus” corresponding to spin configurations where most spins s_z are equal to -1 with small islands of 0 or $+1$, and a phase “plus” corresponding to spin configurations which are a mixture of 0 and $+1$ with small islands of -1 . As we will see below, it can be shown that for $\beta > \beta_c$, where β_c is the critical temperature of the Ising model, the phase minus is stable as long as h is smaller than $h_t = -\beta^{-1} \operatorname{arcsinh}(1/2)$, while the plus phase is stable for $h > h_t$. At $h = h_t$, the infinite volume magnetisation jumps from a value $m_- = m_-(\beta)$ to a value $m_+ = m_+(\beta) > m_-$.

In order to compare the predictions of FSS theory with numerical data, one needs the numerical values of the infinite volume magnetisation $m_{\pm}$ and susceptibility $\chi_{\pm}$ at the point $h = h_t \pm 0$. As shown in [16], these constants can be obtained as follows: Let $Z(V, \beta, h)$ be the partition function of the model (1.2), and let $Z_I(V, \beta, h)$ be the partition function of the standard Ising model. Then Z and Z_I are related by the formula

$$Z(V, \beta, h) = e^{\mu(\beta, h) - h} \beta^V Z_I(V, \beta, h) \quad (2.1a)$$

with

$$\mu(\beta, h) = (1/2\beta) \ln(e^{\beta h} + e^{2\beta h}). \quad (2.1b)$$

As a consequence, we obtain the free energy $f(\beta, h)$ of the model (1.2) in terms of the free energy $f_I(\beta, \mu)$ of an Ising model with magnetic field μ

$$f(\beta, h) = f_I(\beta, \mu(\beta, h)) + h - \mu(\beta, h). \quad (2.1c)$$

In a similar way, it is possible to relate the correlation length ξ and the interface tension σ of the model (1.2) at $h = h_t$ to those of the Ising model at zero field. One finds

$$\xi(\beta) = \xi_I(\beta) \quad \text{and} \quad \sigma(\beta) = \sigma_I(\beta). \quad (2.2)$$

Recalling that phase transitions are associated with singularities of the free energy, we can read off the phase diagram of the model (1.2): It has the same critical temperature as the Ising model, while h_c is just the point where $\mu(\beta, h)$ is zero, i.e.

$$h_c = -\frac{1}{\beta} \operatorname{arsinh}(1/2) = -\frac{0.48121183\dots}{\beta}. \quad (2.3)$$

Differentiating equation (2.1c) with respect to h and taking the limit $h \rightarrow h_c \pm 0$, we finally obtain $m_{\pm}$ and $\chi_{\pm}$. In terms of the magnetisation $m_I = m_I(\beta)$ and the susceptibility $\chi_I = \chi_I(\beta)$ of the Ising model at $\mu = +0$ one gets

$$m_{\pm} = -\frac{1}{2}e^{\theta h_c} \pm \frac{1}{2}(2 - e^{\theta h_c})m_I \quad (2.4a)$$

$$\chi_{\pm} = \frac{1}{2}(\beta \pm \beta m_I)e^{2\theta h_c} + \frac{1}{4}(1 + e^{2\theta h_c})^2 \chi_I. \quad (2.4b)$$

Note that the difference $\chi_+ - \chi_-$ does not involve the constant χ_I ,

$$\chi_+ - \chi_- = (\beta m_I)e^{2\theta h_c}. \quad (2.4c)$$

We have simulated the model (1.2) in $d = 2$, where $m_I(\beta)$ and $\chi_I = 1/2\sigma_I$ are exactly known [19],

$$m_I(\beta) = (1 - (\sinh 2\beta)^{-4})^{1/8} \quad (2.5a)$$

$$\begin{aligned} \xi_I(\beta) &= (4\beta + 2\ln(\tanh \beta))^{-1} \quad \text{for } \beta > \beta_c \\ &= -(2\beta + \ln(\tanh \beta))^{-1} \quad \text{for } \beta < \beta_c. \end{aligned} \quad (2.5b)$$

In order to compare FSS theory to numerical data, we will also need χ_+ and χ_- . Unfortunately, the susceptibility $\chi_I(\beta)$ of the Ising model is not exactly known. We therefore used the low-temperature series for $\chi_I(\beta)$ found in [20] together with the method of Padé approximants [21] to calculate $\chi_I(\beta)$. As in [22], we embodied the knowledge of the exact transition point and critical exponent in our approximation. The result is

$$\chi_I(\beta) = 4\beta u^2 \left[\left(1 + \sum_{n=1}^M p_n u^n \right) / \left((1 - 6u + u^2) \left(1 + \sum_{n=1}^N q_n u^n \right) \right) \right]^{7/4} \quad (2.6)$$

where $u = e^{-4\beta}$, and p_n, q_n are given in table 1. The factor $1 - 6u + u^2$ has a simple zero at $\beta = \beta_c$, ensuring that χ_I diverges as $(\beta - \beta_c)^{7/4}$. It is amusing to note that a Padé approximant of the form (2.6) delivers the exact formula (2.5a) for m_I if used on the strong coupling series for magnetisation.

n	p_n	q_n	n	p_n	q_n
1	-.11.422	-9.9941	4	-0.83867	-41.824
2	42.042	27.744	5	-4.9026	
3	-49.064	-5.1210			

TABLE 1: Low temperature Padé coefficients for the susceptibility of the Ising model.

3. Equal Weight versus Equal Height

In order to obtain the FSS of the order parameter distribution $P_V(s)$ of an order parameter s (e.g. $s = \frac{1}{V} \sum s_i$ is the magnetisation per lattice site for spin models) in a cubic system with periodic boundary conditions and volume $V = L^d$, Binder and Landau approximated the probability distribution $P_V(s)$ by a sum of two Gaussians, centered around the infinite volume magnetisation m_- and m_+ . As for the relative normalisation, in the original version [6] they added up two normalised Gaussians so that at the transition point h_t each of the two coexisting phases contributes with the same weight. Following this “equal weight”-rule [6] one gets

$$P_V(s) \cong \frac{1}{N} \left(\frac{1}{\sqrt{\chi_-}} e^{-V\beta \frac{(s-m_-)^2}{\chi_-}} + \frac{1}{\sqrt{\chi_+}} e^{-V\beta \frac{(s-m_+)^2}{\chi_+}} \right) e^{\beta \epsilon(h-h_t)V}, \quad (3.1a)$$

where $\chi_{\pm}$ are the infinite volume susceptibilities at the point $h_t - 0$ and $h_t + 0$, respectively, and $N = N(\beta, h)$ is chosen in such a way that $P_V(s)$ is normalised. Led by mean-field arguments they revised this version for the relative normalisation, adding up the two Gaussians in such a way that the peak heights are equal at the transition point h_t . Following this “equal height”-rule [7] $P_V(s)$ turns out to be

$$P_V(s) \cong \frac{1}{N} \left(e^{-V\beta \frac{(s-m_-)^2}{\chi_-}} + e^{-V\beta \frac{(s-m_+)^2}{\chi_+}} \right) e^{\beta \epsilon(h-h_t)V}. \quad (3.1b)$$

It is an easy exercise to calculate the FSS for the finite volume magnetisation $m(V, h) = \int s P_V(s) ds$ following from (3.1a) and (3.1b). For the point $h_{\chi}(V)$ where the susceptibility $\chi(V, h) = dm(V, h)/dh$ has its maximum, a lengthy but straight-forward calculation gives the FSS predictions

$$h_{\chi}(V) = h_t + \frac{6(\chi_+ - \chi_-)}{\beta^2(m_+ - m_-)^3} \frac{1}{V^2} + O(V^{-3}) \quad (3.2a)$$

and

$$h_{\chi}(V) = h_t - \frac{\ln(\chi_+/\chi_-)}{2\beta(m_+ - m_-)} \frac{1}{V} + \left(6 - \frac{1}{8} \ln^2(\chi_+/\chi_-) \right) \frac{\chi_+ - \chi_-}{\beta^2(m_+ - m_-)^3} \frac{1}{V^2} + O(V^{-3}), \quad (3.2b)$$

respectively. We emphasize that the shift predicted by the “equal weight”-rule [6] is proportional to $1/V^2$ while the shift predicted by the “equal height”-rule [7] is proportional to $1/V$. (For symmetric phase transitions both rules are equivalent and predict that $h_{\chi}(V) = h_t$.)

Fig. 1a: The position $h_{\chi}(V)$ of the susceptibility maximum for $\beta = 0.45$. Open points are Monte Carlo results, solid points are exact results (see appendix). The solid line is the theoretical prediction (3.2a), while the dashed line is the prediction (3.2b).

We have tested the predictions (3.2) for three different temperatures, namely for $\beta = 0.45$, $\beta = 0.47$ and $\beta = 0.5$. In our simulation we first generated the probability distribution $P_V(s)$ for $L = 4, 5, \dots, 12, 14, 18, 30$ at $h = h_t$ using a multimagnetical version [23] of the heat bath algorithm. Applying the reweighting technique of Ferrenberg and Swendsen [24], we

obtain the probability distribution $P_V(s)$ and hence the magnetisation, $m(V, h) = \sum s P_V(s)$ and the susceptibility, $\chi(V, h) = \beta V \{ \sum s^2 P_V(s) - (\sum s P_V(s))^2 \}$, for h in the vicinity of h_t . In order to use the vector structure of the CRAY X-MP and Y-MP, we calculated 64 independent copies of the system in parallel, with 15 million sweeps for each of them. Grouping 8 of them together to save memory, we obtained 8 statistically independent data sets for each L , and therefore 8 statistically independent estimates for $h_\chi(V)$. Statistical error estimates are obtained from these data in the standard way.

Fig. 1b: The position $h_\chi(V)$ of the susceptibility maximum for $\beta = 0.47$.

We have plotted the theoretical curves (3.2a) and (3.2b) together with our numerical data in Figs. 1a-c. We recall that the coefficient in (3.2a) is known exactly, while the coefficients in (3.2b) have been obtained with the help of a Padé improved low-temperature expansion. Within the resolution of Fig. 1, the errors for the numerically determined points $h_\chi(V)$ are invisible, except for $\beta = 0.45$. Fig. 1 shows overwhelming evidence for the equal weight prescription.

Fig. 1c: The position $h_\chi(V)$ of the susceptibility maximum for $\beta = 0.50$.

Convinced of the “equal weight” rule we tried the new method to determine the transition point h_t of first-order phase transitions recently proposed by C. Borgs and W. Janke [25]. This method just uses the fact that both phases should have equal weight at $h = h_t$. To be precise, one defines a finite volume transition point $h_{EW}(V)$ as the point where

$$\sum_{s < s_{min}} P_V(s) = \sum_{s > s_{min}} P_V(s). \quad (3.3)$$

Here s_{min} , $s_- < s_{min} < s_+$, is the point s where the distribution $P_V(s)$ has its minimum. According to Ref. [25], it is expected that the systematic errors $|h_{EW}(V) - h_t|$ are exponentially small. Here they are so small that $h_{EW}(V)$ agrees with h_t within the statistical errors as soon as L/ξ is noticeably bigger than one (see Table 2 at the end of this paper).

4. Asymptotic First-Order FSS-Scaling

In this section we test the FSS predictions following from (1.1) by

Taylor expanding $f_{\pm}(\beta, h)$ around $h = h_i$,

$$f_{\pm}(\beta, h) = f(\beta, h_i) - m_{\pm}(h - h_i) - \frac{\chi_{\pm}}{2}(h - h_i)^2 + \frac{\chi'_{\pm}}{3!}(h - h_i)^3 + \dots \quad (4.1)$$

For the maximum $\chi_{\max}(V)$ of the susceptibility, its position $h_{\chi}(V)$ and the magnetisation $m(h_{\chi}(V))$ at the point $h_{\chi}(V)$, the FSS predictions following from (1.1) are

$$h_{\chi}(V) = h_i + \frac{6(\chi_+ - \chi_-)}{\beta^2(m_+ - m_-)} \frac{1}{V^2} + O\left(\frac{1}{V^3}\right) \quad (4.2)$$

$$m(h_{\chi}(V)) = \frac{m_+ + m_-}{2} + \frac{3}{2\beta} \frac{\chi_- - \chi_+}{m_- - m_+} \frac{1}{V} + O\left(\frac{1}{V^2}\right) \quad (4.3)$$

$$\chi_{\max}(V) = \beta \left(\frac{m_+ - m_-}{2} \right)^2 V + \frac{\chi_+ + \chi_-}{2} + O\left(\frac{1}{V}\right). \quad (4.4)$$

It is also interesting to analyse the FSS of the susceptibility and magnetisation at the infinite volume transition point h_i (for systems where h_i is not known exactly, one could use the point $h_{EW}(V)$ introduced in [23]). Due to the exponentially small error bound in (1.1), the corrections to the known leading order FSS predictions are exponentially small

$$m(h_i) = \frac{m_+ + m_-}{2} \left(1 + O(Ve^{-L/L_0}) \right) \quad (4.5)$$

$$\chi(h_i) = \left[\beta \left(\frac{m_+ - m_-}{2} \right)^2 V + \frac{\chi_+ + \chi_-}{2} \right] \left(1 + O(Ve^{-L/L_0}) \right). \quad (4.6)$$

Note that the error bounds in (4.5) and (4.6) are only bounds. For the model considered here, the value of $m(h_i)$, e.g., is in fact exactly $(m_+ - m_-)/2$, due to (2.1) and the fact that the finite volume magnetisation with periodic boundary conditions and no external field is zero for the standard Ising model. The corrections for $\chi(h_i)$, on the other hand, are expected to be asymptotically proportional to $L^\alpha e^{-L/\xi}$, with a constant α not necessary equal to the worst case bound (4.6).

In addition to the FSS of the susceptibility and magnetisation, we also analyse the FSS of the cumulant U_V introduced by Binder [5]. It is defined

$$U_V = - \frac{\langle M^4 \rangle_c}{3 \langle M^2 \rangle_c^2} = - \frac{\langle (M - \langle M \rangle)^4 \rangle - 3 \langle (M - \langle M \rangle)^2 \rangle^2}{3 \langle (M - \langle M \rangle)^2 \rangle^2} \quad (4.7)$$

as Here $\langle \cdot \rangle_c$ denotes connected expectation values and $M = \sum_{x \in V} s_x$. By the inequality $\langle F^2 \rangle \geq \langle F \rangle^2$ (applied to $F = (M - \langle M \rangle)^2$) the cumulant U_V cannot exceed the value $2/3$. In the high temperature region, where the asymptotic probability distribution of M is gaussian, U_L goes to 0 in the infinite volume limit for all h , while in the low temperature region considered here

$$U_V(h_i) = \frac{2}{3} \left[1 - \frac{4(\chi_+ + \chi_-)}{\beta(m_+ - m_-)^2} \frac{1}{V} + O\left(\frac{1}{V^2}\right) \right], \quad (4.8)$$

implying that U_V goes to $2/3$ at the first order transition point $h = h_i$ if $V \rightarrow \infty$. At the second order transition point $h = h_i$, $\beta = \beta_c$ the cumulant U_V is expected to go to a non-zero limit strictly smaller than $2/3$.

In most theories h_i is not known. Instead of $U_V(h_i)$, it is therefore more natural to consider the maximum $U_V^{\max}$ of U_V as a function of h , and its position $h_U(V)$. Again, the FSS of these two quantities can be obtained from (1.1). One obtains

$$h_U(V) = h_i + \frac{4(\chi_+ - \chi_-)}{\beta^2(m_+ - m_-)^2} \frac{1}{V^2} + O\left(\frac{1}{V^3}\right) \quad (4.9)$$

$$U_V^{\max} = \frac{2}{3} \left[1 - \frac{4(\chi_+ + \chi_-)}{\beta(m_+ - m_-)^2} \frac{1}{V} + O\left(\frac{1}{V^2}\right) \right], \quad (4.10)$$

Even though (1.1) is rigorous for large β , and probably correct for all $\beta > \beta_c$ provided V is large enough, it is not clear where the large V asymptotics for the first-order FSS predictions (4.2) through (4.10) sets in. In fact, warned by the results of Billaire and co-workers [18], we expect that the asymptotic FSS sets only in once $L \gg \xi$, where $\gg$ might mean five to ten times larger.

We have tested the above finite-size scaling predictions for three different temperatures, namely for $\beta = 0.45$, $\beta = 0.47$ and $\beta = 0.5$ (corresponding to $\xi = 13.5096\dots$, $\xi = 4.35091\dots$ and $\xi = 2.19238\dots$). In our simulation we measured the quantities $h_x(V)$, $m(h_x(V))$, $\chi_{\max}(V)$, $\chi(h_i)$, $h_U(V)$, $U_V^{\max}$ and $U_V(h_i)$ for $L = 4, 5, \dots, 12, 14, 18, 30$, see Section 3 above for the details of the simulation. The results are given in Table 2a-c.

We show both the theoretical prediction according to (4.2) through (4.10) and our numerical results for $h_x(V)$, $h_U(V)$, $m(h_x(V))$, $\chi(h_i)$ and $U_V(h_i)$ in Figs. 2 - 5 below. Monte Carlo data are represented by open circles and open squares, and theoretical predictions are represented by full or dashed lines.

As a first point we want to stress that our theoretical calculations predict that both $h_x(V) - h_i$ and $h_U(V) - h_i$ go to zero like $1/V^2$, a peculiar feature of an asymmetric first order transition with two coexisting phases¹. Indeed, our numerical data conform with this prediction, see Fig. 2a. Since the last points in Fig. 2a are barely distinguishable, we show a blow up of this figure in Fig. 2b. Unfortunately, the statistical errors for the corresponding data points are so big that our data for $h_x(V)$ and $h_U(V)$ agree for the last five lattice sizes $L = 11, \dots, 30$, see Table 2. We therefore decided to present data points from an exact transfer matrix calculation in Fig. 2b. As can be seen from the figure, these data are in perfect agreement with the theoretical predictions (4.2) and (4.9).

Next, we point out that the volumes for which the asymptotic FSS according to (4.2) through (4.10) sets in depends strongly on the considered quantity. While the asymptotic regime corresponds to $L/\xi \geq 1 - 2$ for

¹ More generally, $h_x(V) - h_i \sim 1/V^2$ and $h_U(V) - h_i \sim 1/V^2$ if the number N_- of phases stable for $h < h_i$ and the number N_+ of phases stable for $h > h_i$ are equal. If $N_+ \neq N_-$ the shift in h is proportional to $1/V$.

Fig. 2a: The finite-size transition points $h_x(V)$ (dashed line) and $h_U(V)$ (solid line).

Fig. 2b: $h_x(V)$ and $h_v(V)$ (blow-up).

Fig. 3: The magnetisation at the susceptibility peak $m(h_x)$.

Fig. 4: The susceptibility $\chi(h_t)$ and its asymptotic behaviour.

Fig. 5: The cumulant $U_V(h_t)$ and its asymptotic behaviour.

$h_\ell(V)$ and $h_u(V)$, and to $L/\xi \geq 2 - 3$ for $m(h_\ell(V))$, both $\chi(h_\ell)$ and $U_V(h_\ell)$ require much larger volume ($L/\xi \geq 4 - 5$ and $L/\xi \geq 6 - 8$, respectively). This conforms to the results of Gupta and Irbäck, who observed similar effects for a symmetric field driven transition [26].

A last point of interest is related to the determination of the transition point h_t , as discussed at the end of the last section [25, 27]. In addition to the point $h_{\text{EW}}(V)$ defined in (3.3), one may consider “single peak” magnetisations and susceptibilities $m_\pm(V)$ and $\chi_\pm(V)$ defined as

$$m_\pm(V) = \sum s P_V^\pm(s) \quad (4.11)$$

and

$$\chi_\pm(V) = \beta V \left(\sum s^2 P_V^\pm(s) - \left(\sum s P_V^\pm(s) \right)^2 \right), \quad (4.12)$$

where $P_V^\pm$ is the normalised probability distribution of the left and right peak, respectively. More explicitly,

$$\begin{aligned} m_-(V) &= \sum_{s < s_{\min}} s P_V(s) / \sum_{s < s_{\min}} P_V(s) \\ m_+(V) &= \sum_{s > s_{\min}} s P_V(s) / \sum_{s > s_{\min}} P_V(s) \end{aligned} \quad (4.13)$$

and similarly for $\chi_\pm(V)$.

The corresponding data can be found in Table 2a-c as well. An inspection of these data for $\beta = 0.50$ shows an interesting feature. For small system sizes, $\chi_\pm(V)$ are smaller than the infinite volume values $\chi_\pm$, and $m_+(V) > m_+$ and while $m_-(V) < m_-$. As the system becomes larger, all these quantities approach the infinite volume value, and in fact overshoot, before they finally approach the infinite volume limit from the other side, see Fig. 6. We have observed the same effect for even lower temperatures (where our exact transfer matrix calculation can easily reach the asymptotic regime), and we expect that it occurs for all temperatures $\beta < \beta_c$ provided the lattice is chosen large enough.

We think that this effect is due to the competing influence of two finite-size effects for the probability distribution $P_V(s)$. One of these effects concerns the fact that the single peaks $P_V^\pm$ obtained by just cutting the original distribution at the point $s_{\min}$ contain events which actually should be considered part of the other phase. A minute of reflection shows that this effect decreases the susceptibility, and shifts the magnetisations $m_\pm(V)$ in the direction observed for small lattice systems (if one assumes that the original distribution is of the form (3.1a), this can be shown rigorously). The second, competing effect comes from tunneling events. These events tend to shift the magnetisations towards each other, and to increase the susceptibility. Since overlap effects between the two peaks die out exponentially with the volume $V = L^d$, while tunneling effects die out exponentially with L^{d-1} , the competition of these two effects seems to naturally explain the observed overshooting for $m_\pm(V)$ and $\chi_\pm(V)$.

Fig. 6: The volume dependence of the “single peak” magnetisation $m_+(V)$ for $\beta = 0.5$. The dashed line shows the infinite volume value.

5. The Crossover from First- to Second-Order FSS

We have seen that the formulae of sections 3 and 4 break down when the correlation length ξ becomes comparable with the lattice size L . In this section we try to explain finite size behaviour in this large ξ region.

In Fig. 7 we show the susceptibility at h_t as a function of lattice size for a range of inverse temperatures β in the neighbourhood of β_c . The results for the smaller lattices (up to size 14×14) are taken from exact calculations of the magnetisation distribution (see appendix). The susceptibilities on the larger lattices are Monte Carlo results, either from our own simulations with the three-state model (1.2) or derived (via the formulae in section 2) from simulations of the two dimensional Ising model [23,29]. What we see is that on small lattices the curves are all rather similar, independent of whether β is above, below or at the critical point. This is indeed reasonable, on a finite lattice all physical quantities are analytic functions of the coupling constants so there can be no sharp distinction between the first order phase transition above β_c , the second order phase transition at β_c or the rapid crossover (without a phase transition) seen below β_c .

This suggests that we should try to understand this region by using the results of the finite size scaling theory in the neighbourhood of a second order phase transition. (For reviews see [10,28]). We expect this theory to apply whenever the correlation length ξ and lattice size L are both large compared to the lattice spacing.

The basic hypothesis is that when the correlation length is large enough thermodynamic properties are no longer dependent on the lattice spacing itself. By this is meant that instead of being a function of the three parameters β , h and L the free energy and correlation length become functions of only two variables, i.e.

Fig. 7: $\chi(h_t)/V$ as a function of lattice size. Solid points are exact results, open points are Monte Carlo results.

$$f^{\text{mag}}(\beta, h, L) \approx L^{-d} \tilde{f}(tL^{1/\nu}, HL^{(\gamma+\theta)/\nu}) \quad (5.1a)$$

and

$$\xi(\beta, h, L) \approx L \tilde{\xi}(tL^{1/\nu}, HL^{(\gamma+\theta)/\nu}) \quad (5.1b)$$

where the thermodynamic variables are $t \equiv (\beta_c - \beta)/\beta$ and $H \equiv h - h_c$, and γ, β and ν are the critical exponents of the model (1.2), which (as explained in section 2) are the same as those of the Ising model, namely $\gamma = 7/4$, $\beta = 1/8$ and $\nu = 1$. (It is unfortunate that the traditional symbol for inverse temperature and that for the critical exponent are both β , we hope that the context will always make it clear in which sense β is to be understood.)

Differentiating the free energy leads to a scaling form for the magnetic susceptibility

$$\chi(\beta, h, L) \approx L^{\gamma/\nu} \tilde{\chi}(tL^{1/\nu}, HL^{(\gamma+\theta)/\nu}). \quad (5.2)$$

(The “hyperscaling” relationship $2\beta + \gamma = d\nu$ has been used to simplify this expression.) A simple test of the scaling form (5.2) comes from our data at $h = h_c$. The finite size scaling prediction is

$$\chi(\beta, h_c, L)/L^{\gamma/\nu} \approx \tilde{\chi}(tL^{1/\nu}, 0), \quad (5.3)$$

i.e. that all data should lie on a universal curve if $\chi(\beta, h_c, L)/L^{\gamma/\nu}$ is plotted against $tL^{1/\nu}$. The results are shown in Fig. 8. In place of $tL^{1/\nu}$ we have used the somewhat more physical coordinate

$$(4\beta + 2\ln(\tanh \beta))L^{1/\nu} = (L/\xi(\beta, h_c, \infty))^{1/\nu} \quad \text{for } \beta > \beta_c$$

$$= -2(L/\xi(\beta, h_c, \infty))^{1/\nu} \quad \text{for } \beta < \beta_c. \quad (5.4)$$

This is possible in this model since we have an exact expression (2.5b) for the correlation length. (The factor -2 in the symmetric phase is there to

ensure that the x -coordinate is an analytic function of β .) We see that the data do indeed fall on a universal curve. (Some deviation is seen at $\beta = 0.50$, which is to be expected since the correlation length is only ≈ 2.19). The curves predicted by the ansatz (3.1a) for first order phase transitions are also plotted in this figure. We see that the predictions are reliable as long as $L/\xi > 4$. The asymptotic results calculated in section 3 allow us to say more than the scaling hypothesis (5.1), because they give us the large x asymptote of the scaling function $\tilde{\chi}(x, 0)$.

Fig. 8: A scaling plot of $\chi(h_c)$. Exact results are shown as filled symbols, Monte Carlo results as open. Only data for $L \geq 3$ are plotted. The curves show the expected large volume behaviour. The dashed curves are the large volume asymptotes for $\beta = 0.30, 0.47$ and 0.50 . The solid curves are the large volume asymptotes at $\beta = \beta_c \pm \epsilon$.

The scaling form (5.2) also has implications for h_x , the position of the susceptibility maximum.

$$h_x(\beta, L) - h_c(\beta) \approx L^{-(\gamma+\theta)/\nu} \tilde{H}(tL^{1/\nu}) \quad (5.5)$$

Taking higher derivatives of the free energy leads to scaling expressions

for any cumulant of the magnetisation distribution, in particular to a scaling expression for the Binder parameter U_V

$$U_V(\beta, h, L) \approx \tilde{U}(tL^{1/\nu}, HL^{(\gamma+\theta)/\nu}) \quad (5.6a)$$

and its maximal value

$$U_V^{\max} \approx \tilde{U}^{\max}(tL^{1/\nu}). \quad (5.6b)$$

The magnetic field at which U_V reaches its maximum is given by a relationship of the form (5.5).

The scaling predicted in (5.6b) is tested in Fig. 9. As in section 4 we find that larger lattices are needed to see finite size scaling for U_V than for χ . Scaling only works well on lattices with L larger than about 5, on smaller lattices the value of $U_V^{\max}$ lies noticeably below the scaling curve.

Fig. 9: A scaling plot of $U_V^{\max}$. Exact results are shown as filled symbols, Monte Carlo results as open. Only data for $L \geq 5$ are plotted. The curve is the large volume asymptote (4.10) at $\beta = \beta_c + \epsilon$

In conclusion we see that there is a considerable region of overlap between the domain of validity of the first order finite size scaling relationships

presented in earlier sections and the scaling region results presented here. The first order results require that L be large with respect to the correlation length, but make no demands on what the correlation length must be, while the results of this section require that L and ξ are both large, but do not restrict the ratio L/ξ . The domain of mutual validity is therefore $L \gg \xi \gg 1$, while in the entire range $L \gg 1$ we can always apply at least one of the formulae.

Appendix - Exact Calculations

Naively one might think that exact calculations of the partition function for our model would be completely impractical, since the number of configurations that need to be considered is 3^{L^d} . However by carefully dividing the lattice up into smaller sub-lattices it is possible to construct algorithms in which the amount of calculation grows exponentially with L^{d-1} rather than L^d , which gives spectacular savings in time, especially when $d = 2$.

The essential insight is that the partition function for a lattice spin model can be calculated recursively, see [30,31].

Suppose that the partition function Z_N for the N site sub-lattice shown at the top of Fig. 10 is a known function of the spins on the boundary, (σ_1 to σ_{2L}). The spins $\vec{\sigma}$ in the interior of the lattice (the gray region) have been summed over. Periodic boundary conditions in the x direction are to be understood.

$$Z_N(\sigma_1, \dots, \sigma_L; \sigma_{L+1}, \dots, \sigma_{j-1}; \sigma_j, \dots, \sigma_{2L}) \\ \equiv \sum_{\vec{\sigma}} \exp(-\beta H_N(\vec{\sigma}, \sigma_1, \dots, \sigma_{2L})) \quad (A.1)$$

The lattice size is next increased by one by adding the new spin σ'_j . The energy difference, δH , between the new lattice and the old lattice depends only on the boundary spins, not on any of the interior spins (in the example shown the energy change depends only on the three spins σ_{j-1} , σ_j and σ'_j , and it is also true for the special cases of the first and last spin in a row that only boundary spins are involved in the energy change).

Finally we can find the partition function for the larger lattice by summing over σ_j . Algebraically

$$Z_{N+1}(\sigma_1, \dots, \sigma_L; \sigma_{L+1}, \dots, \sigma'_j; \sigma_{j+1}, \dots, \sigma_{2L})$$

29

Fig. 10: The procedure for adding a spin to the lattice when calculating the partition function recursively. (See the text for a detailed explanation.)

30

$$= \sum_{\sigma_j} \exp(-\beta \delta H(\sigma_{L+1}, \dots, \sigma_L; \sigma'_j)) \\ \times Z_N(\sigma_1, \dots, \sigma_L; \sigma_{L+1}, \dots, \sigma_{j-1}; \sigma_j, \dots, \sigma_{2L}) \quad (A.2)$$

The recursion can be started from the fact that Z_0 (the partition function of a lattice with no spins) is 1.

In our implementation of this procedure the amount of calculation grows as q^{2L} and the amount of storage needed as q^L for a q -state spin model. The basic reason that the calculational burden grows like q^{2L} is that the partial partition function Z_N depends on the $2L$ boundary spins, which means that its arguments can take q^{2L} different values. The number of arguments would be greatly reduced if the boundary conditions in the y direction allow us to sum over the spins in the first row (the spins σ_1 to σ_L). Examples of boundary conditions which permit such a sum are either free boundary conditions or a fixed boundary configuration. These would reduce the computational time to a value $\propto q^L$. Even though they are not the best boundary conditions for exact calculations we have used periodic boundary conditions in both directions to make comparison with Monte Carlo results possible.

The close relationship of our model to the Ising model means that we can calculate partition functions for the Ising model which has $q = 2$ and then transform to the $q = 3$ model at the end, meaning that computation grows as 4^L instead of 9^L .

Even though the calculation grows exponentially we were able to handle surprisingly large lattices, with sizes of up to 14×14 , without needing a supercomputer.

Once we have found the multiplicity of states $N_I(E, n_-)$ for the Ising model it is a simple matter to use the binomial distribution to calculate the corresponding multiplicities for our model (1.2)

$$N(E, n_-, n_0) = N_I(E, n_-) \frac{(L^d - n_-)^{n_0}}{n_0! (L^d - n_- - n_0)!} \quad (A.3)$$

Here E is the number of broken bonds and n_- , n_0 and n_+ are the number of spins of a given type. We have a simple relationship of this type because boundaries between spin 0 and spin +1 entail no energy cost while the energy cost for boundaries between spin -1 and either of the other two spin types is the same in both cases. It is a simple matter to derive relationships between partition functions and magnetisation histograms of the two models by summing (A.3) over (some of) its arguments.

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L	$h_{\chi}(V)$	$m(h_{\chi}(V))$	$\chi_{max}(V)/V$	$\chi(h_{\chi})/V$
4	-1.062853(44)	-.291383(07)	.171484(05)	.171355(06)
5	-1.066617(38)	-.297833(08)	.163352(07)	.163301(06)
6	-1.067975(22)	-.301295(12)	.157308(08)	.157282(08)
7	-1.068570(22)	-.303355(08)	.152602(05)	.152587(05)
8	-1.068925(25)	-.304703(07)	.148814(07)	.148807(06)
9	-1.069085(17)	-.305617(12)	.145699(07)	.145694(07)
10	-1.069174(15)	-.306261(10)	.143083(08)	.143080(09)
11	-1.069264(21)	-.306770(11)	.140857(06)	.140856(06)
12	-1.069300(20)	-.307118(08)	.138951(12)	.138951(12)
14	-1.069340(17)	-.307640(11)	.135837(08)	.135837(08)
18	-1.069338(11)	-.308173(09)	.131496(11)	.131495(11)
30	-1.069360(13)	-.308728(30)	.125416(17)	.125416(18)

L	$h_U(V)$	$U_V^{P^{**}}$	$U_V(h_U)$
4	-1.065260(44)	.598796(07)	.598403(12)
5	-1.067640(38)	.606576(08)	.606414(08)
6	-1.068483(22)	.611434(11)	.611339(12)
7	-1.068851(22)	.614893(07)	.614841(10)
8	-1.069093(25)	.617588(09)	.617564(10)
9	-1.069191(16)	.619833(10)	.619818(11)
10	-1.069244(15)	.621759(16)	.621748(18)
11	-1.069313(21)	.623476(09)	.623473(10)
12	-1.069335(20)	.625052(16)	.625051(16)
14	-1.069359(17)	.627860(11)	.627860(10)
18	-1.069345(11)	.632589(14)	.632587(13)
30	-1.069361(13)	.642953(22)	.642953(36)

L	$h_{EW}(V)$	$m_+(V)$	$m_-(V)$	$\chi_+(V)$	$\chi_-(V)$
4	-1.069540(42)	.28434(02)	-.90380(02)	.2488(01)	.1523(01)
5	-1.069546(38)	.27217(02)	-.89176(02)	.3214(01)	.2229(01)
6	-1.069332(24)	.26350(03)	-.88146(03)	.3967(02)	.3093(01)
7	-1.069321(34)	.25587(21)	-.87381(21)	.4849(24)	.3989(23)
8	-1.069398(26)	.24938(06)	-.86779(08)	.5832(09)	.4918(09)
9	-1.069435(27)	.24364(22)	-.86313(24)	.6948(42)	.5828(43)
10	-1.069362(18)	.24004(18)	-.85797(16)	.7860(41)	.7035(38)
11	-1.069407(22)	.23605(10)	-.85447(09)	.9016(23)	.8081(24)
12	-1.069402(26)	.23292(14)	-.85123(17)	1.0142(48)	.9216(50)
14	-1.069397(17)	.22771(15)	-.84606(14)	1.2488(59)	1.1530(61)
18	-1.069361(11)	.22064(07)	-.83897(10)	1.7316(58)	1.6274(66)
30	-1.069363(13)	.21183(08)	-.82994(09)	3.036(16)	2.944(16)
∞	-1.06935961..	.2087522..	-.8267862..	4.75	4.67

TABLE 2a: Monte Carlo results for $\beta = 0.45$

L	$h_{\chi}(V)$	$m(h_{\chi}(V))$	$\chi_{max}(V)/V$	$\chi(h_{\chi})/V$
4	-1.018099(59)	-.2917063(77)	.1879631(51)	.1878343(56)
5	-1.021451(34)	-.2980674(69)	.1817612(56)	.1817091(46)
6	-1.022640(37)	-.3014688(70)	.1775283(55)	.1775017(61)
7	-1.023182(23)	-.3035004(77)	.174855(53)	.1744708(53)
8	-1.023500(27)	-.3048250(81)	.172426(57)	.1722357(57)
9	-1.023655(16)	-.3057188(65)	.1705276(51)	.1692216(67)
10	-1.023683(14)	-.3063460(64)	.1692254(71)	.1681995(46)
11	-1.023747(20)	-.3068320(83)	.1682018(49)	.1674017(67)
12	-1.023794(23)	-.3071837(68)	.1674027(69)	.1662241(71)
14	-1.023813(14)	-.3076732(72)	.1662249(70)	.1662241(71)
18	-1.023858(09)	-.3082108(91)	.1649093(29)	.1649093(29)
30	-1.023846(06)	-.3087157(58)	.1636313(95)	.1636305(96)

L	$h_U(V)$	$U_V^{P^{**}}$	$U_V(h_U)$
4	-1.020208(59)	.610042(05)	.609678(15)
5	-1.022328(34)	.619780(04)	.619626(06)
6	-1.023067(37)	.626326(06)	.626242(11)
7	-1.023413(23)	.631209(05)	.631161(06)
8	-1.023635(27)	.635124(05)	.635103(08)
9	-1.023740(16)	.638356(05)	.638348(06)
10	-1.023738(14)	.641133(07)	.641119(06)
11	-1.023784(20)	.643530(05)	.643523(05)
12	-1.023820(23)	.645639(06)	.645636(07)
14	-1.023828(14)	.649127(06)	.649124(08)
18	-1.023863(09)	.654133(03)	.654133(04)
30	-1.023847(06)	.661167(06)	.661162(09)

L	$h_{EW}(V)$	$m_+(V)$	$m_-(V)$	$\chi_+(V)$	$\chi_-(V)$
4	-1.024032(60)	.303542(24)	-.922636(09)	.2300(01)	.1273(01)
5	-1.023969(35)	.296404(16)	-.915421(15)	.2807(01)	.1770(01)
6	-1.023816(49)	.291949(99)	-.90913(99)	.3280(11)	.2320(10)
7	-1.023822(22)	.288400(97)	-.906357(92)	.3783(12)	.2830(12)
8	-1.023887(25)	.285761(73)	-.903962(76)	.4281(11)	.3297(12)
9	-1.023896(17)	.283968(58)	-.902061(63)	.4727(12)	.3761(13)
10	-1.023843(15)	.282669(71)	-.900775(64)	.5146(18)	.4171(17)
11	-1.023858(19)	.281679(38)	-.899877(42)	.5533(10)	.4536(12)
12	-1.023872(23)	.281054(34)	-.899190(31)	.5859(11)	.4874(11)
14	-1.023857(14)	.280179(28)	-.898372(28)	.6433(13)	.5400(15)
18	-1.023873(09)	.279668(27)	-.897689(30)	.7050(19)	.6123(16)
30	-1.023848(06)	.279645(23)	-.897639(17)	.7451(10)	.6528(09)
∞	-1.02385495..	.27967484..	-.89770883..	.7378	.6433

TABLE 2b: Monte Carlo results for $\beta = 0.47$

L	$h_s(V)$	$m(h_s(V))$	$\chi_{max}(V)/V$	$\chi(h_s)/V$
4	-.957392(41)	-.2920229(56)	.2112605(43)	.2111255(57)
5	-.960375(48)	-.2982867(54)	.2070919(30)	.2070386(41)
6	-.961410(34)	-.3016339(73)	.2045653(26)	.2045385(27)
7	-.961846(20)	-.3036197(50)	.2029542(30)	.2029382(34)
8	-.962120(32)	-.3049161(23)	.2018638(42)	.2018563(46)
9	-.962230(32)	-.3057789(67)	.2011162(70)	.2011113(60)
10	-.962293(23)	-.3064065(53)	.2005801(45)	.2005767(48)
11	-.962335(20)	-.3068779(35)	.2001908(63)	.2001885(60)
12	-.962391(17)	-.3072287(51)	.1998887(67)	.1998882(69)
14	-.962375(06)	-.3076952(53)	.1994660(22)	.1994642(24)
18	-.962421(14)	-.3082342(72)	.1990105(36)	.1990105(37)
30	-.962427(08)	-.3087290(42)	.1985327(29)	.1985326(26)

L	$h_V(V)$	U_V^{max}	$U_V(h_s)$		
4	-.959187(41)	.6220964(36)	.6217458(11)		
5	-.961108(48)	.6328211(25)	.6326800(12)		
6	-.961761(34)	.6398531(33)	.6397789(09)		
7	-.962034(20)	.6448716(16)	.6448240(05)		
8	-.962230(32)	.6486261(28)	.6486061(09)		
9	-.962298(32)	.6515315(27)	.6515180(07)		
10	-.962337(23)	.6538180(23)	.6538081(06)		
11	-.962365(20)	.6556541(31)	.6556476(04)		
12	-.962413(17)	.6571408(20)	.6571405(03)		
14	-.962387(06)	.6593655(08)	.6593586(03)		
18	-.962425(14)	.6620330(08)	.6620329(07)		
30	-.962428(08)	.6649091(06)	.6649074(18)		

L	$h_{EW}(V)$	$m_+(V)$	$m_-(V)$	$\chi_+(V)$	$\chi_-(V)$
4	-.962517(40)	.325905(14)	-.944591(07)	.20538(04)	.09385(03)
5	-.962494(48)	.323234(09)	-.941739(09)	.23116(02)	.11961(03)
6	-.962411(34)	.321985(05)	-.940006(11)	.25043(05)	.14281(06)
7	-.962386(20)	.321226(27)	-.939220(26)	.26679(35)	.15947(34)
8	-.962440(32)	.320767(09)	-.938869(08)	.27967(08)	.17092(05)
9	-.962428(32)	.320633(17)	-.938654(18)	.28755(27)	.17991(31)
10	-.962423(23)	.320555(16)	-.938597(13)	.29333(28)	.18540(22)
11	-.962423(20)	.320557(10)	-.938592(14)	.29611(12)	.18939(14)
12	-.962454(17)	.320545(10)	-.938622(15)	.29874(13)	.19064(17)
14	-.962409(06)	.320596(07)	-.938641(07)	.30017(11)	.19183(09)
18	-.962434(14)	.320660(09)	-.938720(11)	.29854(08)	.19102(10)
30	-.962429(08)	.320688(08)	-.938717(05)	.29677(09)	.18925(11)
∞	-.96242365..	.32068921..	-.93872320	.29677	.18920

Table 2c: Monte Carlo results for $\beta = 0.50$