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A Green's Function Method for Hadrons in Nuclei

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ABSTRACT

A partial review is given of the Green's function method for calculating inclusive and semi-inclusive cross sections for quasi-free processes where various hadrons are produced in nuclear targets. The general properties of the Green's function related to the energy spectrum of the system is discussed. It is shown that the method provides us with a unique way of studying hadrons in nuclei in the form of the final state interaction between the produced hadron and the residual nucleus. The method is applied to the Σ -hyperon production by (K^-, π^-) reactions on nuclei in connection with the possibility of observing Σ -hypernuclear states, the production of hadronic atoms where the strong interaction interferes with the Coulomb interaction giving rise to interesting phenomena and the high energy electron scattering in the quasi-elastic region associated with the nuclear or color transparency.

KEYWORDS

Green's function method; hadrons in nuclei; UBS (unstable bound state); Σ^- -hypernucleus; Σ^- -atom; deeply bound pionic atom; nuclear transparency.

2.2 The Green's function and the optical potential

We have so far assumed that the nuclear ground state does not contain the produced particle, c , i.e. $\Phi_c(x)/A-1, 0) = 0$. This is not justified when c is a nucleon or when its anti-particle, $\bar{c}$, plays an important role. The latter is the case when we consider pion as the particle, c . In such a case, it is more convenient to introduce the causal Green's function defined by

$$G(x, x') = (A-1, 0 | T[\Phi_c(x)\Phi_c^\dagger(x')] | A-1, 0), \quad (2.12)$$

where $x = (t, \mathbf{x})$ denotes a space-time point. T is the time-ordered product of the operators,

$$T[\Phi_c(x)\Phi_c^\dagger(x')] = \theta(t-t')\Phi_c(x)\Phi_c^\dagger(x') \mp \theta(t'-t)\Phi_c^\dagger(x')\Phi_c(x), \quad (2.13)$$

where the upper (lower) sign refers to fermions (bosons). If the Hamiltonian, H , is time independent, the Green's function depends only on the time difference $t-t'$, i.e.

$$G(x, x') = G(t-t'; \mathbf{x}, \mathbf{x}'). \quad (2.14)$$

Then the Fourier transform of $G(x, x')$ with respect to the time variable yields

$$\begin{aligned} G(\omega; \mathbf{x}, \mathbf{x}') &= \int dt e^{i\omega(t-t')} G(x, x') \\ &= (A-1, 0 | \Phi_c(\mathbf{x}) \frac{1}{\omega - H + E_0 + i\eta} \Phi_c^\dagger(\mathbf{x}') | A-1, 0) \\ &\quad \mp (A-1, 0 | \Phi_c^\dagger(\mathbf{x}') \frac{1}{\omega + H - E_0 - i\eta} \Phi_c(\mathbf{x}) | A-1, 0) \\ &= \sum_n \left\{ \frac{1}{\omega - E_n + E_0 + i\eta} (A-1, 0 | \Phi_c(\mathbf{x}) | n) \langle n | \Phi_c^\dagger(\mathbf{x}') | A-1, 0) \right. \\ &\quad \left. \mp \sum_n \frac{1}{\omega + E_n - E_0 - i\eta} (A-1, 0 | \Phi_c^\dagger(\mathbf{x}') | n) \langle n | \Phi_c(\mathbf{x}) | A-1, 0) \right\}, \end{aligned} \quad (2.15)$$

where use has been made of the relations

$$\begin{aligned} \Phi_c(x) &= e^{iH} \Phi_c(\mathbf{x}) e^{-iHt}, \\ \Phi_c^\dagger(x) &= e^{iHt} \Phi_c^\dagger(\mathbf{x}) e^{-iHt}. \end{aligned} \quad (2.16)$$

The cross section considered in the previous section, or more generally, the nuclear response function for adding a particle, c , can be expressed as

$$S(\omega) = -\frac{1}{\pi} \int dx dx' F^\dagger(\mathbf{x}) \text{Im} G(\omega; \mathbf{x}, \mathbf{x}') F(\mathbf{x}'), \quad (2.17)$$

where the imaginary part of the Green's function is given by

$$\begin{aligned} \text{Im} G(\omega; \mathbf{x}, \mathbf{x}') &= -\pi \left\{ \sum_n \delta(\omega - E_n + E_0) \langle A-1, 0 | \Phi_c(\mathbf{x}) | n \rangle \langle n | \Phi_c^\dagger(\mathbf{x}') | A-1, 0) \right. \\ &\quad \left. \mp \sum_n \delta(\omega + E_n - E_0) \langle A-1, 0 | \Phi_c^\dagger(\mathbf{x}') | n \rangle \langle n | \Phi_c(\mathbf{x}) | A-1, 0) \right\}. \end{aligned} \quad (2.18)$$

The first term represents the nuclear response to the addition of a particle, c , while the second term contributes to its removal. In the usual situation, these two terms do not contribute simultaneously and only the first term contributes in the relevant region of ω . Thus, the causal Green's function and the one introduced by eq. (2.11) in the previous subsection are equivalent for the present purpose. We also introduce the free Green's function, G_0 , where the nuclear ground state is replaced by the vacuum $|0\rangle$, i.e.

$$G_0(x, x') = (0 | T[\Phi_c(x)\Phi_c^\dagger(x')] | 0). \quad (2.19)$$

The optical potential, $U(\omega; \mathbf{x}, \mathbf{x}')$, for the particle, c , in the nucleus is then defined by

$$G(\omega; \mathbf{x}, \mathbf{x}') = G_0(\omega; \mathbf{x}, \mathbf{x}') + \int dx'' dx''' G_0(\omega; \mathbf{x}, \mathbf{x}'') U(\omega; \mathbf{x}'', \mathbf{x}''') G(\omega; \mathbf{x}''', \mathbf{x}') \quad (2.20)$$

or in an abbreviated notation

$$G = G_0 + G_0 U G. \quad (2.21)$$

The optical potential is generally complex due to inelastic and absorptive processes and the imaginary part of the Green's function consists of two terms, i.e.

$$\text{Im} G = (1 + G^I U^I) \text{Im} G_0(1 + UG) + G^I \text{Im} UG. \quad (2.22)$$

The response function, S , which can be expressed in the abbreviated notation as

$$S = -\frac{1}{\pi} F^\dagger \text{Im} G F, \quad (2.23)$$

is correspondingly decomposed into two terms [5, 6], i.e.

$$S = S^I + S^A, \quad (2.24)$$

where

$$S^I = -\frac{1}{\pi} F^\dagger (1 + G^I U^I) \text{Im} G_0(1 + UG) F \quad (2.25)$$

and

$$S^A = -\frac{1}{\pi} F^\dagger G^I \text{Im} UG F. \quad (2.26)$$

S^I is that part of the response function in which the particle c is emitted leaving the nucleus in its ground state, while S^A is the remaining part due to the imaginary part of the optical potential, which can be further decomposed according to its reactive contents, i.e. inelastic scatterings, absorption etc..

3 Singularities of the Green's function

We will discuss here the analytic property of the Green's function as a function of energy, paying a special attention to its singularities and their relation to the spectrum of the system. The subject has been extensively discussed in many text books of quantum scattering theory [7]. We will therefore restrict ourselves to the problems which are directly related to the later applications.

3.1 General considerations

The causal Green's function introduced in the previous section can be expressed as a sum of two boundary values of analytic functions, one from above and the other from below the real axis as one sees from eqs. (2.15) and (2.18). The analytic property of each term can be studied by the resolvent operator,

$$\hat{R}(z) = (z - H)^{-1}, \quad (3.1)$$

where z is the complex energy variable and H is the Hamiltonian of the system. Its spectrum is assumed to consist of a continuum along the real axis, E , starting from the threshold, E_0 , i.e. $E > E_0$, and discrete eigenvalues, E_n ($n = 0, 1, \dots, E_0 < E_1 < \dots < E_n$). The corresponding eigenstates are denoted by $|E\rangle$, and $|n\rangle$, respectively. $\hat{R}(z)$ can be expressed as

$$\hat{R}(z) = \sum_n \frac{|n\rangle\langle n|}{z - E_n} + \int_{E_0}^{\infty} dE \frac{|E\rangle\langle E|}{z - E}. \quad (3.2)$$

$\hat{R}(z)$ thus has simple poles at discrete eigenvalues, E_n , with the projection operator, $|n\rangle\langle n|$, as residues and a branch cut along the real axis starting from the threshold, E_0 , the discontinuity across the cut being given by the projection operator, $|E\rangle\langle E|$. $\hat{R}(z)$ has no other singularities on this complex z plane but it can be analytically continued from the upper half plane to the lower half plane across the cut and the extended function may have singularities in the extended region which is called unphysical Riemann sheet to be distinguished from the original z -plane called physical sheet. They can be physically significant only if they are close to observable region, i.e. just below the real axis or in the neighborhood of the threshold. A pole just below the real axis corresponds to a resonance state and the distance from the real axis gives its half width. Denoting the pole position as $(E_r - i\frac{\Gamma}{2})$, one can approximate the Green's function in its neighborhood as

$$\hat{R}(z) \simeq \frac{|r\rangle\langle r|}{z - E_r + i\frac{\Gamma}{2}}, \quad (3.3)$$

Figure 3.1: The integration paths along which $\hat{R}(z)$ is integrated to give the completeness relation.

where $|j\rangle$ is an eigenfunction of the Hamiltonian, H , with a purely outgoing boundary condition and $|\bar{j}\rangle$ is its conjugate solution with a purely incoming boundary condition. Neither $|j\rangle$ nor $|\bar{j}\rangle$ belongs to the Hilbert space and neither should be included in the complete set of the eigenfunctions of H . In the expression eq. (3.2), the resonance behavior of the Green's function described by eq. (3.3) is produced by the behavior of $|E\rangle$ which has an enhanced interior amplitude proportional to $|j\rangle$ in the vicinity of E_j .

The integral of $\hat{R}(z)$ along a large circle on the z -plane gives us the completeness relation, i.e.

$$\frac{1}{2\pi i} \oint dz \hat{R}(z) = 1 = \sum_n |n\rangle \langle n| + \int_{E_n}^{\infty} dE |E\rangle \langle E|, \quad (3.4)$$

where we have deformed the integration path as illustrated in fig. 3.1 to obtain the second equality. For the extended $\hat{R}(z)$, we can further deform the path into the unphysical sheet to pick up the contribution from resonance poles but the resulting "completeness" relation including resonance states is useful only in some special cases.

Eq. (3.4) gives sum rules for the imaginary part of G . When c is a fermion, we have

$$-\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \text{Im} G(\omega; \mathbf{x}, \mathbf{x}') = \langle A-1, 0 | [\Phi_c(\mathbf{x}), \Phi_c^\dagger(\mathbf{x}')] | A-1, 0 \rangle = \delta(\mathbf{x} - \mathbf{x}'), \quad (3.5)$$

while we have two sum rules for a boson, c , i.e.

$$-\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \text{Im} G(\omega; \mathbf{x}, \mathbf{x}') = \langle A-1, 0 | [\Phi_c(\mathbf{x}), \Phi_c^\dagger(\mathbf{x}')] | A-1, 0 \rangle = 0 \quad (3.6)$$

and

$$-\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega \text{Im} G(\omega; \mathbf{x}, \mathbf{x}') = i(A-1, 0 | [\dot{\Phi}_c(\mathbf{x}), \dot{\Phi}_c^\dagger(\mathbf{x}')] | A-1, 0) = \delta(\mathbf{x} - \mathbf{x}'). \quad (3.7)$$

The resulting sum rules for the response function when F is ω -independent are

$$\int_{-\infty}^{\infty} d\omega S(\omega) = \int d\mathbf{x} F^\dagger(\mathbf{x}) F(\mathbf{x}), \quad (3.8)$$

for a fermion and

$$\int_{-\infty}^{\infty} d\omega S(\omega) = 0, \quad \int_{-\infty}^{\infty} d\omega S(\omega) = \int d\mathbf{x} F^\dagger(\mathbf{x}) F(\mathbf{x}), \quad (3.9)$$

for a boson. In actual cases, however, F usually depends on ω and the above sum rules are of limited use.

3.2 Potential scattering

Let us take a concrete example of single channel complex local potential and study the properties of the Green's function. The radial part of the Schrödinger equation for the l -th partial wave is

$$\left[-\frac{1}{2m} \left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} \right) + U(r) \right] \psi_l(r) = E \psi_l(r), \quad (3.10)$$

where the potential, U , is taken to be complex in general but energy independent. Actual optical potentials are known to satisfy a dispersion relation and their imaginary parts govern the energy dependence of the real parts. Thus, an energy-independent complex potential represents an optical potential only in a limited region but is convenient for discussing response function in that energy region. The second order differential equation, eq. (3.10), has two independent solutions. One can define them in terms of the boundary condition at infinity:

$$\psi_l^{(\pm)}(k, r) \sim h_l^{(\pm)}(kr). \quad (3.11)$$

$h_l^{(+)}$ and $h_l^{(-)}$ are spherical Hankel functions which are related to the ordinary ones by

$$\begin{aligned} h_l^{(+)}(x) &= i h_l^{(1)}(x), \\ h_l^{(-)}(x) &= -i h_l^{(2)}(x). \end{aligned} \quad (3.12)$$

$\psi_l^{(+)} (\psi_l^{(-)})$ is called the outgoing (incoming) solution. S -matrix is then defined by

$$S_l(k) = \lim_{r \rightarrow 0} \frac{\psi_l^{(-)}(k, r)}{\psi_l^{(+)}(k, r)}. \quad (3.13)$$

We define another solution which is regular at the origin by

$$u_l(k, r) = -\frac{1}{2i} \left[\psi_l^{(-)}(k, r) - S_l(k) \psi_l^{(+)}(k, r) \right]. \quad (3.14)$$

For a real positive k , u_l is an eigenfunction with the energy eigenvalue

$$E_k = \frac{k^2}{2\mu}. \quad (3.15)$$

In addition to these continuous eigenvalues, there exist discrete eigenvalues

$$E_{nl} = \frac{k_{nl}^2}{2\mu}, \quad (3.16)$$

if

$$S_l(k_{nl}) = \infty \quad \text{with} \quad \text{Im } k_{nl} > 0. \quad (3.17)$$

For discrete eigenvalues one can define normalized eigenfunctions by

$$\varphi_{nl}(r) = \left[k_{nl}^{-2} \text{res } S_l(k) |_{k=k_{nl}} \right]^{1/2} \psi_l^{(+)}(k_{nl}, r). \quad (3.18)$$

These eigenfunctions satisfy the orthonormal relations

Figure 3.2: A schematic illustration of the poles of the S -matrix in the k -plane for a real attractive potential in the potential scattering. The arrows indicate how they move when the imaginary part of the potential, U_i , is introduced.

$$\int_0^\infty dr r^2 \varphi_m(r) \varphi_m(r) = \delta_{mm},$$

$$\int_0^\infty dr r^2 u_l(k, r) u_l(k', r) = \frac{\pi}{2k^2 S_l(k)} \delta(k - k'),$$

$$\int_0^\infty dr r^2 \varphi_m(r) u_l(k, r) = 0, \quad (3.19)$$

and the completeness relation

$$\sum_n \varphi_m(r) \varphi_n(r') + \frac{2}{\pi} \int_0^\infty dk k^2 S_l(k) u_l(k, r) u_l(k, r') = \frac{1}{r^2} \delta(r - r'). \quad (3.20)$$

In fig. 3.2 various types of poles of S -matrix (or equivalently of the Green's function) are schematically shown in the complex k -plane for a real attractive potential. The arrows indicate how these poles move when absorption is gradually introduced. Bound state poles (closed circle), which without absorption lie on the positive imaginary axis, move into the second quadrant, so that the associated wavefunctions continue to be damping at infinity. Resonance poles (cross), which without absorption occur in the fourth quadrant and are associated with wavefunctions divergent at infinity, move away from the real axis, increasing thereby the width of "states" they may represent, often to the point of losing whatever significance they might have had. The partners of resonance poles (open circle), located in the third quadrant symmetrically to resonance poles with respect to the imaginary axis for a real potential, approach the negative real axis and eventually come up to the second quadrant. The associated wavefunctions, then, become damping and if the locations of poles are close to the real axis, their energies have positive real parts and small negative imaginary parts (hereafter we call minus twice the imaginary part of the energy "width"). Gal' *et al.* called them "unstable bound states" (UBS) and proposed that the experimentally observed narrow states of S -hypernuclei above the S -emission threshold correspond to the UBS [8]. It is known, however, that the UBS is not observed in scattering processes. Therefore a crucial question is whether they show up in formation processes or not.

As has been shown in sec. 2.1, formation probabilities of S -hypernuclei are related to the Green's function for Σ in nuclei. Assuming an energy-independent optical potential, one can expand the Green's function in terms of eigenfunctions

$$G^{(l)}(E; r, r') = \sum_n \frac{\varphi_n(r) \varphi_n(r')}{E - E_n} + \frac{2}{\pi} \int_0^\infty dk k^2 \frac{S_l(k) u_l(k, r) u_l(k, r')}{E - E_k}, \quad (3.21)$$

where the summation over n includes all the poles of the S -matrix in the k -plane with positive imaginary part. Thus, the UBS do appear in the sum. The UBS satisfy the criterion (3.17) and are included in the summation of the first

Table 3.1: The potential strengths, pole momenta and pole energies in the cases a, b and c. (I) ((III)) indicates that the pole lies on the first (second) Riemann sheet of the E -plane.

Case	V (MeV)	W (MeV)	k_{pole} (MeV/c)	E_{pole} (MeV)
a	-26	-2	$-16.5 + 18.2i$	$-3.48 - 1.35i$ (I)
b	-16	0	$52.3 - 14.8i$	$1.16 - 0.72i$ (II)
c	-9	-9	$-144.7 + 1.1i$	$9.70 - 0.15i$ (I)

term in eq. (3.21). Therefore, one might think that a sharp structure appears in the physical energy region if a UBS has a small "width". It turns out, however, that a part of the continuum contribution in the second term exactly cancels the pole contribution of the first term near the physical region for the UBS. This can be easily seen for a local optical potential as follows. By contour integral one finds

$$\frac{2}{\pi} \int_0^\infty dk k^2 \frac{S_l(k) u_l(k, r) u_l(k, r')}{E - E_k} = - \sum_n \frac{\varphi_n(r) \varphi_n(r')}{E - E_n} - 2ik u_l(k, r) u_l(k, r'). \quad (3.22)$$

Substituting eq. (3.22) into eq. (3.21) one obtains another expression for the Green's function

$$G^{(l)}(E; r, r') = -2ik u_l(k, r) u_l(k, r'). \quad (3.23)$$

From this expression it is obvious that the Green's function has the same singularities as the S -matrix, and therefore the UBS do not give rise to any significant physical effect. We expect that the same also holds for a nonlocal potential though there is no simple expression like eq. (3.23).

In the following we will confirm the above observation using a simple model as an illustration. We take a complex square-well potential for the optical potential, U ,

$$U(r) = (V + iW)\theta(R - r), \quad (3.24)$$

and a surface δ -function with a definite angular momentum for the function, F ,

$$F(r) = \sqrt{\pi} \frac{1}{R^2} \delta(r - R) Y_l^{(l)}(\hat{r}). \quad (3.25)$$

Then the response functions S , S^l and S^l are reduced to simple forms. It is sufficient to calculate two of them, for instance, S and S^l . By substituting eqs. (3.24) and (3.25) into (2.23) (or (2.17)) and (2.26), we obtain

$$S(E) = -\text{Im } G(E; R, R) \quad (3.26)$$

and

$$S^l(E) = -iW \int_0^R dr r^2 |G^{(l)}(E; r, R)|^2, \quad (3.27)$$

respectively. In order to roughly simulate the case of S -hypernuclei in the ^{12}C region to be discussed in §4.1, we take $l = 1$, $R = 3$ fm and the reduced mass $\mu = 1080$ MeV, while V and W are varied so as to give several cases with bound states, resonances and/or UBS.

We first study three typical cases. The values of V and W together with the positions of the relevant poles in the k - and E -planes are given in table 3.1. The poles in the cases a, b and c are a bound state pole (which has become unstable due to the imaginary part of the potential), a resonance pole and a UBS pole, respectively. Figs. 3.3-3.5 show the response functions S and S^l for these cases. As is clearly seen, in the cases a and b there are remarkably sharp peaks and both their positions and widths correspond very well to the complex pole energies. On the contrary, in the case c, there is no visible structure around the real part of the pole energy, 9.7 MeV, although there should be such a structure if the UBS has physical significance.

Secondly, the effect of the UBS is more thoroughly investigated. This time, V is fixed to be -16 MeV and W is varied from 0 to -6 MeV. The motion of poles in the k - and E -planes as W changes are shown in figs. 3.6 and 3.7, respectively. Figs. 3.8-3.11 show the response functions S and S^l for $W = 0, -2, -3$ and -6 MeV. When $W = 0$, the resonance

Figure 3.3: The response functions, S (solid curve) and S^1 (dashed curve), in arbitrary units for the case a .

Figure 3.4: The same as fig. 3.3 for the case b .

Figure 3.5: The same as fig. 3.3 for the case c .

Figure 3.6: The positions of the poles in the k -plane. V is fixed to be -16 MeV and W is 0 , -2 , -3 and -6 MeV in the case a , b , c and d , respectively.

pole and the partner pole are located symmetrically with respect to the imaginary k -axis, the response function S has a very sharp peak at the real part of the pole energy (1.2 MeV) and its width is equal to twice the imaginary part of the pole energy (1.4 MeV). As $|W|$ increases, the resonance pole moves down away from the real k -axis while the partner pole moves up and eventually crosses the negative real k -axis. Accordingly, the peaks of the response functions S and S^1 become lower and broader and the ratio S^1/S increases. The peak energy and the width for each case also seem to be determined comparatively well by the pole energies. The UBS appear for $W = -3$ and -6 MeV but the corresponding anomaly is observed in the behavior of neither the strength S nor S^1 though especially for $W = -3$ MeV the UBS should have shown up at the energy about 2.3 MeV with the width about 0.7 MeV. The reason can be traced to the fact that though the pole position of the UBS looks similar to that of the usual resonance in the E -plane, it actually does not lie on the same sheet and is too far from the physical region to produce any observable effect.

We note here that, in a complex potential, the imaginary part of the Green's function is not given by the discontinuity across the real axis, i.e.

$$\begin{aligned} \text{Im } G_i(E; r, r') &= \frac{1}{2i} \{G_i(E + i\eta; r, r') - G_i^*(E - i\eta; r, r')\} \\ &\neq \frac{1}{2i} \text{Disc } G_i(E; r, r') \\ &= \frac{1}{2i} \{G_i(E + i\eta; r, r') - G_i(E - i\eta; r, r')\}, \end{aligned} \quad (3.26)$$

where G_i^* is the Green's function for the conjugate potential U^* . The UBS does contribute significantly to the discon-

Figure 3.7: The positions of the poles in the E -planes. The potential parameters are the same as in fig. 3.6.

Figure 3.8: The response function, $S = S^I$, in arbitrary units for the case a.

Figure 3.9: The response functions, S (solid curve) and S^I (dashed curve), in arbitrary units for the case b.

Figure 3.10: The same as fig. 3.9 for the case c.

Figure 3.11: The same as fig. 3.9 for the case d .

tinuity but does not to the imaginary part and the response functions are given by the latter.

3.3 Coupled channels problem

As was mentioned in the previous subsection, an energy-independent complex potential does not represent an optical potential for the whole energy region. The analytic properties of the Green's function studied there should thus be taken with some caution. We will consider here the problem of coupled channels and discuss the properties of the Green's function for the energy-dependent optical potential obtained by eliminating all but one channel [10].

Denoting the intrinsic state of the channel n by $|n\rangle$, and expanding the state of the total system as

$$\Psi = \sum_n \psi_n |n\rangle, \quad (3.29)$$

we have a coupled channels problem defined by

$$(E - \epsilon_n - H)\psi_n = \sum_{n'} \langle n|V|n'\rangle\psi_{n'}, \quad (3.30)$$

where ϵ_n is the intrinsic energy for the channel n and K is the kinetic energy operator. Introducing the projection operators P and Q by

$$\begin{aligned} P &= |0\rangle\langle 0| \\ Q &= 1 - P = \sum_{n \neq 0} |n\rangle\langle n|. \end{aligned} \quad (3.31)$$

we get an effective single-channel problem for ψ_0 , i.e.

$$\{E - \epsilon_0 - K - U(E)\}\psi_0 = 0, \quad (3.32)$$

where the optical potential, U , is given by

$$U(E) = \langle 0|V|0\rangle + \langle 0|VQ \frac{1}{E - QHQ} QV|0\rangle \quad (3.33)$$

with

$$H = K + V + \sum_n \epsilon_n |n\rangle\langle n|. \quad (3.34)$$

U becomes complex for $E > E_n^* \equiv \min\{\epsilon_n, n \neq 0\}$, and satisfies a dispersion relation

$$U(E + i\eta) = \langle 0|V|0\rangle - \frac{1}{\pi} \int_{E_n^*}^{\infty} dE' \frac{\text{Im } U(E')}{E - E' + i\eta}, \quad (3.35)$$

where $\text{Im } U$ is due to the transitions to the Q -space, i.e.

$$\text{Im } U(E) = -\pi \langle 0|VQ\delta(E - QHQ)QV|0\rangle. \quad (3.36)$$

The full Green's function sandwiched between the intrinsic state $|0\rangle$ is given by the single channel Green's function with the optical potential, U , i.e.

$$G \equiv \langle 0| \frac{1}{E - H + i\eta} |0\rangle = \frac{1}{E - \epsilon_0 - K - U + i\eta} \quad (3.37)$$

and one clearly sees that the response function $S^1(S^1)$ defined by eq. (2.25) ((2.26)) represents the transition to the P -space (Q -space).

In the following, we will restrict ourselves to the simplest case of two channels where Q -space consists of one channel (1) and discuss the properties of the S -matrix and the Green's function in some detail.

In a two-channel case, the S -matrix and the Green's function are no longer single-valued functions of any single-channel momentum contrarily to the single-channel case. We therefore introduce two momenta, k_0 and k_1 , given by $k_0^2 = E - \epsilon_0$ and $k_1^2 = E - \epsilon_1$, respectively and prepare two Riemann sheets for complex k_0 planes. On these sheets we show various types of poles of the 0 channel component of the S -matrix (or equivalently of those of the Green's function) in the case of no coupling between 0 and 1 channels, i.e. $\langle 0|V|1\rangle = 0$ (fig. 3.12). The arrows specify the direction of motion of these poles with the gradual introduction of the coupling $\langle 0|V|1\rangle$. The sheet a corresponds to that of the single-channel case with an absorptive optical potential, while the locations of poles on the sheet b are obtained by reflecting those on the sheet a with respect to the imaginary k_0 axis since the S -matrix and the Green's function satisfy the symmetry relations

$$S(-k_0^*, -k_1^*) = S^*(k_0, k_1), \quad G^{(1)}(-k_0^*, -k_1^*; r, r') = G^{(0)*}(k_0, k_1; r, r'). \quad (3.38)$$

When there is no coupling, there exist further equalities

$$\begin{aligned} S_{00}(k_0, -k_1) &= S_{00}(k_0, k_1), \\ G_{00}^{(1)}(k_0, -k_1; r, r') &= G_{00}^{(0)}(k_0, k_1; r, r'), \end{aligned} \quad (3.39)$$

and

$$S_{00}(-k_0^*, k_1^*) = S_{00}^*(k_0, k_1), \quad G_{00}^{(1)}(-k_0^*, k_1^*; r, r') = G_{00}^{(0)*}(k_0, k_1; r, r'), \quad (3.40)$$

so that poles are located symmetrically with respect to the imaginary k_0 axis within each Riemann sheet. The motion of various types of poles on the sheet a is analogous to that in the single-channel case. The pole A in the figure moves from the positive imaginary axis to the second quadrant. The pole B (a resonance pole) moves away from the real axis in the fourth quadrant. The pole C (the partner of the resonance pole) moves in the third quadrant towards the real axis and eventually comes up to the second quadrant, which corresponds to the UBS in the two-channel case. We should note here that the imaginary part of k_1 is negative in the second quadrant of the sheet a as denoted in the fig. 3.12. It means that though the D -space component of the UBS wave function damps exponentially, its 1-space component diverges exponentially. Thus, these poles do not correspond to eigenstates of the total Hamiltonian. (In fact, this is obvious if one recalls the fact that eigenstates of the coupled-channels hermitian Hamiltonian are true bound states below the lowest threshold and scattering states in each channel.) The pole A , however, shows up as a peak in observables through the S -matrix or the Green's function and have an approximate meaning of states if the location of the pole is close to the imaginary axis. This is the same situation as for resonance poles in the single- and multi-channel cases. On the contrary, the UBS in the multi-channel case are neither eigenstates nor included in the discrete sum of the spectral decomposition of the Green's function. In the following, we will illustrate the above general consideration using a simple coupled-channels model. As a coupled-channels potential, we take real square-well potentials for the diagonal parts and a surface δ -function for the off-diagonal part,

Figure 3.12: A schematic illustration of the poles of the O-channel component of the S -matrix in the k_0 -plane for an attractive diagonal potential, $\langle 0|V|0\rangle$ and no coupling between two channels, i.e. $\langle 0|V|1\rangle = 0$. The arrows indicate how they move when the coupling is introduced. The dashed lines show the cuts and the numbers in parentheses indicate in which quadrant k_1 is located, e.g. (1) means $\text{Re } k_1 > 0$ and $\text{Im } k_1 > 0$.

$$\begin{aligned}\langle 0|V|0\rangle &= v_0\theta(R-r), \\ \langle 1|V|1\rangle &= v_1\theta(R-r), \\ \langle 0|V|1\rangle &= v_0, R\delta(r-R).\end{aligned}\quad (3.41)$$

In the O-space with the angular momentum l , this two-channels model is equivalent to the single-channel model with an effective Hamiltonian, H_{eff} ,

$$H_{eff} = \epsilon_0 + \hat{K}_0 + U(E; r). \quad (3.42)$$

U is the effective potential defined by

$$U(E; r) = v_0\theta(R-r) + v_0^2 R^4 G_1^{(l)}(E; R, R)\delta(r-R), \quad (3.43)$$

where G_1 is the uncoupled Green's function in the 1-space and can be formally written as

$$G_1 = \frac{1}{E - (1/H(1) + i\eta)}. \quad (3.44)$$

This consists of a simple example of explicit construction of the optical potential from a coupled-channels problem. The fact that the optical potential is local is due to the peculiarity of the δ -function in the off-diagonal part of the coupled-channels potential. From eqs. (2.21), (2.22) and (3.18) we get

$$S^l(E) = -v_0^2 R^6 \text{Im } G_1^{(l)}(E; R, R) |G_1^{(l)}(E; R, R)|^2. \quad (3.45)$$

We again choose the case of Σ -hypernuclei to be discussed in §4.2, and take the ΣY^{-1} state as the O-space and the ΛN^{-1} state as the 1-space. Though this would not be a realistic situation, this model serves as an illustrative example for our purpose. In order to roughly simulate the case of Σ^{-} in ^{12}C , we take $l \equiv 1$, $R = 3 \text{ fm}$, $\epsilon_0 - \epsilon_1 = 80 \text{ MeV}$, the reduced masses $\mu_0 = 1080 \text{ MeV}$ and $\mu_1 = 1014 \text{ MeV}$, while v_0 , v_1 and v_01 are varied so as to study several cases.

First, we give an example of a bound state. For the parameters $v_0 = -25 \text{ MeV}$, $v_1 = -25 \text{ MeV}$ and $v_01 = 5 \text{ MeV}$, there is a pole at $k_0 = (-12.6 + 63.6i) \text{ MeV}$ ($k_1 = (398.2 - 1.9i) \text{ MeV}$, $E - \epsilon_0 = (-1.8 + 0.7i) \text{ MeV}$) on the sheet a. Fig. 3.13 shows the response function S (which is equal to S^1 below the threshold of the O channel), where one can see a sharp

Figure 3.13: The response functions, S (solid curve) and S^1 (dashed curve), in arbitrary units for the potential parameters, $v_0 = -25 \text{ MeV}$, $v_1 = -25 \text{ MeV}$ and $v_01 = 5 \text{ MeV}$.

Figure 3.14: The positions of the poles of the O-channel component of the S -matrix in the k_0 -plane, which corresponds to the sheet a of fig. 12. v_0 and v_1 are fixed to be -16 MeV and v_01 is 0, 3, 8 and 9 MeV in the case a, b, c and d, respectively.

Figure 3.15: The response functions, S (solid curve) and S^I (dashed curve), in arbitrary units for the potential parameters, $v_0 = -16$ MeV, $v_1 = -16$ MeV and $v_{01} = 3$ MeV.

Figure 3.16: The same as fig. 3.15 for $v_{01} = 8$ MeV.

Figure 3.17: The same as fig. 3.15 for $v_{01} = 9$ MeV.

peak whose position and width also in this case correspond very well to the pole energy. Secondly, we examine the case of the UBS. Both v_0 and v_1 are fixed to be -16 MeV while v_{01} is varied from 0 to 9 MeV. The motion of poles in the k_0 -plane is shown in fig. 3.14 where only the sheet α is shown. For $v_{01} = 3, 8, 9$ MeV the response functions, S and S^I are shown in figs. 3.15 and 3.16, respectively. The case $v_{01} = 0$ is the same as that of $W = 0$ in sec. 3.1 which is an example of a resonance (fig. 3.8). As the coupling strength v_{01} increases the resonance pole moves down away from the positive real axis, while the partner pole moves up toward the negative real axis, as in the single-channel case, and crosses it between $v_{01} = 8$ and 9 MeV so that a UBS emerges for $v_{01} = 9$ MeV with the pole energy $E_0 - \epsilon = (6.48 - 0.01i)$ MeV. Also, the change of the response functions due to the increase of the coupling strength v_{01} is qualitatively similar to that in the single-channel case. That is, the peaks of the response functions S and S^I become lower and broader, as v_{01} increases, and the ratio S^I/S increases without any anomaly due to the emergence of the UBS. In view of the above, we conclude that the UBS never shows up in the formation process whether one employs the energy-independent optical potential (single-channel approach) or the energy-dependent optical potential (coupled-channels approach).

4 Applications

We will discuss here some applications of the Green's function method to the production and the knock-out processes of hadrons. We first discuss Σ -hypernuclei, secondly hadronic atoms and thirdly nuclear transparency in electron scattering.

4.1 Σ -hypernuclei

The possibility of narrow Σ -hypernuclear states were experimentally suggested in early 80's in the studies of in-flight $(\bar{K}, \pi)$ reactions [11, 12] and the stopped $\bar{K}^-$ captures [13]. Although the anomalously narrow peaks have not been confirmed in the later experiments [14, 15, 16], relatively narrow states may still exist and have been extensively studied both experimentally and theoretically.

A characteristic feature of Σ -hypernuclei is that they are always unstable with respect to the strong interaction due to the Λ -conversion process. The width of Σ in nuclear matter density, become as large as 80 MeV. Several mechanisms which may reduce the width have been proposed [17, 18, 8]. Another point we should keep in mind is that the structures in

the pion spectra reported in most of early experiments lie above the Σ -emission threshold. If these structures are to correspond to Σ -hypernuclear states, the probability of the produced Σ escaping from the nucleus has to be also small.

Recently, the possibility of light Σ -hypernuclei in the bound Σ region was theoretically pointed out in ref. [19, 20, 21]. The results of the succeeding experiments seem to support the above theoretical prediction [22, 23], even though there is still some objection about the above calculation [24], the theoretical interpretation [25] and the reported experimental results do not seem to be final.

Therefore, in order to understand the Σ -nucleus interaction theoretically, one must therefore take full account of the continuum nature of the produced states.

computational procedure

Here we concentrate on the production of Σ -hypernuclei by K^- capture in ^{12}C which was experimentally studied at KEK [13, 14, 15]. We assume that K^- is absorbed from the atomic 3d orbit which is known to give the main contribution [26]. We need the kaon atomic wave function, the pion distorted wave, the nucleon wave functions in addition to the Green's function for Σ in order to obtain the response functions.

Both for the kaon and the pion, we use optical model wave functions which satisfy the equation of the form

$$\left[\left(1 + \frac{\omega}{M_T} \right) \nabla^2 + (\omega - V_C(r))^2 - \mu^2 \right] \chi(r) = 2\omega V(r) \chi(r), \quad (4.1)$$

where V_C and V denote, respectively, the Coulomb and the strong interaction. Since no information on the pion optical potential with the Σ -hypernucleus as a target is available, we use the same strong interaction potential and the same shape of the charge density as those for the target nucleus, ^{12}C . (The strength of the Coulomb potential is normalized to the total charge of the Σ -hypernucleus.) The explicit form of the strong potential is taken as

$$2\omega V(r) = -4\pi(1 + \eta)b_0\rho(r) + 4\pi(1 + \eta)^{-1}c_0 \left(\nabla\rho(r) \nabla - \frac{1}{2}\eta^2 \nabla^2 \rho(r) \right), \quad (4.2)$$

where $\eta = \omega/M_T$ and b_0 is the sum of proton and neutron densities. Parameters b and c are taken from ref. [26] for the kaon and ref. [3] for the pion.

We obtain nucleon-hole wave functions by solving the Schrödinger equation with the potential

$$U^N(r) = V_C^N f(r) + V_{O,1}^N \cdot sr_0^2 \frac{1}{r} \frac{d}{dr} f(r), \quad (4.3)$$

where

$$f(r) = \left[1 + \exp \left(\frac{r-R}{a} \right) \right]^{-1}. \quad (4.4)$$

Parameters r_0 , a and R are taken from ref. [27] while V_C^N and $V_{O,1}^N$ are determined to be -61 MeV and 27 MeV, respectively, so as to reproduce observed separation energies for nucleons in $p_{1/2}$ and $p_{3/2}$ orbits.

In the calculation of the Green's function for Σ , the following assumptions are made.

- (i) The coupling between different core states is not so important. Therefore, the Green's function as well as the optical potential is diagonal.
- (ii) The ground state dominates the sum in eq. (2.9).

Indeed, according to the shell model calculation by Cohen and Kurath [29], the probability of obtaining the ground state of ^{11}B by picking up a nucleon from the orbit, $p_{3/2}$, is almost one order greater than other states.

As a result of the assumptions (i) and (ii), the problem is reduced to a one-body problem.

- (iii) The optical potential has the same form as that for the nucleon with a complex strength for the central part:

$$U^{\Sigma}(r) = (V_C^{\Sigma} + iW_C^{\Sigma})f(r) + V_{O,1}^{\Sigma} \cdot sr_0^2 \frac{1}{r} \frac{d}{dr} f(r) \quad (4.5)$$

Table 4.1: Potential parameters of U^{Σ} in the cases I - VII

Case	V_C^{Σ} (MeV)	W_C^{Σ} (MeV)	V_O^{Σ} (MeV)
I	0	0	0
II	-10	0	0
III	-10	-3	0
IV	-10	-6	0
V	-10	-6	27
VI	-10	-6	54
VII	-10	-9	0

We also use the same shape parameter, f , whereas the potential strengths V_C^{Σ} , W_C^{Σ} and V_O^{Σ} are treated as free parameters.

results and discussions

In ref. [26] the sensitivity of the Σ -production process to the Σ -optical potential parameters was studied. The parameters used are summarized in table 4.1. The case I assumes no strong interaction between Σ and the nucleus and corresponds to the purely quasi-free production of Σ . A real central potential is assumed in the case II and an absorptive potential is added in the case III, which is increased by a factor of 2 in the case IV. Moreover, the spin-orbit term with the same strength as that for a nucleon is included in the case V while the strength is increased by a factor of 2 in the case VI. The parameters for the case VII are chosen so as to reproduce the observed shift and width of the Σ^- -atom level for the present potential shape without spin-orbit term.

The calculated response functions S and S' are shown in fig. 4.1 (a)-(f). S for the case I is shown in all cases for comparison. Just below the threshold (-1 MeV $< E < 0$) there exist infinitely many narrow structures of Σ^- -atom states, whose integrated strength, however, is very small. In fig. 4.1 the response functions which have been smeared in the Σ^- -atom region are shown. Response functions in the Σ^- -atom region will be discussed in detail in sec. 4.2.

One sees that the response functions around the threshold (-5 MeV $< E < 5$ MeV) are very sensitive to the optical potential, U^{Σ} , whereas they become insensitive at high energies ($E > 10$ MeV).

In the case II a bound state in a partial wave s and a sharp resonance in p just above the threshold are seen. One also sees that the ratio S'/S increases notably as absorption increases. This ratio depends also on the energy of Σ^- , as one goes to higher energies the Σ^- -emission process becomes more important. However, for a reasonable strength of W_C^{Σ} the Δ -conversion process still contributes significantly at rather high energies ($E \sim 20$ MeV).

When a strong spin-orbit interaction is included, clear peaks are seen even for rather strong absorption; in the case V a bound state in $p_{3/2}$ is seen and in the case VI a sharp resonance in $d_{3/2}$ and a broad bump in $f_{7/2}$ are additionally seen. In general, a peak with higher angular momentum has a smaller width than that with lower angular momentum, if their energies are the same. This is because a higher partial wave has a higher angular momentum barrier.

The partial wave decomposition of S and S' for cases IV, V and VI is shown in fig. 4.2 (a)-(c). In the low energy region the contributions from low partial waves are dominant; below 5 MeV the contributions from s , p and d explain the major parts of the spectra. These contributions from low partial waves become smaller while higher partial waves become important as the energy becomes higher. As a result, the spectra become quasi-free like. One also sees that the Σ^- -emission becomes more important for higher partial waves, which explains why the structures of S' are less clear than those of S (or S').

In ref. [14] a systematic analysis of the experimental spectra was performed by comparing them with the calculation by using the Green's function method where the above Woods-Saxon form of the optical potential is assumed. Fig. 4.3 shows the case with $V_C^{\Sigma} = 0$, $W_C^{\Sigma} = -9$ MeV and $V_O^{\Sigma} = 0$, which is the best fit within the calculated parameter sets of potential strengths. The best fit parameters are considerably different from those obtained by the analysis of the Σ^- -atom, the case VII above.

Figure 4.1: The response functions, S (solid curves) and S' (dashed curves), for the cases II (a), III (b), IV (c), V (d), VI (e) and VII (f). S for the case I is shown in all figures by dotted curves.

Figure 4.2: Partial wave decompositions of S (solid curves) and S' (dashed curves) for the cases IV (a), V (b) and VI (c).

Figure 4.3: Observed cross sections and the calculated ones with the best fit parameters.

In ref. [30, 31, 32, 33, 34] the Σ -nucleus optical potential is constructed by folding the ΣN effective interaction, which is obtained through the G -matrix calculation. The obtained optical potential has the imaginary part which is roughly consistent with the above (-9 MeV), while its real part has a shape very different from the Woods-Saxon form with an interior repulsion and an exterior attraction. It is found that such an optical potential reproduces both the Σ -atom and Σ -hypernucleus data [30, 31].

examination of the bound state approximation

The bound state approximation has been frequently used in estimating the production rates of unstable states [36, 37, 38]. In this approximation, one takes the optical potential deep enough to make the unstable states bound and calculates the production rates by using the bound-state wave functions. Then, they are compared with experimentally observed strengths which are obtained by integrating the continuous spectrum over the peak energy region with the assumed background subtracted. We will examine here the accuracy of the bound-state approximation using the Green's function method.

Let us start with simple consideration. A naive decomposition of the Green's function into the pole term, G_{pole} , and the background, G_b , would be

$$G^{(l)}(E; r, r') = G_{pole}^{(l)}(E; r, r') + G_b^{(l)}(E; r, r') \quad (4.6)$$

with

$$G_{pole}^{(l)}(E; r, r') = \frac{1}{E - E_{\pi} + i\eta} \text{res } G^{(l)}(E; r, r')|_{E=E_{\pi}}, \quad (4.7)$$

where only a pole term at the energy $E = E_{\pi}$ is taken into account. For a bound-state pole, the pole residue of the Green's function is given by

$$\text{res } G^{(l)}(E; r, r')|_{E=E_{\pi}} = \varphi_{\pi l}(r) \varphi_{\pi l}(r') \quad (4.8)$$

with $\varphi_{\pi l}(r)$ denoting the radial part of the normalized bound-state wave function. If the optical potential is local, the residue of a resonance pole can be written in the same form as in eq. (4.8) with $\varphi_{\pi l}(r)$ defined by

$$\varphi_{\pi l}(r) = [i k_{\pi}^2 \text{res } S_l(k)]_k = k_{\pi l}^{1/2} v_l^{(+)}(k_{\pi}, r), \quad (4.9)$$

where k_{π} is the pole momentum, S_l is the S -matrix and $v_l^{(+)}$ is the outgoing solution. We can then take out the pole contribution in the response function, S_l , as

$$S_{pole}(E) = -\frac{1}{\pi} \text{Im} \int dr dr' r^2 r'^2 f^{(l)*}(r) G_{pole}^{(l)}(E; r, r') f^{(l)}(r'). \quad (4.10)$$

Let us define the strength of the pole contribution by

$$A_{\pi l} = \int dr dr' r^2 r'^2 \text{res } G^{(l)}(E; r, r')|_{E=E_{\pi}} f^{(l)*}(r') f^{(l)}(r). \quad (4.11)$$

For a bound-state pole with a real optical potential, we find

$$A_{\pi l} = \left| \int dr r^2 \varphi_{\pi l}(r) f^{(l)}(r) \right|^2 \quad (4.12)$$

by substituting eq. (4.8) into eq. (4.11). For a resonance pole and/or a bound-state pole with a complex potential, the integration of eq. (4.10) gives

$$\begin{aligned} \int_{-\Gamma/2}^{+\Gamma/2} dE S_{pole}(E) &= 0.5 \text{ Re } A_{\pi l}, \\ \int_{-\Gamma/2}^{+\Gamma/2} dE S_{pole}(E) &= 0.70 \text{ Re } A_{\pi l}, \\ \int_{-\infty}^{\infty} dE S_{pole}(E) &= \text{Re } A_{\pi l}, \end{aligned} \quad (4.13)$$

where $E_{\pi} = \epsilon - \frac{1}{2}i\Gamma$. This means that even if one can successfully subtract the background, the strength in the range Γ (2%) around the peak is only 50 (70) % of that in the bound-state approximation under the condition that $\text{Re } A_{\pi l}$ is a slowly-changing function of the potential strength.

In the actual case, however, eqs. (4.6) and (4.7) do not necessarily give a reasonable decomposition, especially for resonance just above the threshold. In addition, $\text{Re } A_{\pi l}$ might depend considerably on whether the pole corresponds to a bound state or a resonance state. We next examine these points for a simple case.

We take a Woods-Saxon potential for the optical potential, U ,

$$U(r) = (V + iW) \left[1 + \exp\left(\frac{r-R}{a}\right) \right]^{-1} \quad (4.14)$$

with fixed parameter $R = 2.3$ fm, $a = 0.4$ fm and reduced mass $\mu = 1080$ MeV which roughly corresponds to Σ in ^{12}C . For the production amplitude we adopt here the simplified treatment in ref. [9], namely we assume that

$$f_{\sigma}(r) = \sqrt{\frac{1}{R^2}} \delta(r - R) Y_m^{(l)}(\hat{r}). \quad (4.15)$$

We first study how the bound-state strength in a partial wave s depends on the absorption strength W . Table 4.2 shows pole energies for several values of W with V fixed to be -30 MeV and fig. 4.4 shows $\text{Re } A_{\pi l}$ as a function of W . We find that $\text{Re } A_{\pi l}$ is almost constant as a function of W . For two values of W , i.e. -1 and -3 MeV, the calculated response functions are shown in fig. 4.5. Resulting curves have almost Lorentzian forms and the integrated strengths for $W = -1$ MeV are

$$\begin{aligned} \int_{-9.7}^{-11.1} dE S_{pole}(E) &= 0.106, \\ \int_{-9.0}^{-11.8} dE S_{pole}(E) &= 0.148, \end{aligned} \quad (4.16)$$

and for $W = -3$ MeV:

$$\begin{aligned} \int_{-12.4}^{-8.4} dE S_{pole}(E) &= 0.106, \\ \int_{-6.4}^{-14.4} dE S_{pole}(E) &= 0.148, \end{aligned} \quad (4.17)$$

in the same unit as that for $\text{Re } A_{\pi l}$. These values correspond very well to the relation, eq. (4.13), which is consistent with the fact that there exists almost no background below the threshold.

Table 4.2: The relation between W and the pole energy ($V = -30$ MeV)

W (MeV)	E_{pole} (MeV)
0	-10.4
-1	-10.4 - 0.7i
-2	-10.4 - 1.3i
-3	-10.4 - 2.0i

Table 4.3: The relation between V and the pole energy ($W = 0$)

V (MeV)	E_{pole} (MeV)
-40	-2.0
-35	-0.2
-30	0.9 - 0.4i
-25	1.8 - 1.5i

Figure 4.4: The real part of the pole residue for $1s$, $\text{Re } A_{1s}$, as a function of W with $V = -30$ MeV.

Figure 4.5: The response function, S , for s -wave. The solid curve is for $V = -30$ MeV and $W = -1$ MeV and the dashed curve is for $V = -30$ MeV and $W = -3$ MeV.

Figure 4.6: The real part of the pole residue for $1p$, $\text{Re } A_{1p}$, as a function of V with $W = 0$.

Figure 4.7: The response function, S , for p -wave. The solid curve is for $V = -40$ MeV and $W = 0$, the dashed curve is for $V = -30$ MeV and $W = 0$ and the dotted curve is for $V = -25$ MeV and $W = 0$.

We next investigate the change when a bound state becomes a resonance in a partial wave p by making potential strength weaker. Table 4.3 shows pole energies for several values of V with $W = 0$ and fig. 4.6 shows $\text{Re } A_{1p}$ as a function of V . $\text{Re } A_{1p}$ decreases rapidly just before the bound state becomes a resonance so that $\text{Re } A_{1p}$ for $V = -25$ MeV is only about 40% of that for $V = -40$ MeV. For $V = -40, -30$ and -25 MeV, the calculated response functions are shown in fig. 4.7. Also in these cases the obtained strength in the resonance region can be roughly decomposed into a peak plus background. The shape of the peak, however, is considerably different from Lorentzian. If one would obtain the strength shown by dashed and dotted curves in fig. 4.7 by experiment, what values one would extract from these "experiments" as the resonance strength depends entirely on what background one assumes. Integrating the strength in the peak region without subtracting any background, one gets for $V = -30$ MeV:

$$\int_0^2 dE S_{\text{pole}}(E) = 0.115, \quad (4.18)$$

and for $V = -25$ MeV:

$$\int_0^3 dE S_{\text{pole}}(E) = 0.148, \quad (4.19)$$

in the same unit as that for $\text{Re } A_{1p}$. Judging from these values and the usual way of background subtraction, we feel that the experimentally obtained strength would be a little smaller than the theoretical value $\text{Re } A_{1p}$ for a resonance and much smaller than that obtained by the bound state approximation. Indeed, it is at least 20-30 % smaller than that of the bound state just below the threshold.

4.2 Hadronic atoms

4.2.1 Σ^- -atoms

In ordinary hadronic atoms the strong interaction is usually regarded as perturbation to the Coulomb interaction. This is true if the Bohr radius of the atomic state is much larger than the nuclear radius. For heavy hadrons and/or heavy nuclei, however, the Bohr radius is comparable to the nuclear radius and the interplay of the strong and the Coulomb interaction can play a significant role. We discuss here some examples of such phenomena, which are found in the theoretical study of Σ^- -atom.

The first example is *interference patterns* of atomic states. Fig. 4.8 shows the response function in the Σ^- -atom region for the case IV in sec. 4.1 where partial wave contributions from s and p are also shown. Energy eigenvalues for the pure Coulomb interaction with Σ^- and the core nucleus treated as point charges are given by

$$E_n = -\frac{0.71}{n^2} \text{ MeV}. \quad (4.20)$$

In partial waves, s and p , one sees structures corresponding to the energy levels given by eq. (4.20) except for $1s$ which seems too broad to be observed. However, they are very different from usual peaks. The shapes of structures in s are "upside-down peaks". Though the structures in p apparently look like usual peaks, their widths are much larger than expected from the complex eigenvalues which are obtained by taking account of the finite size effect of Σ^- and the core nucleus and the strong interaction. (For instance, the $2p$ state has a complex eigenvalue $(-0.237 - 0.037i)$ MeV.)

This can be understood as follows. From eqs. (4.7), (4.10) and (4.11) the pole term of the response function can be written as

$$\begin{aligned} S_{\text{pole}}(E) &= -\frac{1}{\pi} \text{Im} \frac{A_n}{E - E_n} \\ &= \frac{1}{\pi} \left\{ \text{Re } A_n \frac{\frac{1}{2}\Gamma}{(E - \epsilon)^2 + \frac{1}{4}\Gamma^2} - \text{Im } A_n \frac{E - \epsilon}{\frac{1}{4}\Gamma^2} \right\}. \end{aligned} \quad (4.21)$$

For a real optical potential U , A_n is real and positive as is seen from eq. (4.12). As U acquires a negative imaginary part, A_n acquires a phase. Fig. 4.9 illustrates how S_{pole} behaves in three typical cases, i.e. $\arg A_n = 0$ (a), $\frac{1}{2}\pi$ (b) and π (c). Whereas all structures studied in sec. 4.1 are similar to (a), Σ^- -atom structures in p (s) are similar to (b) ((c)). Under this correspondence widths of structures coincide well with the complex eigenvalues.

From the above observation, it turned out that even "atomic" structures could be drastically modified by the effect of the strong absorption. This seems to imply that special attention is required to extract shifts and widths of Σ^- -atoms

Figure 4.8: The response function, S , for the case IV in sec. 4.1 ($V_0^Z = -10$ MeV, $W_0^Z = -6$ MeV and $V_{SO}^Z = 0$) in the Σ^- -atom region and its partial wave decomposition into s and p -waves.

Figure 4.9: S_{pole} for $\text{Arg} A_{\nu l} = 0$ (a), $\text{Arg} A_{\nu l} = \frac{1}{2}\pi$ (b) and $\text{Arg} A_{\nu l} = \pi$.

Figure 4.10: An example of the atomic and nuclear hybrid state in the case of ^{40}Ar .

from the observed spectra, though the case of existing data might be safer because of their higher angular momenta ($l \geq 2$) [35].

The second example is *hybrids* of atomic and nuclear states. For hadronic atoms of light hadrons and light nuclei the energy scale of the binding energy is of the order of keV, while the energy scale of that for nuclear and hypernuclear states is of the order of MeV. Therefore, atomic states and nuclear states are usually well separated. This is no longer true for heavy hadrons and/or heavy nuclei. For instance, in the case of Σ^- and Ca the binding energy of the atomic $1s$ state with the nucleus treated as a point charge is of the order of 10 MeV. This is comparable to the energy scale of nuclear states, which is even more so if the strong potential for Σ is much shallower than that for nucleon as is suggested by the analysis of recent experimental data [26–29]. If this is the case, it might happen that some eigenstates of the total Hamiltonian cannot be classified as either atomic states or nuclear states: they are *hybrids* of atomic and nuclear states. This possibility was first pointed out in ref. [39]. Fig. 4.10 shows such an example with Si as a nucleus.

If these hybrids exist, their widths are smaller than ordinary Σ -hypernuclear states and still their production rates are larger than ordinary Σ^- atomic states, because their overlap to the nuclear bound states for nucleon is smaller than that of nuclear states and larger than that of atomic states. Thus, even if the Λ -conversion width is so large that Σ -hypernuclear states cannot be observed, there is still a chance of observing hybrid states.

4.2.2 Deeply bound pionic atoms

Deeply bound pionic atoms are atomic states of pions which are bound so deeply that they cannot be produced in the standard method. In the standard method slow negative pions are injected into matter, which eventually get trapped in outermost atomic orbits and lose energy by emitting Auger electrons and then x-rays to cascade down to deeper atomic levels. As the pion cascades down, however, the pion absorption starts to compete with and eventually dominates the x-ray emission. Thus, this method cannot produce deeply bound pionic atoms in heavy nuclei. This does not mean, however, that deeply bound pionic atoms do not exist. In fact, standard pion optical potentials predict the width of the $1s$ level of pionic atoms in heavy nuclei to be of the order of 1 MeV, which is appreciably smaller than the level spacing. In view of the above, many theoretical [40, 41, 42, 43] and experimental attempts [44, 45, 46] are made concerning deeply bound pionic atoms. Here, we discuss the application of the Green's function method to the calculation of the production cross section of deeply bound pionic atoms.

We first discuss the introduction of the electro-magnetic interaction to the Green's function method. Let us start with

an effective Lagrangian density for the Klein-Gordon equation,

$$\mathcal{L}^{\pi/l} = \varphi^\dagger(x) \left\{ -\partial_\mu \partial^\mu - m^2 - 2mU \right\} \varphi(x), \quad (4.22)$$

where U is the optical potential for the pion and has the form,

$$2mU(\mathbf{x}) = S(\mathbf{x}) - \nabla P(\mathbf{x}) \nabla. \quad (4.23)$$

By introducing the electro-magnetic field by minimal substitution, $\partial_\mu \rightarrow \partial_\mu - ieA_\mu$, we obtain the total Lagrangian density which includes both the strong and electro-magnetic interactions:

$$\begin{aligned} \mathcal{L}^{\pi/l} = & \varphi^\dagger(x) \left[-(\partial^\mu - ieA^\mu)(\partial_\mu - ieA_\mu) - m^2 \right. \\ & \left. - \{S - (\nabla - ieA)P(\nabla - ieA)\} \right] \varphi(x). \end{aligned} \quad (4.24)$$

Then the effective Hamiltonian is given by

$$H^{\pi/l} = H_{op}^{\pi/l} + H_{int}^{\pi/l}, \quad (4.25)$$

where

$$H_{op}^{\pi/l} = \int d^3x \varphi^\dagger(x) \left\{ -\partial_0^2 - \nabla^2 + m^2 + 2mU + e^2 A_0^2 \right\} \varphi(x) \quad (4.26)$$

and

$$H_{int}^{\pi/l} = \int d^3x (1 - P) \left\{ -ieA (\varphi \nabla \varphi^\dagger - \varphi^\dagger \nabla \varphi) + e^2 A^2 \varphi^\dagger \varphi \right\}. \quad (4.27)$$

The Green's function for a pion interacting with a nucleus by the strong and the Coulomb interaction is expressed in terms of $H_{op}^{\pi/l}$ as

$$G(\omega; \mathbf{x}, \mathbf{x}') = \langle 0 | \varphi(\mathbf{x}) \frac{1}{\omega - H_{op}^{\pi/l} + i\eta} \varphi^\dagger(\mathbf{x}') | 0 \rangle - \langle 0 | \varphi^\dagger(\mathbf{x}') \frac{1}{\omega + H_{op}^{\pi/l} - i\eta} \varphi(\mathbf{x}) | 0 \rangle. \quad (4.28)$$

Before presenting the results of the detailed calculation for the production of deeply bound pionic atoms by means of a particular reaction, we consider the response function, S , defined by eq. (2.17) for a simple choice of the function, F ,

$$F(\mathbf{r}) = \sqrt{\pi} \delta(\mathbf{r} - R) Y^{(l)}(i), \quad (4.29)$$

in order to study the relative importance of the pole term and the background term. Physically, the choice of eq. (4.29) means that the pion is produced at a distance R from the center of the nucleus in a partial wave l . The response function is then expressed simply in terms of the imaginary part of the diagonal matrix element of the Green's function in the coordinate space representation:

$$S(\omega) = -R^2 \text{Im} G^{(l)}(\omega; R, R). \quad (4.30)$$

In fig. 4.11 we plot $S(\omega)$ as a function of ω for $R = 2, 4, 6, 8$ fm and $l = 0, 1$ in the case of Pb as a nucleus. Note that the nuclear radius of Pb is 6.51 fm.

Fig. 4.11 clearly shows that inside the nucleus the continuous background is dominant, while around the nuclear surface and outside the nucleus the atonic peaks give most of the strength below the pion threshold. There is no qualitative difference between partial waves $l = 0$ and $l = 1$.

application to $(\pi^- \gamma)$ reaction

As a theoretically simple example of the processes producing the deeply bound pionic atoms, we choose here the (π^-, γ) reaction proposed in ref. [47].

Figure 4.11: The response functions, S , for $R = 2$ fm (dotted curve), 4 fm (dash-dotted curve), 6 fm (dashed curve) and 8 fm (solid curve) in the case of s-wave (a) and p-wave (b).

The inclusive double differential cross section for the $(\pi^-\gamma)$ reaction is given in the lowest order of the electromagnetic coupling constant, e , by

$$\begin{aligned}\frac{d^2\sigma}{d\Omega dk_\gamma} &= \sum_f \delta(\omega_f + E_i - k_\gamma - E_f) \left(\frac{d\sigma}{d\Omega} \right)_{f, k_\gamma, k_*} \\ &= \sum_f \delta(\omega_f + E_i - k_\gamma - E_f) \sum_\lambda \frac{k_\lambda^2}{(2\pi)^3 v_\gamma} \left| \langle f, k_\lambda | H_{int} | i, k_\lambda \rangle \right|^2 \\ &= -\frac{1}{4\pi^3 v_\gamma} \text{Im} \sum_\lambda \langle i, k_\lambda | \mathcal{O}^\dagger \frac{1}{E_i - k_\gamma - H_0 + i\eta} \mathcal{O} | i, k_\lambda \rangle.\end{aligned}\quad (4.31)$$

H_0 is the Hamiltonian for the pion-nucleus system including the Coulomb interaction, H_{int} is the interaction Hamiltonian which describes the electromagnetic interaction between the pion-nucleus system and the transverse photon,

$$\mathcal{O}_\lambda = \int d^3x \frac{\delta H_{int}}{\delta A(\mathbf{x})} \Big|_{A=0} \epsilon_\lambda e^{i\mathbf{k}_\gamma \cdot \mathbf{x}} \equiv \int d^3x j(\mathbf{x})_\lambda e^{i\mathbf{k}_\gamma \cdot \mathbf{x}} \quad (4.32)$$

and ϵ_λ is the polarization vector. In terms of the effective Hamiltonian the cross section is written as

$$\frac{d^2\sigma}{d\Omega dk_\gamma} = -\frac{1}{4\pi^3 v_\gamma} \text{Im} \sum_\lambda F_\lambda(\mathbf{k}_\gamma, \mathbf{k}_* | \mathbf{x}) G(\omega_f + E_i - k_\gamma, \mathbf{x}) F_\lambda(\mathbf{k}_\gamma, \mathbf{k}_* | \mathbf{x}'), \quad (4.33)$$

where the Green's function, G , is given by eq. (4.28) in terms of the effective Hamiltonian and

$$F_\lambda(\mathbf{k}_\gamma, \mathbf{k}_* | \mathbf{x}) = \langle i, \mathbf{x} | \frac{\delta H_{eff}}{\delta A(\mathbf{x})} \Big|_{A=0} \epsilon_\lambda e^{i\mathbf{k}_* \cdot \mathbf{x}} | i, \mathbf{k}_* \rangle, \quad (4.34)$$

Results and discussions

Figs. 4.12 and 4.13 show our final results for the γ -ray spectra from $^{208}\text{Pb}(\pi^-\gamma)$ near the threshold of π^- emission calculated with the optical potentials of refs. [48] and [5], respectively. The incident pion kinetic energy is 10 MeV and the two angles, $\theta_\gamma = 45^\circ$ and 60° are chosen to favor $2p$ and $1s$ transitions, respectively. Peaks corresponding to π^- atomic states are clearly seen in the spectra. The partial wave decomposition indicates that s , p and d -waves dominate the spectra. The two optical potentials give essentially the same results. In the following, we will therefore show only the results with the optical potential of ref. [48], which is chosen to facilitate the comparison between the results of the present work and those of ref. [47].

One of new features of the present calculation is the use of the electromagnetic current including the contribution of the non-local term in the optical potential (i.e. the factor $(1-P)$ in H_{int}^{eff} of eq. (4.27)). Fig. 4.14 shows the spectra calculated with the impulse current which does not include this contribution. The other ingredients in the calculation are the same as those for fig. 4.12. One sees that the use of the full current hardly changes the peak components of the spectra but largely reduces the smooth components. This can be understood by the factor $(1-P)$ shown in fig. 4.15 which suppresses the contribution from the internal region, and the behavior of $\text{Im } G(\omega; R, R)$ shown in fig. 4.11, which indicates that the background spectra are very important in the nuclear interior.

In order to extract the energies and the widths of the atomic states and also to make a detailed comparison with the results of the bound state approximation used in ref. [47], we have analyzed the peaks corresponding to $1s$ and $2p$ states by the expression given by

$$\begin{aligned}\frac{d^2\sigma}{d\Omega dk_\gamma} &= \text{Im} \left\{ -\frac{1}{\pi} \frac{A_{nl}}{E - (\epsilon_{nl} - i\frac{\Gamma_{nl}}{2})} + B_{nl} \right\} \\ &= |A_{nl}| \cos \theta_{nl} \frac{1}{\pi} \frac{\frac{\Gamma_{nl}}{2}}{(E - \epsilon_{nl})^2 + \frac{\Gamma_{nl}^2}{4}} - |A_{nl}| \sin \theta_{nl} \frac{1}{\pi} \frac{E - \epsilon_{nl}}{(E - \epsilon_{nl})^2 + \frac{\Gamma_{nl}^2}{4}} + \text{Im } B_{nl},\end{aligned}\quad (4.35)$$

where $E = \omega_\pi + E_i - k_\gamma - m_\pi$. In addition to the usual Breit-Wigner form, we have considered the background term, B_{nl} , and the interference described by the phase, θ_{nl} . The impulse current was used in ref. [47] and therefore we have studied both the case with the full current and that with the impulse current. Tables 4.4 and 4.5 show the results of such analyses for $1s$ and $2p$, respectively, and fig. 4.16 shows the fit to the $1s$ peak for $\theta_{nl} = 60^\circ$. The extracted energies

Figure 4.12: The double differential cross section for the $(\pi^-\gamma)$ reaction for the optical potential of ref. [47] with the full current for $\theta_\gamma = 45^\circ$ (a) and $\theta_\gamma = 60^\circ$ (b).

Figure 4.13: The same as fig. 4.12 for the optical potential of ref. [5]

Figure 4.14: The double differential cross section for the (π^- , γ) reaction for the optical potential of ref. [37] calculated with the impulse current for $\theta_1 = 45^\circ$ (a) and $\theta_1 = 60^\circ$ (b).

Figure 4.15: The plot of $1 - P(R)$ as a function of R . The real and imaginary parts are shown by the solid and the dashed curves, respectively.

are in good agreement with those of ref. [47], while the extracted widths are larger than those of ref. [47] by about 25 % for 1s and 35 % for 2p. This might be due to the difference in the determination of the energies and widths. The third entry in the tables, $|A_{n1}|$, is the magnitude of the cross section for the peaks. The values for the case c, ref. [47], are obtained from the peak values of the cross section and their widths. The quantity, $2|A_{n1}|/\pi\Gamma_{n1}$, shown in the table is to be compared with the peak value of the cross section given in ref. [47]. A direct comparison can only be made for the case of the impulse current. The difference of the present results and those of ref. [47] stay within at most 35 %. The use of the full current enhances the calculated cross section by about 10 %. The last entry of the table gives the signal-to-noise ratios of the peaks and one clearly sees how the background parts are suppressed by the use of the full current.

The main difficulty in the experimental observation of the atomic states in this reaction is the subtraction of backgrounds in detecting γ -rays. They will be greatly reduced if the emitted π^- 's are also detected in coincidence with the γ -rays in the spectra above the threshold. A merit of using the Green's function method is that it enables us to decompose the spectra above the threshold into the π^- emission part and the remaining (absorption) part. The actual calculation of the spectra above the threshold is technically much more difficult than those below it due to the long-range contribution

Table 4.4: The results of the fit by eq. (4.35) for 1s. A and B are the cases of the full and the impulse current, respectively, and C is the case of ref. [47].

	$\theta_\gamma = 45^\circ$			$\theta_\gamma = 60^\circ$		
	A	B	C	A	B	C
$\epsilon_n(\text{MeV})$	-6.68	-6.68	-6.8	-6.68	-6.68	-6.8
$\Gamma_n(\text{MeV})$	0.52	0.52	0.41	0.52	0.52	0.41
$ A_{n1} (\mu\text{b/sr})$	25.0	22.7	21.4	31.0	28.2	24.1
$\theta_{n1}(^\circ)$	2.6	9.8	—	2.0	8.9	—
$\text{Im} B_{n1}(\mu\text{b/sr/MeV})$	0.04	1.90	—	0.03	2.40	—
$\frac{2 A_{n1} }{\pi\Gamma_n}(\mu\text{b/sr/MeV})$	30.8	28.0	33.4	38.3	34.8	37.6
$\frac{2 A_{n1} }{\pi\Gamma_n\text{Im}B_{n1}}$	776	30	—	1470	15	—

	$\theta_\gamma = 45^\circ$			$\theta_\gamma = 60^\circ$		
	A	B	C	A	B	C
$\epsilon_n(\text{MeV})$	-4.98	-4.98	-5.1	-4.98	-4.98	-5.1
$\Gamma_n(\text{MeV})$	0.37	0.37	0.27	0.37	0.37	0.27
$ A_{n1} (\mu\text{b/sr})$	24.6	23.8	27.0	37.4	33.1	24.4
$\theta_{n1}(^\circ)$	-2.7	-6.4	—	0.5	2.4	—
$\text{Im} B_{n1}(\mu\text{b/sr/MeV})$	1.10	6.84	—	0.80	7.59	—
$\frac{2 A_{n1} }{\pi\Gamma_n}(\mu\text{b/sr/MeV})$	42.6	41.2	62.5	64.6	57.1	51.8
$\frac{2 A_{n1} }{\pi\Gamma_n\text{Im}B_{n1}}$	39	6	—	81	8	—

Table 4.5: The same as table 4.4 for 2p.

Figure 4.16: An example of the fit of the double differential cross section (solid curve) by the formula given by eq. (4.35) (dashed curve).

Figure 4.17: The double differential cross section for the (π^-, γ) reaction for the optical potential of ref. [47] smeared with the energy resolution of 0.5 MeV.

of the bremsstrahlung in the Coulomb field. The calculation is under way and the results will be reported in the near future [49].

We have to take account of an experimental energy resolution in comparing the calculated spectra with the observed ones. Fig. 4.17 shows the calculated spectra smeared with the energy resolution of 0.5 MeV. Most of the fine structures close to the threshold disappear and what remain to be observed are the 2p peak and a structure corresponding to the sum of 3p and 3d peaks. An experiment is being carried out in TRIUMF, Vancouver and we will see in near future whether the background subtraction can be performed well enough to observe the expected structures in the spectra.

4.3 Nuclear transparency in electron scattering

The nuclear response to electron scattering in the quasi-elastic region at large momentum transfers has recently attracted the attention of many researchers in connection with a speculated phenomenon of nuclear or color transparency [50, 51]. It was predicted about a decade ago that the nuclear medium would be transparent for a proton involved in a large momentum transfer semi-inclusive process such as $(e, e'p)$ or $(p, 2p)$ on a nuclear target, since such a process would be dominated by a point-like configuration of the proton which was expected to interact weakly with the other nucleons.

The transparency for electron scattering can be defined as the ratio of the $(e, e'p)$ response to the inclusive (e, e') response in the quasi-elastic region and is conveniently formulated by the Green's function method. We will give here a way of calculating the response function and the transparency based on the Glauber approximation [56] for the Green's function, without taking account of the proton internal structure. This will serve as a reference for observing the color transparency due to the internal dynamics in the $(e, e'p)$ experiments.

The differential cross section for (e, e') inclusive process on a target nucleus, A , can be expressed as

$$\frac{d\sigma_{eA}}{dE_e d\Omega'} = Z \left\langle \frac{d\sigma_{ep}}{d\Omega} \right\rangle S(\omega, \mathbf{q}), \quad (4.36)$$

where E_e is the energy of the scattered electron, $\mathbf{q}$ and ω are the transferred momentum and energy, respectively, $\langle \rangle$ implies the average over spins and isospins as well as the Fermi averaging and the difference between the longitudinal

and the transverse responses is neglected. The response function, S , is then given by

$$\begin{aligned} S(\omega, \mathbf{q}) &= \sum_X |\langle X | O(\mathbf{q}) | A \rangle|^2 \delta(\omega - E_X) \\ &= -\frac{1}{\pi} \text{Im} \langle A | O'(\mathbf{q}) \frac{1}{\omega - H_A + i\eta} O(\mathbf{q}) | A \rangle, \end{aligned} \quad (4.37)$$

where the energy, E_X , of the final state, $|X\rangle$, is measured from the ground state $|A\rangle$ of the target. The operator, $O(\mathbf{q})$, is given by

$$O(\mathbf{q}) = \frac{1}{\sqrt{2}} \sum_{i=1}^Z e^{i\mathbf{q}\cdot\mathbf{r}_i}. \quad (4.38)$$

Neglecting the interference terms among different i 's in $O(\mathbf{q})$ and $O'(\mathbf{q})$, we can restrict ourselves to the case where the struck nucleon is 1, i.e. $O(\mathbf{q}) = e^{i\mathbf{q}\cdot\mathbf{r}_1}$. H_A is then decomposed as

$$H_A = K_1 + H_{A-1} + V_{1,A-1}, \quad (4.39)$$

where K_1 is the kinetic energy for the nucleon 1, H_{A-1} is the Hamiltonian for the residual nucleus and $V_{1,A-1}$ is the interaction between 1 and the residual nucleus. The struck nucleon is considered to become energetic enough to justify the fixed-scatterer approximation for the final state interaction due to $V_{1,A-1}$. The response function, S , in this approximation is given by

$$\begin{aligned} S(\omega, \mathbf{q}) &= -\frac{1}{\pi} \text{Im} \int d\mathbf{r}_1 d\mathbf{r}_2 \cdots d\mathbf{r}_A \Psi_A^*(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_A) \Psi_A(\mathbf{r}_1', \mathbf{r}_2', \dots, \mathbf{r}_A) e^{i\mathbf{q}\cdot(\mathbf{r}_1' - \mathbf{r}_1)} G(\omega - \hat{\epsilon}; \mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \dots, \mathbf{r}_A) \end{aligned} \quad (4.40)$$

where $\hat{\epsilon}$ is the average nucleon separation energy and G is the Green's function for the nucleon 1 in the residual nucleus in the fixed-scatterer approximation, i.e.

$$G(\omega - \hat{\epsilon}; \mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \dots, \mathbf{r}_A) = \langle \mathbf{r}_1 | \frac{1}{\omega - \hat{K}_1 - V_{1,A-1} + i\eta} | \mathbf{r}_1' \rangle \quad (4.41)$$

with $V_{1,A-1}$ given by

$$V_{1,A-1}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_A) = \sum_{j=2}^A V_{NN}(\mathbf{r}_1 - \mathbf{r}_j). \quad (4.42)$$

Introducing the free Green's function, G_0 , by

$$G_0(\omega; \mathbf{r}_1, \mathbf{r}_1') = \langle \mathbf{r}_1 | \frac{1}{\omega - \hat{K}_1 + i\eta} | \mathbf{r}_1' \rangle, \quad (4.43)$$

we can write $\text{Im } G$ in eq. (4.40) as

$$\begin{aligned} \text{Im } G &= \text{Im } G^1 + \text{Im } G^1, \\ \text{Im } G^1 &= (1 + G^1 V_{1,A-1}^1) \text{Im } G_0(1 + V_{1,A-1}^1 G), \\ \text{Im } G^1 &= G^1 \text{Im } V_{1,A-1}^1 G. \end{aligned} \quad (4.44)$$

Note that V_{NN} is complex in the energy region we are interested in due to the inelastic processes such as $NN \rightarrow \pi NN$, $NN \rightarrow NN^*$, etc. $\text{Im } G^1$ is the contribution of the processes in which the struck nucleon gets out of the nucleus without undergoing the inelastic processes, which contribute to $\text{Im } G^1$. $\text{Im } G^1$ thus corresponds to the $(e, e'p)$ reaction with the momentum of the emitted proton integrated. The response function, S , can be correspondingly decomposed as

$$S(\omega, \mathbf{q}) = S^1(\omega, \mathbf{q}) + S^1(\omega, \mathbf{q}), \quad (4.45)$$

and the transparency, T , is defined by

$$T(\mathbf{q}) = \frac{\int d\omega S^1(\omega, \mathbf{q})}{\int d\omega S(\omega, \mathbf{q})}, \quad (4.46)$$

where the integral over ω is restricted to the quasi-elastic region.

The Green's function defined by eq. (4.41) is still too complicated to be of practical use. We will here introduce the eikonal approximation for the Green's function in order to make the response functions calculable. The eikonal expression of the Green's function is,

$$G(\omega; \mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \dots, \mathbf{r}_A) = -\frac{i}{v} \delta(\mathbf{b}_1 - \mathbf{b}_1') \theta(z_1 - z_1') \exp \left\{ i p z_1 - z_1' - \frac{i}{v} \int_{z_1}^{z_1'} dz_1'' \sum_{j=2}^A V_{NN}(r_1'' - r_j) \right\} + (\text{backward piece}), \quad (4.47)$$

where v and p are, respectively, the velocity and the momentum of the nucleon 1 determined by the energy, ω , the z -axis is arbitrarily chosen at this stage though it will be taken to be parallel to the momentum transfer $\mathbf{q}$, later, $\mathbf{b}_1$ is orthogonal to the z -axis and $\mathbf{r}_1' \equiv (\mathbf{b}_1, z_1')$. The backward piece becomes important only when we discuss the sum rule and will be omitted in the following. $\text{Im } G$ and $\text{Im } G^*$ in this approximation are given by

$$\begin{aligned} \text{Im } G(\omega; \mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \dots, \mathbf{r}_A) &= \frac{1}{2i} \{ G(\omega; \mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \dots, \mathbf{r}_A) - G^*(\omega; \mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \dots, \mathbf{r}_A) \} \\ &= -\frac{1}{v} \delta(\mathbf{b}_1 - \mathbf{b}_1') e^{i p(z_1 - z_1')} \left[\theta(z_1 - z_1') \exp \left\{ -\frac{i}{v} \int_{z_1}^{z_1'} dz_1'' \sum_{j=2}^A V_{NN}(r_1'' - r_j) \right\} \right. \\ &\quad \left. + \theta(z_1' - z_1) \exp \left\{ -\frac{i}{v} \int_{z_1'}^{z_1} dz_1'' \sum_{j=2}^A V_{NN}^*(r_1'' - r_j) \right\} \right] \end{aligned} \quad (4.48)$$

and

$$\begin{aligned} \text{Im } G^*(\omega; \mathbf{r}_1, \mathbf{r}_1'; \mathbf{r}_2, \dots, \mathbf{r}_A) &= -\frac{1}{v} \delta(\mathbf{b}_1 - \mathbf{b}_1') \exp \left\{ i p z_1 + \frac{i}{v} \int_{z_1}^{\infty} dz_1'' \sum_{j=2}^A V_{NN}(r_1'' - r_j) \right\} \exp \left\{ -i p z_1' - \frac{i}{v} \int_{z_1'}^{\infty} dz_1'' \sum_{j=2}^A V_{NN}^*(r_1'' - r_j) \right\}, \quad (4.49) \end{aligned}$$

respectively. Substituting these into the r.h.s. of eq. (4.40), we obtain the response function S and S^* in the Glauber approximation. The resulting expressions still contain multi-dimensional integrals over $\mathbf{r}_1, \dots, \mathbf{r}_A$. In order to simplify them, we assume that the many-body density given by the ground state wave function, $\Psi_A(\mathbf{r}_1, \dots, \mathbf{r}_A)$, can be written as a product of one-body densities, $\rho(\mathbf{r}_i)$, and the two-body correlation functions $g(\mathbf{r}_{ij})$, which is equivalent to assuming the wave function, Ψ_A , to be effectively of the form,

$$\Psi_A(\mathbf{r}_1, \dots, \mathbf{r}_A) = \prod_{i < j} g(\mathbf{r}_{ij}) \prod_i \sqrt{\rho(\mathbf{r}_i)}. \quad (4.50)$$

Without the two-body correlation, i.e. $g(\mathbf{r}_{ij}) = 1$, the relevant part of the integral for S becomes

$$\begin{aligned} &\int d\mathbf{r}_2 \dots d\mathbf{r}_A \exp \left\{ -\frac{i}{v} \sum_{j=2}^A V_{NN}(r_1'' - r_j) \right\} \Psi_A(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_A) \Psi_A^*(\mathbf{r}_1', \mathbf{r}_2, \dots, \mathbf{r}_A) \\ &= \rho(\mathbf{r}_1, \mathbf{r}_1') \int d\mathbf{r}_2 \rho(\mathbf{r}_2) \exp \left\{ -\frac{i}{v} V_{NN}(r_1'' - r_2) \right\} \Bigg|^{A-1} \\ &\simeq \rho(\mathbf{r}_1, \mathbf{r}_1') \exp \left\{ -\frac{i}{v} (A-1) \int d\mathbf{r}_2 \rho(\mathbf{r}_2) V_{NN}(r_1'' - r_2) \right\}. \end{aligned} \quad (4.51)$$

Including the two-body correlation to the lowest order in $1 - g^1$, we obtain an expression similar to the above with $V_{NN}(r_1'' - r_2)$ multiplied by $g(r_1' - r_2)g(r_1 - r_2)$. The final expression for the inclusive (e, e') response function, S , is

$$\begin{aligned} S(\omega, \mathbf{q}) &= \frac{1}{\pi v} \int d\mathbf{b}_1 d\mathbf{z}_1 d\mathbf{z}_1' \rho(\mathbf{r}_1, \mathbf{r}_1') e^{i p(z_1 - z_1')} \\ &\quad \times \left[\theta(z_1 - z_1') \exp \left\{ -\frac{i}{v} (A-1) \int_{z_1}^{z_1'} dz_1'' \int d\mathbf{r}_2 \rho(\mathbf{r}_2) V_{NN}(r_1'' - r_2) g(r_1' - r_2) g(r_1 - r_2) \right\} \right. \\ &\quad \left. + \theta(z_1' - z_1) \exp \left\{ -\frac{i}{v} (A-1) \int_{z_1'}^{z_1} dz_1'' \int d\mathbf{r}_2 \rho(\mathbf{r}_2) V_{NN}^*(r_1'' - r_2) g(r_1' - r_2) g(r_1 - r_2) \right\} \right], \end{aligned} \quad (4.52)$$

where $\mathbf{r}_1 = (\mathbf{b}_1, z_1)$, $\mathbf{r}_1' = (\mathbf{b}_1, z_1')$, $\mathbf{r}_1'' = (\mathbf{b}_1, z_1'')$ and p is determined by $\omega - \epsilon = E_p = \sqrt{p^2 + m_N^2}$ with $v = p/E_p$. In the case of nuclear matter, we get an expression for S similar to that of Benhar *et al.* [32], with the following modifications.

¹The recent calculation in ref. [57] shows that the higher order terms in $(1 - g)$ are important.

Figure 4.18: Nuclear transparency, $T(q)$, for $(e, e'p)$ reaction in the region $q = 5 \sim 10$ GeV/c. The solid, dash-dotted and dashed curves correspond to the full calculation with the correlation, the calculation without the correlation and the conventional calculation with the total cross section.

1. $\text{Im } V_{NN}$ gives the nucleon-nucleon reaction cross section, σ_{NN}^t , instead of the total cross section, σ_{NN}^t , used in ref. [52].
2. The two-body correlation shows up in the form $g(r_1' - r_2)g(r_1 - r_2)$ rather than the form $g(r_1 - r_2)^2$ employed in ref. [52].

A calculation based on eq. (4.52) is under way by Kohama *et al.* and will be reported in the near future [55].

The corresponding expression for S^* is

$$\begin{aligned} S^*(\omega, \mathbf{q}) &= \frac{1}{\pi v} \int d\mathbf{b}_1 d\mathbf{z}_1 d\mathbf{z}_1' \rho(\mathbf{r}_1, \mathbf{r}_1') e^{i p(z_1' - z_1 - q \cdot \mathbf{r}_1 - \mathbf{r}_1')} \\ &\quad \times \exp \left\{ \frac{i}{v} (A-1) \int d\mathbf{r}_2 \rho(\mathbf{r}_2) g(\mathbf{r}_1 - r_2) g(\mathbf{r}_1' - r_2) \left\{ \int_{z_1}^{\infty} dz_1'' V_{NN}(r_1'' - r_2) - \int_{z_1'}^{\infty} dz_1'' V_{NN}^*(r_1'' - r_2) \right\} \right\}. \end{aligned} \quad (4.53)$$

In calculating the transparency $T(q)$, defined by eq. (4.46), we assume that the integral over ω is determined by the region $p \sim q$ and neglect the ω -dependence of the integrands except the oscillating factor $e^{i p(r_1 - r_1')}$. The ω integration then gives

$$\int d\omega S(\omega, \mathbf{q}) = 1 \quad (4.54)$$

and

$$\int d\omega S^*(\omega, \mathbf{q}) = \int d\mathbf{r}_1 \rho(\mathbf{r}_1) \exp \left[\frac{2}{v} (A-1) \int d\mathbf{r}_2 \rho(\mathbf{r}_2) g(\mathbf{r}_1 - r_2)^2 \int_{z_1}^{\infty} dz_1'' \text{Im } V_{NN}(r_1'' - r_2) \right]. \quad (4.55)$$

With the zero-range approximation for $\text{Im } V_{NN}$,

$$\text{Im } V_{NN}(r) \simeq -\frac{v}{2} \sigma_{NN}^t \delta(r), \quad (4.56)$$

we get [53, 54]

$$T(q) = \int d\mathbf{r}_1 \rho(\mathbf{r}_1) \exp \left[-(A-1) \sigma_{NN}^t \int_{z_1}^{\infty} dz_1'' g(z_1 - z_1')^2 \rho(z_1'') \right]. \quad (4.57)$$

Fig. 4.18 shows the calculated $T(q)$ as a function of the target mass number, A , in the region where the nucleon-nucleon cross sections are practically energy-independent, i.e. $q = 5 \sim 10$ GeV/c [55]. The correlation function $g(r)$ is taken to be

$$g(r) = 1 - ce^{-r^2/l^2} \quad (4.58)$$

with $c = 0.84$ and $a = 0.7$ fm, which simulates the one used in ref. [52]. The solid line corresponds to the full calculation with correlation. The case without the correlation ($g(r) = 1$) is shown by the dash-dotted line for comparison. The expression with $g(r) = 1$ and σ_{NN}^{tot} replaced by σ_{NN}^{rel} corresponds to the conventional mean-free-path estimate of the transparency and the results are also shown by the dashed line. The cross section used in the calculation are

$$\sigma_{NN}^{rel} = 40 \text{ mb}, \sigma_{NN}^t = 28 \text{ mb}. \quad (4.59)$$

One sees that the transparency is greatly enhanced by the use of the reaction cross section instead of the total cross section and the correlation also increases it appreciably. For large A , ($A \gtrsim 100$), the transparency is roughly proportional to $A^{-1/3}$ times the proton mean-free-path in nuclear matter estimated with the effective cross section. The correlation reduces effective cross section by about 20 %. The results of the full calculation using the reaction cross sections and including the correlation are to be compared with the $(e, e'p)$ experiments to find the effects of the proton internal dynamics.

5 Concluding remarks

The optical potential model is a standard way of studying the interaction of a particle with nuclei. It allows us to represent the complicated interactions with the many-body systems in the form of one-body potentials and has been quite successful in analyzing the experiments on the scattering of the particle with nuclear targets and providing us with a starting point of further theoretical studies on the many-body effects and the basic interaction of the particle with a nucleon. We have shown in this paper that the Green's function method gives us a convenient way of using the same optical potential in analyzing the final state interaction of the particle with the nucleus even in the cases where the particle is not explicitly observed. This is particularly useful when the particle is not available as a beam or when we want to study those features of its interaction with the nucleus that are not directly obtainable from the scattering observables.

A notable feature of the optical potential lies in its absorptive part which consists of the contributions from all possible inelastic processes, gives rise to the energy dependence of the real part through the dispersion relation and thus represents all the complicated many-body effects in the interaction with nuclei. The Green's function method enables us to decompose the inclusive spectrum, which is proportional to the nuclear response to the reaction, into the quasi-free part in which the particle in question gets out of the nucleus and the rest which is given by the absorptive part of the optical potential. The decomposition corresponds to that of the total cross section into the elastic and reaction cross sections in the scattering processes, and is especially useful in determining the absorptive part from the experiments.

The idea has been applied to $(\bar{A}, \pi)$ reactions producing Σ -hyperons in §4.1 and useful information on Σ -nucleus interaction has been obtained. The main interest in the application to hadronic atoms in §4.2 lies in the anomalous spectra due to the interference with the Coulomb interaction and the feasibility of experimentally observing the deeply bound orbits for which the strong interaction effects cannot be treated as a perturbation to the dominant Coulomb field. Hadronic atoms were the oldest sources of information on the optical potential and have been considered to probe it mainly in region near and outside of the nuclear surface. The possibility of observing them by the methods other than the atomic X-rays may make them a new source of information on the optical potential in the nuclear interior.

The Green's function method is not to be restricted to the cases where the interaction is described by the optical potential. An example of the electron scattering in the quasi-elastic region has been studied in §4.3 where the final state interaction between the struck proton and the residual nucleus is treated by the multiple scattering theory in the Glauber approximation. The method has been found to give a unified way of analyzing the electron inclusive spectra from (e, e') and $(e, e'p)$ reactions and calculating the nuclear transparency including nuclear many-body effects as accurately as possible but neglecting the internal dynamics of the nucleon. The calculated transparency is thus expected to serve as a reference in observing the color transparency due to the internal dynamics.

The subjects discussed in this article are only few of the many that are related to the Green's function method. We still hope to have made it clear that the method facilitates parametrizing the complicated many-body effects in a simple and tractable way, keeping at the same time the intriguing aspects of quantum mechanical systems.

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A Green's function in second quantization

In this appendix we discuss the Green's function in the second quantization formalism, which enables us to treat nonrelativistic and relativistic particles on the same footing.

A.1 Schrödinger field [3]

First we consider a nonrelativistic particle with spin 1/2 in a potential $V(\mathbf{x})$, which is described by the Schrödinger equation,

$$\left\{ i \frac{\partial}{\partial t} + \frac{1}{2M} \nabla^2 - V(\mathbf{x}) \right\} \psi(\mathbf{x}) = 0. \quad (A.1)$$

It should be noted that the strong and the Coulomb interactions can be incorporated in a single form of the potential, $V(\mathbf{x})$, in the Schrödinger equation. The Lagrangian which gives the above Schrödinger equation as the Euler-Lagrange equation is

$$\mathcal{L} = \int d^3x \bar{\psi}(\mathbf{x}) \left\{ i \frac{\partial}{\partial t} + \frac{1}{2M} \nabla^2 - V(\mathbf{x}) \right\} \psi(\mathbf{x}). \quad (A.2)$$

The momentum conjugate to $\bar{\psi}(\mathbf{x})$ is defined by

$$\pi(\mathbf{x}) = i\bar{\psi}(\mathbf{x}). \quad (A.3)$$

The equal time anticommutation relation is

$$\{\bar{\psi}(\mathbf{x}), \pi(\mathbf{x}')\} = i\delta^3(\mathbf{x} - \mathbf{x}') \quad \text{where} \quad t = t'. \quad (A.4)$$

Then, the Hamiltonian is given by

$$\hat{H} = \int d^3x \bar{\psi}(\mathbf{x}) \left\{ \frac{1}{2M} \nabla^2 \bar{\psi}(\mathbf{x}) \nabla \psi(\mathbf{x}) + \bar{\psi}(\mathbf{x}) V(\mathbf{x}) \psi(\mathbf{x}) \right\}. \quad (A.5)$$

Let us consider the eigenfunction of the Schrödinger equation, eq. (A.1),

$$\bar{\psi}_n(\mathbf{x}) = \bar{\psi}_n(\mathbf{x}) e^{-iE_n t}, \quad (A.6)$$

where $\bar{\psi}_n(\mathbf{x})$ satisfies

$$\left\{ E_n + \frac{1}{2M} \nabla^2 - V(\mathbf{x}) \right\} \bar{\psi}_n(\mathbf{x}) = 0. \quad (A.7)$$

The eigenfunctions are normalized as

$$\int d^3x \bar{\psi}_n^\dagger(\mathbf{x}) \bar{\psi}_m(\mathbf{x}) = \delta_{nm}. \quad (A.8)$$

We expand the field operators by the eigenfunctions as

$$\begin{aligned} \bar{\psi}(\mathbf{x}) &= \sum_n \bar{a}_n \bar{\psi}_n(\mathbf{x}), \\ \psi(\mathbf{x}) &= \sum_n \bar{a}_n^\dagger \psi_n(\mathbf{x}). \end{aligned} \quad (A.9)$$

These equations can be solved for $\bar{a}_n$ and $\bar{a}_n^\dagger$ as

$$\begin{aligned} \hat{a}_n &= \int d^3x w_n^\dagger(x) \hat{\psi}(x), \\ \hat{a}_n^\dagger &= \int d^3x \hat{\psi}^\dagger(x) w_n(x). \end{aligned} \quad (\text{A.10})$$

The anticommutation relations of $\hat{a}_n$ and $\hat{a}_n^\dagger$ can be calculated by using eqs. (A.4) and (A.8) to give

$$\{\hat{a}_n, \hat{a}_m^\dagger\} = \delta_{nm}. \quad (\text{A.11})$$

The consistency condition of eqs. (A.9) and (A.10) provides us with the following completeness relation for the eigenfunctions

$$\sum_n \psi_n(x) \psi_n^\dagger(x') = \delta(x - x'). \quad (\text{A.12})$$

By substituting eq. (A.9) into the Hamiltonian, eq. (A.5), and using eqs. (A.7) and (A.8) we get

$$\hat{H} = \sum_n E_n \hat{a}_n^\dagger \hat{a}_n. \quad (\text{A.13})$$

The Green's function is defined by

$$\begin{aligned} G(x, x') &= -i \langle 0 | T [\hat{\psi}(x) \hat{\psi}^\dagger(x')] | 0 \rangle \\ &= -i \sum_n e^{i E_n (x - x')} \theta(x - x') \psi_n(x) \psi_n^\dagger(x') \end{aligned} \quad (\text{A.14})$$

and satisfies

$$\left\{ i \frac{\partial}{\partial t} + \frac{1}{2M} \nabla^2 - V(x) \right\} G(x, x') = \delta^4(x - x'). \quad (\text{A.15})$$

Its Fourier transform is

$$\begin{aligned} G(\omega; x, x') &= \int dt e^{i\omega t} G(x, x') \\ &= \sum_n \frac{\psi_n(x) \psi_n^\dagger(x')}{\omega - E_n + i\eta} \end{aligned} \quad (\text{A.16})$$

and satisfies

$$\left\{ \omega + \frac{1}{2M} \nabla^2 - V(x) \right\} G(\omega; x, x') = \delta^3(x - x'). \quad (\text{A.17})$$

A.2 Complex Klein-Gordon field [58]

Secondly we consider a relativistic charged particle with spin 0 in the Coulomb potential $V_c(x)$ and the strong potential $V(x)$, which is described by the Klein-Gordon equation,

$$\left\{ \left(i \frac{\partial}{\partial t} - V_c(x) \right)^2 + \nabla^2 - m^2 - 2mV(x) \right\} \hat{\psi}(x) = 0. \quad (\text{A.18})$$

The Lagrangian which gives the above Klein-Gordon equation as the Euler-Lagrange equation is

$$\mathcal{L} = \int d^3x \hat{\psi}^\dagger(x) \left\{ \left(i \frac{\partial}{\partial t} - V_c(x) \right)^2 + \nabla^2 - m^2 - 2mV(x) \right\} \hat{\psi}(x). \quad (\text{A.19})$$

The momenta conjugate to $\hat{\psi}(x)$ and $\hat{\psi}^\dagger(x)$ are defined by

$$\begin{aligned} \hat{\pi}(x) &= \left(\frac{\partial}{\partial t} - iV_c(x) \right) \hat{\psi}^\dagger(x) \equiv D_0 \hat{\psi}^\dagger(x), \\ \hat{\pi}^\dagger(x) &= \left(\frac{\partial}{\partial t} + iV_c(x) \right) \hat{\psi}(x) \equiv D_0 \hat{\psi}(x). \end{aligned} \quad (\text{A.20})$$

The equal-time commutation relations are

$$\begin{aligned} [\hat{\psi}(x), \hat{\pi}(x)] &= i\delta^3(x - x'), \\ [\hat{\psi}^\dagger(x), \hat{\pi}^\dagger(x)] &= i\delta^3(x - x'), \quad \text{where } i = i' \\ [\hat{\psi}(x), \hat{\psi}(x)] &= [\hat{\psi}^\dagger(x), \hat{\psi}^\dagger(x)] = [\hat{\psi}^\dagger(x), \hat{\psi}(x)] = 0, \\ [\hat{\pi}(x), \hat{\pi}(x)] &= [\hat{\pi}^\dagger(x), \hat{\pi}^\dagger(x)] = [\hat{\pi}^\dagger(x), \hat{\pi}(x)] = 0. \end{aligned} \quad (\text{A.21})$$

Then, the Hamiltonian is given by

$$\hat{H} = \int d^3x \hat{\psi}(x) \left\{ -\frac{\partial^2}{\partial t^2} - V_c(x)^2 - \nabla^2 + m^2 + 2mV(x) \right\} \hat{\psi}^\dagger(x). \quad (\text{A.22})$$

Let us consider the eigenfunction of the Klein-Gordon equation, eq. (A.18)

$$\varphi_n(x) = \varphi_n(x) e^{-i\omega_n t}, \quad (\text{A.23})$$

where

$$\left\{ \omega_n - V_c(x)^2 + \nabla^2 - m^2 - 2mV(x) \right\} \varphi_n(x) = 0. \quad (\text{A.24})$$

The eigenfunctions are normalized as

$$\int d^3x \varphi_n^\dagger(x) \left\{ \omega_n + \omega_n - 2V_c(x) \right\} \varphi_m(x) = \pm \delta_{nm}, \quad (\text{A.25})$$

where the upper (lower) sign is for the positive (negative) ω_n . Hereafter we use the suffices n and m to specify eigenstates of positive and negative energies, respectively. We expand the field operators by the eigenfunctions

$$\begin{aligned} \hat{\psi}(x) &= \sum_n \hat{a}_n \varphi_n(x) + \sum_m \hat{d}_m^\dagger \varphi_m(x), \\ \hat{\psi}^\dagger(x) &= \sum_n \hat{d}_n^\dagger \varphi_n^\dagger(x) + \sum_m \hat{a}_m \varphi_m^\dagger(x), \\ \hat{\pi}(x) &= \sum_n \hat{a}_n^\dagger D_0 \varphi_n^\dagger(x) + \sum_m \hat{d}_m D_0 \varphi_m^\dagger(x) \\ &= i \sum_n \hat{a}_n^\dagger (\omega_n - V_c(x)) \varphi_n^\dagger(x) + i \sum_m \hat{d}_m (\omega_m - V_c(x)) \varphi_m^\dagger(x), \\ \hat{\pi}^\dagger(x) &= \sum_n \hat{a}_n D_0 \varphi_n(x) + \sum_m \hat{d}_m^\dagger D_0 \varphi_m(x) \\ &= -i \sum_n \hat{a}_n (\omega_n - V_c(x)) \varphi_n(x) - i \sum_m \hat{d}_m^\dagger (\omega_m - V_c(x)) \varphi_m(x). \end{aligned} \quad (\text{A.26})$$

These equations can be solved for $\hat{a}_n$, $\hat{a}_n^\dagger$, $\hat{d}_n$ and $\hat{d}_n^\dagger$ as

$$\begin{aligned} \hat{a}_n &= \int d^3x \varphi_n^\dagger(x) \{ (\omega_n - V_c(x)) \hat{\psi}(x) + i \hat{\pi}^\dagger(x) \}, \\ \hat{a}_n^\dagger &= \int d^3x \varphi_n(x) \{ (\omega_n - V_c(x)) \hat{\psi}^\dagger(x) - i \hat{\pi}(x) \}, \\ \hat{d}_n &= \int d^3x \varphi_m(x) \{ (\omega_m - V_c(x)) \hat{\psi}^\dagger(x) - i \hat{\pi}(x) \}, \\ \hat{d}_m^\dagger &= \int d^3x \varphi_m^\dagger(x) \{ (\omega_m - V_c(x)) \hat{\psi}(x) + i \hat{\pi}^\dagger(x) \}. \end{aligned} \quad (\text{A.27})$$

The commutation relations of $\hat{a}_n$, $\hat{a}_n^\dagger$, $\hat{d}_n$ and $\hat{d}_n^\dagger$ can be calculated by using eqs. (A.21) and (A.26) to give

$$\begin{aligned} [\hat{a}_n, \hat{a}_n^\dagger] &= \delta_{nn}, \\ [\hat{d}_m, \hat{d}_m^\dagger] &= \delta_{mm}, \\ [\hat{a}_n, \hat{a}_m] &= [\hat{a}_n^\dagger, \hat{a}_m^\dagger] = [\hat{d}_n, \hat{d}_m] = [\hat{d}_n^\dagger, \hat{d}_m^\dagger] = [\hat{a}_n, \hat{d}_m] = [\hat{a}_n^\dagger, \hat{d}_m^\dagger] = [\hat{a}_n, \hat{d}_m^\dagger] = [\hat{a}_n^\dagger, \hat{d}_m] = 0. \end{aligned} \quad (\text{A.28})$$

The consistency conditions of eqs. (A.21) and (A.26) then provides us with the following completeness relations for the eigenfunction

$$\begin{aligned} \sum_n \varphi_n(x) \varphi_n^\dagger(x') - \sum_m \varphi_m(x) \varphi_m^\dagger(x') &= 0, \\ \sum_n \omega_n \varphi_n(x) \varphi_n^\dagger(x') - \sum_m \omega_m \varphi_m(x) \varphi_m^\dagger(x') &= \delta^3(x - x'), \\ \sum_n \omega_n^2 \varphi_n(x) \varphi_n^\dagger(x') - \sum_m \omega_m^2 \varphi_m(x) \varphi_m^\dagger(x') &= \delta^3(x - x')^2 V_c(x). \end{aligned} \quad (\text{A.29})$$

By substituting eq. (A.26) into the Hamiltonian and using eqs. (A.23), (A.24) and (A.25) we get

$$\hat{H} = \sum_{\mathbf{n}} \omega_{\mathbf{n}} \hat{a}_{\mathbf{n}}^\dagger \hat{a}_{\mathbf{n}} - \sum_{\mathbf{m}} \omega_{\mathbf{m}} \hat{d}_{\mathbf{m}}^\dagger \hat{d}_{\mathbf{m}}. \quad (\text{A.30})$$

The Green's function for the Klein-Gordon field is defined by

$$\begin{aligned} G(x, x') &= -i \langle 0 | T [\phi(x) \phi^*(x')] | 0 \rangle \\ &= -i \theta(t - t') \sum_{\mathbf{n}} e^{i\omega_{\mathbf{n}}(t-t')} \langle 0_{\mathbf{n}} | x \rangle \phi_{\mathbf{n}}^*(x') + i \theta(t' - t) \sum_{\mathbf{m}} e^{i\omega_{\mathbf{m}}(t-t')} \langle 0_{\mathbf{m}} | x \rangle \phi_{\mathbf{m}}^*(x') \end{aligned} \quad (\text{A.31})$$

and satisfies

$$\left[\left(i \frac{\partial}{\partial t} - V_{\ell}(x) \right)^2 + \nabla^2 - m^2 - 2mV(x) \right] G(x, x') = \delta^4(x - x'). \quad (\text{A.32})$$

Its Fourier transform is

$$\begin{aligned} G(\omega; \mathbf{x}, \mathbf{x}') &= \int dt e^{i\omega t} G(x, x') \\ &= \sum_{\mathbf{n}} \frac{\varphi_{\mathbf{n}}(\mathbf{x}) \varphi_{\mathbf{n}}^*(\mathbf{x}')}{\omega - \omega_{\mathbf{n}} + i\eta} - \sum_{\mathbf{m}} \frac{\varphi_{\mathbf{m}}(\mathbf{x}) \varphi_{\mathbf{m}}^*(\mathbf{x}')}{\omega - \omega_{\mathbf{m}} - i\eta} \end{aligned} \quad (\text{A.33})$$

and satisfies

$$\left\{ (\omega - V_{\ell}(x))^2 + \nabla^2 - m^2 - 2mV(x) \right\} G(\omega; \mathbf{x}, \mathbf{x}') = \delta^4(\mathbf{x} - \mathbf{x}'). \quad (\text{A.34})$$

One sees that the first and the second of eq. (A.29) are a special case of the first and the second of eq. (3.9), respectively, while the third has no counter part in the general case.

The Green's function for the Klein-Gordon equation in the first quantization, eq. (A.34), looks very different from that for the Schrödinger field, eq. (A.7) and its relation to the response function is unclear. The former, however, has the same form as the latter in the second quantization formalism except for the anti-particle contributions and the relation becomes clearer in this formalism.

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