\( N_{H} \) is a function of the incoming energy \( E \) of the null matter swallowed by the black hole. The null matter is not only swallowed, it is also replaced by a new null matter, which we denote \( N_{H} \).

In this paper, we study the case of quantum black hole formation using the semi-classical, boson-field model. We find analytically that the shape of the incoming null matter follows the Hawking temperature function.

In certain two-dimensional models, the collapsing matter forms a black hole if

\[ T \ll \Theta_{\text{Hawking}} \]

**Abstract**

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**Abstract**

Formation and coalescence of an arbitrarily small black hole. The distance from criticality, and hence the solution describing the critical black hole, is described via a universal formula in terms of the critical temperature. At the critical temperature, the black hole mass is given by a universal formula in terms of the Hawking temperature. Near the critical temperature, the Hawking temperature function is

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**Introduction**

There appears to be little analytic or conceptual understanding of these interesting phenomena. One would like to understand the behavior of black holes as functions of the initial data set, as functions of the mass \( M \) or \( Q \). We consider the case where all of the initial data is fixed, and the only variable is the initial entropy.

This result suggests that, in the limit of low entropy, the black hole mass is given by

\[ T \ll \Theta_{\text{Hawking}} \]

where the critical temperature is numerically found to be near 37. This result suggests that \( M \) is a function of the incoming energy \( E \) of the null matter swallowed by the black hole. The null matter is not only swallowed, it is also replaced by a new null matter, which we denote \( N_{H} \).

In this paper, we study the case of quantum black hole formation using the semi-classical, boson-field model. We find analytically that the shape of the incoming null matter follows the Hawking temperature function.

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Formation and coalescence of an arbitrarily small black hole. The distance from criticality, and hence the solution describing the critical black hole, is described via a universal formula in terms of the Hawking temperature. At the critical temperature, the black hole mass is given by a universal formula in terms of the Hawking temperature. Near the critical temperature, the Hawking temperature function is

\[ T \ll \Theta_{\text{Hawking}} \]
the black hole during its lifetime and $\lambda$ is the dimensionful parameter of the model. The $O(\delta^3)$ term depends universally on $\alpha$, the second derivative of the energy density at the point where the critical threshold is exceeded\footnote{It would be interesting to determine if the $O(\delta^3)$ term of (1) has a similar universal dependence.}. The $O(\delta^{1/2})$ corrections depend non-universally on the shape of the incoming pulse. We also find a scaling solution near criticality which corresponds to the formation and evaporation of an arbitrarily small black hole. We have not determined whether the scaling (2) is universal with respect to small changes in the coupling constants of the theory. Presumably numerical work is required to answer this interesting question.

There are obvious similarities between our results and those of Ref. [1], but there are also apparent differences. First of all, our critical exponent is a rational number whereas Choquette's at least appears to be irrational. Second, there is no analog of the self-similar oscillations in our work. This could be a special feature arising from the linear nature of the RST equations, while more general two-dimensional models (which are not analytically soluble) might exhibit such oscillations.

We now present a derivation of the scaling relation (2). The semi-classical effective action for the RST model\footnote{For more details on this model consult [4,6]. We use the conventions of [6].} is

$$S = \frac{1}{2\pi} \int d^2z \sqrt{-g}(e^{-2\phi} - \frac{N}{24}\phi) R + 4e^{-2\phi}((\nabla \phi)^2 + \lambda^2) - \frac{1}{2} \sum_{i=1}^{N} (\nabla f_i)^2$$

$$- \frac{N}{28\pi} \int d^2z \sqrt{-g}(z) \int d^2z' \sqrt{-g(z')} R(z;z') G(z;z') R(z'),$$

where $g_{\mu\nu}$ is the two-dimensional metric, $\phi$ is a scalar field called the dilaton, $f_i$ are $N$ minimally coupled scalar matter fields and $G$ is a Green function for the operator $\nabla^2$. The effective action includes the one-loop Liouville term due to the matter fields and if $N$ is large this term provides the dominant quantum back-reaction on the geometry. This model differs from the original CGHS model [7] by a finite local counterterm, which restores a global symmetry of the classical theory and enables writing down exact semi-classical solutions in a rather simple form. Numerical analyses of the original model [8] indicate, however, that the two models are similar, i.e. that the qualitative behavior of semi-classical solutions is not sensitive to the existence of the global symmetry.

It is convenient to work in conformal gauge, $g_{++} = -\frac{1}{2} e^{2\phi}$, $g_{+-} = g_{-+} = 0$, and use the global symmetry to further fix the coordinates to "Kruskal gauge", where $\rho = \phi + \frac{1}{4} \log \frac{N}{24}$. This eliminates the conformal factor from the discussion. If we define a new dilaton field,

$$\Omega = \frac{12}{N} e^{-2\phi} + \frac{1}{2} \phi + \frac{1}{4} \log \frac{N}{48},$$

the semi-classical equations reduce to

\begin{align}
\partial_\pm \partial_\pm f_i &= 0, \\
\partial_\pm \partial_\Omega &= - \lambda^2, \\
-\partial_\Omega^2 \Omega &= T_{\pm}^T + t_\pm,
\end{align}

where $T_{\pm}^T = \frac{6}{N} \sum_{i=1}^{N} (\partial_\pm f_i)^2$. The function $t_+(z^+)$ takes the value $t_+ = -\frac{1}{4} e^{\phi} \rho$ in Kruskal coordinates for any incoming matter energy flux which vanishes sufficiently rapidly at asymptotic early and late times. The field redefinition (4) is degenerate at $\Omega = \frac{1}{4}$ and $\Omega < \frac{1}{4}$ does not correspond to a real value of $\phi$. The curve $\Omega = \frac{1}{4}$ is the analog of the origin of radial coordinates in higher dimensional gravity\footnote{This analogy can be made precise when the model is interpreted as an effective theory for radial modes of near extremal magnetic dilaton black holes in four dimensional gravity [3].} and solutions should not be continued beyond it. Instead, RST impose the following boundary conditions at this curve, wherever it is timelike [4],

$$\partial_\pm \Omega \big|_{\Omega = \frac{1}{4}} = 0.$$
Title in the caption:

1. The only condition that the higher terms in the Taylor expansion of \( f(x) \) do not contain

\[
(2) \quad (\cdots + \varepsilon^{\infty} x^{\infty} \varepsilon^0 - 1)(y + \frac{1}{\varepsilon} x) = (\varepsilon^0 + \varepsilon^1 x)
\]

near \( x = 0 \).

Hint: if \( \varepsilon \rightarrow 0 \), the infinite integral of this type may be parametrized as follows.

1.1. The boundary curve becomes more regular when \( \varepsilon \) is small.

2. The boundary curve becomes more regular when \( \varepsilon \) is small.

Now consider a block hole formation just above threshold. For convenience,

\[
(3) \quad \left( \frac{\partial x}{\partial z} \right) \text{of} \quad \frac{1}{\varepsilon} = H \rho W
\]

according to (11) and (g) follows, and that

\[
(4) \quad (\frac{\partial x}{\partial z}) \text{of} \quad \frac{1}{\varepsilon} = H \rho W
\]

where the block hole defines the block hole position of the block hole.

In the case of the unconfined water table, the block hole position approaches the singularity. The curve of the point on the boundary curve from which the block hole position approaches the singularity.

\[
(5) \quad \left( \frac{\partial x}{\partial z} \right) \text{of} \quad \frac{1}{\varepsilon} = H \rho W
\]

The solution corresponding to incoming matter energy, which lies on the

current front expansion boundary, or the current front expansion boundary.

\[
(6) \quad \left( \frac{\partial x}{\partial z} \right) \text{of} \quad \frac{1}{\varepsilon} = H \rho W
\]

The boundary conditions ensure current expansion boundary conditions.

\[
(7) \quad \left( \frac{\partial x}{\partial z} \right) \text{of} \quad \frac{1}{\varepsilon} = H \rho W
\]
The only reference made to the parameter $\alpha$ is in the definition of the scaling variables (13). Higher orders in the Taylor expansion of $T_{\delta^+}(\sigma^+)$ in (12) only contribute terms carrying positive powers of $\delta$, which can be ignored in the scaling region.

It is straightforward to show that the singularity curve is also universal in this limit. Expressing (7) in terms of $(a^+, a^-)$ coordinates and applying $\frac{d^2}{da^+}$ to both sides yields the following differential equation for the $\Omega = \frac{1}{4}$ curve,

$$
\left(a_s^+ - a_H(a_s^-)\right) \frac{da^-_s}{da_s^+} = -a_s^+ + a_H(a_s^-). \quad (15)
$$

This is a non-linear differential equation and we do not have an analytic solution, but there is no explicit dependence on the flux parameters so the shape of the curve must be universal. However, we do not need the whole singular curve in order to determine the black hole mass (11). It is sufficient to locate the endpoint of evaporation and this can be achieved by the following geometric argument, which is illustrated in figure 2.

FIGURE 2. The two shaded regions have equal area for any point $(a_p^+, a_s^-)$ on the singularity curve.
The dashed curve is the apparent horizon (4.3).

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FIGURE 3. Singularity curve obtained from numerical calculation.
5. A. Strominger and S. Trivedi, unpublished.


