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Abstract.— We show the existence of Lax equations which does not correspond to the vanishing of the Lie or covariant derivative of a $(1,1)$ -type tensor field with respect to a vector field, but they have a different origin, and we provide a more general setting for this kind of equations.

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1. INTRODUCTION

Lax equations [1] were introduced when studying isospectral problems. More specifically, if $L(t)$ is a family of linear operators depending on a parameter t and the evolution equation in t can be written as $L_t = [M, L]$, then the eigenvalues $\lambda(t)$ of $L(t)$ are constant and their eigenvectors ψ , i.e. $L(t)\psi = \lambda\psi$, evolve according to $\psi_t = M\psi$. In that way the Korteweg de Vries equation is associated with Schrödinger equation and the existence of an infinite number of constants of motion in involution is related with the possibility of such description as a Lax equation. When it was discovered that many interesting physical systems described by partial differential equations can be dealt with in the framework of infinite-dimensional Hamiltonian systems, the interest in looking for a geometrical setting for the abovementioned Lax equations was evident and several papers [2,3] proposed a geometrical interpretation as corresponding to the vanishing of the Lie derivative of a type $(1,1)$ tensor field Λ w.r.t a vector field X , $\mathcal{L}_X \Lambda = 0$. This fact is very important because there exist many ways of construction of such tensor fields invariant under a vector field X . This is for instance the case when X is a locally-Hamiltonian vector field in a symplectic manifold (M, ω) and there exists another X -invariant two form ω' . Such an invariant form can be found when we have a symmetry, either a finite φ or an infinitesimal one Y , of X which is not a symmetry of ω . In fact, it suffices to

consider the relation $\varphi^*[\mathcal{L}_{\varphi^*(X)}\omega] = \mathcal{L}_X\varphi^*\omega$, or $[\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X]\omega = 0$, in order to see that $\omega' = \varphi^*\omega$ in the first case or $\omega' = \mathcal{L}_Y\omega$ in the second one is such invariant 2-form. Then, the $(1,1)$ tensor field $\Lambda : \hat{\omega}^{-1} \circ \hat{\omega}'$ obtained by composition of the inverse of the map $\hat{\omega} : \mathfrak{X}(M) \rightarrow \wedge^1(M)$ defined by contraction of the vector fields with the symplectic form, which is invertible because ω is nondegenerate, and the corresponding map for the 2-form ω' , is X -invariant. We will get in this way the so-called non-Noether constants of motion [4]. The same can be said when there exists X -invariant 2-form ω' in a differentiable manifold M endowed with a Poisson structure, playing the role of $\hat{\omega}^{-1}$.

Another particular example is that of a second order differential equation vector field (SODE for short) admitting two alternative Lagrangian descriptions and therefore giving rise to a bi-Hamiltonian system (see e.g. [4,5,6]). In all these cases one can find a $(1,1)$ tensor field Λ , usually called recursion operator, that may be used to generate new symmetry transformations, and the coordinate expression of the condition of the vanishing of the Lie derivative of the tensor Λ , $\mathcal{L}_X\Lambda = 0$, reduces to a Lax equation [3]. In fact, if A and B are the matrix representations of Λ and $\mathcal{L}_X$ with respect to a local basis $\{E_i\}$ of vector fields in M , namely, $\Lambda(E_i) = A_i^j E_j$, $\mathcal{L}_X E_i = B_i^j E_j$, then the X -invariance condition for Λ becomes $\dot{A} = [A, B]$, where $\dot{A}_i^j$ is a short notation for $\Gamma(A_i^j)$. The point to be remarked is that only the property of $\mathcal{L}_X$ being a derivation is used when writing it as a Lax equation. This fact suggests us to consider a similar situation but with $\mathcal{L}_X\Lambda = 0$ replaced by $\mathbb{D}_X\Lambda = 0$, with reference to a linear connection $\mathbb{D}$ [7,8]. For instance, the quantum evolution of a physical system can be seen as a parallel transport [9] and so the quantum time evolution, when expressed in terms of the density matrix, is but a Lax equation. Bi-Lagrangian connections were also investigated as a method of looking for Lax equations and their reductions [10]. We recall that if (M, ω) is a $2n$ -dimensional symplectic manifold, we say that the manifold M is endowed with a Lagrangian bifoliation if there exist two integrable Lagrangian subbundles F_1 and F_2 such that $TM = F_1 \oplus F_2$. By Lagrangian subbundle we mean a n -dimensional subbundle such that the restriction to it of the symplectic form ω vanishes. Given such a Lagrangian splitting there will be an associated bi-Lagrangian connection. If the splitting is compatible with dynamical vector field Γ admitting a bihamiltonian structure, then we can construct a $(1,1)$ tensor field that is parallel along the dynamical vector field Γ with respect to the abovementioned connection. For more details see e.g. [10]. However we remark that only the properties of the maps $\mathcal{L}_X$ and $\mathbb{D}_X$ being derivations are used in the derivation of the Lax equation from the invariance condition, and therefore we can generalize the technique for a more general situation, as for instance that of supermechanics.

In this paper we aim to show that other Lax type equations can arise that are neither associated with X -invariance of a tensor field of type $(1,1)$, nor

with the parallel transport with respect to a linear connection, and we will propose a more general geometrical interpretation of Lax equations. We hope we will get in this way a better understanding of the meaning of Lax equations and that this process of algebraization of the condition for the existence of Lax equations will allow the generalization to other problems.

2. SODES AND LAX EQUATIONS

We begin by analysing some geometrical structures associated to a second order differential equation which giving rise to Lax equations are neither of the type $\mathcal{L}_X\Lambda = 0$, nor of the $\mathbb{D}_X\Lambda = 0$ one.

A SODE vector field Γ on a n -dimensional differentiable manifold Q determines a (nonlinear) Ehresmann connection in the tangent bundle $\tau: TQ \rightarrow Q$ (see [11,12,13,14,15]). We first recall that tangent bundle structures are characterized by an integrable almost tangent structure, defined by a $(1,1)$ tensor field S such that $\text{Ker } S = \text{Im } S$, and therefore $S^2 = 0$. In particular the tangent bundle TQ has a canonical $(1,1)$ -tensor field S called the vertical endomorphism [11], which defines the natural almost tangent structure on TQ . Its action on a vector tangent to TQ corresponds to first projecting onto the base TQ and then lifting it vertically to the original point. In terms of natural coordinates (q^i, v^i) of the tangent bundle TQ , S is given by

$$S = \frac{\partial}{\partial v^i} \otimes dq^i.$$

The other remarkable geometric object in TQ is the Liouville vector field Δ generating dilations along the fibres. In coordinates as above its local expression is

$$\Delta = v^i \frac{\partial}{\partial v^i}$$

Both objects can be used to characterize SODE fields: Γ is a SODE iff $S(\Gamma) = \Delta$.

We recall that if T is a $(1,1)$ -tensor field on M , the Nijenhuis tensor N_T of T is a $(1,2)$ skew-symmetric tensor field defined by

$$N_T(X_1, X_2) = [TX_1, TX_2] + T^2([X_1, X_2]) - T([TX_1, X_2]) - T([X_1, TX_2])$$

where X_1 and X_2 are arbitrary vector fields in M . The tensor field T is integrable when N_T is zero. As an interesting example, the Nijenhuis tensor N_S of the vertical endomorphism S vanishes, and therefore S is integrable.

A fundamental property of SODE fields Γ is that the Lie derivative of the vertical endomorphism, $\mathcal{L}_\Gamma S$, satisfies $(\mathcal{L}_\Gamma S)^2 = I$, and therefore it allows us to construct two projection operators, the first one $P_V = \frac{1}{2}(I + \mathcal{L}_\Gamma S)$, with range

in the τ -vertical subbundle, and the second one $P_H = I - P_V = \frac{1}{2}(I - \mathcal{L}_\Gamma S)$. They give us a splitting of the tangent space providing a generalized non-linear connection. If the local expression of Γ in natural coordinates (x^i, v^i) on the tangent bundle, is

$$\Gamma = v^i \frac{\partial}{\partial x^i} + f^i(x, v) \frac{\partial}{\partial v^i}, \quad (2.1)$$

then the coefficients of the associated connection are given by

$$\Gamma_j^i = -\frac{1}{2} \frac{\partial f^i}{\partial v^j}. \quad (2.2)$$

A local basis for the linear space of vector fields on TQ adapted to this connection is

$$H_i = \frac{\partial}{\partial x^i} - \Gamma_i^j \frac{\partial}{\partial v^j} \quad V_i = \frac{\partial}{\partial v^i}, \quad (2.3)$$

the corresponding dual basis for $\wedge^1(TQ)$ being

$$H^i = dx^i \quad V^i = dv^i + \Gamma_j^i dx^j. \quad (2.4)$$

We will throughout use this base.

It is an easy matter to check that

$$[H_i, H_j] = R_{ij}^k V_k \quad [H_i, V_j] = \Gamma_{ij}^k V_k \quad [V_i, V_j] = 0, \quad (2.5)$$

where R_{ij}^k and Γ_{ij}^k are given by

$$\begin{aligned} R_{ij}^k &= \frac{\partial \Gamma_i^k}{\partial x^j} - \frac{\partial \Gamma_j^k}{\partial x^i} - \frac{\partial \Gamma_i^l}{\partial v^j} \Gamma_j^l + \frac{\partial \Gamma_j^l}{\partial v^i} \Gamma_i^l, \\ \Gamma_{ij}^k &= \frac{\partial \Gamma_i^k}{\partial v^j} = \frac{\partial \Gamma_j^k}{\partial v^i}. \end{aligned} \quad (2.6)$$

We also have

$$\begin{aligned} [\Gamma, H_i] &= \Gamma_j^i H_j + \Phi_j^i V_j \\ [\Gamma, V_i] &= -H_i + \Gamma_j^i V_j, \end{aligned} \quad (2.7)$$

where the functions Φ_j^i are defined by

$$\Phi_j^i = -\frac{\partial f^i}{\partial x^j} - \Gamma_k^i \Gamma_j^k - \Gamma(\Gamma_j^i). \quad (2.8)$$

The coordinate expression of the vertical endomorphism and its Lie derivatives are

$$\begin{aligned} S &= V_i \otimes H^i \\ \mathcal{L}_\Gamma S &= -H_i \otimes H^i + V_i \otimes V^i \\ \mathcal{L}_\Gamma \mathcal{L}_\Gamma S &= -2H_i \otimes V^i - 2\Phi_j^i V_i \otimes H^j, \end{aligned} \quad (2.9)$$

From this expressions it is easy to see that $S \circ \mathcal{L}_\Gamma S = -S$ and $\mathcal{L}_\Gamma S \circ S = S$, where the symbol $\circ$ denotes the usual composition law.

The theory of the symmetry for SODE fields Γ and Lagrangian mechanics makes an extensive use of the set $\mathfrak{X}_\Gamma$ of Newtonoid vector fields [16, 17]:

$$\mathfrak{X}_\Gamma = \{ X \in \mathfrak{X}(TQ) \mid S(\mathcal{L}_\Gamma X) = 0 \}, \quad (2.10)$$

That is, X is Newtonoid with respect to Γ if it is of the form

$$X = X^i H_i + (\Gamma^i X^i + \Gamma_j^i X^j) V_i. \quad (2.11)$$

The product $f \star X = fX + \Gamma f S(X)$ endows $\mathfrak{X}_\Gamma$ with a $C^\infty(TQ)$ -structure.

As a representation for the dual of $\mathfrak{X}_\Gamma$, Sarti [16] proposed the following set

$$\mathfrak{X}_\Gamma^* = \left\{ \alpha \in \wedge^1(TQ) \mid \alpha - \mathcal{L}_\Gamma(S^*(\alpha)) = 0 \right\}. \quad (2.12)$$

The product by functions in $\mathfrak{X}_\Gamma^*$ is defined by $f \star \alpha = f\alpha + \Gamma f S^*(\alpha)$, and the duality is given by the bilinear $(X, \alpha) = (S(X), \alpha)$. An element of $\mathfrak{X}_\Gamma^*$ is of the form $\alpha = \alpha_i H^i + \{\Gamma \alpha_i - \alpha_j \Gamma_j^i\} V^i$.

There exists other relevant set of 1-forms which can also be considered as the dual of $\mathfrak{X}_\Gamma$:

$$\mathfrak{M}_\Gamma^* = \left\{ \alpha \in \wedge^1(TQ) \mid S^*(\mathcal{L}_\Gamma \alpha) = 0 \right\}, \quad (2.13)$$

whose elements are of the form $\alpha = \alpha_i H^i - \{\Gamma \alpha_i - \alpha_j \Gamma_j^i\} V^i$. It is endowed with a $C^\infty(TQ)$ -module structure when the product by functions is defined by $f \odot \alpha = f\alpha - \Gamma f S^*(\alpha)$.

For the sake of completeness we will mention the set of vector fields in TQ

$$\mathfrak{M}_\Gamma = \{ X \in \mathfrak{X}(TQ) \mid X + \mathcal{L}_\Gamma(S(X)) = 0 \} \quad (2.14)$$

which can be endowed with a $C^\infty(TQ)$ -module structure by defining the product $f \odot X = fX - \Gamma f S(X)$. Its elements are of the form $X = X^i H_i - \{\Gamma^i X^i + X^j \Gamma_j^i\} V_i$.

Finally, we mention that $\mathcal{L}_\Gamma S$ is an isomorphism between $\mathfrak{X}_\Gamma$ and $\mathfrak{M}_\Gamma$, and that $\mathcal{L}_\Gamma S^*$ is an isomorphism between $\mathfrak{X}_\Gamma^*$ and $\mathfrak{M}_\Gamma^*$. Another important fact is that $\mathfrak{X}_\Gamma$ contains the set of symmetries of Γ and $\mathfrak{X}_\Gamma^*$ contains the set of Γ -invariant 1-forms.

We will now show that some tensor fields leaving invariant the set of Newtonoid vector fields give rise to Lax equations.

Theorem 2.1. *A $(1,1)$ type tensor field R on TQ will preserve the set $\mathfrak{X}_\Gamma$ if and only if*

$$[S, R] = 0 \quad \text{and} \quad S \circ \mathcal{L}_\Gamma R = 0. \quad (2.15)$$

The proof can be found in [18].

Theorem 2.2. *Let R be a $(1,1)$ -tensor preserving $\mathfrak{X}_\Gamma$ and commuting with $\mathcal{L}_\Gamma S$. Then, $\text{Tr}(R^k)$ is a constant of motion, for every natural number $k \in \mathbb{N}$.*

Dem: If R preserves $\mathfrak{X}_\Gamma$, then $[S, R] = 0$ and $S \circ \mathcal{L}_\Gamma R = 0$. If, furthermore, $[R, \mathcal{L}_\Gamma S] = 0$, then

$$\mathcal{L}_\Gamma[S, R] = [\mathcal{L}_\Gamma S, R] + [S, \mathcal{L}_\Gamma R] = -\mathcal{L}_\Gamma R \circ S,$$

so that $\mathcal{L}_\Gamma R \circ S = 0$. Therefore we have that $P_H \circ \mathcal{L}_\Gamma R$ and $\mathcal{L}_\Gamma R \circ P_V$ vanish. Since $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B)$, $\text{Tr}(A \circ B) = \text{Tr}(B \circ A)$ and $\mathcal{L}_\Gamma(\text{Tr}(A)) = \text{Tr}(\mathcal{L}_\Gamma A)$, we obtain

$$\mathcal{L}_\Gamma(\text{Tr}(R)) = \text{Tr}(\mathcal{L}_\Gamma R \circ (P_H + P_V)) = \text{Tr}(P_H \circ \mathcal{L}_\Gamma R) + \text{Tr}(\mathcal{L}_\Gamma R \circ P_V) = 0.$$

Let us remark that for any natural number k , R^k preserves $\mathfrak{X}_\Gamma$ and commutes with $\mathcal{L}_\Gamma S$ too, from which the result follows. $\square$

The property $[R, \mathcal{L}_\Gamma S] = 0$ is equivalent to say that R commutes with the horizontal and vertical projections. Thus, R does not mix horizontal and vertical components.

The expression in the basis (H, V) of a $(1,1)$ -tensor preserving $\mathfrak{X}_\Gamma$ is :

$$[R]_{(H,V)} = \begin{pmatrix} U & 0 \\ \Gamma U + [\Gamma, U] & U \end{pmatrix}, \quad (2.16)$$

where Γ is the matrix whose elements are Γ_j^i . Then, R commutes with $\mathcal{L}_\Gamma S$ if and only if

$$\Gamma U + [\Gamma, U] = 0 \quad (2.17)$$

which is but a Lax equation. The trace of U^k will therefore be a (locally defined) constant of motion. Note that $\text{Tr}(R^k) = 2 \text{Tr}(U^k)$ and that the dimension of the Lax equation so obtained is $\dim M$ and not $2 \dim M$.

In general, the tensor R is not invariant. In the base (H, V) the matrix representation of $\mathcal{L}_\Gamma R$ is

$$[\mathcal{L}_\Gamma R]_{(H,V)} = \begin{pmatrix} 0 & 0 \\ [\Phi, A] & 0 \end{pmatrix}, \quad (2.18)$$

from where we see that R will be invariant iff $[\Phi, A] = 0$, which is the local expression of the following global result.

Theorem 2.3. *Let R be a $(1,1)$ -tensor preserving $\mathfrak{X}_\Gamma$. Then R is invariant iff R commutes with $\mathcal{L}_\Gamma S$ and $\mathcal{L}_\Gamma \mathcal{L}_\Gamma S$.*

Dem: If R is invariant and preserves $\mathfrak{X}_\Gamma$, then taking Lie derivatives WRT Γ in $[R, S] = 0$ we get

$$0 = \mathcal{L}_\Gamma[R, S] = [R, \mathcal{L}_\Gamma S],$$

and taking Lie derivatives WRT Γ again,

$$0 = \mathcal{L}_\Gamma[R, \mathcal{L}_\Gamma S] = [R, \mathcal{L}_\Gamma \mathcal{L}_\Gamma S].$$

Conversely, if R preserves $\mathfrak{X}_\Gamma$ and commutes with $\mathcal{L}_\Gamma S$ and $\mathcal{L}_\Gamma \mathcal{L}_\Gamma S$, we know from the proof of the preceding Theorem that this implies that $P_H \circ \mathcal{L}_\Gamma R$ and $\mathcal{L}_\Gamma R \circ P_V$ vanish. Taking Lie derivatives WRT Γ in $[R, \mathcal{L}_\Gamma S] = 0$ we will obtain $[\mathcal{L}_\Gamma R, \mathcal{L}_\Gamma S] = 0$, from where $\mathcal{L}_\Gamma R$ commutes with the horizontal and vertical projections. Hence

$$\mathcal{L}_\Gamma R = \mathcal{L}_\Gamma R \circ (P_H + P_V) = P_H \circ \mathcal{L}_\Gamma R + \mathcal{L}_\Gamma R \circ P_V = 0.$$

$\square$

Notice that if R preserves $\mathfrak{X}_\Gamma$, then R^* preserves $\mathfrak{X}_\Gamma^*$. It is also worth to note that not every Γ -invariant tensor field preserves $\mathfrak{X}_\Gamma$ since in general they do not commute with the vertical endomorphism.

We will show an explicit example. Let $Q = \mathbb{R}^2$ and Γ be the SODE

$$\Gamma = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} - 2(v_1 + v_2) \frac{\partial}{\partial v_1} - 2v_2 \frac{\partial}{\partial v_2}.$$

The tensor field R on TQ given in the base (H, V) by

$$R = \begin{pmatrix} \left(1 + v_2 e^{-\frac{1}{v_2}}\right)^2 & -\frac{1}{2} v_1 v_2 e^{-2\frac{1}{v_2}} & 0 & 0 \\ 0 & 1 + 2v_2 e^{-\frac{1}{v_2}} & 0 & 0 \\ 0 & 0 & \left(1 + v_2 e^{-\frac{1}{v_2}}\right)^2 & -\frac{1}{2} v_1 v_2 e^{-2\frac{1}{v_2}} \\ 0 & 0 & 0 & 1 + 2v_2 e^{-\frac{1}{v_2}} \end{pmatrix}$$

preserves $\mathfrak{X}_\Gamma$ and commutes with $\mathcal{L}_\Gamma S$ but it is not Γ -invariant.

In Lagrangian Mechanics one of the ways for obtaining Lax equations is from a bi-Lagrangian system. This situation can be generalized as follows. We will say that a 2-form Ω in TQ maps $\mathfrak{X}_\Gamma$ in $\mathfrak{M}_\Gamma^*$ when

$$\hat{\Omega}: \mathfrak{X}(TQ) \longrightarrow \wedge^1(TQ) \\ X \longmapsto i_X \Omega \quad (2.19)$$

maps $\mathfrak{X}_\Gamma$ in $\mathfrak{M}_\Gamma^*$.

Proposition 2.4. A 2-form Ω in TQ maps $\mathfrak{X}_\Gamma$ in $\mathfrak{M}_\Gamma^2$ if and only if

- (1) The vertical distribution is isotropic with respect to Ω , i.e., the 2-form Ω vanishes on any couple of vectors of the vertical distribution.
- (2) $\mathcal{L}_\Gamma \Omega$ is a semibasic 2-form.

Dem: Let us suppose that $\hat{\Omega}(\mathfrak{X}_\Gamma) \subset \mathfrak{M}_\Gamma^2$ and let $X \in \mathfrak{X}_\Gamma$. Then, for any function f in TQ , $f \circ \hat{\Omega}(X)$ and $\hat{\Omega}(f \star X)$ lie into $\mathfrak{M}_\Gamma^2$, and the same is true for their difference

$$\hat{\Omega}(f \star X) - f \circ \hat{\Omega}(X) = (\Gamma f)(\hat{\Omega} \circ S + S^* \circ \hat{\Omega})(X).$$

Since this is true for any $f \in C^\infty(TQ)$ and $X \in \mathfrak{X}_\Gamma$, we get $\hat{\Omega} \circ S + S^* \circ \hat{\Omega} = 0$, which implies that the vertical distribution is isotropic wrt Ω . Moreover, if $X \in \mathfrak{X}_\Gamma$ and $\hat{\Omega}(X) \in \mathfrak{M}_\Gamma^2$, then $\mathcal{L}_\Gamma(\hat{\Omega}(X))$ is semibasic:

$$\begin{aligned} S^*(\mathcal{L}_\Gamma(\hat{\Omega}(X))) &= S^*((\mathcal{L}_\Gamma \hat{\Omega})(X)) + S^*(\hat{\Omega}(\mathcal{L}_\Gamma X)) \\ &= S^*((\mathcal{L}_\Gamma \hat{\Omega})(X)) + \hat{\Omega}(S(\mathcal{L}_\Gamma X)) \\ &= S^*((\mathcal{L}_\Gamma \hat{\Omega})(X)), \end{aligned}$$

from where we see that $(\mathcal{L}_\Gamma \hat{\Omega})(X)$ is semibasic, for any $X \in \mathfrak{X}_\Gamma$. This implies that $\mathcal{L}_\Gamma \Omega$ is semibasic too.

Conversely, if $X \in \mathfrak{X}_\Gamma$ and $Y \in \mathfrak{X}(TQ)$, then

$$S^*(\mathcal{L}_\Gamma(\hat{\Omega}(X)))(Y) = (\mathcal{L}_\Gamma \Omega)(X, S(Y)) + \Omega(\mathcal{L}_\Gamma X, S(Y)),$$

and if Ω satisfies (1) and (2), both terms vanish and therefore $\hat{\Omega}(X) \in \mathfrak{M}_\Gamma^2$. $\square$

From the proof we see that conditions (1) and (2) are equivalent to

$$\hat{\Omega} \circ S + S^* \circ \hat{\Omega} = 0 \quad \text{and} \quad S \lrcorner \mathcal{L}_\Gamma \Omega = 0. \quad (2.20)$$

Theorem 2.5. If a 2-form Ω maps $\mathfrak{X}_\Gamma$ in $\mathfrak{M}_\Gamma^2$ then

- (1) $\hat{\Omega} \circ \mathcal{L}_\Gamma S + \mathcal{L}_\Gamma S^* \circ \hat{\Omega} = 0$.
- (2) The horizontal distribution is isotropic wrt Ω .
- (3) For each couple of vector fields in the base M the following relation holds:

$$\Omega(X^H, Y^V) = \Omega(Y^H, X^V). \quad (2.21)$$

Dem:

- (1) It suffices to take Lie derivatives wrt Γ on $\hat{\Omega} \circ S + S^* \circ \hat{\Omega} = 0$ and we get

$$\hat{\Omega} \circ \mathcal{L}_\Gamma S + \mathcal{L}_\Gamma S^* \circ \hat{\Omega} = -\mathcal{L}_\Gamma \hat{\Omega} \circ S - S^* \circ \mathcal{L}_\Gamma \hat{\Omega}.$$

The two terms on the right hand side vanishes because $\mathcal{L}_\Gamma \Omega$ is semibasic.

- (2) Taking into account that $(\mathcal{L}_\Gamma S)(H) = -H$ for any horizontal vector field H ,

$$\hat{\Omega}(H) = -\hat{\Omega}(\mathcal{L}_\Gamma S)(H) = (\mathcal{L}_\Gamma S)^*(\hat{\Omega}(H)).$$

and then, for any pair H, H' of horizontal vector fields

$$\begin{aligned} \Omega(H, H') &= \langle H', \hat{\Omega}(H) \rangle = \langle H', (\mathcal{L}_\Gamma S)^*(\hat{\Omega}(H)) \rangle \\ &= \langle (\mathcal{L}_\Gamma S)(H'), \hat{\Omega}(H) \rangle = -\langle H', \hat{\Omega}(H) \rangle \\ &= -\Omega(H, H'). \end{aligned}$$

- (3) Let be X and $Y \in \mathfrak{X}(M)$. Since Ω vanishes on any pair of vertical vector fields, $\Omega(X^V, Y^V) = 0$, by taking Lie derivatives wrt Γ we get

$$\mathcal{L}_\Gamma \Omega(X^V, Y^V) + \Omega(\mathcal{L}_\Gamma X^V, Y^V) + \Omega(X^V, \mathcal{L}_\Gamma Y^V) = 0.$$

The first term is zero because $\mathcal{L}_\Gamma \Omega$ is semibasic. The property follows by taking into account that $\mathcal{L}_\Gamma X^V = X^c - 2X^H = -X^H + (X^c - X^H)$ and that $X^c - X^H$ is vertical. $\square$

It follows that if Ω is regular, i.e., the map $\hat{\Omega} : \mathfrak{X}(\mathfrak{M}) \rightarrow \wedge^1(\mathfrak{M})$ defined by contraction is invertible, both the horizontal and vertical distribution are Lagrangian distributions, because they are isotropic distributions of dimension n . Notice also that as a consequence of $\hat{\Omega} \circ \mathcal{L}_\Gamma S + (\mathcal{L}_\Gamma S)^* \circ \hat{\Omega} = 0$, $\hat{\Omega}$ maps $\mathfrak{M}_\Gamma$ in $\mathfrak{X}_\Gamma^*$.

In the base (H, V) , the expression of Ω is

$$[\Omega]_{(H,V)} = \begin{pmatrix} 0 & -g_{ij} \\ g_{ij} & 0 \end{pmatrix}. \quad (2.22)$$

where g_{ij} is a symmetric matrix satisfying

$$\Gamma g_{ij} + g_{ik} \Gamma_j^k + g_{kj} \Gamma_i^k = 0. \quad (2.23)$$

and the expression of $\mathcal{L}_\Gamma \Omega$ reads as follows

$$[\mathcal{L}_\Gamma \Omega]_{(H,V)} = \begin{pmatrix} g_{ik} \Phi_j^k + \Phi_i^k g_{kj} & 0 \\ 0 & 0 \end{pmatrix}, \quad (2.24)$$

Therefore, Ω is Γ -invariant iff $g_{ik} \Phi_j^k + \Phi_i^k g_{kj} = 0$, which is but the local expression of the following statement.

Theorem 2.6. *Let Ω be a 2-form mapping $\mathfrak{X}_\Gamma$ in $\mathfrak{M}_\Gamma^*$. Then*

$$\mathcal{L}_\Gamma \Omega = 0 \quad \text{if and only if} \quad \hat{\Omega} \circ \mathcal{L}_\Gamma(\mathcal{L}_\Gamma S) + \mathcal{L}_\Gamma(\mathcal{L}_\Gamma S)^* \circ \hat{\Omega} = 0. \quad (2.25)$$

Dem: Taking Lie derivatives w.r.t Γ in $\hat{\Omega} \circ \mathcal{L}_\Gamma S + (\mathcal{L}_\Gamma S)^* \circ \hat{\Omega} = 0$ we obtain

$$\mathcal{L}_\Gamma \hat{\Omega} \circ \mathcal{L}_\Gamma S + (\mathcal{L}_\Gamma S)^* \circ \mathcal{L}_\Gamma \hat{\Omega} = -\hat{\Omega} \circ \mathcal{L}_\Gamma(\mathcal{L}_\Gamma S) - \mathcal{L}_\Gamma(\mathcal{L}_\Gamma S)^* \circ \hat{\Omega}.$$

If Ω is Γ -invariant, the left hand side vanishes and therefore $\hat{\Omega} \circ \mathcal{L}_\Gamma(\mathcal{L}_\Gamma S) + \mathcal{L}_\Gamma(\mathcal{L}_\Gamma S)^* \circ \hat{\Omega} = 0$.

Conversely, if this last relation holds, then

$$\begin{aligned} (\mathcal{L}_\Gamma \Omega)(H, H') &= -(\mathcal{L}_\Gamma \Omega)(\mathcal{L}_\Gamma S)(H), H') \\ &= (\mathcal{L}_\Gamma \Omega)(H, \mathcal{L}_\Gamma S)(H') \\ &= -(\mathcal{L}_\Gamma \Omega)(H, H'), \end{aligned}$$

and consequently, $\mathcal{L}_\Gamma \Omega$ is zero on the horizontal distribution. It vanishes since $\mathcal{L}_\Gamma \Omega$ is also semibasic. $\square$

It is to be remarked that when Ω is the Cartan 2-form $\omega_L = -d(S^*(dL))$ defined by a Lagrangian L , then it maps $\mathfrak{X}_\Gamma$ in $\mathfrak{M}_\Gamma^*$. This fact is a consequence of the vanishing of the Nijenhuis tensor of S and the Γ -invariance of ω_L . More generally, consider the case of a non-conservative system described by a Lagrangian L and a system of non-conservative forces depending only on positions. The non-conservative forces are described by a basic 1-form $\alpha \in \wedge^1(Q)$ and the dynamical field Γ is defined by the equation $i_\Gamma \omega_L = dE_L - \tau^* \alpha$. Then ω_L maps $\mathfrak{X}_\Gamma$ in $\mathfrak{M}_\Gamma^*$.

An interesting case occurs when there exist two 2-forms mapping $\mathfrak{X}_\Gamma$ in $\mathfrak{M}_\Gamma^*$, when at least one of them is regular. Then we can define a $(1, 1)$ -tensor field preserving $\mathfrak{X}_\Gamma$ and commuting with $\mathcal{L}_\Gamma S$.

Theorem 2.7. *Let Ω_1 and $\Omega_2 \in \wedge^2(TQ)$ be such that they map $\mathfrak{X}_\Gamma$ in $\mathfrak{M}_\Gamma^*$, and assume that Ω_1 is regular. There exists a $(1, 1)$ -type tensor R such that $\hat{\Omega}_2 = \hat{\Omega}_1 \circ R$ and satisfies*

- (1) $\hat{\Omega}_1 \circ R = R^* \circ \hat{\Omega}_1$,
- (2) R preserves $\mathfrak{X}_\Gamma$, and
- (3) $[R, \mathcal{L}_\Gamma S] = 0$.

Conversely, if $\hat{\Omega}_1$ is a 2-form mapping $\mathfrak{X}_\Gamma$ in $\mathfrak{M}_\Gamma^*$ and R is a $(1, 1)$ -tensor satisfying (1) and (2), then $\hat{\Omega}_2$ defined by $\hat{\Omega}_2 = \hat{\Omega}_1 \circ R$ is a 2-form mapping $\mathfrak{X}_\Gamma$ in $\mathfrak{M}_\Gamma^*$. Moreover, if $\hat{\Omega}_1$ is regular, then (3) is a consequence of (1) and (2).

Dem: Since Ω_1 is regular, we define $R = \hat{\Omega}_1^{-1} \circ \hat{\Omega}_2$ we get that R is such that $\hat{\Omega}_2 = \hat{\Omega}_1 \circ R$ and R preserves $\mathfrak{X}_\Gamma$. The skew-symmetry of Ω_2 implies

(1). Taking into account that $\hat{\Omega}_1 \circ \mathcal{L}_\Gamma S + (\mathcal{L}_\Gamma S)^* \circ \hat{\Omega}_1 = 0$, we will obtain $\hat{\Omega}_1^{-1} \circ (\mathcal{L}_\Gamma S)^* + \mathcal{L}_\Gamma S \circ \hat{\Omega}_1^{-1} = 0$, and then

$$\begin{aligned} [R, \mathcal{L}_\Gamma S] &= \hat{\Omega}_1^{-1} \circ \hat{\Omega}_2 \circ \mathcal{L}_\Gamma S - \mathcal{L}_\Gamma S \circ \hat{\Omega}_1^{-1} \circ \hat{\Omega}_2 \\ &= -\hat{\Omega}_1^{-1} \circ (\mathcal{L}_\Gamma S)^* \circ \hat{\Omega}_2 + \hat{\Omega}_1^{-1} \circ (\mathcal{L}_\Gamma S)^* \circ \hat{\Omega}_2 = 0. \end{aligned}$$

Conversely, when R satisfies the abovementioned conditions, if we define $\hat{\Omega}_2$ by $\hat{\Omega}_2 = \hat{\Omega}_1 \circ R$, then (1) implies that $\hat{\Omega}_2$ is a skew-symmetric tensor. From $\hat{\Omega}_2 = \hat{\Omega}_1 \circ R$ we see that $\hat{\Omega}_2$ maps $\mathfrak{X}_\Gamma$ in $\mathfrak{M}_\Gamma^*$ because R preserves $\mathfrak{X}_\Gamma$ and $\hat{\Omega}_1$ maps $\mathfrak{X}_\Gamma$ in $\mathfrak{M}_\Gamma^*$. Furthermore, if $\hat{\Omega}_1$ is regular, then (1) implies that R^* preserves $\mathfrak{M}_\Gamma^*$. Therefore, R^* and $(\mathcal{L}_\Gamma S)^* \circ R^* \circ (\mathcal{L}_\Gamma S)^*$ preserve $\mathfrak{X}_\Gamma$. Since

$$S^* \circ (R^* - (\mathcal{L}_\Gamma S)^* \circ R^* \circ (\mathcal{L}_\Gamma S)^*) = S^* \circ R^* - R^* \circ S^* = 0$$

we obtain that $R - \mathcal{L}_\Gamma S \circ R \circ \mathcal{L}_\Gamma S$ takes vertical values and preserves $\mathfrak{X}_\Gamma$. Now, $\mathfrak{X}_\Gamma \cap \mathfrak{X}^V(TQ) = \{0\}$, and then $R = \mathcal{L}_\Gamma S \circ R \circ \mathcal{L}_\Gamma S$, that is to say, $[R, \mathcal{L}_\Gamma S] = 0$. $\square$

3. SELF-DUAL DERIVATIONS AND LAX EQUATIONS

We have seen in the last section that there are Lax equations which are not of the form $\mathcal{L}_X \Lambda = 0$ or $\mathbb{D}_X \Lambda = 0$. In this section we will propose a more general geometrical setting where it is possible to include these more general type of Lax equations.

Let $\pi: E \rightarrow M$ be a vector bundle, and denote by $\Sigma(\pi)$ the $C^\infty(M)$ -module of sections of such bundle. By a derivation of $\Sigma(\pi)$ we mean a map $D: \Sigma(\pi) \rightarrow \Sigma(\pi)$ satisfying

- (1) $D(\sigma_1 + \lambda\sigma_2) = D\sigma_1 + \lambda D\sigma_2$, for $\lambda \in \mathbb{R}$ and $\sigma_1, \sigma_2 \in \Sigma(\pi)$; and
- (2) there exists a map $\bar{D}: C^\infty(M) \rightarrow C^\infty(M)$ such that $D(f\sigma) = (\bar{D}f)\sigma + f(D\sigma)$ for $f \in C^\infty(M)$ and $\sigma \in \Sigma(\pi)$.

It follows that $\bar{D}$ is a derivation of the algebra $C^\infty(M)$, and thus there exists a vector field X on M such that $\bar{D}f = Xf$. We will simplify the notation and we will write just D instead of $\bar{D}$.

The action of D can be extended to $\Sigma(\pi^*)$, the set of sections of the dual bundle $\pi^*: E^* \rightarrow M$, by imposing the following duality rule:

$$D(\alpha, \sigma) = \langle \alpha, D\sigma \rangle + \langle D\alpha, \sigma \rangle, \quad (3.1)$$

for $\sigma \in \Sigma(\pi)$, $\alpha \in \Sigma(\pi^*)$, and where $\langle \cdot, \cdot \rangle$ stands for the usual pairing between vectors and covectors.

Moreover, D can be extended to a derivation of the $C^\infty(M)$ -module of sections of the tensor bundle $\pi^{p,q}: E \otimes \dots \otimes E \otimes E^* \otimes \dots \otimes E^* \rightarrow M$, by imposing the Leibnitz rule for the tensor product

$$D(T_1 \otimes T_2) = (DT_1) \otimes T_2 + T_1 \otimes (DT_2). \quad (3.2)$$

Such derivations are not the more general derivations of $\Sigma(\pi^{p,q})$ and we will call them self-dual derivations.

A self dual derivation is determined by its action on $C^\infty(M)$ and $\Sigma(\pi)$. The action on $C^\infty(M)$ is given by the associated vector field X . If we fix a local base $\{E_\alpha\}_{\alpha=1}^m$ of $\Sigma(\pi)$, then the action of D on $\Sigma(\pi)$ is completely determined by $m \times m$ functions D_α^β on M , called the coefficients of the derivation in the base $\{E_\alpha\}$, and defined by

$$DE_\alpha = D_\alpha^\beta E_\beta. \quad (3.3)$$

The action on $\Sigma(\pi^*)$ is given by the action on the dual base $\{E^\alpha\}$, which is

$$DE^\alpha = -D_\beta^\alpha E^\beta. \quad (3.4)$$

Indeed, from $(E^\alpha, E_\beta) = \delta_\beta^\alpha$ and the duality rule we have

$$0 = D(E^\alpha, E_\beta) = \langle DE^\alpha, E_\beta \rangle + \langle E^\alpha, DE_\beta \rangle = D_\beta^\alpha + \langle E^\alpha, DE_\beta \rangle,$$

from where the result follows.

We propose the following definition of a (global) Lax equation: by a Lax equation on a manifold M we mean an equation of the form

$$DU = 0, \quad (3.5)$$

where D is a self-dual derivation of the set of sections of some bundle $\pi: E \rightarrow M$, and U is a section of the (1,1) type tensor bundle $\pi^{1,1}: E \otimes E^* \rightarrow M$.

To justify this definition we should prove that, locally, our equation is of the form $\dot{L} = [L, M]$. If $\{E_\alpha\}$ is a local base of $\Sigma(\pi)$, then the expression of U is of the form $U = U_\alpha^\beta E_\alpha \otimes E^\beta$, and thus

$$\begin{aligned} DU &= (DU_\alpha^\beta) E_\alpha \otimes E^\beta + U_\alpha^\beta (DE_\alpha) \otimes E^\beta + U_\alpha^\beta E_\alpha \otimes (DE^\beta) \\ &= (XU_\alpha^\beta + D_\gamma^\alpha U_\beta^\gamma - U_\gamma^\alpha D_\beta^\gamma) E_\alpha \otimes E^\beta, \end{aligned} \quad (3.6)$$

where X is the vector field associated to D and D_α^β are the coefficients of the derivation in the base $\{E_\alpha\}$. Thus $DU = 0$ is equivalent to

$$XU_\alpha^\beta = U_\gamma^\alpha D_\beta^\gamma - D_\gamma^\alpha U_\beta^\gamma, \quad (3.7)$$

which is of the form of a local Lax equation.

Given a covariant derivative $\mathbb{D}$ on the bundle $\pi: E \rightarrow M$, every self-dual derivation D of $\Sigma(\pi)$ can be uniquely decomposed as a sum [20]

$$D = \mathbb{D}_X + \mu_A, \quad (3.8)$$

where $X \in \mathfrak{X}(M)$ is the vector field associated to D and $A \in \Sigma(\pi^{1,1}) = \text{End}(E)$. Here μ_A is defined on $\Sigma(\pi)$ by $\mu_A \sigma = A(\sigma)$ for $\sigma \in \Sigma(\pi)$, and hence $\mu_A \sigma = -A^*(\sigma)$. It follows that $\mu_A U = [U, A]$, and then $DU = 0$ reads

$$\mathbb{D}_X U = [U, A]. \quad (3.9)$$

When X is a vector field without singular points (i.e. $X(x) \neq 0 \forall x \in M$), then we can find a covariant derivative $\mathbb{D}'$ such that $D = \mathbb{D}'_X$, so that our Lax equation is of the form $\mathbb{D}'_X U = 0$. Indeed, if X has not singular points there exists $\rho \in \bigwedge^1(M)$ such that $\rho(X) = 1$. Then defining $\mathbb{D}'$ by

$$\mathbb{D}'_Z \sigma = \mathbb{D}_Z \sigma + \rho(Z) A(\sigma), \quad (3.10)$$

we have that $\mathbb{D}'_X \sigma = \mathbb{D}_X \sigma + A(\sigma) = (\mathbb{D}_X + \mu_A) \sigma$, so that $D = \mathbb{D}'_X$.

We consider now the case explained in Section 2. We consider the induced bundle $\tau^* \tau: \tau^*(TQ) \rightarrow TQ$, i.e. $E = \tau^*(TQ) = TQ \times_Q TQ$ and $\pi = \tau^* \tau = \text{pr}_1$. A section of π is called a vector field along τ , and more generally, a section of $\pi^{p,q}$ is called a (p,q) -type tensor field along τ [21]. Every vector field W on TQ can be decomposed uniquely in the form

$$W = X^H + Y^V, \quad (3.11)$$

where X and Y are vector fields along τ , and where the superscripts H and V means horizontal and vertical lift, respectively (see [22,23] for the details). Using this decomposition we can define a derivation ∇ of $\mathfrak{X}(\tau)$ and a (1,1)-type tensor field Φ along τ by means of the equation [24]

$$\mathcal{L}_\tau X^H = \{\nabla X\}^H + \{\Phi(X)\}^V. \quad (3.12)$$

The coefficients of the derivation ∇ are just the coefficients of the connection, and the matrix elements of Φ , which is called Jacobi endomorphism associated to Γ [24], are just the functions Φ_j^i defined in Section 2.

For a (1,1)-type tensor field U along τ we can define two tensor fields on TQ , U^H and U^V , called respectively the horizontal and vertical lift of U , by means of the relations

$$U^H(X^H) = \{U(X)\}^H \quad U^H(X^V) = \{U(X)\}^V, \quad (3.13)$$

and

$$U^V(X^H) = \{U(X)\}^V \quad U^V(X^V) = 0. \quad (3.14)$$

Then, a tensor field R preserves $\mathfrak{X}_\Gamma$ iff it is of the form $R = U^H + \{\nabla U\}^V$, and R commutes with $\mathcal{L}_\Gamma S$ iff $\nabla U = 0$. The last equation is the Lax equation (2.17) we got in Section 2. Moreover, such tensor field is Γ -invariant iff $[U, \Phi] = 0$.

Finally we mention that a SODE on Q also defines a linear connection $\tilde{\mathbb{D}}$ on the bundle $\tau^*\tau: \tau^*(TQ) \rightarrow TQ$ (see [25]). In terms of this connection we have that $\nabla = \tilde{\mathbb{D}}_\Gamma + \mu_t$, where t is the tension of the non-linear connection, so that $\nabla U = 0$ reads $\tilde{\mathbb{D}}_\Gamma U = [t, U]$. We recall that the tension t of the nonlinear connection defined by Γ , also called homogeneity torsion, is a vector valued 1-form that vanishes when the connection is linear and it is basic when the connection is affine [24].

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