Phase structure of the Higgs-Yukawa systems with chirally invariant lattice fermion actions

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We develop analytical technique for examining phase structure of $Z_2$, $U(1)$, and $SU(2)$ lattice Higgs-Yukawa systems with radially frozen Higgs fields and chirally invariant lattice fermion actions. The method is based on variational mean field approximation. We analyse phase diagrams of such systems with different forms of lattice fermion actions and demonstrate that it crucially depends both on the symmetry group and on the form of the action. We discuss location in the diagrams of possible non-trivial fixed points relevant to continuum physics, and argue that the candidates can exist only in $Z_2$ system with SLAC action and $U(1)$ systems with naive and SLAC actions.

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1. Introduction

Since discovering the first evidence of complexity of phase structure of Higgs-Yukawa systems [1], [2], [3] much has been done to understand them better (for a review and list of references see [4], [5]). There were studied phase structure and continuum limits of such systems with different lattice formulations and symmetry groups; proposed hypotheses on possible non-trivial fixed points and checked some of them. There were used numerical, analytical, and combined methods. Much has been understood about this, much is still to be answered. In particular, such questions as: which features of the phase diagrams are inherent properties of the Higgs-Yukawa model, and which are lattice artifacts; whether non-trivial fixed points exist at intermediate values of the Yukawa coupling; if so, what does distinguish such points on the phase diagrams and whether this depends on group of symmetry of the system, have not yet definite answers.

This paper, not claiming to solve definitely those unanswered problems, rather pursues the following aims: to give a simple analytical technique for examining on the same basis the phase structure of a wide class of lattice formulations of the Higgs-Yukawa systems with various symmetry groups; to compare the phase diagrams for some of such formulations for $Z_2$, $U(1)$, and $SU(2)$ symmetrical systems; to give additional arguments pro et contra those previous results which seem to be not well-established, and to draw one’s attention to interesting points which were left being not understood well.

We shall use variational mean field approximation and analyse the systems with radially frozen Higgs fields and chirally invariant lattice fermion actions possessing real fermionic determinants. These conditions enables us to calculate contributions of fermionic determinant into mean field free energy — the main problem of the method, in a ladder approximation [6], in other words, to sum up the contributions of leading and some of those of next-to-leading orders in inverse space-time dimension $1/D$ knowing only form of the free fermion propagator in the momentum space. Compared with other mean field calculations [7], [8], [9], [10], [11] this gives us two advantages: to analyse the phase diagrams of the systems with a wide class of lattice fermion actions, and, in fact, for any values of the Yukawa coupling, including the most interesting region of intermediate ones. On the other hand the above conditions do not allow us to discuss interesting features revealed in a model with non-frozen Higgs field [12] and to compare our results with those obtained with staggered fermions [3], [13] (except formulations of [9], [10], [11] with a number of staggered fermions is a multiple of $2^{D/2}$), and with results for the Wilson fermions [7], [14].
By virtue of the Nielsen-Ninomiya theorem [15], chirally invariant lattice fermion actions (providing they are bilinear in fermion fields and lattice translation invariant, the conditions we shall respect) must be either non-Hermitean, or non-local, otherwise they involve an equal number of the left-handed and right-handed Weyl fermions. Using our technique we cannot take non-Hermitean actions, being able to consider naive, non-local, and mirror fermion actions. In all the cases the fermions coupled to the Higgs fields in the same local way.

The outline and the results of the paper are as follows.

The systems under consideration are defined in Sect. 2. We use their usual lattice parameterization in terms of scalar hopping parameter $\kappa$ and Yukawa coupling $y$. In Sect. 3 we describe the method and approximations, and obtain the explicit formulae for second order phase transition lines $\kappa_{cr}(y)$ for the symmetry groups $Z_2$, $U(1)$, and $SU(2)$.

In Sect. 4 we apply these formulae to the systems with naive, SLAC, a maximally non-local (the Weyl), and a mirror fermion actions. We find that the phase diagram for all the systems show a kind of universality at $\kappa \geq 0$, while at $\kappa < 0$ they crucially depend on both fermion action and symmetry group. In particular, ferrimagnetic phase does not appear in any $Z_2$ systems, while appearing in $U(1)$ ones with the Weyl and mirror fermion actions, and in all $SU(2)$ systems.

In Sect. 5 we make mean field estimations of fermionic propagators and condensates along the ferromagnetic-paramagnetic phase transition lines. Both show different behaviour in weak and strong coupling regimes in all the systems. This enables us to speculate on locations of possible non-trivial fixed points. We conclude that the most interesting ones can exist in $Z_2$ system with SLAC action and in $U(1)$ systems with naive and SLAC actions. We also note impossibility of defining a chiral theory in paramagnetic phase of mirror fermion model at strong coupling.

Sect. 6 is a summary and discussion.

2. The system

The system is defined on a hypercubic $D$-dimensional ($D$ is even) lattice $\Lambda$ with sites numbered by $n = (n_1, ..., n_D)$, $-N/2 + 1 \leq n_\mu \leq N/2$ ($N$ is even, and eventually tends to infinity) and with lattice spacing $a = 1$; $\hat{\mu}$ is the unit vector along a lattice link in the positive $\mu$-direction. Dynamical variables are Dirac fermion fields $\psi_n$, $\bar{\psi}_n$, and scalar fields
\( \Phi_n \in G \), where group \( G = Z_2, U(1), \) or \( SU(2) \), so that \( \Phi_n^4 = 1 \). We use the following representations for the group elements:

\[
\Phi = T \phi \equiv \sum_{s=0}^p T^s \phi^s,
\]

with real \( \phi^s \) such that \( \sum_{s=0}^p (\phi^s)^2 = 1 \). Hence we have

\[
\begin{align*}
    p &= 0, \quad T = 1, \quad Z_2, \\
    p &= 1, \quad T = (1, i), \quad U(1), \\
    p &= 3, \quad T = (1, i \tau^1, i \tau^2, i \tau^3), \quad SU(2).
\end{align*}
\]

We imply antiperiodic boundary conditions for the fermion and periodic for the scalar fields.

The system is defined by functional integrals

\[
Z[J] = \int \prod_n d\Phi_n d\psi_n d\bar{\psi}_n e^{-A[\Phi, \psi, \bar{\psi}]} + \sum_{n,s} J_n^s \phi_n^s
\]

with action

\[
A[\Phi, \psi, \bar{\psi}] = -2\kappa \sum_{n, \mu, s} \phi_n^s \phi_{n+\mu}^s + \sum_{m,n} \bar{\psi}_m \Phi_{mn} + y (P_L \Phi_{mn}^4 + P_R \Phi_{nm}) \delta_{mn} \psi_n,
\]

where \( d\Phi_n \) is the Haar measure on \( G \); \( \kappa \in (-\infty, \infty) \) is hopping parameter; \( y \) is the Yukawa coupling which without loss of generality will be considered non-negative; \( P_{L,R} = (1 \pm \gamma_D + 1)/2 \) are chiral projecting operators; \( \partial \) is a lattice Dirac operator determining the form of the fermion action (the systems with mirror fermions which we consider in Sect. 4.4 is reduced to this form).

We shall consider actions with operators \( \partial \) satisfying the properties

\[
\begin{align*}
    \phi_{mn} &= -\phi_{nm}, \\
    \phi_{mn} &= \int e^{ip(m-n)} \sum_{\mu} i \gamma_{\mu} L_{\mu}(p), \\
    L_{\mu}^*(p) &= L_{\mu}(p), \quad L_{\mu}(-p) = -L_{\mu}(p),
\end{align*}
\]

where \( \int_p \equiv \int d^D p/(2\pi)^D, \) \( p_\mu \in (-\pi, \pi) \), and we use the Hermitean \( \gamma \)-matrices: \( [\gamma_\mu, \gamma_\nu]_+ = 2\delta_{\mu\nu} \).

Action (2.4) is invariant under \( G \times G \) global chiral transformations

\[
\begin{align*}
    \psi_n &\to (h_L P_L + h_R P_R) \psi_n, \\
    \bar{\psi}_n &\to \bar{\psi}_n (P_R h_L^4 + P_L h_R^4), \\
    \Phi_n &\to h_L \Phi_n h_R^4,
\end{align*}
\]

with \( h_{L,R} \in G \).
3. Method and approximations

To analyze the phase diagrams of the system we use the variational mean field approximation [16] which becomes applicable to (2.3) after integrating out the fermions

\[ Z[J] = e^{-W[J]}\]

\[ = \int \prod_n d\Phi_n e^{2\kappa \sum_{n,\mu, a} \phi_n^a \phi_{n+\mu}^a + \ln \det[\Theta + y\bar{\Phi}]} + \sum_{n, a} J_n^a \phi_n^a, \]

where \( \bar{\Phi} \equiv (P_L \Phi_1 + P_R \Phi). \) Then for free energy of the system \( F = W[0] \) the method yields inequality

\[ F \leq F_{MF} = \inf \limits_{h_n^a} \left[ \sum_n u(h_n) + \sum_{n, a} h_n^a \langle \phi_n^a \rangle_h \right. \]

\[ \left. - \langle 2\kappa \sum_{n, \mu, a} \phi_n^a \phi_{n+\mu}^a + \ln \det[\Theta + y\bar{\Phi}] \rangle_h \right], \]

where \( h_n^a \) is the mean field with radial component \( h_n = \left[ \sum_a (h_n^a)^2 \right]^{1/2}, \)

\[ u(h_n) = -\ln \int d\Phi_n e^{\sum a h_n^a \phi_n^a} \]

\[ = -\ln \cosh h_n - \frac{1}{2} h_n^2 + \frac{1}{12} h_n^4 + O(h_n^6), \quad Z_2, \]

\[ = -\ln I_0(h_n) = -\frac{1}{4} h_n^2 + \frac{1}{64} h_n^4 + O(h_n^6), \quad U(1), \]

\[ = -\ln \frac{2}{h_n} I_1(h_n) = -\frac{1}{8} h_n^2 + \frac{1}{384} h_n^4 + O(h_n^6), \quad SU(2), \]

is the test free energy per lattice site \( [I_0(z) \text{ is the modified Bessel functions}] \), and

\[ \langle O[\phi] \rangle_h = e^{\sum_n u(h_n)} \int \prod_n d\Phi_n O[\phi] e^{\sum_{n, a} h_n^a \phi_n^a}. \]

The inequality (3.2) is expected to tend to equality in the limit of \( D \to \infty \) [16]. So, we can get some idea of the system at \( D = 4 \) studying \( F_{MF} \). From (3.3) it immediately follows that

\[ \langle \phi_n^a \rangle_h = -\frac{\partial}{\partial h_n^a} u(h_n) \]

\[ = h_n + O(h_n^3), \quad Z_2, \]

\[ = \frac{1}{2} h_n^2 + O(h_n^3), \quad U(1), \]

\[ = \frac{1}{4} h_n^2 + O(h_n^3), \quad SU(2); \]
In our case it has the following representations:

\[
\langle \phi_n^a \phi_n^b \rangle_h = \frac{\partial}{\partial h_m^a} u(h_m) \frac{\partial}{\partial h_m^b} u(h_m) + \delta_{mn} \frac{\partial}{\partial h_m^a} \frac{\partial}{\partial h_m^b} u(h_m) \\
= h_m h_n + \delta_{mn} (1 - h_n^2) + O(h_n^4), \quad Z_2, \\
= \frac{1}{4} h_m^a h_n^b + \frac{1}{2} \delta_{mn} \left[ \epsilon^a_b (1 - \frac{1}{8} h_n^2) - \frac{1}{4} h_m^a h_n^b \right] + O(h_n^4), \quad U(1), \\
= \frac{1}{16} h_m^a h_n^b + \frac{1}{4} \delta_{mn} \left[ \epsilon^a_b (1 - \frac{1}{24} h_n^2) - \frac{1}{12} h_m^a h_n^b \right] + O(h_n^4), \quad SU(2);
\]

and so on. Therefore the main problem is calculation of expectation value \( \langle \ln \det [\hat{\theta} + y\hat{\phi}] \rangle_h \).

In our case it has the following representations:

\[
\langle \ln \det [\hat{\theta} + y\hat{\phi}] \rangle_h = \ln \det [\theta] - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} y^{2n} \sum_{i_1, \ldots, i_{2n}} \text{tr} \left[ (\hat{\theta}_{i_1}^{-1} \hat{\theta}_{i_2}^{-1} \ldots \hat{\theta}_{i_{2n}}^{-1}) \langle \hat{\Phi}_{i_1}^\dagger \hat{\Phi}_{i_2} \ldots \hat{\Phi}_{i_{2n}} \rangle_h \right] \\
= c_2 2^{D/2} N^D \ln y \\
- \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} y^{2n} \sum_{i_1, \ldots, i_{2n}} \text{tr} \left[ (\hat{\phi}_{i_1} \hat{\phi}_{i_2} \hat{\phi}_{i_3} \ldots \hat{\phi}_{i_{2n}}) \langle \hat{\Phi}_{i_1} \hat{\Phi}_{i_2} \ldots \hat{\Phi}_{i_{2n}} \rangle_h \right],
\]

where \( \text{tr} \) stands for the trace over spinorial indices, as well as for group ones for \( SU(2) \) in which case factor \( c_2 = 2 \), otherwise \( c_2 = 1 \), and it has been taken into account that the trace of an odd number of \( \gamma \)-matrices vanishes; correlators of the Higgs fields in (3.7) in terms of their real components read as

\[
\langle \hat{\Phi}_{i_1}^\dagger \hat{\Phi}_{i_2} \ldots \hat{\Phi}_{i_{2n}} \rangle_h = \sum_{a_1, \ldots, a_{2n}} T^{a_1} \dagger T^{a_2} \ldots T^{a_{2n}} \langle \phi_{i_1}^{a_1} \phi_{i_2}^{a_2} \ldots \phi_{i_{2n}}^{a_{2n}} \rangle_h.
\]

We consider \( F_{MF} \) on translation invariant ansatz for \( h_n^a \)

\[
h_n^a = h^a + \epsilon_n h_{st}^a, \quad \epsilon_n = (-1) \sum_{\mu} n^\mu.
\]

Then the mean field equations are reduced to

\[
\frac{\partial}{\partial h} F_{MF} = 0, \quad \frac{\partial}{\partial h_{st}} F_{MF} = 0.
\]

In view of (3.5), the solutions of Eqs. (3.10), \( h^a \) ("magnetization") and \( h_{st}^a \) ("staggered magnetization"), in fact are the order parameters distinguishing the ferromagnetic (FM): \( h^a \neq 0, h_{st}^a = 0 \); antiferromagnetic (AM): \( h^a = 0, h_{st}^a \neq 0 \); paramagnetic (PM): both are zero, and ferrimagnetic (FI): both are nonzero, phases in the system.
Further simplification comes from the observation (see, for example [8]), that as the values \( h = 0, h_{st} = 0 \) are always solutions of Eqs. (3.10), and, therefore, the second order phase transition lines are determined by equation

\[
\frac{\partial^2}{\partial h^2} F_{MF} \frac{\partial^2}{\partial h_{st}^2} F_{MF} - \frac{\partial^2}{\partial h \partial h_{st}} F_{MF} = 0
\]  

(3.11)

at \( h = 0, h_{st}^*, h^*_{st} = 0 \), in order to find the critical lines not far from the PM phase, it is sufficient to know \( F_{MF} \) to terms of order of \( h_{st}^2 \).

If the problem could be solved exactly both of two representations (3.7) of the fermionic determinant would yield the same answer. But correlations of \( \phi \)'s at coinciding arguments make the problem unsolvable exactly. Indeed, the contributions of order of \( h_{st}^2 \) to (3.7) come not only from terms \( \propto u'' \) [first terms in Eq. (3.6)], but also from terms of any orders in \( u'' \) [terms proportional to \( \delta_{m,n} \) in (3.6)], as well as from higher correlators. Some of such contributions shown schematically in Fig. 1. Therefore, we are forced to use some approximations, and, in particular, to use two representations of (3.7) separately for “weak” and “strong” coupling regimes of \( y \), though the exact meaning of this can only be clear a posteriori.

Our approximation involves summing up all diagrams of Fig.1 (a) (proper ladder diagrams) and (b) (crossed ladder diagrams), so we call it ladder approximation. Diagrams (a) and (b) correspond to contributions to \( \langle \ln \text{det} [\hat{\theta} + y\hat{\Phi}] \rangle_h \) of the form \( u^i_{i_1} u^j_{i_{n+1}} (u'')^{2n-2} \delta_{i_2 i_3} \cdots \delta_{i_n i_{n+2}} \), and \( (u'')^{2n} \delta_{i_1 i_{n+1}} \delta_{i_2 i_3} \cdots \delta_{i_n i_{n+2}} \), respectively.

Terms \( \propto h h_{st} \) do not contribute to \( F_{MF} \) (due to momentum conservation). Hence, Eq. (3.11) in neighbourhood of the PM phase turns to pair independent equations

\[
\frac{\partial^2}{\partial h^2} F_{MF} \bigg|_{h=h_{st}=0} = 0, \quad \frac{\partial^2}{\partial h_{st}^2} F_{MF} \bigg|_{h=h_{st}=0} = 0.
\]  

(3.12)

Then, using properties (2.5) of the Dirac operator, formulae (3.6), and the properties of operators \( T \):

\[
\sum_{a=0}^{p} T^a \dagger T^b T^a \dagger = 1, \quad Z_2,
\]

\[
= 0, \quad U(1),
\]

\[
= -2 T^{b \dagger}, \quad \text{tr} T^a \dagger T^b = 2 \delta^{ab}, \quad SU(2),
\]

we find that the second order phase transition lines for the system (2.3), (2.4) in our approximation are determined by the expressions:
$Z_2$:

\[
\kappa_{cr}^{F,M(W)}(y) = \frac{1}{4D} \left\{ 1 - 2^{D/2} \left[ \frac{y^2 G^W(0)}{1 + y^2 G^W(0)} \right. \\
- \left. \int_q \left( \frac{y^2 G^W(q)}{(1 + y^2 G^W(q))^2 + y^4 G^W^2(q)} \right) \right] \right\},
\]

\[
\kappa_{cr}^{A,M(W)}(y) = -\frac{1}{4D} \left\{ 1 - 2^{D/2} \left[ \frac{y^2 G^W(\pi)}{1 + y^2 G^W(\pi)} \right. \\
- \left. \int_q \left( \frac{y^2 G^W(q)}{(1 + y^2 G^W(q))^2 + y^4 G^W^2(q)} \right) \right] \right\},
\]

\[
\kappa_{cr}^{F,M(S)}(y) = \frac{1}{4D} \left\{ 1 - 2^{D/2} \left[ \frac{G^S(0)}{y^2 + G^S(0)} \right. \\
- \left. \int_q \left( \frac{y^2 G^S(q)}{(y^2 + G^S(q))^2 + \frac{1}{y^4} G^S^2(q)} \right) \right] \right\},
\]

\[
\kappa_{cr}^{A,M(S)}(y) = -\frac{1}{4D} \left\{ 1 - 2^{D/2} \left[ \frac{G^S(\pi)}{y^2 + G^S(\pi)} \right. \\
- \left. \int_q \left( \frac{y^2 G^S(q)}{(y^2 + G^S(q))^2 + \frac{1}{y^4} G^S^2(q)} \right) \right] \right\};
\]

$U(1)$:

\[
\kappa_{cr}^{F,M(W)}(y) = \frac{1}{2D} \left[ 1 - 2^{D/2} \frac{y^2}{2} G^W(0) \right],
\]

\[
\kappa_{cr}^{A,M(W)}(y) = -\frac{1}{2D} \left[ 1 - 2^{D/2} \frac{y^2}{2} G^W(\pi) \right],
\]

\[
\kappa_{cr}^{F,M(S)}(y) = \frac{1}{2D} \left[ 1 - 2^{D/2} \frac{1}{2y^2} G^S(0) \right],
\]

\[
\kappa_{cr}^{A,M(S)}(y) = -\frac{1}{2D} \left[ 1 - 2^{D/2} \frac{1}{2y^2} G^S(\pi) \right];
\]
\[ SU(2) : \]

\[
\kappa_{cr}^{FM(W)}(y) = \frac{1}{D} \left\{ 1 - 2^{D/2} \left[ \frac{y^2 G^W(0)}{2 - y^2 G^W(0)} - \int_q \left( \frac{2 y^2 G^W(q)}{(2 - y^2 G^W(q))^2} - \frac{1}{4} y^4 G^W(q)^2 \right) \right] \right\},
\]

\[
\kappa_{cr}^{AM(W)}(y) = -\frac{1}{D} \left\{ 1 - 2^{D/2} \left[ \frac{y^2 G^W(\pi)}{2 - y^2 G^W(\pi)} - \int_q \left( \frac{2 y^2 G^W(q)}{(2 - y^2 G^W(q))^2} - \frac{1}{4} y^4 G^W(q)^2 \right) \right] \right\},
\]

\[
\kappa_{cr}^{FM(S)}(y) = \frac{1}{D} \left\{ 1 - 2^{D/2} \left[ \frac{G^S(0)}{2y^2 - G^S(0)} - \int_q \left( \frac{2 y^2 G^S(q)}{(2y^2 - G^S(q))^2} - \frac{1}{4} y^4 G^S(q)^2 \right) \right] \right\},
\]

\[
\kappa_{cr}^{AM(S)}(y) = -\frac{1}{D} \left\{ 1 - 2^{D/2} \left[ \frac{G^S(\pi)}{2y^2 - G^S(\pi)} - \int_q \left( \frac{2 y^2 G^S(q)}{(2y^2 - G^S(q))^2} - \frac{1}{4} y^4 G^S(q)^2 \right) \right] \right\},
\]

where

\[
G^W(q) = \int_p \frac{L(p)L(p+q)}{L^2(p)L^2(p+q)},
\]

\[
G^S(q) = \int_p L(p)L(p+q),
\]

\[
\int_q \equiv \int \frac{dDq}{(2\pi)^D}, \quad q_\mu \in (-\pi, \pi].
\]

In (3.14) – (3.16) functions \( \kappa^{FM}(y) \) describe the critical lines between the FM and PM phases, and \( \kappa^{AM}(y) \) those between the AM and PM phases, while superscripts \( W \) and \( S \) mean strong and weak coupling regimes, respectively. The contributions to (3.14) – (3.16) which are proportional to \( G(0) \) or \( G(\pi) \) come from the diagrams of Fig. 1(a), while the integral terms from those of Fig. 1(b). The second terms in the integrands take into account that the first crossed ladder diagram enters with a factor 1, rather than 2 (in preprint version of [6] and in [17] such terms in the equations for \( Z_2 \) system were erroneously missed). In the case of \( U(1) \) due to the specific property of operator \( T \) (3.13) both ladder and crossed ladder diagrams vanish, and all the contribution of the fermionic determinant is due to the first diagram of Fig. 1(a). In the case of \( n_f \) fermions couple to the Higgs fields with the same \( y \), factor \( n_f 2^{D/2} \) appears instead of \( 2^{D/2} \) in Eqs. (3.14) – (3.16).
The diagrams of Fig. 1(a) are generalization of “double chain” diagrams of Refs. [8], [10] to any configurations of the same topology. These ladder diagrams coincide with the double chain ones only in the case of strict locality of the Dirac operator (or its inverse operator). In fact, this is the case only for the strong coupling regime for the systems with naive action (see Sect. 4.1). Crossed ladder diagrams of Fig. 1(b) are generalization of the double chain diagrams with coinciding ends, which have not been taken into account in previous calculations. Although the crossed ladder diagrams are of $O(D^{-1})$ compared with the proper ones, they can become dominating at “intermediate” ones, when $y$ is close to the singular points of the integrands in (3.14) and (3.16).

are basically determined by four constants $G^W(0)$, $G^W(\pi)$, $G^S(0)$, and $G^S(\pi)$, which, in their turn, are determined by the form of the free fermion propagators. It is these constants that determine the singular points of the expressions for $Z_2$ and $SU(2)$, thereby determining domains of the weak and strong coupling regimes in these cases. As $G(0) > 0$, $G(\pi) < 0$, these domains are:

$$ Z_2 : \quad y < y^W \equiv |G^W(\pi)|^{-\frac{1}{2}}, \quad y > y^S \equiv |G^S(\pi)|^\frac{1}{2}; \quad (3.18) $$

$$ SU(2) : \quad y < y^W \equiv \left[\frac{1}{2}G^W(0)\right]^{-\frac{1}{2}}, \quad y > y^S \equiv \left[\frac{1}{2}G^S(0)\right]^\frac{1}{2}. \quad (3.19) $$

There can be the following possibilities:

(i) $y^W < y^S$. Although we have no analytical expressions describing the system in the region $y^W \leq y \leq y^S$, we can get, in fact, complete picture of the phase diagrams if the integral terms diverge at the points $y^W$ and $y^S$ and no FI phases appear. These are the cases of $Z_2$ systems with naive and mirror fermions [Figs 2(a), (d)]

(ii) $y^W > y^S$. The domains of the weak and strong coupling regimes are overlapped, so we can continue the lines $\kappa^W(y)$ and $\kappa^S(y)$ until they intersect each other. In these cases we know the phase diagrams in fact at any $y$ [an example is $U(1)$ system with naive action, Fig. 3(a)] (see, however, remark at the end of Sect. 6).

(iii) Lines $\kappa^{FM}$ intersect lines $\kappa^{AM}$ forming the FI phases before cases (i) or (ii) are realized. In such a case formulae (3.14) - (3.16) describe the phase diagrams only in a neighbourhood of the PM phases (examples are all $SU(2)$ systems, Figs. 4). The important fact is that the lines together with their first derivatives are smooth in the points of intersection. This follows from Eqs. (3.12).
For $U(1)$ systems we continue lines $\kappa^W(y)$ and $\kappa^S(y)$ until either case (ii) or (iii) is realized. In this case, however, such a procedure need to be justified. The natural and simple way is checking that the intersections of the curves occur at the points at which quantities $(y^2/2)G^W$ and $G^S/(2y^2)$ are obviously less than 1.

The proper ladder diagrams give the contributions beginning from the order $O(y^{\pm 2})$, the crossed ladder ones from $O(y^{\pm 4})$ (note that $\int_q G(q) = 0$). Contributions of other diagrams [Fig. 1(c)] come into play in higher orders in $y^{\pm 2}$, at least from the order of $y^{\pm 6}$. So, we expect that these contributions are non-singular and numerically suppressed.

4. Phase diagrams

In this section we apply formulae (3.14) - (3.16) to four-dimensional systems with different chirally invariant formulations of fermions on a lattice. All the phase diagrams are shown for $n_f = 1$, unless other is indicated.

4.1. Naive fermion action

In this case operator $\hat{\phi}$ is local

$$\hat{\phi}_{mn} = \sum_{\mu} \gamma_{\mu} \frac{1}{2} (\delta_{m+\mu n} - \delta_{m-\mu n}),$$

$$L_{\mu}(p) = \sin p_{\mu},$$

and therefore produces species doubling. The system is invariant under transformations:

$$(\psi, \bar{\psi})_n \rightarrow \exp(i\epsilon_n \pi/4)(\psi, \bar{\psi})_n, \phi_n \rightarrow \epsilon_n \phi_n, \kappa \rightarrow -\kappa, y \rightarrow -iy$$ [8], that, in particular, results in $G(\pi) = -G(0)$. Numerically one has

$$G^W(0) = -G^W(\pi) \approx 0.620, \quad G^W(0) = -G^S(\pi) = 2. \quad \tag{4.2}$$

Phase diagrams for $Z_2$, $U(1)$, and $SU(2)$ at $n_f = 2$ are shown in Figs. 2(a), 3(a), and 4(a), respectively. Some comments follow.

$Z_2$: Fig. 2(a). Domains of the weak and strong coupling regimes are determined by

$$y^W \approx 1.27, \quad y^S \approx 1.41. \quad \tag{4.3}$$

The lines $\kappa_{cr}^{FM}(y)$ and $\kappa_{cr}^{AM}(y)$ do not intersect each other, so no FI phase appears, the case (i) being realized. The lines form two disconnected domains with PM phases, as well as
with AM phases; the funnel with FM phase is formed by logarithmic dropping of the critical lines near the points $y^W$ and $y^S$: \( \kappa_{cr}^{FM(W)} \propto \ln[1 + y^2 G^W(\pi)] \), \( \kappa_{cr}^{FM(S)} \propto \ln[1 + G^S(\pi)/y^2] \), and therefore extends up to $\kappa \to -\infty$.

$U(1), n_f = 2$: Fig. 3(a). The lines form three connected domains with FM, PM, and AM phases; no FI phase appears. These features are hold for any $n_f$. Both curves intersect each others (point A and its counterpart on PM-AM line) at $y \approx 1.34$, where the construction is justified. The result agrees with previous analytical ones [9], [11], but disagrees with numerical results of Ref. [2], where an evidence of FI phase has been revealed. This fact, as well as point A which is a candidate for non-trivial fixed point, is discussed in Sect. 5 and 6.

$SU(2), n_f = 2$: Fig. 4(a). Domains of the weak and strong coupling regimes are overlapped:

\[
y^W \approx 1.80, \quad y^S = 1,
\]

the case \((iii)\) being realized. The FI phase appears from $n_f = 1$ extending with $n_f$. In the domain of applicability of our formulae (solid lines) the phase diagram is in a quantitative agreement with the Monte Carlo results of Ref. [7].

4.2. SLAC fermion action

This action is defined by operator \( \bar{\phi} \) of the form [18]

\[
\bar{\phi}_{mn} = \sum_{\mu} \gamma_\mu \sum_{l>0} (-1)^{l+1} \frac{1}{l} (\delta_{m+l\mu, n} - \delta_{m-l\mu, n}),
\]

\[
L_\mu(p) = p_\mu, \quad p_\mu \in (-\pi, \pi).
\]

It represents an action with a moderate non-locality as \( \bar{\phi}_{mn} \) drops with distance like \( |m - n|^{-1} \). In this case we have

\[
G^W(0) \approx 0.109, \quad G^W(\pi) \approx -0.0544, \quad G^S(0) = \frac{4}{3} \pi^2, \quad G^S(\pi) = -\frac{2}{3} \pi^2.
\]

Phase diagrams are shown in Figs. 2(b), 3(b), and 4(b).

$Z_2$: Fig. 2(b). Domains of the weak and strong coupling regimes are overlapped:

\[
y^W \approx 4.29, \quad y^S \approx 2.57.
\]

There are three connected domains with FM, PM, and AM phases, and no FI phase. These features are hold for any $n_f$.

$U(1)$: Fig. 3(b). The phase diagram for $n_f = 1$ looks like in the case of $Z_2$. But for $n_f \geq 2$ the FI phase appears. Points of intersection of the critical lines located at $y \approx 3.31$, where the construction is justified too.

$SU(2)$: Fig. 4(b). Domains of the weak and strong coupling regimes coincide with (4.7) (for $y^W$ within calculational errors). FI phase appears for any $n_f \geq 1$. 
4.3. Weyl fermion action

This action is defined by the finite dimensional approximation of functional integrals for Weyl quantization [19]. In this case we have

$$\phi_{mn} = \sum_{\mu} \gamma_{\mu} \sum_{l \geq 0} (-1)^{l+1} \frac{2}{l!} \left( \delta_{m+l \mu n} - \delta_{m-l \mu n} \right),$$

$$L_\mu(p) = 2 \tan \frac{1}{2} p_\mu.$$ (4.8)

This is a maximally non-local action in the sense that $\phi_{mn}$ does not drop at all with increasing $|m-n|$. But a remarkable fact is that this action can be transform to a local form if we introduce variables $\psi^d$ defined on the centres of D-cells of the lattice, i.e. on the sites of the dual lattice, leaving fields $\overline{\psi}$ being defined on sites of the original one. Then change of variables, which in momentum space looks like $\psi_p = F(p)\psi^d_p$, with $F(p) = \prod_{\mu} \cos \frac{1}{2} p_\mu$, leads to a local action with $L_\mu(p) = 2 \sin \frac{1}{2} p_\mu \prod_{\nu \neq \mu} \cos \frac{1}{2} p_\nu$. Although now $L_\mu(p)$ has additional zeroes at the Brillouin zone boundary, the system of course is not changed: contributions of the additional species to the partition function are canceled by Jacobian coming from the change of variables, while their coupling to the Higgs field is suppressed by the factor $F(p)$ (in this point this looks similar to the Zaragoza proposal [20]).

The more non-locality of the action, the less $|G^W|$, and the greater $|G^S|$. In this extremal case we have (at $N \to \infty$)

$$G^W(0) \approx 0.0450, \quad G^W(\pi) \approx -0.00739, \quad G^S(0) \to \infty, \quad G^S(\pi) = -16.$$ (4.9)

The divergence of $G^S(0)$ means that in this case terms of the strong coupling expansion diverge, and cannot be summed up into the finite expression in the ladder approximation (in [17], because of missing the second term in the integrand of (3.14), the wrong conclusion has been made on this point). Thus, we can analyse the phase diagram only in the weak coupling regime; the results are in Figs. 2(c), 3(c), and 4(c).

$Z_2$: Fig. 2(c). Formally domains of the weak and strong coupling regimes are overlapped:

$$y^W \approx 11.6, \quad y^S = 4.$$ (4.10)

Despite a tendency, no FI phase appears.

$U(1)$: Fig. 3(c). The FI phase appears at $y \approx 5.16$, where our condition is satisfied.

$SU(2)$: Fig. 4(c). In this case formally we have

$$y^W \approx 6.67, \quad y^S \to \infty.$$ (4.11)

The phase diagram is similar to that of $U(1)$ case, however, in view of (4.11) it looks plausible that at $y > y^W$ only the FM phase exists.
4.4. Mirror fermion action

We consider the simplest variant of the mirror fermion action \cite{21} with zero (bare)
mixing parameter between fermion field \( \psi \) and its mirror counterpart \( \chi \) and with only \( \chi \) 
coupled to the Higgs field

\[
A = \sum_{m,n} \left[ \bar{\psi}_m (\bar{\phi}_m^N \psi_n + W_{mn} \chi_n) + \bar{\chi}_m (\bar{\phi}_m^N \psi_n + W_{mn} \psi_n) \right] \\
+ \sum_n y \bar{\chi}_n (P_L \Phi_m + P_R \Phi_m^\dagger) \chi_n, 
\]

(4.12)

where \( \bar{\phi}^N \) is Dirac operator for naive fermions (4.1), while \( W \) is the Wilson operator

\[
W_{mn} = -\frac{1}{2} (\delta_{m+\mu,n} + \delta_{m-\mu,n} - 2 \delta_{mn}).
\]

(4.13)

The action has the mirror symmetry

\[
\psi_n \rightarrow (h_L P_L + h_R P_R) \psi_n, \quad \bar{\psi}_n \rightarrow \bar{\psi}_n (P_R h_R^\dagger + P_L h_L^\dagger), \\
\chi_n \rightarrow (h_R P_L + h_L P_R) \chi_n, \quad \bar{\chi}_n \rightarrow \bar{\chi}_n (P_R h_R^\dagger + P_L h_L^\dagger), \\
\Phi_n \rightarrow h_L \Phi_n h_R^\dagger.
\]

(4.14)

We can apply our formulae to this system after integrating out \( \psi \). Then, as \( \psi \) does not 
couple to \( \phi \) and therefore its determinant is an irrelevant constant, we come to effective 
non-local action in terms of fields \( \chi \) and \( \phi \) of the form of (2.3) – (2.5) with

\[
\bar{\phi}_{mn} = [\bar{\phi}^N - W (\bar{\phi}^N)^{-1} W]_{mn},
\]

\[
L_\mu(p) = \sin p_\mu \left[ 1 + \frac{\sum_\nu (1 - \cos p_\nu)^2}{\sum_\nu \sin^2 p_\nu} \right].
\]

(4.15)

This non-locality is of a new type compared with two preceding cases as \( L_\mu(p) \) now 
involves also all \( p_\nu \) with \( \nu \neq \mu \). Now we have

\[
G^W(0) \approx 0.0259, \quad G^W(\pi) \approx -0.00734, \quad G^S(0) \approx 348, \quad G^S(\pi) \approx -159.
\]

(4.16)

Phase diagrams are shown in Figs. 2(d), 3(d), and 4(d).

Z2: Fig. 2(d). Domains of the weak and strong coupling regimes are not overlapped:

\[
y^W \approx 11.7, \quad y^S \approx 12.6,
\]

(4.17)
so this case incorporates features of both local and non-local actions. The phase diagram, except the scale of $y$, has the same features as that for naive fermion action.

$U(1), \ n_f = 2$: Fig. 3(d). Domain with FI phase appears from $n_f = 1$, expanding with $n_f$. For $n_f = 2$ it appears at $y \approx 5.19$ and $y \approx 9.72$, so that according to our criterion we can trust the picture. The phase diagram qualitatively agrees with the result of Ref. [22] (there the phase diagrams examined in a region of parameter space that is different from ours).

$SU(2)$: Fig. 4(d). The weak and strong coupling domains are determined by

$$y^W \approx 8.79, \quad y^S \approx 13.2.$$  \hspace{1cm} (4.18)

As it happened in all other $SU(2)$ systems, FI phase appears.

To show the relative role of the proper and crossed ladder diagrams, we display in Figs. 2 (by thin dashed lines) contributions of the proper ones. This demonstrates that crossed ladder diagrams play the important role only when the case (i) is realized, being in fact negligible in other cases. We expect that contributions of diagrams that we did not take into account are actually invisible, at least in the cases (ii) and (iii).

5. Fermion correlators

In order to learn more about structure of various phases of Figs. 2 – 4 we now make a mean field estimation of fermion correlators $\langle \psi_m \overline{\psi}_n \rangle$ for the above systems. We mainly concentrate on an neighbourhood of the PM-FM critical lines, as it is this domain that is expected to be the most interesting for the continuum physics.

We shall evaluate the quantity

$$\langle \psi_m \overline{\psi}_n \rangle_{MF} \equiv \langle \left[ \hat{\Theta} + y \hat{\Phi} \right]^{-1}_{mn} \rangle_{h^*},$$  \hspace{1cm} (5.1)

where expectation value in the r.h.s. is defined in (3.4), while $h^*$ is a solution of the mean field equation (3.10). Such an expression appears very naturally as a variational mean field approximation for the correlator (see [9]), though there is no strict relation similar to Eq. (3.2) for free energy. By definition this is a quenched estimation.
We have the following representations for \( \langle \psi_m \overline{\psi}_n \rangle_{MF} \):

\[
\langle \psi_m \overline{\psi}_n \rangle_{MF} = \phi_{mn}^{-1} + \sum_{l=1}^{\infty} (-1)^l y^l \sum_{i_1, \ldots, i_l} \langle \phi_{m_i^{-1}} \phi_{m_{i_1}^{-1}} \phi_{m_{i_2}^{-1}} \cdots \phi_{m_{i_{l-1}}^{-1}} \phi_{m_{i_l}^{-1}} \rangle_{h^*} \tag{5.2}
\]

\[
\langle \psi_m \overline{\psi}_n \rangle_{MF} = \frac{1}{y} \langle \phi_{m_1} \rangle_{h^*} \delta_{mn} + \sum_{l=1}^{\infty} \frac{(-1)^l}{y^{l+1}} \sum_{i_1, \ldots, i_{l-1}} \langle \phi_{m_1} \phi_{m_{i_1}^{-1}} \phi_{m_{i_2}^{-1}} \cdots \phi_{m_{i_{l-1}}^{-1}} \phi_{m_{i_l}^{-1}} \rangle_{h^*}.
\]

We shall use these two representations for weak and strong coupling regimes which have been determined by our preceding considerations.

Consider first \( \langle \psi_m \overline{\psi}_n \rangle_{MF} \) in FM phase in the approximation of uncorrelated Higgs fields. Choose their expectation values to be real: \( \langle \phi_{m_1} \rangle_{h^*} = \delta^{a_0} \langle \phi \rangle \), so that

\[
\langle \phi_{m_1} \phi_{m_2} \cdots \phi_{m_l} \rangle_{h^*} = \langle \phi_{m_1} \rangle_{h^*} \langle \phi_{m_2} \rangle_{h^*} \cdots \langle \phi_{m_l} \rangle_{h^*} = \langle \phi \rangle^l.
\]

Then, from (5.2) it follows:

\[
\langle \psi_m \overline{\psi}_n \rangle_{MF} = \left( \phi + y \langle \phi \rangle \right)^{-1}_{mn},
\]

\[
= \left( \phi + \frac{y}{\langle \phi \rangle} \right)^{-1}_{mn}.
\]

This reproduces the well-known result, which has been obtained by various methods (for the first references see [23], [1], [13]), that the behavior of the fermion masses with \( \langle \phi \rangle \) (and, therefore, that of renormalized \( y \)) is completely different in the weak and strong coupling regimes. While in the weak regime one has the usual perturbative Higgs mechanism (with the Gaussian fixed point for \( y \)), the fermion masses at strong coupling do not vanish on the critical lines and in PM phase. In this approximation they tend to infinity, that is, the fermions decouple there. We shall refer hereafter to PM (FM) phases at weak and strong coupling regimes as PM(W) and PM(S) [FM(W) and FM(S)], respectively.

In [17] it was contemplated a possibility to use this feature of the Higgs-Yukawa systems for defining a chiral theory in PM(S) phase of the mirror fermion model (4.12). The idea was that the mirror fermions \( \chi \) decouple there leaving behind massless chirally invariant fermions \( \psi \). Using the above approximation, however, it is easy to show that though all goes in such a way, the goal cannot be reached: \( \psi \) turn to naive fermions.

Although approximation (5.3) yields correct qualitative picture, it is too rough for a quantitative analysis: it has been shown by numerical calculations [1], [2], [7], that the
fermion masses at strong coupling indeed increase with decreasing \( \langle \phi \rangle \), but remain finite
on the critical line, rather than tend to infinity.

To proceed further we make the mean field estimation of the fermion condensate

\[
\langle \bar{\psi} \psi \rangle_{MF} \equiv N^{-D} \sum_n \langle \bar{\psi}_n \psi_n \rangle_{MF}
\]

(5.5)
along the FM-PM critical lines where it has the form:

\[
\langle \bar{\psi} \psi \rangle_{MF} = -2^{D/2} C(y) \langle \phi \rangle + O(\langle \phi \rangle^2).
\]

(5.6)
This allows us to use the ladder approximation, that is, to sum up the contributions to
(5.6) of the diagrams of Fig. 5 (a). Then, from (3.5), (3.6) and (5.2) we find

\[
Z_2:
C^{(W)}(y) = \frac{y G^W(0)}{1 + y^2 G^W(0)},
\]

(5.7)
\[
C^{(S)}(y) = \frac{y}{y^2 + G^S(0)};
\]

\[
U(1):
C^{(W)}(y) = y G^W(0),
\]

(5.8)
\[
C^{(S)}(y) = \frac{1}{y};
\]

\[
SU(2):
C^{(W)}(y) = \frac{2y G^W(0)}{2 - y^2 G^W(0)},
\]

(5.9)
\[
C^{(S)}(y) = \frac{2y}{2y^2 - G^S(0)}.
\]

In the case of \( U(1) \) only the first diagram of Fig. 5 (a) gives a contribution to (5.8): the
situation is very similar to that for critical lines (3.15) (see also [9]).

condensates are smooth functions along both PM(W)-FM(W) and PM(S)-FM(S) crit-
ical lines. In the cases when FI phase appears it prevents us to follow far beyond the
points of intersection of PM-FM and PM-AM lines. The important fact, however, is that
in a neighbourhood of those points condensates remain smooth functions of both their
arguments \( y \) and \( \langle \phi \rangle \). This is due to the fact that the staggered magnetization gives no
contributions of order \( O(\langle \phi \rangle) \) to \( \langle \bar{\psi} \psi \rangle_{MF} \) (because of momentum conservation).

In the cases when the PM-FM line is continuous and no FI phase appear, the function
\( C(y) \) can be discontinuous at the points of intersection of FM(W) and FM(S) lines. We
have only three such systems: \( Z_2 \) with SLAC fermions, and \( U(1) \) with naive and SLAC
(n_f = 1) fermions [points A in Figs. 2(b) and 3(a),(b), respectively]. Figs. 6 and 7(a),(b) show that this is indeed the case. This means that the condensate, being zero in both PM phases and on the whole critical line, is discontinuous in FM phase. The discontinuity of the fermion condensate is an evidence of the first order phase transition. Indeed, the condensate can be defined as the first order derivative of the free energy in respect to an infinitesimal fermion mass (see also [8]). As the condensate is an order parameter of the systems, points A in Figs. 2(b) and 3(a),(b) look like tricritical points in which the first order phase transition turns to the second order one. This allows us to identify the points A with point A in Fig. 2 of Ref. [1], thereby considering them as candidates for non-trivial fixed points.

6. Summary and discussion

We derived the explicit formulae [Eqs. (3.14) – (3.16)] describing phase diagrams of a wide class of $Z_2$, $U(1)$, and $SU(2)$ Higgs-Yukawa systems, and applied them to the systems with naive, SLAC, the Weyl, and mirror fermion actions (Figs. 2 – 4). The phase diagrams turned out to be very different for different symmetry groups and fermion actions at $\kappa < 0$, being of the same form at $\kappa \geq 0$.

The difference between the phase diagrams for different formulation of lattice fermions shows a lack of universality in the systems at $\kappa < 0$, and can be interpreted as a lattice artifact. It is well known that in this region the sufficient condition of reflection positivity is not satisfied, and, therefore, relevance of the systems to well-defined quantum field theories is under the question. If, however, in that region physical positivity is fulfilled, the systems are still interesting from the point of view of continuum physics. Analysis of scalar propagators at $\kappa < 0$ in $SU(2)$ system with naive fermions [5], [24] gives some evidence that this is indeed the case.

All the points which are candidates for non-trivial fixed points lies in the region of negative $\kappa$. There are two types of such points: those where PM critical lines intersect AM lines forming FI phase, and the points where we can expect a phase transition separating weak and strong coupling regimes. The points of the first type present in all $SU(2)$ systems, Figs. 4, and in some of $U(1)$ systems, Figs. 3(c), (d). This case, however, seems to be excluded: the numerical investigations of such points in $SU(2)$ system with naive fermions [7], [5], [24] gave no evidence of non-trivial behaviour of the system. Our analysis speaks in favour that too: we applied fermionic condensate calculated along the PM-FM critical
lines [Eqs. (5.5) – (5.9)] as an order parameter sensitive to first order phase transitions, and showed that the condensate, as well as both critical lines, does not feel these points, remaining smooth functions of their arguments.

Among the points of the second type only those are relevant to continuum physics that border on PM phase: at other points either $\langle \phi \rangle$ or $\langle \phi_{st} \rangle$ is not zero. Therefore, the most interesting are systems which have continuous PM-FM critical lines and no FI phase. In our examples these are $Z_2$ system with SLAC action and $U(1)$ systems with naive and SLAC ($n_f = 1$) actions. Indeed, in FM phase near such points the fermion condensate is discontinuous, Figs. 6 and 7(a),(b), that is an evidence of the first order phase transition separating the FM(W) and FM(S) phases. We therefore identify the points A in Figs. 1(b), and 2(a),(b) with point A in Fig. 2 of Ref. [1]. If these points are really non-Gaussian, than non-trivial continuum theories with two relevant parameters, Higgs and fermion masses, can be defined approaching the points from FM(W) phase.

The $SU(2)$ systems we considered have no such points. The latter, however, can exist in $SU(2)$ systems with other formulations of lattice fermions, in particular, in those with staggered fermions coupled to the Higgs fields in a local way, when number of the fermions is not too big [10], [11].

We expect that disagreement of the phase diagram for $U(1)$ system with naive fermion action, Fig. 3(a), with results of Ref. [2], where an evidence of FI phase has been found, is due to finite lattice effects in the numerical calculations. Our results for $SU(2)$ system with naive fermions are in quantitative agreement with numerical calculations of Ref. [7], and results for $U(1)$ system with mirror fermions are in qualitative agreement with those of Ref. [22] (we cannot compare them directly, because the latter have been obtained in different region of parameter space of the model). In both cases the appearance of FI phases has very clear reasons. Therefore we do not see any reasons why our approximation could fail in this case. The more so, that in this case it is most stable against $1/D$ corrections (cf. [9]). Otherwise, there must be some underlying physics which by unknown reasons is not taken into account by our approximation, and which worth further investigation. The only change of the above picture which we cannot exclude is that the FM-PM (perhaps together with AM-PM) critical line is actually discontinuous, being teared up by the first order phase transition line at some value of $y$ not far from $y^A \approx 1.34$. Then the point $A$ splits into two first order phase transition points where critical lines $\kappa_{cr}^{FM(W)}$ and $\kappa_{cr}^{FM(S)}$ end up. In this case, in view of the results of Ref. [12], one can hardly expect the existence
of non-trivial fixed points in the system. Examination of this issue, however, requires another technique.

Therefore, it would be very interesting to repeat thorough numerical investigation of the $U(1)$ system with naive fermion action near the point $(y^4, \kappa^4) \approx (1.34, -0.43)$ (for $n_f = 2$). Another interesting issue is to trace the evolution of the phase diagrams of such system at finite scalar self-coupling $\lambda$ [12] with increasing $\lambda$.

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References

Figure captions

Fig. 1. Diagrams contributed to the expectation value of the fermion determinant (3.7) to the order $\hbar_n^2$; (a) is the ladder, (b) the crossed ladder diagrams. Solid lines stand for $\bar{\theta}$ or $\bar{\theta}^{-1}$, solid circles for $u'$, dashed lines for $u''$.

Fig. 2. Phase diagrams of $Z_2$ systems with (a) naive, (b) SLAC, (c) the Weyl, (d) mirror fermion actions. Dashed lines show the contribution of only diagram of Fig. 1(a). Point A in (b) is discussed in the text as possible non-trivial fixed point.

Fig. 3. Phase diagrams of $U(1)$ systems with (a) naive ($n_f = 2$), (b) SLAC, (c) the Weyl, (d) mirror fermion ($n_f = 2$) actions. Dashed lines in (d) is extrapolation of the formulae (3.15) to FI region. Points A in (a) and (b) are discussed in the text as possible non-trivial fixed points.

Fig. 4. Phase diagrams of $SU(2)$ systems with (a) naive ($n_f = 2$), (b) SLAC, (c) the Weyl, (d) mirror fermion actions. Dashed lines is extrapolation of the formulae (3.16) to FI region.

Fig. 5. Diagrams contributed to the fermion condensate (5.6) to the order $\langle \phi \rangle$; (a) is the ladder diagrams. Solid circles stand for $\langle \phi \rangle$, crosses for site $n$ in (5.5); other notations as in Fig. 1.

Fig. 6. Fermion condensate [function $C(y)$ in (5.6)] along the PM-FM critical line for $Z_2$ system with SLAC fermions.

Fig. 7. The same as in Fig. 6, but for $U(1)$ systems with (a) naive, (b) SLAC fermions.