Is there any Scaling in the Cluster Distribution 

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Abstract

We apply fractal analysis methods to investigate the scaling properties in the Abell and ACO catalogs of rich galaxy clusters. The methods are adapted to account for the incompleteness of the samples by appropriately using the known selection functions in the partition sums. We also discuss different technical aspects of the method when applied to data sets with small number of points as the cluster catalogs. Results are compared with simulations based on the Zel'dovich approximation.

We limit our analysis to scales less than 100 \( h^{-1} \) Mpc. The cluster distribution show a scale invariant multifractal behavior in a limited scale range. For the Abell catalog this range is 15–60 \( h^{-1} \) Mpc, while for the ACO sample it extends to smaller scales. Despite this difference in the extension of the scale–range where scale–invariant clustering takes place, both samples are characterized by remarkably similar multifractal spectra in the corresponding scaling regime. In particular, the correlation dimension turns out to be \( D_2 \simeq 2.2 \) for both Abell and ACO clusters.

Although it is difficult to point out the scale at which homogeneity is reached with the present size of these redshift surveys, our results indicate that the cluster distribution shows a tendency to homogeneity at large scales, disproving the picture of a pure scale invariant fractal structure extending to arbitrarily large distances.

Subject headings: Galaxies: clusters; large–scale structure


1 Introduction

The study of the large scale matter distribution is one of the fundamental problems in modern cosmology. Extended redshift surveys of galaxies and galaxy clusters have revealed the existence of filaments and sheet-like structures up to the maximum considered depths \( \gtrsim 100 \, h^{-1} \, \text{Mpc} \) (Kirshner et al. 1981; de Lapparent et al. 1986; Geller & Huchra 1989; Broadhurst et al. 1990; Postman, Huchra & Geller 1992). In the standard framework of the galaxy formation theory these patterns arise as a result of the gravitational interactions in the expanding medium, with primordial fluctuations of very low amplitude acting as seeds for the subsequent structure formation.

It is then clear that the study of large-scale distribution of cosmic structures is of extreme interest since it should give us information about the shape and statistics of the primordial fluctuation spectrum. Catalogs of different extragalactic objects have been studied with various statistical methods. One of the most popular approach is based on the estimate of the correlation functions for both galaxy and cluster distributions. In this respect, the most striking result is that the 2-point correlation function, \( \xi(r) \), turns out to closely follow a power-law shape,

\[
\xi(r) = \left( \frac{r_o}{r} \right)^\gamma,
\]

with \( \gamma \approx 1.8 \) for both galaxies and clusters, although holding on different scale ranges and with remarkably different correlation length, \( r_o \); for galaxies it is \( r_o,g \approx 5-6 \, h^{-1} \, \text{Mpc} \) (Peebles 1980; Davis & Peebles 1983; Vogeley et al. 1992) for \( 0.1 \lesssim r \lesssim 10 \, h^{-1} \, \text{Mpc} \), while it is \( r_o,c \approx 20 \, h^{-1} \, \text{Mpc} \) for rich Abell clusters in the scale-range \( 10 \lesssim r \lesssim 60 \, h^{-1} \, \text{Mpc} \) (Bahcall & Soneira 1983, Klypin & Kopylov 1983; Postman, Huchra & Geller 1992). Although the exact value of the cluster correlation length still represents a widely debated issue, with several authors claiming a smaller value, \( r_o,c \approx 14 \, h^{-1} \, \text{Mpc} \) (Sutherland 1988, Sutherland & Efstathiou 1991, Dalton et al. 1992), nevertheless it is firmly established that clusters are much more strongly correlated than galaxies and the amount of this enhanced clustering represents a challenge for any theoretical model of large-scale structure formation (e.g., White et al. 1987; Bahcall & Cen 1987; Holtzman & Primack 1993; Mann, Heavens & Peacock 1993; Croft & Efstathiou 1994; Borgani, Coles & Moscardini 1994). In the framework of the biased model for structure formation (e.g., Kaiser 1984; Bardeen et al. 1986) the enhanced clustering of rich galaxy systems is nothing but

\[ h \] is the Hubble constant in units of 100 Km sec\(^{-1}\) Mpc\(^{-1}\).
the consequence of their occurrence in correspondence of high-density peaks of the underlying matter distribution.

By adopting a completely different point of view, Coleman, Pietronero & Sanders (1988) argued that the existing relation between the correlation properties of galaxies and clusters is the probe for a self-similar (fractal) structure of the Universe, extending up to arbitrarily large scales. The corresponding fractal dimension, $D$, turns out to be related to the slope of the 2-point correlation function according to $D = 3 - \gamma$. In this picture, the absence of any characteristic scale, beyond which homogeneity is reached, makes the correlation length $r_s$ a meaningless quantity, since it turns out not to deal with intrinsic properties of the clustering, but depends only on the size $R_s$ of the sampled volume, according to $r_s \propto R_s$ (see Coleman & Pietronero 1992, for a review about a fractal Universe). Therefore, this should explain the depth dependence of the galaxy clustering (Einasto, Klypin & Saar 1986; Davis et al. 1988; Martínez et al. 1993), as well as the large value of the cluster correlation length, as due to the larger volumes encompassed by cluster samples with respect to galaxy samples.

As for the galaxy distribution, Martínez & Jones (1990) have shown that any self-similarity should be confined to small scales, $r_s \lesssim 4 h^{-1} Mpc$ (see also Martínez et al. 1990), while homogeneity should be reached at larger scales. It is not clear whether or not such scales are encompassed by presently available redshift samples. The view of a small-scale fractality followed by a smooth transition toward large-scale homogeneity has also been supported by scaling analyses of the non-linear clustering developed by N-body simulations (Valdarnini, Borgani & Provenzale 1992; Colombi, Bouchet & Schaeffer 1992; Yepes, Domínguez-Tenreiro & Couchman 1992; Murante et al. 1994, in preparation). These analyses consistently show that at small scales of few Mpc's non-linear gravitational dynamics acts to create a self-similar clustering. The resulting fractal dimension turns out to take a characteristic value, $D \simeq 1$, quite independent of the choice for the initial fluctuation spectrum. Therefore, the detected value of the galaxy correlation length must not be interpreted as the scale above which homogeneity is attained. Instead, it separates the small-scale regime of non-linear gravitational clustering from the larger-scale one, where mildly non-linear dynamics preserves a more clear imprint of initial conditions.

Due to the similarity between the clustering properties of galaxies and clusters, one may ask whether the observed self-similarity for $r \lesssim r_s$ also holds for the cluster distribution. If this were the case, it is clear that it cannot be explained on the ground of non-linear gravitational clustering, since it occurs at scales ($\gtrsim 10 h^{-1} Mpc$) where the fluctuation evolution should still
be near to the linear regime.

Actually, Borgani, Plionis & Valdarnini (1993a) analyzed the scaling properties for the projected distributions of cluster samples having different richness, selected with an overdensity criterion from the Lick galaxy map (Plionis, Barrow & Frenk 1991). As a result they found that the distribution of the richest clusters displays a well defined scaling behavior, remarkably similar to that of galaxies, for angular scales \( \theta \lesssim 7^\circ \) (corresponding to physical scales \( \lesssim 25 h^{-1} Mpc \) at the depth of the Lick map). At larger scales, the self-similarity breaks down and homogeneity, at least in the projected distribution, is attained. Furthermore, it has also been shown that simulating a cluster distribution by projecting a three-dimensional scale-invariant structure does not introduce characteristic scales in the angular distribution, thus indicating that the breaking of self-similarity in the angular distribution reflects the presence of a characteristic scale even in the spatial cluster distribution. Borgani et al. (1994) generated synthetic angular samples by projecting three-dimensional N-body simulations based on both Gaussian and skewed initial conditions for a CDM spectrum. After reproducing the observational setup of the Plionis et al. (1991) cluster samples, they addressed the question whether the detected small scale self-similarity for clusters imposes a stringent constraint on the initial conditions. As a result they found that, at least for the CDM model, the initial Gaussian statistics does not succeed at explaining the small-scale fractality, while some non-Gaussian models fare much better.

In the present paper we realize a detailed scaling analysis for large redshift samples of Abell and ACO clusters and will compare them with synthetic cluster samples extracted from numerical simulations based on Zel'dovich simulations with Gaussian initial conditions. This kind of analysis should allow us to answer the following questions: a) is the scale-invariance detected for angular samples at small scales a real feature imprinted in the three-dimensional distribution ?; b) If yes, can it be explained simply on the ground of a peak selection procedure on a Gaussian background or does it require something more ?; c) Do the present cluster samples encompass the scale of homogeneity or should the fractal picture of the Universe be not still disproved ?

The plan of the paper is as follows. In section 2 we present the real cluster samples and the simulations of the cluster distributions. In section 3 we study the scaling in the high density regions by means of the correlation integral method for real catalogs as well as simulations. Section 4 is devoted to the analysis of the scaling in the low density regions by means of the \( n \)-nearest neighbor distances. General conclusions are drawn in section 5.
2 The cluster samples

2.1 Real data

The analysis is done on subsets of the Abell and ACO cluster catalogs (Abell 1958; Abell, Corwin and Olowin 1989). The subsample parameters are those already used in previous papers (Plionis & Valdarnini 1991). For the Abell catalog the geometrical boundaries are $|b^{II}| \geq 30^\circ$, $\delta \leq -27^\circ$ and $z \leq 0.1$, with a total area of 4.8 sr. These constraints result in a total number of 206 clusters distributed in the Northern (NGC) and Southern (SGC) Galactic Cap. For the ACO catalog the selection criteria are $m_{10} \leq 16.4$, $b^{II} \leq -20^\circ$ and $\delta \geq -17^\circ$. The subsample has an area of 1.8 sr and 103 clusters. In this subset 19 clusters have redshift estimated from the $m_{10} - z$ relation used by Plionis & Valdarnini 1991, otherwise redshift are directly taken from the original catalog or complemented with those measured by a number of authors (Plionis, Valdarnini & Jing 1992, hereafter PVJ). In the above selection criteria we included all clusters with richness $R \geq 0$. According to Postman, Huchra & Geller (1992) the density distribution and the clustering properties of the $R = 0$ and $R \geq 1$ cluster samples are similar out to $z \approx 0.2$. In a different context, PVJ have checked the robustness of their statistical analysis by applying the same tests to the $R \geq 1$ and $R = 0$ sample. The results are not very much different and we decided to include clusters with $R = 0$ in our studies. The selection criteria we used allow us to have defined subsets of the original cluster catalogs almost volume-limited, excluding projection effects due to poorly sampled, rich and distant clusters.

The selection function we introduce are both in galactic latitude and redshifts. The former has the functional form

$$P(b^{II}) = 10^{a(1 - z_{c}/[b^{II}] )},$$

where $a = 0.3$ (0.2) for the Abell (ACO) subsample. As for the redshift selection function, we take the expression

$$P(z) = \begin{cases} 
1 & ; \quad z < z_c \\
A \exp^{-z/z_{0}}, & ; \quad z \geq z_c 
\end{cases}$$

where $z_c$ is the maximum redshift at which the cluster distribution follows that of a uniform one. The choice of $z_c$ is given by a least-square fit of $P(z > z_c)$ vs. $\log N_{obs}/N_{exp}$, where $N_{obs}$ and $N_{exp}$ are the number of clusters which are observed and which are expected for a uniform distribution, respectively. The redshift are $z_c = 0.081, 0.063$ and 0.066 for NGC, SGC and ACO, respectively. The details of the selection function determination are described in PVJ.
In the following, the Abell North and Abell South samples will be merged together, while the ACO clusters will be considered separately, since they have been shown to display some different clustering properties (PVJ; Cappi & Maurogordato 1992).

### 2.2 Simulations

The simulated cluster catalogues that we use are the same of PVJ. We refer to this paper for more details. We randomly distribute $N_p$ points in a cube of size $640 \ Mpc^{-1}$ with $N_p^3 = 64^3$ grid points. The points are then displaced from their positions according to the Zeldovich approximation (Zel’dovich 1970; Shandarin & Zel’dovich 1989). The Zeldovich approximation has been shown to be an adequate prescription to follow gravitational clustering in the mildly non–linear regime (e.g., Coles, Melott & Shandarin 1993). Its relevance to the study of the distribution of galaxy clusters has been stressed by Blumenthal, Dekel & Primack (1988) and, more recently, by Mann et al. (1993) and Borgani et al. (1994).

The density fluctuation spectrum $\delta_k$ is chosen to have a Gaussian distribution with random phases and power spectrum

$$P(k) = \langle \delta_k^2 \rangle = A k^n \exp(-|k|^2/\Lambda^2) \Theta(|k|). \quad (4)$$

Following Postman et al. (1989), we take $\Lambda^{-1} = 0.1 \ Mpc^{-1}$, $\Theta(|k|) = 0$ for $|k| > (80 \ Mpc^{-1})^{-1}$ and $\Theta(|k|) = 1$ otherwise.

To each particle is tagged a $\nu$ value, such that $\nu = \delta/\sigma$, being $\delta$ the value at the nearest grid point of the fluctuation of the linear density field smoothed with a Gaussian window of size $R = 10 \ Mpc$, and $\sigma$ the corresponding rms fluctuation value.

The number of points in the simulations, $N_p$, must be chosen so that, after having applied to the simulated samples the survey boundary limits and all the selection functions of the real data, we have the same number of clusters as in the real samples. Taking $N_p = 73000$ results to about $\sim 12000$ points being associated with peaks with $\delta > \nu \sigma$ with $\nu = 1.3$.

The parameters of the simulations, namely $A$, $n$ and $\nu$, are chosen in such a way to reproduce the observed cluster 2–point correlation function in the appropriate scale range. PVJ generated several cluster simulations, all having the same slope $\Gamma \simeq 1.8$ for $\xi(r)$, and clustering lengths ranging from $\simeq 14 \ Mpc$ to $\simeq 25 \ Mpc$. In the present analysis we will only consider the simulation with $r_s = 20 \ Mpc$, which has been shown to better reproduce several clustering features of the Abell and ACO samples. A set of 50 different initial phase assignments is
considered, so to have a large ensemble over which to estimate the effects of the cosmic variance for the cluster distribution. PVJ have shown that these cluster simulations, which are called to reproduce the observed 2-point function, also reproduces as an extra bonus even the 3-point function, thus supporting their reliability to give a faithful representation of the clustering of rich galaxy clusters.

3 Scaling from the correlation integral analysis

3.1 The method

Fractal analysis methods are based on the determination of the so-called spectrum of generalized dimension, which characterizes the scaling properties of the system. In order to introduce the concept of fractal structure and fractal dimension, let us consider a point distribution and suppose to cover it with cubic cells all having the same size $r$. Let also $\bar{p}_j(r)$ be the probability measure associated to the $j$-th cell, that is the fraction of all the points inside there, and $N_c(r)$ the number of non-empty cells of side $r$. Accordingly, for a fractal structure it is possible to define the sequence of Rényi dimensions

$$D_q = \frac{1}{q-1} \lim_{r \to 0} \frac{\log \sum_{j=1}^{N_c(r)} [\bar{p}_j(r)]^q}{\log (1/r)}$$

(Rényi 1970), $q$ being a generic real number. For $q = 0$, eq.(5) provides the box-counting (capacity) dimension, which only depends on the number of non-empty cells, thus accounting only for the geometry of the distribution. Note that for $q > 0$ the sum in eq.(5) mostly weights the overdensities, while $q < 0$ is for the underdense parts of the distribution. Under general conditions it is possible to show that $D_q$ is a non-increasing function of $q$. The simplest case occurs when $D_q = const$. In this case the structure is said to be monofractal and it is described just by a single scaling index (see Paladin & Vulpiani 1987, for a review about multifractals).

The interpretation of large-scale clustering in terms of fractal dimensions has been shown to provide a comprehensive description, the $D_q$ dimension spectrum of eq.(5) being strictly connected to fundamental statistical descriptors, such as the $N$-point correlation functions and the void probability function (e.g., Balian & Schaeffer 1989, Jones, Coles & Martínez 1992; Borgani 1993).

Although the definition (5) of $D_q$ dimension spectrum is given in the limit of infinitesimally small scales, in practical estimates one usually deals with a finite amount of data points, so
that only finite scale-ranges can be probed. Therefore, one is forced to resort to approximate algorithms, which are based on different approximations to the formal definition of fractal dimensions and suffers by different degrees for the presence of poor statistics (see, e.g., Martínez et al. 1990; Valdarnini, Borgani & Provenzale 1992; Borgani et al. 1993b). One of such methods is based on the evaluation of the correlation integrals (Grassberger & Procaccia 1983; Paladin & Vulpiani 1984): for each point $i$, one evaluates the probability measure $p_i(r)$, which represents the fraction of all the points within the sphere of radius $r$ and centered on $i$. Accordingly, the statistics is described by the partition function

$$Z(q, r) = \frac{1}{N} \sum_{i=1}^{N} [p_i(r)]^{q-1} = \frac{\langle n^{q-1} \rangle}{N^{q-1}} \sim r^{\tau_q}. \quad (6)$$

Here $N$ is the total number of points in the sample and $\langle n^q \rangle$ is the $q$–th order moment for the neighbor counts. For a fractal structure the scaling index $\tau_q$ is strictly independent of the scale and defines the $q$–th order dimension as

$$D_q = \frac{\tau_q}{q-1}. \quad (7)$$

Although this method is rather reliable to estimate fractal dimensions when a high number of points is available, spurious estimates are originated by limited statistics. This problem is expected to be particularly important for the distribution of clusters, whose mean separation ($d \simeq 40 h^{-1} \text{Mpc}$) is roughly twice the corresponding correlation length ($r_s \simeq 20 h^{-1} \text{Mpc}$), but not as important for the galaxy distribution, for which $d \simeq r_s \simeq 5 h^{-1} \text{Mpc}$.

A subtle point concerning the evaluation of the $p_i(r)$ probabilities is whether or not including in it the center point $i$. It is clear that, when the number of neighbors is sufficiently large, no difference is expected, while much more care is needed at the smallest sampled scales. Grassberger (1988) strongly recommended that that "the central point must obviously not included". In this way, for $q > 1$ the sum in eq.(6) always weights only the more clustered structures, thus limiting the bias in the dimension estimate due to the poorly sampled part of the distribution. However, as smaller and smaller scales are considered, more and more points give no contribution to $Z(q, r)$. Consequently, the partition function steepen at small scales and the resulting dimension diverges.

On the other hand, by including the center point in the estimate of $p_i(r)$, at very small scales most spheres will take only the contribution from the center. As a consequence, the value of the partition function changes only slowly by increasing $r$, so that $D_q \simeq 0$. Although
spurious, in this case this result has a simple geometrical interpretation, since at small scales we are measuring nothing but the topological dimension of each single point, which is indeed zero.

An efficient method to correct dimension estimates from undersampling lies in subtracting the shot-noise contributions, which generates the underestimate of the local dimension. Let us denote by $d\mu(x)$ the local measure associated to the fractal structure. Therefore, the “mass” contained within a ball of radius $r$, centered on the point $i$ belonging to the fractal, is $\mu_i(r) = \int_{<r} d\mu$. If $m_q(r) = \langle \mu^q(r) \rangle$ is the corresponding $q$-th order moment, then, under the usual assumption that this sampling is purely Poissonian, the moments of counts for the sampling points, $\langle n^q \rangle$, are related to the “true” moments, $m_q$, according to suitable recurrence relations. At the first orders these read

\[
\begin{align*}
\langle n^2 \rangle & = \langle n \rangle + m_2; \\
\langle n^3 \rangle & = \langle n \rangle + 3m_2 + m_3; \\
\langle n^4 \rangle & = \langle n \rangle + 7m_2 + 6m_3 + m_4 \quad (8)
\end{align*}
\]

(e.g., Peebles 1980), while more cumbersome expressions are expected at higher orders. The recursive application of the above expressions allows one recover the background statistics from that of the Poissonian realizations. The reliability of this prescription in fractal analysis to recover the correct dimension estimates has been checked by Borgani & Murante (1994) and shows that the expected scaling is nicely recovered even when heavily diluting the original sample. It is however clear that such a procedure to correct for poor sampling does not always guarantee a meaningful result. Typical examples are when the mean neighbor count is much less than unity or when the points are located in correspondence of density peaks, which clearly do not represent a Poissonian sampling.

A further problem in the application of the correlation–integral method resides in the treatment of boundaries, when the point distribution is confined in a finite volume. A first possibility, which makes no assumptions about the distribution outside the sample volume, consists in discarding from the partition sum of eq.(6) at the scale $r$ all the centers whose distance from the nearest boundary is less than $r$. However, this kind of procedure severely limits the statistics and, moreover, the analyzed sample is statistically different at different scales. This represents a serious problem in our case, since the geometry of the boundaries and the limited number of clusters does not allow to consider scales larger than $\sim 70 h^{-1} Mpc$. A further possibility is to consider for each center $i$ the fraction $f_i(r)$ of the sphere of radius $r$ centered on it, which falls
within the boundaries. In this way, if \( \tilde{n}_i(r) \) is the counted number of neighbors, the corrected values is \( n_i(r) = \tilde{n}_i(r)/f_i(r) \). Actually, this procedure also allows to account for other systematics of the cluster sample, such as the dependence of the local cluster density on redshift and galactic latitude, as provided by the selection functions of eqs. (2) and (3).

However, this border correction method relies on the assumption that the behaviour of the cluster distribution inside the sample boundaries is statistically equivalent to that inside such boundaries. Therefore, it assumes, rather than verifies, that the analyzed distribution represents a fair sample. Therefore, one must be sure that, at the considered scales, no serious effects of homogeneization are spuriously introduced by boundary corrections.

The corrected local count around the \( i \)-th point is

\[
\begin{align*}
n_i(r) &= \frac{1}{f_i(r)} \sum_{j=1}^{N} \frac{\theta(|x_i - x_j| - r)}{P_j(b) P_j(z)}
\end{align*}
\]

where \( P_j(b) \) and \( P_j(z) \) are the values of the galactic latitude and redshift selection functions at the position of the \( j \)-th cluster. In order to implement this correction, we realized a Monte Carlo sampling with rejection, according to sample boundaries and selection functions as described in section 2, so to measure the corrected count \( n_i(r) \). Note, however, that at distances much larger than the completeness redshift, \( P(z) \) rapidly declines, with a subsequent increase of the noise in the correction procedure. For this reason, we will mainly present results based on clusters within the distance \( d = 200 \, h^{-1} \, Mpc \), although we will also show the effect of correcting for selection functions, when their value becomes much smaller than unity.

In order to check that our analysis method correctly detects the presence of a characteristic scale in the distribution, we resort to the \( \beta \)-model prescription (e.g., Paladin & Vulpiani 1987) to generate a scale-dependent monofractal structure with \( D = 1 \) below some homogeneity scale \( L_h \) and \( D = 3 \) above it. We choose a box-counting homogeneity scale \( L_h = 80 \, h^{-1} \, Mpc \), which roughly corresponds to the homogeneity scale \( \simeq 40 \, h^{-1} \, Mpc \) as seen by the correlation-integral algorithm. Therefore, we introduce the same boundaries and selection effects as for real samples, so to generate synthetic data sets. In Figure 1 we plot the local dimension \( D_q(r) \), obtained from a five point log-log local linear regression on the partition function shape, for the simulated Abell North sample. From top to bottom we show results for progressively increasing \( q \) values (\( q = 2, 3 \) and \( 5 \)). Circles and squares correspond to including and excluding the center object, respectively, in the estimate of the probability \( p_i(r) \). Triangles refer to the shot-noise corrections, that, according to eq.(8), can be applied for \( q \geq 3 \). For \( q = 2 \), including the center
increases the shot noise effects associated to the discrete nature of the distribution; $D_2$ is heavily underestimated at small scales and no scale range is detected where the expected fractal scaling takes place. Going to higher $q$ values, the method based on including the center improves to some extent, although it still provides a biased estimate of the local dimension. However, the reliability of this method is largely improved after the removal of the shot–noise terms (see also Borgani & Murante 1994); in this case the scaling regime is fairly detected in the expected range and the correct fractal dimension value is recovered. This shows the reliability of the method to detect the presence of a fractal scaling regime. In a similar way, also excluding the center allows a fair determination of the scaling regime.

Although the presence of the characteristic scale at $L_h \simeq 40 h^{-1} Mpc$ is well detected, nevertheless even at the largest sampled scale the local dimension does not attain the homogeneity value $D_q = 3$. Instead, at $r \simeq 90 h^{-1} Mpc$ it is $D_q \simeq 2$, quite independent of the multifractal order $q$ and on the analysis method. This indicates some inertia of the $Z(q, r)$ partition function to follow the change of scaling when applied to a point distribution as large as the cluster samples we are dealing with.

Furthermore, we verify that the boundary corrections do not introduce characteristic scales in a structure which is otherwise self-similar at all the scales. To do this, we generate a synthetic Abell North sample from a $D = 1$ monofractal structure without large-scale homogeneity. In Figure 2 we plot the resulting local dimensions for $q = 2, 3$ and 5. In this case, only results based on excluding the center point are presented, although no significant differences between the methods have been detected, due to the strong clustering associated to this pure fractal. It is apparent that quite large fluctuations appear due to the lacunarity (e.g., the presence of big voids) induced by the $\beta$-model. However, no homogenization is spuriously induced, at least at the scales of interest, which, instead, would turn into $D_q \simeq 3$ at the largest considered scales.

Based on the analyses presented in this section about the reliability of the implementation of the correlation–integral method, we draw the following conclusions.

**a)** Excluding the center points in the estimate of $p_i(r)$ does usually provide a fair dimension estimate. However, as we will see in the following, a spurious increase of $D_q(r)$ should be expected in some cases at the small scales, where undersampling becomes severe.

**b)** Including the center points could heavily underestimate the dimension in a broad scale range and pollute the presence of a scaling regime. However, after suitably removing the shot–noise terms, a reliable estimate of the fractal dimension is provided and the scaling
is correctly recovered.

c) Our prescription to correct boundary effects is shown to fairly detect the presence of a characteristic scale. However, the local dimension fails to detect homogeneity at the largest scales sampled by the available cluster samples, even for a structure which is known a priori to be homogeneous at such scales. This shows that the condition $D_q \simeq 3$ is sufficient but not necessary to claim homogeneity in the cluster distribution.

Based on these results, in the following we will discuss the results of our fractal analysis on real as well as simulated cluster samples.

### 3.2 The results

In Figures 3 and 4 we plot the local dimension for different $q$ values, for Abell and ACO clusters, respectively. Each panel is for a given $q$ value and report the results for the partition function and for the corresponding local dimension, as obtained by excluding the contribution of the $i$-th cluster in the estimate of $p_i(r)$. For $q = 3$ and 5 we also show the local dimension as obtained after correcting for shot-noise effects, according to eq.(8). The errorbars in the partition function represents the 1σ scatter between 100 bootstrap resamplings. Uncertainties in the local dimension are three times the standard deviations in the 5-point weighted least-square fitting in the partition function slope. For reason of clarity we do not plot errorbars for shot-noise correction results, which are always comparable to the plotted ones.

For the Abell clusters, both methods consistently show that the profile of the local dimension does not remain flat over the whole considered scale-range, thus disproving the picture of a purely fractal structure of the large-scale clustering. At small scales, $D_q(r)$ suffers from the limited statistics and decreases toward small values ($D_q(r) \lesssim 1$), due to the flattening of $Z(q,r)$ at $r \lesssim 10 \, h^{-1} \text{Mpc}$. On the other hand, the dimension increases at $r \gtrsim 50 \, h^{-1} \text{Mpc}$ and reaches $D_q \gtrsim 2$ at the largest considered scales ($\approx 100 \, h^{-1} \text{Mpc}$). In Table 1, we show the dimension values $D_q$ for $q = 2, 3$ and 5 calculated in the range $15-60 \, h^{-1} \text{Mpc}$, by excluding the center point in the $n_h(r)$ local count. Also reported are the 1σ uncertainties.

Quite different results are found for the ACO clusters, for which both methods do not indicate a smooth increasing of $D_q(r)$. For $q = 2$, excluding the centers in the $p_i$ estimates causes a spurious increase of the $D_q$ at small scales. For higher $q$’s, more dense regions are weighted and undersampling effects are less important. Correspondingly, the same method detect the presence of a wider scale range, where the local dimension remains nearly flat. Note
that for \( q = 3 \) correcting for shot-noise reveals the existence of scaling regime, extending down to the smallest considered scales, thus much wider than that detected for the Abell clusters. Correspondingly, the \( Z(q, r) \) partition function never flattens even at \( r < 10 \, h^{-1} \, Mpc \). This result calls for a substantial difference between the clustering properties of the two samples. The self-similarity is broken at scales \( \simeq 50 \, h^{-1} \, Mpc \) after which the dimension starts increasing, quite similarly to what found for the Abell sample.

[Despite this difference in the extension of the scaling regime, it is worth noting that the value of \( D_2 \approx 2.2 \) obtained for both catalogs agrees remarkably well with the slope \( \gamma_2 \) of the function \( 1 + \xi(r) \propto r^{\gamma_2} \) obtained by Calzetti, Giavalisco & Meiksin (1992) for the Abell clusters (note the \( \gamma_2 \approx 3 - D_2 \)).] Despite this difference in the extension of the scaling regime, it is worth noting that the values of the \( D_q \) dimensions are always quite similar. For \( q = 2 \) the resulting correlation dimension, \( D_2 \approx 2.2 \), obtained for both catalogs agrees remarkably well with the slope \( \gamma_2 \) of the function \( 1 + \xi(r) \propto r^{\gamma_2} \) obtained by Calzetti, Giavalisco & Meiksin (1992) for the Abell clusters (note the \( \gamma_2 \approx 3 - D_2 \)).

Therefore, the overall emerging picture for the cluster distribution indicates the presence of a well defined scaling behavior in the finite scale range \( 15 \leq r \leq 60 \, h^{-1} \, Mpc \) for Abell sample, and more extended to smaller scales for the ACO sample. This self-similar clustering is followed by a breaking at larger scales. Despite this fact, we never succeed to detect \( D_3 \approx 3 \), which would be the unique imprint of large-scale homogeneity. However, this should not surprise, since, as previously shown in §3.1, the limited amount of statistics induces some inertia in the large-scale detection of homogeneity, even for structures which are intrinsically homogeneous at such scales. In any case, even adopting a conservative point of view, we can reliably reject the hypothesis of a purely fractal structure for the cluster distribution, extending up to arbitrarily large scales.

The next question to be addressed in whether the different behavior of Abell and ACO clusters is a spurious effect induced by limited statistics and/or sample geometry, or they actually traces different populations of cosmic structures.

To answer this question, we realized the same scaling analysis for the set of simulated cluster samples presented in §2.2. Figures 5 and 6 are the same as Figures 3 and 4, respectively, but for the artificial Abell and ACO samples, so that they can be directly compared. The plotted data are obtained by averaging over 50 different realization. In no cases is there evidence of scale-invariant behavior extended on a scale-range as large as that observed for the ACO sample. In the regime where the analysis methods reliably work, the local dimension exhibits a smooth
increasing trend, similarly to the Abell case. Quite remarkably, the simulations, which are intrinsically homogeneous at large scales, provide dimension estimates at $r > 50 \, h^{-1} \, Mpc$, which are quite similar to that displayed by observational data. This further confirm that the observed cluster distribution is consistent with a picture in which the clustering develops induced by gravitational instability from nearly homogeneous initial conditions. As for the method based on excluding the self-point, it is apparent in several cases the effect of overestimating the dimension at small scales for $q = 2$, which suggests that this method could have trouble at small $q$ values, when dealing with such a limited statistics.

Therefore, the conclusion that we draw from the analysis of the synthetic cluster samples is that the different scaling properties of Abell and ACO clusters are likely to be real. While the Abell distribution can be interpreted simply on the ground of a peak selection on a Zel’dovich perturbed Gaussian background, the ACO statistics requires something more. Conclusions go in the same direction as suggested by previous comparison between simulated and observed cluster distributions. PVJ found some significant differences between the clustering patterns of Abell and ACO samples, the latter being systematically at variance with respect to the clustering provided by the evolution of an initially Gaussian fluctuation spectrum. A similar finding has also been found by Plionis, Valdarnini & Coles (1992) from the analysis of the topological genus. They found a systematic “meatball” shift for ACO clusters, which is observed neither in the Abell sample nor in the same set of simulations. The conservative assumption about the different statistics of the two samples is that for ACO the higher sensitivity of the emulsion plates used to construct the catalogue implies that it traces the cluster distribution in a better way than Abell. Otherwise, one should be willing to accept the idea of a real difference in the statistical properties of clusters in different regions of the sky.

In Figure 7 we show the effects of including also clusters with $d > 200 \, h^{-1} \, Mpc$ (left panel) and of not correcting for finite volume effects (right panel) on the estimate of the correlation dimension $D_2$. In the first case, we note some difference in the dimension estimate is introduced at small scales. This shows that our procedure to correct for redshift selection works until $P(z)$ takes very small values. In this regime, a larger number of objects is needed to suppress noise effects. On the other hand, no significant difference is induced by redshift selection at large scales. In this regime, instead, the effects of border corrections are more important. As expected, taking $f_i(r) = 1$ in eq.(9) causes a spurious underestimate of $D_2(r)$ for $r > 40 \, h^{-1} \, Mpc$. 

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4 Scaling from the \( n \)-nearest neighbor analysis

4.1 The method

Although the method described in the previous section is in principle valid for any real \( q \), in practice the scaling relation shown in eq.(6) does not always hold for \( q < 2 \), even for unambiguous multifractal sets. As we have already said, low \( q \) values emphasizes the low density regions of the point set. The scaling in this region is better followed by means of the following scaling law,

\[
W(\tau,p) = \frac{1}{N} \sum_{i=1}^{N} r_i(p)^{-\tau} \sim p^{1-q},
\]

where \( r_i(p) \) is the radius of a sphere centered at point \( i \), which encompasses \( n \) neighbors, and \( p = n/N \). In an unlimited and complete point set, \( r_i(p) \) is just the distance from point \( i \) to its \( n \)-th nearest neighbor. However, due to the sample boundaries and the redshift and galactic latitude selection functions of the surveys, the values of \( r_i(p) \) have to be corrected. Let us call \( d_i^0 \) the actual distance from cluster \( i \) to its \( n \)-th nearest neighbor within the survey. As we are missing clusters at large distances and low latitudes, the distance to the \( n \)-th nearest neighbor in a complete sample, \( r_i(p = n/N) \) should be less or equal to \( d_i^0 \). The correction may be performed in the following way.

We compute \( d_i^j \) for \( j = 1, ..., n \). Then we consider the number of neighbors, \( n(d_i^j) \), within a ball of radius \( d_i^j \), taking into account the selection functions, but not the border corrections, as in eq.(9). Note that \( n(d_i^j) \) in general is not an integer. The next step is to find the neighbors \( j_1 \) and \( j_2 \) satisfying the inequality \( n(d_i^{j_1}) \leq n \leq n(d_i^{j_2}) \). Then the value \( \tilde{r}_i(p = n/N) \) corrected for selection functions is obtained just by the linear interpolation

\[
\tilde{r}_i(p) = d_i^{j_1} + \left( d_i^{j_2} - d_i^{j_1} \right) \frac{n - n(d_i^{j_1})}{n(d_i^{j_2}) - n(d_i^{j_1})}.
\]

The radius has still to be corrected for the edge effects due to the finite volume of the survey. This correction is performed taking into account the fraction of the sphere of radius \( \tilde{r}_i(p) \), \( f(\tilde{r}_i(p)) \), falling within the boundaries of the sample. The corrected radius is

\[
r_i(p) = \tilde{r}_i(p) \sqrt{f(\tilde{r}_i(p))}.
\]

The corrected radius enters into eq.(10), in order to obtain \( q \) as a function of \( \tau \).
The reliability of the $n$–nearest neighbor method to recover the scaling properties of a fractal distribution has been discussed in detail by Borgani et al. (1993). After applying this method to the analysis of several fractal distributions, it has been found that this algorithm provides a considerable improvement in the estimate of the fractal dimension in the $q < 2$ regime.

### 4.2 The results

In Figs. 8 and 9 we show the scaling behavior of the function $W(\tau, p)$ for $\tau = -2, -5, -10$. The plotted errorbars are evaluated in the same way as for the correlation integral method.

The analysis is just for the real cluster samples. The scaling is well observed for both the Abell and ACO cluster samples. The oscillating behavior of the dimension estimates observed in the plots have to be interpreted as the fingerprint of the presence of lacunarity in the catalogs. As $\tau$ decreases, the dimension increases showing multifractality. The exact dimension values as well as the corresponding statistical errors, are shown in Table 2. The scaling range for $p = n/N$ in the linear regression fit has been $n = 5 - 50$. Figure 10 shows that this method is less affected by the border correction than the correlation integral method, since the dimensions shown on the top panels b and c are quite similar. The correction for the selection functions explained in the previous paragraph works well when the method is applied to the whole catalog with depth $284 \ h^{-1}Mpc$. The dimension estimates obtained for the closer and nearly complete sample (up to $200 \ h^{-1}Mpc$) agrees well with the values obtained for the deepest and less complete sample (see Figure 10).

As for the capacity dimension, we note that consistent values are detected for Abell and ACO samples, with a value, $D_0 \simeq 2.6$, which is intermediate between that, $D_0 \simeq 2.2$, detected for optical galaxies (e.g., Martínez et al. 1990; Domínguez-Tenreiro, Gómez-Flechoso & Martínez 1993) and that, $D_0 \simeq 2.9$ estimated for QDOT–IRAS galaxies (Martínez & Coles 1994). Even going to more and more negative $\tau$ values, no significant differences are detected. This indicates that, apart from the different extension of the scaling regime detected for $q \geq 2$, the two cluster samples display a consistent clustering in both overdense and underdense regions.

### 5 Discussion and conclusions

We presented the results of a detailed scaling analysis for the spatial distribution of Abell and ACO clusters. To this purpose, we resorted to the fractal analysis approach, which has been
shown to be a reliable tool to characterize the statistics of the large-scale distribution of cosmic structures (e.g., Martínez et al. 1990; Valdarnini, Borgani & Provenzale 1992). The major problem in this kind of analysis is the rather low amount of statistical information allowed by the available cluster samples. For this reason, much care must be payed to disentangle the effects of poor statistics in the interpretation of the results.

To properly face these difficulties, we decided to apply two different algorithms to estimate fractal dimensions, which are expected to work in two different regimes; the correlation–integral method of eq.(6), which is expected to give reliable dimension estimates within overdensities, and the $n$–nearest neighbor method of eq.(10), which works better inside the rarefied parts of the distribution. However, the limited statistics does not represent the only problem. Further complications arise due to the observational biases present in the cluster samples, such as the finite size of the surveyed volumes and the selection effects both in galactic latitude and in redshift. To account for these, we introduced suitable corrections in the fractal dimension estimators and verified their reliability on simulated cluster samples, with a priori known scaling properties.

Based on the results presented in the previous sections, we can properly answer the three questions addressed in the Introduction.

a) The spatial cluster distribution displays a scale–invariant behavior, which is however confined to a limited scale range. The extension of this scaling regime is found to be different for Abell and ACO clusters, respectively. The first develops in the $15 \lesssim r \lesssim 60 \, h^{-1} \, Mpc$ scale range, while the second extends down to the smallest considered scales ($r \gtrsim 5 \, h^{-1} \, Mpc$). In this scale range, the cluster distribution shows a marked multifractal behavior. This is apparent from Figure 11, where we plot the $D_q$ spectrum of multifractal dimensions, obtained from a log–log linear regression over the scale–range $(15 \lesssim r \lesssim 60 \, h^{-1} \, Mpc)$, where both Abell and ACO clusters develop the scaling behavior. For $q < 2$ the estimate is based on the $n$–nearest neighbor method, while for $q \geq 2$ we resort to the correlation–integral approach.

Note that in the former case the physical scale is not fixed a priori. Instead, fixing the range of neighbor orders, larger and larger scales are preferentially weighted in the $W(\tau, p)$ estimate as more and more negative $q$ values are considered. Therefore, the decreasing trend of the $D_q$ curve for $q < 2$ could be not only due to multifractal scaling, but also to the presence of scale–dependent clustering (see also Borgani et al. 1993). However, the rather smooth behavior of $D_q$ around $q = 2$ indicates that both analysis methods provide
consistent dimension estimates inside the overdensities.

For both samples the value of the correlation dimension is $D_2 \approx 2.2 h^{-1} Mpc$ in agreement with the behavior of $1 + \xi(r)$ detected by Calzetti et al. (1992). Despite the different extension of the scaling regime inside the overdensities, Abell and ACO clusters shows a remarkably consistent multifractal spectrum in both $q > 2$ and $q < 2$ regimes.

b) The comparison with the cluster simulations based on the Zel’dovich approximation shows that these substantially account for the scaling of Abell clusters, while they do not reproduce the more extended scaling regime displayed by the ACO sample. This suggests that the distribution of ACO clusters requires something more than a peak selection procedure on Zel’dovich perturbed Gaussian fluctuations, which were otherwise able to generate the correct 2- and 3-point cluster correlation function (see also PVJ and Plionis, Valdammini & Coles 1992, for similar conclusions about the different clustering of Abell and ACO redshift samples).

c) The evidence for a limited scale-range of self–similar clustering disproves the picture of a purely fractal structure of the Universe, extending up to arbitrarily large scales (see also Peebles (1993) for other indirect tests reaching to the same conclusion). A different, but related, question is whether we are able to detect the homogeneity scale within the volume encompassed by the cluster samples. It is clear that, if we define the homogeneity scale $L_h$ as that where $D_q(L_h) = 3$, the answer is no. Instead, at the largest considered scales ($\approx 100 h^{-1} Mpc$) we find $D_q \geq 2$. However, we find the same result even analyzing point distributions, which are intrinsically homogeneous at large–scales, when using the same amount of statistics allowed by the cluster samples. Therefore, our conclusion is that the spatial cluster distribution behaves like a structure which becomes homogeneous at large scales around $L_h \approx 100 h^{-1} Mpc$.

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**Figure Captions**

**Figure 1.** The local dimension for the artificial Abell cluster sample, extracted from a scale-dependent fractal structure, with homogeneity scale $L_h \approx 40 \ h^{-1} \ Mpc$ and $D = 1$ for $r < L_h$ (see text). The results are based on the correlation integral method. From top to bottom we plot results for $q = 2, 3$ and 5. For each $r$ value, $D_q(r)$ is evaluated by means of a 5–point log-log local linear regression on the slope of the $Z(q,r)$ partition function. We compare results obtained when including the center point in the evaluation of the probability $p_i(r)$ (circles), when excluding it (squares) and when correcting for shot-noise effects (triangles).

**Figure 2.** The same as in Figure 1, but for clusters extracted from a purely fractal structure without large-scale homogeneity and dimension $D = 1$. Only results based on excluding the center points are shown.

**Figure 3.** The $Z(q,r)$ partition function and the $D_q(r)$ local dimension are plotted for the real Abell cluster sample. The analysis is here realized by taking only clusters within $d < 200 \ h^{-1} \ Mpc$. From left to right we show results for $q = 2, 3$ and 5. Filled dots refer to the analysis done by excluding the center points. The open dots for $q = 3$ and 5 show the local dimension obtained by correcting for shot–noise effects. Errorbars for $Z(q,r)$ are evaluated as $1\sigma$ scatter between 100 bootstrap resamplings. Errorbars in the local dimension are $3\sigma$ uncertainties in the 5–point weighted least square fit.

**Figure 4.** The same as in Figure 3, but for the ACO sample.

**Figure 5.** The same as in Figure 3, but for the simulated Abell sample (see text).

**Figure 6.** The same as in Figure 3, but for the simulated ACO sample (see text).

**Figure 7.** Effects of corrections for boundaries and redshift selection functions on the estimate of the correlation dimension $D_2$ for the Abell sample. The left panel is for the whole sample, extending up to $d = 284 \ h^{-1} \ Mpc$ and correcting for redshift selection. Central panel is the same as in Figure 3. The right panel includes only clusters with $d < 200 \ h^{-1} \ Mpc$ but without correcting for boundary effects.

**Figure 8.** The $W(\tau,p)$ partition function, along with the corresponding local dimension, for the Abell sample. From left to right we report results for $\tau = -2, -5$ and $-10$. Errors in
$W(\tau,p)$ are estimated from 100 bootstrap resamplings.

**Figure 9.** The same as in Figure 8, but for the ACO sample.

**Figure 10.** The same as in Figure 7, but for the $W(\tau,p)$ partition function at $\tau = -2$.

**Figure 11.** The multifractal dimension spectrum $D_q$ for Abell (solid line) and ACO (dashed line) samples. For $q \leq 2$, the estimate is based on the $W(\tau,p)$ partition function, while for $q \geq 2$ we resort to the correlation integral method. In the first case the dimension is evaluated over the range of $p = n/N$ values corresponding to $n = 5 - 50$, while in the second case it is evaluated over the scale range $15 \leq r \leq 60$ h$^{-1}$ Mpc. Errorbars are standard deviations in the weighted log-log linear regression of the corresponding partition function.
Table 1: $D_q$ dimensions for the Abell and ACO cluster samples calculated by means of the correlation integral method. Also reported are the corresponding standard deviations for the weighted linear regression. The fit is realized over the scale-range $15 \leq r \leq 60 \, h^{-1} \, Mpc$.

<table>
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<tr>
<th>Sample</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_5$</th>
</tr>
</thead>
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<tr>
<td>Abell</td>
<td>2.19 ± 0.05</td>
<td>1.91 ± 0.04</td>
<td>1.74 ± 0.03</td>
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<tr>
<td>ACO</td>
<td>2.21 ± 0.07</td>
<td>1.95 ± 0.05</td>
<td>1.80 ± 0.04</td>
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</table>

Table 2: $D_q$ dimensions for the Abell and ACO cluster samples calculated by means of the $n$-nearest neighbor method. For each value of $\tau$ the obtained $q$ and $D_q$ is given, along with the corresponding statistical errors. The estimate of the capacity dimension $D_0$ is shown in the first column. The fit is realized over the considered range of $p$ values, with $p = n/N$, $n = 5 - 50$. of neighbor order.

<table>
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<th>$D_0$</th>
<th>$\tau = -2$</th>
<th>$\tau = -5$</th>
<th>$\tau = -10$</th>
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<td></td>
<td></td>
<td>$q$ $D_2$</td>
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<td>$q$ $D_2$</td>
</tr>
<tr>
<td>Abell</td>
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<tr>
<td>ACO</td>
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<td>-0.79 2.80 ± 0.07</td>
<td>-2.33 3.00 ± 0.10</td>
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