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THE POST NEWTONIAN SPIN OCTUPOLE MOMENT

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Generation of Gravitational Waves: The Post Newtonian Spin Octupole Moment

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Abstract

Using the gravitational wave formalism developed by Blanchet, Damour and Iyer, this note computes the post-Newtonian-accurate spin octupole moment. The result is checked by explicitly verifying the transformation of the radiative spin octupole moment under shifts of the spatial origin.

The prospects of gravitational wave interferometers operating by the turn of this century have made more urgent the need to have as accurate a knowledge of the gravitational waveform as possible (see e.g. Ref. [1]). The possibility to convert the gravitational wave interferometers into an ‘observatory’ and the observations into an astronomy requires a good treatment of the generation problem which aims to relate the far zone waveforms to details and motions of the sources. Building on the Multipolar-Post-Minkowskian formalism of Blanchet and Damour[2], a mathematically well defined post-Newtonian (PN) generation formalism has been developed recently by Blanchet, Damour and Iyer[3, 4, 5]. The first in this series of papers[3] provided compact support expressions for the PN mass moments while the second[4] displayed the corresponding expressions for the PN spin moments. The third paper in this series[5] computed the leading hereditary contributions in the generation of

gravitational waves (1.5 PN “tail” contributions to the radiative mass and spin quadrupole moments). Further, the work[4] provided an explicit expression for the post Newtonian spin quadrupole. This expression was used by Wiseman in studies of linear momentum carried by gravitational radiation from coalescing binaries in the PN approximation[6] as also, together with the results of[5], in the computation of waveforms up to 2.5 post Newtonian order including wave tails[7]. Recently, Iyer and Will[8] have discussed the PN gravitational radiation reaction for 2-body systems using PN expressions for energy and angular momentum flux to infinity.

To go beyond these works, *i.e.* to study post-post Newtonian (PPN) fluxes of energy, angular momentum and linear momentum, will require a PPN generation formalism, *i.e.* $1/c^4$ -accurate expressions for mass quadrupole, mass octupole and spin quadrupole. Though such a PPN generation formalism is still in the future, there is one contribution to the PPN-accurate linear momentum flux coming from the existing PN generation formalism which can be already derived: namely the contribution coming from the beating of the PN spin octupole moment J_{ijk} against the PN mass octupole moment I_{ijk} , *i.e.* a term proportional to $\epsilon_{ijk} I_{jab}^{(4)} J_{kab}^{(4)}$. The purpose of the present note is to provide the explicit expression of the PN-accurate spin octupole moment needed to compute the latter contribution. The calculations, though long, are straightforward. The correctness of various numerical coefficients in this formula is verified by a consideration of the transformation of the spin octupole under a shift of the spatial origin.

In [4] the implementation of the multipolar gravitational wave generation formalism yielded a well-defined expression for the PN spin source moments $J_L[\text{source}] + O(c^{-4})$. Specializing to $l = 3$ the above expression reduces to

$$\begin{aligned}
J_{ijk} = & \frac{STF}{ijk} \left\langle \epsilon_{kab} \left[\int d^3 y_1 y_1^{aij} \left(1 + \frac{4U}{c^2} \right) \sigma_1^b - \right. \right. \\
& - \frac{G}{2c^2} \int d^3 y_1 d^3 y_2 \left\{ \sigma_1^s \sigma_2^t \left(4Y_{,bs}^{aij} - 3Y_{,sb}^{aij} \right) - 4\sigma_1^b \sigma_2^t Y_{,ss}^{aij} \right\} + \\
& + \frac{1}{18c^2} \frac{d}{dt} \left\{ \int d^3 y_1 y_1^{assij} \sigma_1^t \frac{\partial U}{\partial y_1^b} + \frac{7G}{5} \int d^3 y_1 d^3 y_2 \sigma_1^s \sigma_2^t Y_{,bs}^{assij} + \right. \\
& \left. \left. + \frac{2}{5} \int d^3 y_1 \left[7y_1^{assij} T^{bj} - 2y_1^{assij} T^{bs} \right] \right\} \right] \right\rangle + O(c^{-4}) \quad (1)
\end{aligned}$$

where STF denotes a symmetric trace-free projection and where

$$\begin{aligned}
\sigma(\mathbf{y}_1, t) &= \frac{T^{00}(\mathbf{y}_1, t) + T^{ss}(\mathbf{y}_1, t)}{c^2}, \quad \sigma^a(\mathbf{y}_1, t) = \frac{T^{0a}(\mathbf{y}_1, t)}{c} \\
\sigma_i &= \sigma(\mathbf{y}_i, t), \quad \sigma_i^a = \sigma^a(\mathbf{y}_i, t), \quad r_{12} = |\mathbf{y}_1 - \mathbf{y}_2| \\
U(\mathbf{y}_1, t) &= G \int \frac{d^3 y_2 \sigma_2}{r_{12}} \\
\mathbf{y}_\alpha &= \alpha \mathbf{y}_1 + (1 - \alpha) \mathbf{y}_2 \\
Y^L(\mathbf{y}_1, \mathbf{y}_2) &= r_{12} \int_0^1 d\alpha y_\alpha^{<L>} \\
Y_{,ab}^L &= \frac{\partial^2}{\partial y_1^a \partial y_2^b} Y^L.
\end{aligned}$$

The calculation needs an evaluation of the $Y_{,bs}^{aij}$ and $Y_{,bs}^{asij}$ like terms. Noting (with angular brackets denoting a symmetric trace-free projection)

$$\begin{aligned}
\int_0^1 d\alpha y_\alpha^{asij} &= y_1^{<asij>} - 2 y_1^{<asi>} y_{12}^{j>} \\
&+ 2 y_1^{<as>} y_{12}^{ij>} - y_1^{<a>} y_{12}^{sij>} + \frac{1}{5} y_{12}^{<asij>} \quad (2)
\end{aligned}$$

$$\begin{aligned}
\int_0^1 d\alpha y_\alpha^{aij} &= y_1^{<aij>} - \frac{3}{2} y_1^{<ai>} y_{12}^{j>} + y_1^{<a>} y_{12}^{ij>} \\
&- \frac{1}{4} y_{12}^{<aij>} \quad (3)
\end{aligned}$$

using (see e.g. Ref. [9])

$$\begin{aligned}
T_{<ijkl>} &= T_{(ijkl)} - \frac{1}{7} \left(\delta_{ij} T_{(klaa)} + \delta_{ik} T_{(jl aa)} + \delta_{il} T_{(jk aa)} \right. \\
&+ \delta_{jk} T_{(il aa)} + \delta_{jl} T_{(ik aa)} + \delta_{kl} T_{(ij aa)} \Big) \\
&+ \frac{1}{35} \left(\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) T_{(aabb)} \quad (4)
\end{aligned}$$

and transforming from variables $(\mathbf{y}_1, \mathbf{y}_2)$ to $(\mathbf{y}_1, \mathbf{y}_{12})$ we obtain after a long calculation

$$\begin{aligned}
STF_{ijk} \epsilon_{kab} Y_{,bs}^{asij} &= -STF_{ijk} \epsilon_{kab} \left[\frac{1}{r_{12}} \left(-y_1^{saij} n_{12}^{sb} + \frac{2}{7} y_1^{ssaj} n_{12}^{bi} \right) - \right. \\
&- \frac{10}{7} y_1^{aij} n_{12}^b + \frac{5}{7} y_1^{sai} n_{12}^{sbj} - \frac{1}{7} y_1^{ass} n_{12}^{bij} + \\
&+ \left. \frac{17}{7} r_{12} y_1^{aj} n_{12}^{bi} - \frac{1}{7} r_{12} y_1^{as} n_{12}^{sbij} - r_{12}^2 y_1^a n_{12}^{bij} \right] \quad (5)
\end{aligned}$$

$$\begin{aligned}
{}^{STF}_{ijk} \epsilon_{kab} Y_{,bs}^{aij} &= - {}^{STF}_{ijk} \epsilon_{kab} \left[\frac{1}{r_{12}} (\delta^{bs} y_1^{aij} - y_1^{aij} n_{12}^{bs}) - \right. \\
&\quad - \left\{ \delta^{bs} y_1^{ai} n_{12}^j + \delta^{si} y_1^{aj} n_{12}^b - y_1^{ai} n_{12}^{sbj} \right\} + \\
&\quad + \left. \frac{r_{12} y_1^a}{3} \left\{ \delta^{sb} n_{12}^{ij} + 2 \delta^{si} n_{12}^{bj} - n_{12}^{sbij} \right\} \right] \quad (6)
\end{aligned}$$

$$\begin{aligned}
{}^{STF}_{ijk} \epsilon_{kab} Y_{,sb}^{aij} &= - {}^{STF}_{ijk} \epsilon_{kab} \left[\frac{\delta^{bs} y_1^{aij} - y_1^{aij} n_{12}^{bs}}{r_{12}} + \right. \\
&\quad + \delta^{as} y_1^{ij} n_{12}^b + \delta^{is} y_1^{aj} n_{12}^b + y_1^{ai} n_{12}^{bsj} - \delta^{bs} y_1^{ai} n_{12}^j - \\
&\quad - r_{12} \left\{ \frac{1}{3} \delta^{is} y_1^a n_{12}^{bj} + \delta^{as} y_1^i n_{12}^{bj} + \frac{1}{3} (y_1^a n_{12}^{bsij} - \delta^{bs} y_1^a n_{12}^{ij}) \right\} + \\
&\quad + \left. \frac{r_{12}^2}{3} \delta^{as} n_{12}^{bij} \right] \quad (7)
\end{aligned}$$

$$\begin{aligned}
{}^{STF}_{ijk} \epsilon_{kab} Y_{,ss}^{aij} &= - {}^{STF}_{ijk} \epsilon_{kab} \left[\frac{2y_1^{aij}}{r_{12}} - 2y_1^{ai} n_{12}^j - \right. \\
&\quad - \left. y_1^{ij} n_{12}^a + r_{12} y_1^a n_{12}^{ij} + 2r_{12} y_1^i n_{12}^{aj} - r_{12}^2 n_{12}^{aij} \right] . \quad (8)
\end{aligned}$$

Using equations (5-8) in equation (1), and simplifying one obtains the main result of this note, i.e. an explicit formula for the spin octupole moment. It may be written as

$$\begin{aligned}
J_{ijk} &= {}^{STF}_{ijk} \left\langle \epsilon_{kab} \int d^3x \left[\sigma^b x^{aij} + \right. \right. \\
&\quad + \frac{1}{c^2} \left\{ \frac{1}{2} \sigma^b x^{aij} P_{(-1)} - \frac{1}{2} \sigma^s x^{aij} P_{(-1)}^{sb} + \frac{7}{2} \sigma^b x^{ai} P_{(0)}^j - \frac{7}{2} \sigma^i x^{aj} P_{(0)}^b - \right. \\
&\quad - \frac{7}{2} \sigma^a x^{ij} P_{(0)}^b + \frac{1}{2} \sigma^s x^{ai} P_{(0)}^{sbj} - \frac{11}{6} \sigma^b x^a P_{(1)}^{ij} + \frac{11}{6} \sigma^i x^a P_{(1)}^{bj} + \\
&\quad + \frac{11}{2} \sigma^a x^i P_{(1)}^{bj} - \frac{1}{6} \sigma^s x^a P_{(1)}^{sbij} + \frac{3}{2} \sigma^a P_{(2)}^{bij} \left. \right\} + \\
&\quad + \frac{1}{c^2} \frac{d}{dt} \left[\sigma \left\{ -\frac{1}{18} x^{ssaij} P_{(-2)}^b + \frac{7}{90} x^{saij} P_{(-1)}^{sb} - \frac{1}{45} x^{ssaj} P_{(-1)}^{bi} + \frac{1}{9} x^{aij} P_{(0)}^b - \right. \right. \\
&\quad - \frac{1}{18} x^{sai} P_{(0)}^{sbj} + \frac{1}{90} x^{ssa} P_{(0)}^{bij} - \frac{17}{90} x^{aj} P_{(1)}^{bi} + \frac{1}{90} x^{sa} P_{(1)}^{sbij} + \frac{7}{90} x^a P_{(2)}^{bij} \left. \right\} + \\
&\quad + \left. \left. \frac{7}{45} x^{ssai} T^{bj} - \frac{2}{45} x^{saij} T^{sb} \right] \right] \rangle \quad (9)
\end{aligned}$$

In the above,

$$P_{(\alpha)}^N(\mathbf{y}_1) \equiv P_{(\alpha)}^{i_1 i_2 \dots i_n} \equiv G \int d^3 y_2 \sigma_2 r_{12}^\alpha n_{12}^{i_1} n_{12}^{i_2} \dots n_{12}^{i_n}$$

It may also be mentioned that the form of the term from this explicit calculation indeed matches with the general form of J_L , - equation (5.38)- discussed in reference[4].

To verify the numerical coefficients that appear in the above equation (9) we examine the transformation of the spin octupole under shifts of the spatial origin. If the world line $\mathcal{L}$ is shifted by $-\vec{\epsilon}$ in space to give $\mathcal{L}_\epsilon$ and if the changes in M_L and S_L induced by that shift are denoted as:

$$\begin{aligned} \delta_\epsilon M_L &:= M_L[\mathcal{L}_\epsilon] - M_L[\mathcal{L}] \\ \delta_\epsilon S_L &:= S_L[\mathcal{L}_\epsilon] - S_L[\mathcal{L}] \end{aligned} \quad (10)$$

then one can obtain ‘shift’ formulas for the mass and spin moments by writing that the metric remains invariant under the combination of the shifts (10) with $\delta\vec{r} = +\vec{\epsilon}$ [4]. These general formulas when specialized to the spin octupole yield

$$\begin{aligned} \delta_\epsilon J_{ijk} &= STF_{ijk} \left[\frac{8}{3} \epsilon_k S_{ij} + \frac{1}{3} \epsilon_a M_{bij}^{(1)} \epsilon_{kab} \right. \\ &\quad \left. + \frac{4}{45c^2} \epsilon_s S_{sijk}^{(2)} \right] + O\left(\frac{1}{c^4}\right). \end{aligned} \quad (11)$$

Writing $\delta_\epsilon J_{ijk} = \epsilon^s K_{s<ijk>}$ and decomposing $K_{s<ijk>}$ into irreducible pieces (see [9] for the needed technology), equation (11) reduces to

$$K_{<sijk>} = \frac{4}{45c^2} S_{sijk}^{(2)} \quad (12)$$

$$STF_{ijk} \epsilon_{kab} K_{abij} = -\frac{4}{9} M_{ijk}^{(1)} \quad (13)$$

$$K_{aaij} = \frac{56}{15} S_{ij} \quad (14)$$

The $K_{s<ijk>}$ is made up of terms that arise from every term of J_{ijk} (except the twelfth) through to a shift in the coordinate position vector, $\vec{x} \rightarrow \vec{x} + \vec{\epsilon}$. The implementation of this procedure is long and tedious, but straightforward

and we summarise the main strategy of this computation. It generates a multitude of term of the following kinds

$$\begin{aligned}\mathcal{I}^{L,a} &= \int d^3x \, \sigma^a x^L \\ I_{(\alpha)}^{L,a,M} &= \int d^3x \, \sigma^a x^L P_{(\alpha)}^M \\ J_{(\alpha)}^{L,M} &= \frac{d}{dt} \int d^3x \, x^L \sigma P_{(\alpha)}^M \\ Q^{M,ab} &= \frac{d}{dt} \int d^3x \, T^{ab} x^M.\end{aligned}$$

The verification of equation (14) uses the existence of identities between some J terms, *e.g.*,

$$2 \, \epsilon_{iab} \, J_{(0)}^{aj,b} = \epsilon_{iab} \, J_{(1)}^{a,jb}.$$

Verification of equation (13) is the most involved and requires the use of 1PN accurate continuity equation in the leading term

$$\frac{\partial}{\partial t} \left(\frac{T^{00}}{c^2} \right) + \frac{\partial}{\partial x^i} \left(\frac{T^{0i}}{c} \right) = -\frac{1}{c^2} \sigma \frac{\partial U}{\partial t} + O\left(\frac{1}{c^4}\right).$$

One needs to implement the time derivatives explicitly in the J terms, use the above continuity equation (in the newtonian limit) and convert the terms to I -type terms and new V -type terms that look like

$$V_{(\alpha)}^{L,a,M} = \int d^3y_1 \, \sigma_1 y_1^L \sigma_2^a n_{12}^M r_{12}^\alpha.$$

The V terms are linear combinations of I terms and taking account of these identities the relation equation (13) is finally verified. The verification of equation (12) is comparatively straightforward and needs no new inputs. Note that the spin moment S_{sijk} is needed only at the newtonian level in this verification.

It should be pointed out that all the numerical coefficients are checked by this procedure except the twelfth term $\sigma^a P_{(2)}^{bij}$, which is translation invariant.

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