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GENERALIZED FREDHOLM DETERMINANTS AND
SELBERG ZETA FUNCTIONS FOR AXIOM A
DYNAMICAL SYSTEMS

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Generalized Fredholm Determinants and Selberg Zeta Functions for Axiom A Dynamical Systems.

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Abstract

We consider a generalized Fredholm determinant $d(z)$ and a generalized Selberg zeta function $\zeta(\omega)^{-1}$ for Axiom A diffeomorphisms of a surface and Axiom A flows on 3-dimensional manifolds, respectively. We show that $d(z)$ and $\zeta(\omega)^{-1}$ extend to entire functions in the complex plane. That the functions are entire and not only meromorphic is proved by a new method, identifying removable singularities by a change of Markov partitions.

For Axiom A systems (diffeomorphisms or flows) hyperbolic properties are conveniently expressed in terms of canonical coordinates, constructed from the stable/unstable foliations [23]. Under the strong assumption that these foliations are real-analytic, using a spectral theory of Ruelle for expanding maps, Ruelle [17] and Fried [6] proved the meromorphicity in the entire complex plane of a weighted ζ -function constructed from periodic orbits. That the underlying assumption imposes serious restrictions on the possible applications of this result is indicated by a result of Ghys [10] stating that a C^∞ geodesic flow on a surface of negative curvature with C^2 weak stable foliations is conjugate to a geodesic flow on a surface with constant negative curvature.

In [21] we introduced a spectral theory for real-analytic hyperbolic maps which a priori treats a different set-up, but makes no special assumptions on the stable/unstable foliations. The coordinates used there are just 'close' to the s/u foliations. In this way analyticity of the map is preserved and at the same time a suitable splitting of the tangent space into approximately contracting and expanding

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directions is maintained. We demonstrated how to use periodic orbits to construct a generalized Fredholm determinant which is entire in the complex plane.

Here we shall apply the latter result to general axiom A dynamical systems and prove the following :

Theorem 1 : *Let $\mathcal{M}$ be a 2-dimensional manifold and let $f : \mathcal{M} \rightarrow \mathcal{M}$ be a diffeomorphism, real-analytic and Axiom A on the compact basic set $\Lambda \subset \mathcal{M}$. Let g be a complex-valued function (weight), analytic on Λ . Then the generalized weighted Fredholm determinant :*

$$d(z) = \exp \left(- \sum_{n \in \mathbb{N}} \frac{z^n}{n} \sum_{x \in \text{fix } f^n|_{\Lambda}} \frac{g(f^{n-1}) \cdots g(x)}{|\det(Df^n(x) - 1)|} \right) \quad (1)$$

extends to an entire function in the complex plane.

Theorem 2 : *Let $\mathcal{M}$ be a 3-dimensional manifold and let $\phi^t : \mathcal{M} \rightarrow \mathcal{M}$, $t \in \mathbb{R}$ be a flow, real-analytic and Axiom A on the compact basic set Λ . Let a be a complex-valued function (weight), analytic on Λ . By $\exp_t(a)$ we denote its exponential under the flow. Let $\mathcal{P}$ denote the prime periodic orbits of ϕ^t and $\lambda(\gamma)$ the return time for each $\gamma \in \mathcal{P}$. Finally, let $P_* \phi_*^{\lambda(\gamma)}(\gamma)$ be the Jacobian of the return map with the direction of the flow projected out. One has $\text{sp}(P_* \phi_*^{\lambda}) \setminus \{0\} = \text{sp}(\phi_*^{\lambda}) \setminus \{1\}$.*

Then the generalized Selberg zeta function :

$$\zeta(\omega)^{-1} = \exp \left(- \sum_{m \in \mathbb{N}} \frac{1}{m} \sum_{\gamma \in \mathcal{P}} \frac{\exp_{\lambda(\gamma)}(\omega m a)}{|\det(P_* \phi_*^{m\lambda(\gamma)}(\gamma) - 1)|} \right) \quad (2)$$

extends to an entire function in the complex plane.

Remarks :

- The fact that the functions are *entire* and not only meromorphic is proved introducing a new method of identifying removable singularities by a change of Markov partitions. The method is not restricted to the analytic set-up but applies equally well to the case of C^k expanding maps of the interval for which the method in dimension one provides a simplified proof of similar results (cf. [19, 20]).

- We propose the following

Conjecture : *The manifolds in the two theorems may be of any finite dimensions (≥ 2 and ≥ 3 , respectively).*

Throughout the paper the manifolds are treated as general d-dimensional

manifolds. The constraint on the dimension of the manifold enters in the application¹ of the theorem of [21] and in the proof of entireness.

Example : Let ϕ^t be the unit velocity flow on a compact surface of constant negative curvature -1 . Let $a \equiv -1$ be the constant weight. For $\gamma \in \mathcal{P}$ we have $\text{sp}(f_*^{\lambda(\gamma)}(\gamma)) = \{e^{\lambda(\gamma)}, e^{-\lambda(\gamma)}, 1\}$ and hence

$$\begin{aligned} \zeta(\omega)^{-1} &= \exp \left(- \sum_{m \in \mathbb{N}} \frac{1}{m} \sum_{\gamma \in \mathcal{P}} \frac{e^{-\omega m \lambda(\gamma)}}{e^{m \lambda(\gamma)} (1 - e^{-m \lambda(\gamma)})^2} \right) \\ &= \exp \left(- \sum_{m \in \mathbb{N}} \frac{1}{m} \sum_{\gamma \in \mathcal{P}} e^{-\omega m \lambda(\gamma)} \sum_{k \in \mathbb{N}} k e^{-k m \lambda(\gamma)} \right) \\ &= \prod_{\gamma \in \mathcal{P}} \prod_{k \in \mathbb{N}} \left(\exp - \sum_{m \in \mathbb{N}} \frac{1}{m} [e^{-(\omega+k)\lambda(\gamma)}]^m \right)^k \\ &= \prod_{\gamma \in \mathcal{P}} \prod_{k \in \mathbb{N}} \left(\exp \log(1 - e^{-(\omega+k)\lambda(\gamma)}) \right)^k \\ &= \prod_{\gamma \in \mathcal{P}} \prod_{k \in \mathbb{N}} (1 - e^{-(\omega+k)\lambda(\gamma)})^k = \prod_{l \in \mathbb{N}} S(\omega + l) , \end{aligned}$$

where

$$S(\omega) = \prod_{\gamma \in \mathcal{P}} \prod_{k=0}^{\infty} (1 - e^{-(\omega+k)\lambda(\gamma)})$$

is the usual Selberg zeta function [22], which in general is entire but for which examples are known [7], where $S(\omega)$ has a simple pole at $\omega = 0$ and a zero at $\omega = 1$. Noting that these cancels out in the product one concludes, in agreement with the theorem, that $\zeta(\omega)^{-1}$ indeed is entire.

Other examples (diffeomorphisms) are the n -disk repeller [9, 4] and piecewise analytic perturbations of the linear Smale horseshoe [21].

Finally, we note the curious result that for a linear hyperbolic diffeomorphism f on the d -torus, with constant weight $g \equiv 1$, the number of fixed points of f^n equals $|Df^n - 1|$ (show that the density of fixed points in the universal covering is $1/|Df^n - 1|$, [14]) and hence,

$$d(z) = 1 - z .$$

¹Recently, Fried [8], found a generalization of the spectral theory for real-analytic hyperbolic maps enabling him to prove the meromorphicity part of the conjecture.

The result is reflected in the easily proved fact that in this special case correlation functions of analytic observables (see [21] for definitions) decay faster than any exponential.

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1 Axiom A diffeomorphisms.

We shall start by giving a brief definition of an Axiom A diffeomorphism. For details on Axiom A and basic sets etc. cf. [16, 23]. Let d be a Riemann metric on a d -dimensional manifold $\mathcal{M}$ and let $|\cdot|_\mu$ be the associated norm, continuous with respect to the base point $\mu \in \mathcal{M}$, on the complex tangent bundle $T\mathcal{M}$.

Let $f : \mathcal{M} \rightarrow \mathcal{M}$ be a diffeomorphism on $\mathcal{M}$ which is Axiom A over the basic set Λ , i.e. it is compact, f -invariant and periodic points are dense in Λ . Λ is hyperbolic, meaning that the restriction of the tangent bundle to Λ splits continuously as a sum of unstable and stable bundles $T_\Lambda \mathcal{M} = E^u \oplus E^s$. Each of these bundles are invariant under Tf and there exists constants $C_0 > 0, 1 > \theta_0 > 0$ such that :

$$\|Tf^n|E^s\| < C_0\theta_0^n, \quad \|Tf^{-n}|E^u\| < C_0\theta_0^n.$$

Bowen [1, 3] showed that for an Axiom A diffeomorphism f one has a Markov partition of the basic set. It consists of a finite cover $\mathcal{C} = \{R_i^A\}_{i \in I}$ of Λ with so-called rectangles having pairwise disjoint interior (relative to Λ). For the definition we refer to the above citations. The cover has the following properties :

- For any $\delta > 0$ the Markov partition can be chosen so that

$$\text{diam} R_i^A < \delta, \quad i \in I.$$

- Associated with the Markov partitioning there is a subshift of finite type $t : I \times I \rightarrow \{0, 1\}$ defined as follows :

$$t_{ji} = \begin{cases} 1 & \text{if } f \text{Int} R_j^A \cap \text{Int} R_i^A \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

- There is a finite number of subshifts of finite type τ_α corresponding to transitions between pairs of adjacent A-rectangles such that :

$$\text{Card Fix } f^n = \sum (-1)^\alpha \text{Card Fix } \tau_\alpha^n.$$

This is the Manning counting formula [15] (cf. also [17]) relating the 'true' periodic orbits on the manifold to periodic orbits of subshifts of finite type.

In the section below in which we proof entireness of the functions $d(z)$ and $\zeta^{-1}(\omega)$ (dimension 2 and 3, respectively) we shall also give a brief prescription for constructing the rectangles in order to obtain a Markov partitioning.

2 Real-analytic hyperbolic maps.

For $\alpha \in I \equiv \{1, \dots, S\}$, $S < \infty$, let $\mathcal{D}_\alpha^1 \subset \mathbb{C}^{d_s}$ and $\mathcal{D}_\alpha^2 \subset \mathbb{C}^{d_u}$ be connected compact sets having C^1 -boundaries.

Given a transition matrix $t_{ij} : I \times I \rightarrow \{0, 1\}$, we associate with each non-zero entry a map $\hat{f}_{ji} = (\hat{f}_{ji}^1, \hat{f}_{ji}^2)$ which is said to be *real-analytic hyperbolic* with respect to $\{\mathcal{D}_\alpha^{1,2}\}_{\alpha \in I}$ provided the following two conditions are fulfilled :

1. $\hat{f}$ is contracting in its first coordinates :

$$\hat{f}_{ji}^1 : Cl(\mathcal{D}_i^1) \times Cl(\mathcal{D}_j^2) \rightarrow Int(\mathcal{D}_j^1) .$$

2. $\hat{f}$ is expanding in its second coordinates, i.e. there exist real-analytic functions,

$$\phi_{ij} : Cl(\mathcal{D}_i^1) \times Cl(\mathcal{D}_j^2) \rightarrow Int(\mathcal{D}_i^2) ,$$

such that for each pair $w_1 \in Cl(\mathcal{D}_i^1)$, $z_2 \in Cl(\mathcal{D}_j^2)$, there is a unique solution (the *partial inverse*) $w_2 = \phi_{ij}(w_1, z_2)$ in $Int(\mathcal{D}_i^2)$ to :

$$\hat{f}_{ji}^2(w_1, w_2) = z_2 .$$

From the above, one proves [21] that to each symbolsequence $(i_0, \dots, i_n)$ with $t_{i_k i_{k+1}} = 1$, $k = 0, \dots, n-1$, there is a composite map $\hat{f}_{(i_0, \dots, i_n)}^n$ which is again real-analytic hyperbolic.

Theorem : Let $\hat{f}_{ij}$ be real-analytic hyperbolic with respect to $\{\mathcal{D}_\alpha^{1,2}\}_{\alpha \in I}$ (as specified above). Then fixed points of $\hat{f}^n$, $n \in \mathbb{N}$ are in 1-1 correspondence with the symbolic cycles, i.e. n -tuples $(i_1, \dots, i_n) \in I^n$ with $t_{i_1 i_2} t_{i_2 i_3} \dots t_{i_n i_1} = 1$. Let g be a weight, analytic in each domain $\mathcal{D}_i^1 \times \mathcal{D}_i^2$. If furthermore, each complex domain $\mathcal{D}_\alpha^k$ ($k = 1, 2$, $\alpha \in \{1, \dots, S\}$) is a polydomain², the generalized Fredholm determinant :

$$d(z) = \exp \left(- \sum_n \frac{z^n}{n} \sum_{x \in \text{Fix}(\hat{f}^n)} \frac{g(\hat{f}^{n-1}(x)) \dots g(x)}{|\det(D\hat{f}^n(x) - 1)|} \right)$$

²Polydomain = direct product of one (complex) dimensional domains. [12]

extends to an entire function in the complex plane.

For the proof when $g \equiv 1$ see [21]. The extension to a non constant, but analytic weight is trivial.

3 Real analytic Axiom A diffeomorphisms.

Let $(\mathcal{A}, \Psi) = \{(A_i \subset \mathbb{R}^d, \psi_i : A_i \rightarrow \mathcal{M})\}_{i \in I}$ be a finite real-analytic subatlas of $\mathcal{M}$, where the A_i 's are compact sets (charts) of small diameter in M such that their interior covers the basic set Λ , i.e. :

$$\Lambda \subset \bigcup_{i \in I} \psi_i \text{Int}(A_i) .$$

We have chosen compact rather than open charts to avoid a special treatment of points close to the boundary. For $x \in A_i$ we let $\tilde{x} = \psi_i x$ denote the corresponding point in M . Also, let $\Lambda_i = A_i \cap \psi_i^{-1}\Lambda$ denote the preimage of the basic set in the chart A_i . For $\mu \in A_i$ we identify the complexified tangent space $T_{\tilde{\mu}}\mathcal{M}$ with $\mathbb{C}^d$. The chart A_i may then be identified with a subset of this tangent space by the natural injection

$$\begin{aligned} \pi_{i,\mu} & : A_i \rightarrow T_{\tilde{\mu}}\mathcal{M} , \\ \pi_{i,\mu}(x) & = x - \mu . \end{aligned}$$

The expression $|v|_{i,\mu} = |T(\psi_i \circ \pi_i^{-1})|_{\mu} v|_{\mu}$, $\mu \in A_i$, $v \in \mathbb{C}^d$ abbreviates the norm on $T_{\tilde{\mu}}\mathcal{M}$ induced by the Riemann metric. When the charts have small diameter this norm is equivalent with the Riemann distance on the manifold, i.e. there are constants $C_1 > c_1 > 0$ such that

$$c_1 d(\tilde{x}, \tilde{y}) \leq |x - y|_{i,\mu} \leq C_1 d(\tilde{x}, \tilde{y}) \quad , x, y \in A_i .$$

The above definitions allow us to talk of induced real-analytic maps between tangent spaces. More precisely, if $\lambda_i \in A_i$, $\lambda_j \in A_j$ and $\psi_j^{-1}f\psi_i\lambda_i \in A_j$ there is a real-analytic induced map $\hat{f}_{ji}$ between respective tangent spaces defined in a neighborhood of the origin by the following diagram :

$$\begin{array}{ccccc} T_{\lambda_i}\mathcal{M} & \xleftarrow{\pi_i} & A_i & \xrightarrow{\psi_i} & \mathcal{M} \\ \hat{f}_{ji} \downarrow & & & & \downarrow f \\ T_{\lambda_j}\mathcal{M} & \xleftarrow{\pi_j} & A_j & \xrightarrow{\psi_j} & \mathcal{M} \end{array} \quad (3)$$

Below we shall show that we can choose the λ_i 's preserving at the same time real-analyticity and hyperbolicity in the sense of the Theorem of the previous section.

We require that the diffeomorphism is real-analytic on the basic set Λ . Using the compactness of the charts this implies that we can find constants $\Delta, \Delta' > 0$ such that :

- For $\lambda_i \in \Lambda_i$, $\lambda_j \in \Lambda_j$ and $\psi_i \lambda_i = \psi_j \lambda_j$ there are analytic extensions (given the same symbols) of ψ_i, ψ_j such that :

$$\psi_j^{-1} \circ \psi_i : B(\lambda_i, \Delta) \subset \mathbb{C}^d \rightarrow B(\lambda_j, \Delta') \subset \mathbb{C}^d$$

is analytic and bijective on its image.

- For $\lambda_i \in \Lambda_i$, $\lambda_j \in \Lambda_j$, $f \tilde{\lambda}_i = \tilde{\lambda}_j$, there is a real-analytic extension of the induced map f_{ji} , with

$$f_{ji} : B(\lambda_i, \Delta) \subset \mathbb{C}^d \rightarrow B(\lambda_j, \Delta') \subset \mathbb{C}^d$$

analytic and bijective on its image. We shall often write f for f_{ji} .

- For $\lambda_i \in \Lambda_i$, $f \lambda_i \in \Lambda_j$ and $x \in B(\lambda_i, \Delta)$ we have :

$$\begin{aligned} |fx - f\lambda_i|_{j, f\lambda_i} &\leq \Delta'/\Delta |x - \lambda_i|_{i, \lambda_i} \\ |fx - f\lambda_i - Df_{\lambda_i}(x - \lambda_i)|_{j, f\lambda_i} &\leq \epsilon_1(|x - \lambda_i|) |x - \lambda_i|_{i, \lambda_i} \end{aligned}$$

where $\epsilon_1 : [0, \Delta] \rightarrow [0, \infty[$ is continuous with $\epsilon_1(0) = 0$.

In the above, f may be replaced by f^{-1} .

We let P_μ^s and P_μ^u ($\mu \in \Lambda$) denote the projections onto the stable and unstable tangent bundle, respectively and with respect to the charts we write :

$$P_\mu^s T\psi_{i|\mu} = T\psi_{i|\mu} P_{i,\mu}^s, \quad P_\mu^u T\psi_{i|\mu} = T\psi_{i|\mu} P_{i,\mu}^u.$$

Thus $P_{i,\mu}^s$ and $P_{i,\mu}^u$ are linear operators in $\mathbb{C}^d$ representing the projections in $A_i \times \mathbb{C}^d$. From transversality of the splitting they are uniformly bounded operators :

$$\|P_{i,\mu}^s\| \leq C_2, \quad \|P_{i,\mu}^u\| \leq C_2.$$

Hyperbolicity and invariance can be stated as follows :

For $\rho \in T_\mu \mathcal{M}$, $k \in \mathbb{N}$ and $\mu \in \Lambda$:

$$\begin{aligned} |Tf_\mu^k P_\mu^s \rho|_\mu &\leq C_0 \theta_0^k |P_\mu^s \rho|_\mu & Tf_\mu P_\mu^s &= P_{f\mu}^s Tf_\mu \\ |Tf_\mu^{-k} P_\mu^u \rho|_\mu &\leq C_0 \theta_0^k |P_\mu^u \rho|_\mu & Tf_\mu^{-1} P_\mu^u &= P_{f^{-1}\mu}^u Tf_\mu^{-1} \end{aligned}$$

Mather norms. We obtain strict contractions for $Tf|E^s$ and $Tf^{-1}|E^u$ by introducing the Mather (semi-) norms [13]. Choose θ_1 , $\theta_0 < \theta_1 < 1$ and define for $\lambda \in \Lambda$, $\rho \in T_\lambda \mathcal{M}$ the semi-norm (similar definition for unstable direction replacing f by f^{-1} , s by u) :

$$(\|\rho\|_\lambda^s)^2 = \sum_{k \in \mathbb{N}_0} \theta_1^{-2k} |Tf_\lambda^k P_\lambda^s \rho|_{f^k \lambda}^2$$

Then :

$$\|\rho\|_\lambda = \max\{\|\rho\|_\lambda^s, \|\rho\|_\lambda^u\}$$

is a norm. As above we set $\|v\|_{i,\lambda}^s \equiv \|T\psi_{i|\lambda} v\|_\lambda^s$ for $\lambda \in \Lambda_i$ and $v \in \mathbb{C}^d$. We shall use the following properties :

1. $|\cdot|_\lambda$ and $\|\cdot\|_\lambda$ are equivalent norms (uniformly in λ).

$$\frac{1}{2}|\rho|_\lambda \leq \|\rho\|_\lambda \leq \frac{C_0 C_2}{\sqrt{1 - (\theta_0/\theta_1)^2}} |\rho|_\lambda .$$

2. $Tf|E^s$ (similarly for unstable) is strictly contracting in the seminorm $\|\cdot\|_\lambda^s$:

$$\|Tf_\lambda \rho\|_{f\lambda}^s \leq \theta_1 \|\rho\|_\lambda^s .$$

3. $\|\cdot\|_\lambda^s$ (similarly for unstable) is continuous with respect to the basepoint λ .

In particular, there is a continuous function $\epsilon_2 : [0, \infty[\rightarrow [0, \infty[$, with $\epsilon_2(0) = 0$, such that for $\lambda, \mu \in \Lambda$, $v \in \mathbb{C}^d$ we have

$$\|v\|_\lambda^s \leq \|v\|_\mu^s + \epsilon_2(d(\lambda, \mu)) \|v\|_\mu .$$

Proofs:

1)

$$|\rho|_\lambda^2 = |P_\lambda^s \rho + P_\lambda^u \rho|_\lambda^2 \leq 4 \max\{|P_\lambda^s \rho|_\lambda^2, |P_\lambda^u \rho|_\lambda^2\} \leq 4 \|\rho\|_\lambda^2 ,$$

$$\|\rho\|_\lambda^2 \leq \sum_k \theta_1^{-2k} C_0^2 \theta_0^{2k} C_2^2 |\rho|_\lambda^2 \leq \frac{(C_0 C_2)^2}{1 - (\theta_0/\theta_2)^2} |\rho|_\lambda^2 .$$

2)

$$(\|Tf_\lambda \rho\|_{f\lambda}^s)^2 = \theta_1^2 \sum_{k \geq 0} \theta_1^{-2k-2} |Tf_\lambda^{k+1} P_\lambda^s \rho|_{f^{k+1}\lambda}^2 \leq \theta_1^2 (\|\rho\|_\lambda^s)^2$$

- 3) Each term in the expression for $(\|\rho\|_\lambda^s)^2$ is a continuous section in $T^*\mathcal{M} \otimes T^*\mathcal{M}$ and as the sum is uniformly convergent (cf. proof of property 1), the statement follows.

Using the coordinates induced by the stable/unstable splitting of the tangent space at a point in the basic set for $\Delta/2 > \delta > 0$ we define an ω -rectangle $R_{i,\mu}^\omega(\delta)$ by :

$$R_{i,\mu}^\omega(\delta) = \{y \in \mathbb{C}^d \mid \|y - \mu\|_{i,\mu} \leq \delta\} .$$

Lemma : *There exist $\beta > 0$, $\delta > 0$ such that whenever $d(\mu, \lambda) < \beta$ and $d(f\mu, f\lambda) < \beta$ for two points $\mu, \lambda \in \Lambda_i$, $f\mu, f\lambda \in \Lambda_j$, f is real-analytic hyperbolic with respect to the ω -rectangles $R_{i,\mu}^\omega(\delta)$ and $R_{j,f\lambda}^\omega(\delta)$.*

Proof : The idea in the proof is the following :

Evidently, if μ and λ were equal, the derivative of f at that point (taking it to its image) would be hyperbolic with respect to the Mather norms and the coordinates induced by the stable/unstable projections. We want to show that this property is preserved (in the sense of real-analytic hyperbolicity) when λ and μ are not too far

apart.

Let $\theta_1 < \theta_2 < 1$, $\lambda, \mu \in \Lambda_i$, $f\lambda, f\mu \in \Lambda_j$, $x \in B(\mu, \Delta)$ and $fx \in B(f\lambda, \Delta')$.

Then

$$\|fx - f\lambda\|_{j,f\lambda}^s \leq \|f\mu - f\lambda\|_{j,f\lambda}^s + \epsilon_2(d(f\tilde{\lambda}, f\tilde{\mu}))|fx - f\mu|_{j,f\mu} + \|fx - f\mu\|_{j,f\mu}^s$$

where

$$\|fx - f\mu\|_{j,f\mu}^s \leq \|fx - f\mu - Df_\mu(x - \mu)\|_{j,f\mu}^s + \|Df_\mu(x - \mu)\|_{j,f\mu}^s.$$

Hence, the estimate :

$$\begin{aligned} \|fx - f\lambda\|_{f\lambda}^s &\leq \frac{C_0 C_2}{\sqrt{1 - (\theta_0/\theta_2)^2}} C_1 d(f\tilde{\mu}, f\tilde{\lambda}) + \epsilon_2(d(f\tilde{\mu}, f\tilde{\lambda}))(\Delta'/\Delta)2\delta + \\ &\quad \frac{C_0 C_2}{\sqrt{1 - (\theta_0/\theta_2)^2}} \epsilon_1(|x - \mu|) |x - \mu| + \theta_1 \|x - \mu\|_{i,\mu}^s \end{aligned}$$

Noting that $d(\tilde{x}, \tilde{\mu}) \leq c_1^{-1}|x - \mu|_{i,\mu}$ we see that we can find $\delta \gg \beta > 0$ such that $\|x - \mu\|_{i,\mu} \leq \delta$ implies $x \in B(\mu, \Delta)$, $fx \in B(f\lambda, \Delta')$ and by the above $\|fx - f\lambda\|_{j,f\lambda}^s \leq \theta_2 \delta$. Replacing f by f^{-1} and stable by unstable, the constants can be chosen so that for $d(\tilde{\mu}, \tilde{\lambda}) < \beta$ and $d(f\tilde{\mu}, f\tilde{\lambda}) < \beta$ we have the following implications

$$\begin{aligned} \|x - \mu\|_{i,\mu} &\leq \delta \Rightarrow \|fx - f\lambda\|_{j,f\lambda}^s < \theta_2 \delta \text{ and} \\ \|fx - f\lambda\|_{j,f\lambda} &\leq \delta \Rightarrow \|x - \mu\|_{i,\mu}^u < \theta_2 \delta. \end{aligned} \quad (4)$$

Finally we note that from the above $\lambda \in \text{Int} R_\mu^\omega(\delta)$ and $f\mu \in \text{Int} R_{f\lambda}^\omega(\delta)$.

We shall show that the above properties imply a real-analytic ‘partial inverse’ and thus real-analytic hyperbolicity by using topological³ properties of holomorphic maps. In particular, one has the following *boundary mapping principle* :

Proposition : *Let $U \subset \mathbb{C}^d$ be open and $f : U \rightarrow \mathbb{C}^d$ holomorphic. Then for any compact $K \subset U$:*

$$\partial f(K) \subset f(\partial K) .$$

Proof [5]: Let $y \in \partial f(K)$ and take $x \in f^{-1}y \cap K$. Either x is on the boundary of K and we are finished or it is interior in K . In the latter case it cannot be isolated (in $f^{-1}y \cap K$) or by the open mapping principle f would be open in a neighborhood of x and y was not on the boundary of $f(K)$ in the first place. Let $H \subset f^{-1}y \cap K$ be

³It is perhaps worth noting that the arguments to follow can be avoided by refining the partitioning further. The proof is cumbersome but evidently needed if one wants to extend to a non analytic set-up.

the connected component of x . From the above H is an analytic set [11] of complex dimension *at least 1*. By the maximum principle it cannot be bounded in $\text{Int}(K)$ and thus it meets ∂K . $\square$

For $\mu \in \Lambda_i$, consider the linear subspaces $P_{i,\mu}^s \mathbb{C}^d$ and $P_{i,\mu}^u \mathbb{C}^d$ identified with $\mathbb{C}^{d_s}$ and $\mathbb{C}^{d_u}$, respectively and equipped with the induced topology. We define :

$$\begin{aligned} D_\mu^1 &= D_\mu^1[\delta] = \{x \in P_{i,\mu}^s \mathbb{C}^d \mid \|x - \mu\|_\mu^s < \delta\} \quad \text{and} \\ D_\mu^2 &= D_\mu^2[\delta] = \{x \in P_{i,\mu}^u \mathbb{C}^d \mid \|x - \mu\|_\mu^u < \delta\} \quad . \end{aligned}$$

Note, that both sets are open in their respective topology. In the following we shall suppress the indices, labelling the charts. We identify :

$$R_\mu^\omega[\delta] = D_\mu^1 \oplus D_\mu^2 \quad \text{and} \quad R_{f\lambda}^\omega[\delta] = D_{f\lambda}^1 \oplus D_{f\lambda}^2 \quad .$$

The equations (4) above imply

$$\begin{aligned} \text{pr}_{f\lambda}^s f &: Cl(R_\mu^\omega) \rightarrow \text{Int}(D_{f\lambda}^1) \quad , \\ \text{pr}_\mu^u f^{-1} &: Cl(R_{f\lambda}^\omega) \rightarrow \text{Int}(D_\mu^2) \quad . \end{aligned}$$

Existence of a partial inverse : Let $x \in Cl R_\mu^\omega$ and $\text{pr}_{f\lambda}^u f(x) \in Cl D_{f\lambda}^2$. Then $f(x) \in R_{f\lambda}^\omega$ and henceforth $\text{pr}_\mu^u x \in \text{Int} D_\mu^2$. We have shown that

$$\text{pr}_{f\lambda}^u f(Cl D_\mu^1 + \partial D_\mu^2) \cap Cl D_{f\lambda}^2 = \emptyset \quad .$$

and for fixed $x^1 \in D_\mu^1$, the boundary mapping principle above implies :

$$\partial \text{pr}_{f\lambda}^u f(x^1 + D_\mu^2) \cap Cl D_{f\lambda}^2 = \emptyset \quad .$$

As $D_{f\lambda}^2$ and D_μ^1 are connected (they are convex and have non-empty interiors), either

$$Cl D_{f\lambda}^2 \subset \text{Int} \text{pr}_{f\lambda}^u f(x^1 + D_\mu^2)$$

for all $x^1 \in D_\mu^1$ or the two sets are always disjoint. Since $f\mu \in \text{Int} R_{f\lambda}^\omega$ we have the first alternative and thus,

$$\text{pr}_{f\lambda}^u f(x^1 + \cdot) : \text{Int} D_\mu^2 \rightarrow Cl D_{f\lambda}^2$$

is surjective.

Uniqueness and real-analyticity of the partial inverse:

For fixed $x^1 \in CID_\mu^1$ and $y^2 \in CID_{f\lambda}^2$ consider the contracting map :

$$g : Cl(D_{f\lambda}^1) \times Cl(D_\mu^2) \rightarrow Int(D_{f\lambda}^1) \times Int(D_\mu^2) .$$

$$(g(y^1, x^2) \equiv (\text{pr}_{f\lambda}^s f(x^1, x^2), \text{pr}_\mu^u f^{-1}(y^1, y^2)) . \quad (5)$$

By a contracting mapping lemma (cf. eg. [17, 21]) g has a unique fixpoint,

$$g(y^1, x^2) = (y^1, x^2) ,$$

which depends real-analytically on $x^1 \in Cl(D_\mu^1)$ and $y^2 \in Cl(D_{f\lambda}^2)$. We note, that for any partial inverse x^2 , the pair $(\text{pr}_{f\lambda}^s f(x^1, x^2), x^2)$ is automatically a (hence the unique) fixpoint of g .

We have shown, that f is hyperbolic analytic between the rectangles R_μ^ω and $R_{f\lambda}^\omega$. $\square$

Proof of Theorem 1:

Let $\{R_k^A\}_{k \in S}$ be a finite Markov partitioning of the basic set Λ into axiom A rectangles of diameter less than $\min\{\frac{\beta}{C_1+1}, \frac{\sqrt{1-(\theta_0/\theta_1)^2}}{C_0 C_2} \delta\}$ (the constants as in the proof of the lemma above). Let $t_{k,l} : S \times S \rightarrow \{0, 1\}$ be the transition matrix associated with the subshift.

For each R_k^A pick a $\lambda_k \in R_k^A$ and some chart $A_{i_k} \supset \psi_{i_k}^{-1} R_k^A$. For $t_{k,l} = 1$ there is an element :

$$\lambda \in f Int(R_{\lambda_k}^A) \cap Int(R_{\lambda_l}^A)$$

and we get :

$$|f\lambda_k - \lambda_l| \leq |f\lambda_k - \lambda| + |\lambda - \lambda_l| \leq C_1 |\lambda_k - f^{-1}\lambda| + |\lambda - \lambda_l| < C_1 \frac{\beta}{C_1+1} + \frac{\beta}{C_1+1} = \beta .$$

Since furthermore :

$$R_k^A \subset B_{\lambda_k} \left(\frac{\sqrt{1-(\theta_0/\theta_1)^2}}{C_0 C_2} \delta \right) \subset R_{\lambda_k}^\omega(\delta)$$

$\{R_{\lambda_k}^\omega\}_{k \in S}$ is a cover of Λ with ω -rectangles on which the induced map $\hat{f}$ is real-analytic hyperbolic.

As fixpoints of $\hat{f}^n$ are in 1-1 correspondance with symbolic cycles, i.e. n -tuples $(i_1, \dots, i_n) \in S^n$ with $t_{i_1 i_2} t_{i_2 i_3} \cdots t_{i_n i_1} = 1$, it follows that $\text{Fix} \hat{f}_\omega^n = \text{Fix} \hat{f}_A^n \xrightarrow{\Psi} \Lambda$

In case that the hyperbolic manifold is 2 dimensional the rectangles are by construction polydomains in $\mathbb{C}^2$ and the Theorem on analytic hyperbolic maps applies, showing that the Fredholm determinant $\hat{d}(z)$ is entire.

In order to get the counting of periodic orbits right we employ the Manning counting formula stated earlier. Each subshift τ_α gives rise to a real-analytic hyperbolic map (only the transition matrix changes) and hence an entire Fredholm determinant $d_\alpha(z)$. Thus, the ratio of finite products,

$$d(z) = \frac{\prod d_{2\alpha}(z)}{\prod d_{2\alpha+1}(z)}$$

is meromorphic. Showing that it is, in fact, entire is postponed to the end of the proof of Theorem 2.

Proof of Theorem 2: The idea is to construct induced maps on small segments transverse to the flow and weights associated with these maps such that the associated Fredholm determinant will give the required generalized Selberg trace formula. Most of the proofs are omitted, the proposition on orbit counting being proved in Fried [6] (which is again based on Bowen [2]) and the others being similar to the corresponding proofs in the discrete map case.

For μ in the basic set define a disk A_μ as the analytic submanifold of co-dimension 1 being the intersection of a small closed ball $B_\mu(\rho)$ of radius ρ , centered at μ and the solution to $\alpha(z) = 0$ where α is a real-analytic non-singular map from a neighborhood of $B_\mu(\rho)$ to $\mathbb{C}$. We choose the disk A_μ transverse to the flow, i.e. $|\alpha_* \dot{\phi}_x^t| \geq \epsilon > 0$, $x \in B_\mu(\rho)$.

For $\epsilon > 0$ a small constant, let

$$\mathcal{N}_\mu^\epsilon = \bigcup_{t \in [0, \epsilon]} \phi^t A_\mu$$

be a tube spanned by A_μ along the flow. P_μ denotes the projection along the flow of $\mathcal{N}_\mu^\epsilon$ onto the disk A_μ

$$0 \rightarrow \mathcal{N}_\mu^\epsilon \xrightarrow{P_\mu} A_\mu \xrightarrow{\alpha_\mu} 0$$

By compactness of the basic set, we may find a finite set $\{\mu_i, A_i\}_{i \in I}$ of points $\mu_i \in \Lambda$ and pairwise disjoint disks $A_i = A_{\mu_i}$ such that $\{\mathcal{N}_i^\epsilon\}_{i \in I}$ covers Λ . The constants ρ and ϵ may be chosen arbitrarily small.

Let $\tau_0 = \inf\{t | A_j \cap \mathcal{N}_i^t = \emptyset, \text{ all } i \neq j\} > 0$ be the minimal return time.

Whenever $A_j \cap \mathcal{N}_i^\epsilon \neq \emptyset$, there are induced real-analytic maps (Poincaré maps and return time functions)

$$\begin{aligned} \hat{f}_{ji} &: A_i \cap \phi^{-\epsilon} \mathcal{N}_j^\epsilon \rightarrow A_j \cap \phi^\epsilon \mathcal{N}_i^{-\epsilon} \\ \lambda_{ji} &: A_i \cap \phi^{-\epsilon} \mathcal{N}_j^\epsilon \rightarrow [\tau_0, \epsilon] \end{aligned}$$

defined through

$$\hat{f}_{ji}(x) \equiv P_j \phi_x^t \equiv \phi_x^{\lambda_{ji}(x)}$$

Let $T_\Lambda \mathcal{M} = E^c \oplus E^s \oplus E^u$ be the splitting of the tangent bundle over the Axiom A set Λ into center, stable and unstable subbundles. Denote by P^c , P^s and P^u the corresponding bounded projection operators, continuous with respect to the base point. One has :

- (1) E^c is one-dimensional and spanned by $\dot{\phi}^t$.
- (2) $\|T\phi_{|E^s}^t\| < C_0\theta_0^t$, $\|T\phi_{|E^u}^{-t}\| < C_0\theta_0^t$, $t > 0$ for some $C_0 > 0$, $0 < \theta_0 < 1$.

For $\theta_0 < \theta_1 < 1$, we define the Mather s-seminorm (the u-seminorm defined similarly)

$$(\|\rho\|_\lambda^s)^2 = \int_0^\infty dt \theta_1^{-2t} |T\phi_\lambda^t P_\lambda^s|_{\phi_\lambda^t}^2$$

which is continuous with respect to the base point $\lambda \in \Lambda$. Then

$$\|\rho\|_\lambda = \max\{\|\rho\|_\lambda^s, \|\rho\|_\lambda^u\}$$

will be a seminorm in $T_\lambda \Lambda$. However, by transversality of the splitting it will be a norm on the co-dimension 1 tangent space $T_\lambda A_\mu$ tangent to the disk A_μ at the point $\lambda \in A_\mu \cap \Lambda$. Furthermore, it is equivalent with the induced Riemann-norm. (Note, that here the flow direction has effectively disappeared).

The induced map $\hat{f} = \hat{f}_{ij}$ is hyperbolic on the basic set intersected with the disks. In particular, with respect to the Mather s-seminorm (τ_0 defined above)

$$\|T\hat{f}_\mu \rho\|_\mu^s \leq \theta_1^{\tau_0} \|\rho\|_{\hat{f}_\mu}^s$$

for $\mu \in A_{\mu_j} \cap \phi^{-\epsilon} \mathcal{N}_{\mu_i}^c \cap \Lambda$, $\rho \in T_\mu \mathcal{M}$.

As in the case of discrete maps we equip the disks with local real-analytic coordinates (we suppress the explicit maps) thus considering them as subsets of $\mathbb{C}^{d-1}$ and identify the trivialized tangent space with $\mathbb{C}^{d-1}$. Then ω -rectangles (polyhedrons) may be defined for $\delta > 0$ as

$$R_\lambda^\omega(\delta) = \{y \in \mathbb{C}^{d-1} \mid \|y - \mu\|_\mu \leq \delta\} .$$

We have the following

Lemma :

There exist $\beta > 0$, $\delta > 0$ such that whenever $d(\mu, \lambda) < \beta$ and $d(f\mu, f\lambda) < \beta$ for two points $\mu, \lambda \in A_j \cap \Lambda$, $\hat{f}_{ij}\mu, \hat{f}_{ij}\lambda \in A_i \cap \Lambda$, $\hat{f}_{ij}$ is real-analytic hyperbolic with respect

to the ω -rectangles $R_\mu^\omega(\delta)$ and $R_{f_\lambda}^\omega(\delta)$.

We shall use this lemma to construct global real-analytic hyperbolic maps. Again, in order to get the right counting of periodic orbits we shall need the existence of a Markov partitioning. We rely on Bowen [2] and Fried [6] to whom we refer when taking the following Markov properties for granted :

With $\epsilon > 0$ sufficiently small and $\beta' > 0$ arbitrarily small we can find analytic disks as above and a finite set of compact rectangles⁴ $\{R_k^A\}_{k \in K}$, each rectangle having diameter less than β' and lying in some disk $R_k^A \subset A_{i(k)} \cap \Lambda$. The set of rectangles has pairwise disjoint interior (relative to Λ) and associated with the rectangles there is a subshift of finite type $\{\tau_{kl}\}_{k,l \in K}$ and real-analytic hyperbolic maps $\hat{f} = \hat{f}_{kl} = \hat{f}_{i(k)i(l)}$. And for each transition we may define the correspondence

$$C_{kl} = \bigcup \{ \phi_x^t \mid x \in R_k^A \cap \hat{f}_{kl}^{-1} R_l^A \text{ and } t \in [0, \lambda_{i(k)i(l)}(x)] \} .$$

The union over k, l covers Λ and their interiors (relative to Λ) are pairwise disjoint.

Consider now the correspondence C_{ij} and a point x in a small neighborhood of the intersection of C_{ij} with A_i . Let $t = \lambda_{ji}(x)$. We may define the map (equivalent with the definition above) and a weight as follows :

$$\begin{aligned} \hat{f}_{ji}(x) &\equiv P_j \circ \phi^t \circ P_i(x) = P_j \circ \phi^t(x) \\ \hat{g}_{ji}(x, \omega) &\equiv \exp_t(\omega a_{P_i(x)}) \end{aligned}$$

(Recall the definition of the exponential $\frac{d}{dt} \exp_t(a_x) = a(\phi_x^t) \exp_t(a_x)$, $\exp_0 \equiv 1$). We also let $\hat{\epsilon}_{ji} \in \{1, -1\}$ be a sign to be specified below associated with the transition.

We may fix the value of t and take derivatives to obtain

$$\hat{f}_{ji*} = P_{j*} \phi_*^t P_{i*} = P_{j*} \phi_*^t$$

(where in general ϕ_*^t is taken at different points in the last equality).

As by transversality, $|\alpha_{i*} \phi_x^t| > \epsilon > 0$ is bounded from below the projections $\{P_i\}_{i \in I}$ are real-analytic. It follows that for a given transition $t_{ji} = 1$ and for $x \in A_i$ intersected with a neighborhood of C_{ij} , the weight $\hat{g}_{ji}$ as well as the map $\hat{f}_{ji}$ are real-analytic and the stable/unstable manifolds in $\mathcal{M}$ project under P_i to stable/unstable manifolds of $\hat{f}$ on A_i (giving rise to the rectangle R_i^A). Imposing the restriction that for each μ in the basic set the itinerary is determined by the transition matrix t_{ji} above, the proof of Theorem 1 carries over and shows that the generalized Fredholm $\hat{d}(z, w)$ associated with the weight g and the hyperbolic real-analytic map $\hat{f}$ is entire.

⁴The precise definition of these rectangles (cf. Fried [6]) is not needed here as the Lemma above ensures a stronger form of hyperbolicity.

For $n \in \mathbb{N}$ consider a periodic allowed symbolic itinerary $(i i_1 \dots i_n i)$. Associated with this there is a unique fixed point $x \in A_i \cap C_{i i_1}$ of $\hat{f}^n$. Let $\tau = \sum \lambda_{i_{k+1} i_k}(\hat{f}_{i_{k+1} i_k})$ be its return time. The fixed point corresponds to a periodic, m -times repeated prime orbit γ of the flow ϕ^τ , $\tau = m\lambda(\gamma)$. From the definitions we get

$$\hat{f}_*^n = P_{i*} \phi_*^\tau P_{i*} = P_{i*} \phi_*^\tau ,$$

now taken at identical base points. Let $\eta \in \text{sp}(\phi_*^\tau)$ and let $u \in \text{Ker}(\phi_*^\tau - \eta)$. Then

$$\hat{f}_*^n P_{i*} u = P_{i*} \phi_*^\tau u = \eta P_{i*} u$$

and thus $P_{i*} u \in \text{Ker}(\hat{f}_*^n - \eta)$. Since $P_{i*} u = 0$ if and only if $u \in \text{span}\{\dot{\phi}^{t=0}\} = \text{Ker}(\phi_*^\tau - 1)$ we have shown that :

$$\text{sp}(\hat{f}_*^n) \setminus \{0\} = \text{sp}(\phi_*^\tau) \setminus \{1\} .$$

In particular, we get the formula :

$$|\det(\hat{f}_*^n - 1)| = |\det(P_* \phi_*^{m\lambda(\gamma)} - 1)| .$$

For the signed weight, we have :

$$\prod \hat{\epsilon}_{i_{k+1} i_k} g_{i_{k+1} i_k} = \hat{\epsilon}(\hat{f}_x^n) \exp_\tau(\omega a_x) = \hat{\epsilon}(\gamma^m) \exp_{\lambda(\gamma)}(\omega m a_\gamma)$$

where the base point may be taken anywhere along the orbit.

Writing

$$\theta_\omega(\gamma^m) = \theta_\omega(\hat{f}_x^n) = \frac{\exp_{\lambda(\gamma)}(\omega m a_\gamma)}{|\det(P_* \phi_*^{m\lambda(\gamma)} - 1)|}$$

it follows that

$$\hat{d}(z, \omega) = \exp \left(- \sum_{n \in \mathbb{N}} \frac{z^n}{n} \sum_{x \in \text{Fix}(\hat{f}^n)} \hat{\epsilon}(\hat{f}_x^n) \theta_\omega(\hat{f}_x^n) \right)$$

extends to an entire function in z and ω .

Orbit counting :

Consider a subshift τ of the above type and let ϵ be an associated matrix of signs and $\hat{f}$ be the real-analytic hyperbolic map. Denote by $\hat{P}$ its prime periodic orbits as $\mathcal{P}$ is the set of prime periodic orbits of the flow. Note, that $\epsilon(\delta^p) = \epsilon(\delta)^p$, for $\delta \in \hat{P}$, $p \in \mathbb{N}$. Write $\delta^p \rightarrow \gamma^m$ if the p -repeat of $\delta \in \hat{P}$ corresponds to the m -repeat of $\gamma \in \mathcal{P}$. In particular, $\lambda(\gamma^m) = \lambda(\delta^p)$.

Proposition (orbit counting [2, 6]) : One can find a finite set of subshifts τ_n and signs ϵ_n such that for any γ^m , $\gamma \in \mathcal{P}$, $m \in \mathbb{N}$:

$$\frac{1}{m} = \sum_n (-1)^n \sum_{\delta^p \rightarrow \gamma^m} \frac{\epsilon^{(n)}(\delta^p)}{p} .$$

Using this identity we can express a sum over repeats of prime periodic orbits on the manifold as a sum over symbolic cycles. In other words,

$$\begin{aligned} \sum_m \sum_{\gamma \in \mathcal{P}} \frac{1}{m} \theta(\gamma^m) &= \\ \sum_k (-1)^k \sum_p \sum_{\delta \in \hat{P}_k} \frac{1}{p} \epsilon^{(k)}(\delta^p) \theta(\delta^p) &= \\ \sum_k (-1)^k \sum_n \sum_{x \in \hat{f}_k^n} \frac{1}{n} \epsilon^{(k)}(\hat{f}_k^n) \theta(\hat{f}_k^n) , \end{aligned}$$

and hence

$$\begin{aligned} \zeta(\omega)^{-1} &= \exp \left(- \sum_m \frac{1}{m} \sum_{\gamma \in \mathcal{P}} \theta_\omega(\gamma^m) \right) \\ &= \frac{\prod d_{2k}(z=1, \omega)}{\prod d_{2k+1}(z=1, \omega)} \end{aligned}$$

is meromorphic in ω .

Recognizing removable singularities in $d(z)$ and $\zeta(w)^{-1}$.

We shall now proceed to show that the two functions extend to entire and not only meromorphic functions. The proof is based on the following key observation on Markov partitions for an Axiom A diffeomorphism (flow) on a 2 (3) dimensional manifold :

Proposition :

We may find two Markov partitions $\mathcal{C}_1$ and $\mathcal{C}_2$ whose set of prime periodic boundary orbits, $\mathcal{B}_1$ and $\mathcal{B}_2$, are disjoint and finite. In particular, each of these orbits is isolated in $\mathcal{B}_1 \cup \mathcal{B}_2$.

This follows from the construction of Markov Partitions [1, 2]. In short, using that periodic points are dense in Λ one picks a finite number of periodic orbits⁵ (δ -dense in Λ) and cuts the manifold into pieces using local stable and unstable manifolds (and transverse segments in the case of flows), thereby obtaining a Markov partition in rectangles.⁶ As no two periodic orbits can lie on the same piece of stable nor unstable manifold the picked upon finite set of orbits will contain the (not necessarily prime) periodic boundary orbits of any subshift constructed from this Markov partition. Picking two such distinct set of periodic orbits the proposition follows.

⁵It suffices to take one orbit in each transitive component of Λ .

⁶Note, that this procedure relies on the dimensionality of the space, in so far as the cutting has to be done by transverse co-dimension 1 submanifolds.

As diffeomorphisms can be treated as a special case of flows we shall consider the latter case first.

Replace the weight ωa_x by $\omega a_x + s c_x$, $s \in \mathbb{C}$, $c_x \in C^\omega(\Lambda)$. The generalized Selberg zeta function, being meromorphic in ω and s , may be written as a product over prime periodic orbits

$$\zeta(\omega, s)^{-1} = \prod_{\gamma \in \mathcal{P}} \zeta_\gamma(\omega, s)^{-1}$$

and a straight forward computation gives for a periodic (not necessarily prime) orbit

$$\zeta_\gamma(\omega, s)^{-1} = \prod_{m_1, m_2 \geq 0} \left(1 - \exp_{\lambda(\gamma)}(\omega a_\gamma + s c_\gamma) \frac{\lambda_2^{m_2}}{|\lambda_1| |\lambda_1^{m_1}|} \right),$$

where λ_1, λ_2 ($|\lambda_1| > 1 > |\lambda_2| > 0$) are the two non zero eigenvalues of $\mathcal{P}_* \phi_*^{\lambda(\gamma)}$.

As the product converges uniformly on compacts and as zeroes of a not everywhere vanishing analytic function are isolated it has the following property, either

1. $\exp_{\lambda(\gamma)}(c_\gamma) = 1$ and $\zeta_\gamma(\omega, 0)^{-1} = \zeta_\gamma(\omega, s)^{-1}$, $s \in \mathbb{C}$ or
2. $\exp_{\lambda(\gamma)}(c_\gamma) \neq 1$ and for any $\omega \in \mathbb{C}$, there is an open neighborhood $\mathcal{N}_\gamma(\omega)$ of 0 such that $\zeta_\gamma(\omega, s)^{-1} \neq 0$, $s \in \mathcal{N}_\gamma(\omega) \setminus \{0\}$.

From the construction of finite subshifts it follows that one may write

$$\zeta(\omega, s)^{-1} = \frac{\theta(\omega, s)}{\chi(\omega, s)}$$

where $\theta(\omega, s)$ is entire and $\chi(\omega, s)$ is a finite product of $\zeta_\gamma(\omega, s)^{-1}$ over (not necessarily prime) periodic boundary orbits $\gamma \in \mathcal{B}$ of the Markov partition. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two Markov partitions with $\mathcal{B}_1$ and $\mathcal{B}_2$ the finite and disjoint sets of periodic boundary orbits.

As functions in $C^\omega(\Lambda)$ separate points we can find⁷ $c_x \in C^\omega(\Lambda)$ such that

1. $\exp_{\lambda(\gamma)}(c_\gamma) \neq 1$ for $\gamma \in \mathcal{B}_1$ and
2. $\exp_{\lambda(\gamma)}(c_\gamma) = 1$ for $\gamma \in \mathcal{B}_2$.

Consequently, the zeta function has the following decomposition

$$\zeta(\omega, s)^{-1} = \frac{\theta_1(\omega, s)}{\chi_1(\omega, s)} = \frac{\theta_2(\omega, s)}{\chi_2(\omega, s)} = \frac{\theta_2(\omega, s)}{\chi_2(\omega, 0)}$$

⁷Compare with [19, 20] in which a partition of unity over the Markov partition was required.

Let $\omega = \omega^*$ be a zero of $\chi_2(\omega, 0)$ of order $p \in \mathbf{N}$. Then $\mathcal{N} = \bigcap_{\gamma \in \mathcal{B}_1} \mathcal{N}_\gamma(\omega^*)$ is an open neighborhood of zero such that $\chi_1(\omega^*, s) \neq 0$ for $s \in \mathcal{N} \setminus \{0\}$. It follows that $\zeta(\omega^*, s)^{-1} = \theta_1(\omega^*, s)/\chi_1(\omega^*, s)$ is holomorphic for $s \in \mathcal{N} \setminus \{0\}$ and hence, that $\omega = \omega^*$ is a zero of order at least p of $\theta_2(\omega, s)$ for $s \in \mathcal{N} \setminus \{0\}$ and then by analytic continuation also of $\theta_2(\omega, 0)$. Hence, the singularity at $\omega = \omega^*$ is removable and $\zeta(\omega, 0)^{-1}$ is holomorphic at this point.

As zeroes of $\chi_2(\omega, 0)$ are countable we conclude that

$$\zeta(\omega)^{-1} = \zeta(\omega, s = 0)^{-1}$$

extends to an entire function in the complex plane.

Substituting $\omega a_\gamma \rightarrow n \log z + \log g$ for the weight, one retrieves the diffeomorphi case. $\square$

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