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Induced Chern-Simons terms and intrinsic polarization of the Dirac sea in 1+2 dimensions

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ABSTRACT

In (1+2)-dimensional gauge theories the Hall effect takes place even for the vacuum through the induced Chern-Simons term. We study physical mechanisms underlying such a quantum phenomenon. It is shown that the Chern-Simons term is induced as a response of the Dirac sea to external electric and magnetic fields; in particular, it is the intrinsic polarization of the Dirac sea, both electric and magnetic, that is responsible for the vacuum Hall current. In the presence of a magnetic field, the Dirac sea gets Zeeman-shifted and asymmetry arises in the spectrum. We discuss its consequences and establish a connection between the Hall conductance and spectral asymmetry.

1. Introduction

There has recently been renewed interest in induced Chern-Simons (CS) terms in quantum field theory [1-5] in connection with the quantum Hall effect [6]. The integer quantum Hall effect is concerned with the exactness of the induced CS term. The CS term also provides a realization of fractional statistics [7] which plays an important role in the fractional quantum Hall effect.

The CS term gets induced through vacuum polarization in quantum electrodynamics in (1+2) dimensions [2]. This implies that the Hall effect would take place for the *vacuum* in the (1+2)-dimensional world. This somewhat surprising phenomenon has a firm mathematical basis and is related to chiral anomalies in (1+1) dimensions [3, 4]. Still it is equally desirable and important to understand this phenomenon from a physical point of view.

The CS term is related to two apparently different physical processes, (i) induced Hall current in response to an applied electric field and (ii) induced charge in response to an applied magnetic field. The purpose of this paper is to clarify the physical mechanisms responsible for such phenomena. An early investigation [4] in this direction revealed the importance of the electron spin from consideration of scattering in an applied field. In this paper we show that the CS term is induced as a response of the Dirac sea (+ physical electrons) to external fields; in particular, it is the intrinsic polarization of the Dirac sea, both electric and magnetic, that is responsible for induced current and charge. In the presence of a magnetic field the induced charge is related to asymmetry [3] in the spectrum of the system. We point out that spectral asymmetry is a consequence of the Zeeman shift of the Dirac sea and that Fermi statistics is essential for relating the Hall conductance to the spectral asymmetry.

In Secs II and III we present, for noninteracting electrons in (1+2) dimensions, a derivation of the CS term through the Berry phase and show that the Hall conductance is related to the geometry of the allowed phase space of electrons. The underlying physical mechanisms are clarified in Sec. IV. In Secs. V and VI we consider electrons in the presence of a magnetic field and study the physical origin of spectral asymmetry and its consequences. Sect. VII is devoted to a summary and discussion.

II. Free Dirac electrons

In this section we present, for free Dirac electrons in 1+2 dimensions, a derivation of the induced Chern-Simons (CS) term through the Berry phase [8]. Let us consider electrons moving in a plane (of size $L_x \times L_y$) and coupled to an external vector potential $A^\mu(x)$, described by the Dirac Lagrangian

$$L = \psi^\dagger (i\partial_0 - H[A])\psi, \quad (2.1)$$

$$H[A] = m\sigma_3 + \Pi^1\sigma_1 + \Pi^2\sigma_2 - eA_0, \quad (2.2)$$

where $\Pi^k \equiv p^k + eA^k(t)$; we have taken the Dirac matrices $\gamma^0 = \sigma_3, \gamma^1 = i\sigma_2, \gamma^2 = -i\sigma_1$. For the derivation of the induced CS term ($\propto \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$) the full machinery of relativistic perturbation theory is unnecessary. One may simply ask how the electron system responds, e.g., to slowly-varying spatially-constant potentials $A^\mu(t)$. The underlying physics is the Hall effect: An applied electric field $E^1(t) = -\partial_0 A^1(t) \equiv -\dot{A}^1(t)$ would induce the Hall current flowing in the y direction, which is detected by $A^2(t)$. The Hall conductance σ_{yx} thereby is read from (the space-independent form of) the CS term in the effective action:

$$\Gamma_{\text{eff}} = \frac{1}{2} \int dt dx dy \sigma_{yx} (A^1 \dot{A}^2 - A^2 \dot{A}^1) + \dots, \quad (2.3)$$

where other terms $\dots$ are higher orders in the time derivative; $\dot{A}^\mu \equiv \partial_0 A^\mu$.

The effective action is nothing but the phase shift of the wave function of the ground state (the vacuum) when external perturbations are adiabatically turned on and then off. In particular, the CS term, being linear in the time derivative, is uniquely given by Berry's geometric phase [8].

The instantaneous eigenmodes of the Hamiltonian $H[A(t)]$ are plane waves of momentum $\mathbf{p} = (p^1, p^2)$ and energy $\pm\epsilon(\mathbf{p})$ with $\epsilon \equiv \epsilon(\mathbf{p}) = (m^2 + \Pi^2)^{1/2}$, where we have set $A_0 \rightarrow 0$; in particular, the eigenmode $|k(\mathbf{p})\rangle$ has the spinor structure $\propto [m + \epsilon, \Pi^1 + i\Pi^2]^t$ while $|- \epsilon(\mathbf{p})\rangle \propto [-\Pi^1 + i\Pi^2, m + \epsilon]^t$. The evolution of these eigenmodes with time yields the associated Berry phases:

$$\begin{aligned} \gamma^{(\pm)}(\mathbf{p}) &\equiv i \int dt \dot{A}^k < \pm\epsilon(\mathbf{p}) | \frac{\delta}{\delta \dot{A}^k} | \pm\epsilon(\mathbf{p}) > \\ &= \mp \int dt \frac{e}{2\epsilon(\epsilon + m)} \epsilon_{jk} (p^j + eA^j) \dot{A}^k, \end{aligned} \quad (2.4)$$

with the normalization $< \pm\epsilon(\mathbf{p}) | \pm\epsilon(\mathbf{p}) > = 1$; $\epsilon_{12} = -\epsilon_{21} = 1$ and $\epsilon_{11} = \epsilon_{22} = 0$.

The ground state (the vacuum) of the present relativistic system is the Dirac sea with all the negative-energy levels filled. Accordingly, collecting $\gamma^{(-)}(\mathbf{p})$ for all negative-energy modes yields the Berry phase associated with the Dirac sea:

$$\Gamma = L_x L_y \int \frac{d^2 p}{(2\pi)^2} \gamma^{(-)}(\mathbf{p}). \quad (2.5)$$

This momentum integral would vanish if the shift $\mathbf{p} \rightarrow \mathbf{p} - e\mathbf{A}$ were allowed. Unfortunately such a shift is not allowed since the integral is linearly UV-divergent. Actually a surface term survives if we cut off the integral so that $|\mathbf{p}| < \Lambda$ and let $\Lambda \rightarrow \infty$ in the end, with the result:

$$\Gamma = \int dt \int d^2 x (e^2/8\pi) \epsilon_{ijk} A^j \dot{A}^k, \quad (2.6)$$

where we have set $L_x L_y = \int d^2 x$. The coefficient $e^2/(8\pi)$ derives from the integral

$$\int \frac{d^2 p}{(2\pi)^2} \frac{e^2 m}{4\epsilon_0^3} = \frac{e^2 m}{8\pi} \int_m^\infty d\epsilon_0 \frac{1}{\epsilon_0^2} = \frac{e^2}{8\pi}, \quad (2.7)$$

which is obtained by expanding $\gamma^{(-)}(\mathbf{p})$ to first order in $\mathbf{A}$; $\epsilon_0 \equiv (m^2 + \mathbf{p}^2)^{1/2}$.

Here the Chern-Simons term, obtained earlier by a field-theoretic calculation [2], is correctly reproduced as the Berry phase associated with the vacuum.

The nonzero Hall conductance following from Eq. (2.6),

$$\sigma_{yx}^{\text{vac}} = e^2/(4\pi) = \frac{1}{2} e^2/h, \quad (2.8)$$

implies that, in the present 1+2 dimensional world, the Hall effect takes place for the vacuum, i.e., without an applied magnetic field and “physical” (positive-energy) electrons. The Hall current is carried by the Dirac sea, as will be discussed in Sec. IV.

III. Geometric expression for the Hall conductance

In this section we point out that the Hall conductance (2.8) has an interesting geometric interpretation. Let us first note that the present electron system is mapped into a well-known example worked out by Berry [8]: The Hamiltonian (2.2) is formally regarded as representing a spin 1/2 particle coupled to a slowly-varying three-dimensional “magnetic field” $\mathbf{R}(t) = (\Pi^1, \Pi^2, m)$, where $\Pi^k = p^k + eA^k(t)$. As $\mathbf{A}(t)$ is adiabatically turned on and off (i.e., $\mathbf{A}(t) \rightarrow 0$ as $t \rightarrow \pm\infty$) with p^k kept fixed, $\mathbf{R}(t)$ varies slowly along a closed path C in $\mathbf{R} = (R^1, R^2, R^3)$ space. Under such circumstances, the Berry phases associated with the (instantaneous) eigenmodes of energy $\pm|\mathbf{R}| = \pm\epsilon$ are written as

$$\gamma^{(\pm)}[C] = \int_S \mathbf{F}^{(\pm)}[\mathbf{R}] \cdot d\mathbf{S}, \quad (3.1)$$

$$\mathbf{F}^{(\pm)}[\mathbf{R}] = \mp \mathbf{R} / (2|\mathbf{R}|^3), \quad (3.2)$$

where S denotes an arbitrary surface spanned by the contour C in $\mathbf{R}$ space. The Berry curvature $\mathbf{F}^{(\pm)}$ represents a “magnetic” field of a monopole of magnetic charge $\mp 2\pi$ located at the origin of $\mathbf{R}$ space; thus $\gamma^{(\pm)}[C] = \mp \frac{1}{2}\Omega(C)$ in terms of the solid angle $\Omega(C)$ subtended by C at the origin $\mathbf{R} = 0$.

In the model of our interest, the area element $d\mathbf{S}$ is equal to $e^2 dA^1 dA^2$ normal to the plane $R_3 = m$, and $\mathbf{F}^{(\pm)}[\mathbf{R}] \rightarrow F_3^{(\pm)}[\mathbf{R}] = \mp m / (2\epsilon^3)$. As before, the Berry phase associated with the vacuum is obtained by collecting $\gamma[C]$ for all negative-energy eigenmodes:

$$\Gamma = e^2 \int d^2x \int \frac{d^2p}{(2\pi)^2} \int_S dA^1 dA^2 F_3^{(-)}[\mathbf{p} + e\mathbf{A}, m]. \quad (3.3)$$

Let us here perform the $\mathbf{p}$ integral first:

$$\int d^2p F_3^{(-)}[\mathbf{p} + e\mathbf{A}, m] = \pi, \quad (3.4)$$

which is equal to one half of the solid angle subtended by the entire plane $p^3 = m$ at point $(-e\mathbf{A}, 0)$ in $(\mathbf{p}, p^3)$ space. Note that Eqs. (3.4) and (2.7) are equivalent. Finally, rewriting the surface integral $\int d^2A$ into a line integral recovers Eq. (2.6):

$$\Gamma = (e^2/8\pi) \int d^2x \oint (A^1 dA^2 - A^2 dA^1), \quad (3.5)$$

where $dA^k = dt \dot{A}^k$.

The above derivation reveals a connection between the Hall conductance and the geometry of the phase space of the Dirac sea, $\Sigma^{(-)} = \{|\mathbf{p}| < \Lambda, p^3 = m\}$. It is readily generalized to include the effect of physical electrons filling levels up to the Fermi energy $E_F = \sqrt{m^2 + p_F^2}$, i.e., with the phase space $\Sigma^{(+)} = \{|\mathbf{p}| < p_F, p^3 = m\}$. The result is

$$\begin{aligned} \sigma_{yx}(E_F) &= (e^2/8\pi^2) \left(\Omega[\Sigma^{(-)}] - \Omega[\Sigma^{(+)}] \right) \\ &= (e^2/4\pi) m / E_F, \end{aligned} \quad (3.6)$$

where $\Omega[\Sigma^{(\pm)}]$ denotes the solid angle subtended by the phase space $\Sigma^{(\pm)}$ at the origin $\mathbf{p} = p^3 = 0$ in $(\mathbf{p}, p^3)$ space. Note that $\sigma_{yx}(E_F)$ decreases as the number of physical electrons increases.

IV. Physical picture of Dirac electrons on a plane

In this section we clarify the physical mechanism responsible for the vacuum Hall effect specific to (1+2) dimensions. Consider Dirac electrons on a plane (the xy plane) in (1+3) dimensions, or more precisely, suppose that everything is homogeneous in the z direction. They are described by the Dirac Hamiltonian with its reference to the z direction suppressed:

$$H_4 = \alpha \cdot (\mathbf{p} + e\mathbf{A}) + \beta m - eA_0, \quad (4.1)$$

where β and α^k ($k = 1, 2$) are the 4×4 Dirac matrices in standard notation.

Note that this dimensionally-reduced Hamiltonian commutes with the matrix

$$\Xi = \gamma_0 \sigma^{12} = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}, \quad (4.2)$$

with eigenvalues $\Xi = \pm 1$. The H_4 and Ξ can therefore be diagonalized simultaneously. In particular, the $\Xi = 1$ spinor $(\phi_1, 0, 0, \phi_4)^T$ assembled in the 2-component form $\Psi(+)= (\phi_1, \phi_4)^T$ obeys the Dirac equation with the Hamiltonian (2.2). Accordingly, the electron in (1+2) dimensions is nothing but the Dirac electron polarized in the z direction. In the rest frame ϕ_1 describes a spin-up ($s_z = 1/2$) positive-energy electron and ϕ_4 describes a spin-down negative-energy electron; they are charge conjugates since $\phi_1 \leftrightarrow \phi_4^\dagger$ under charge conjugation.

Similarly, the $\Xi = -1$ spinor $\Psi(-) = (\phi_2, \phi_3)^T$, which in its rest frame consists of a spin-down positive-energy component ϕ_2 and a spin-up negative-energy component ϕ_3 , is governed by the Hamiltonian (2.2) with $\Pi^2 \rightarrow -\Pi^2$. Interestingly

these $\Xi = \pm 1$ Hamiltonians are unitarily equivalent:

$$\sigma_2 (m\sigma_3 + \Pi^1 \sigma_1 + \Pi^2 \sigma_2) \sigma_2 = -(m\sigma_3 + \Pi^1 \sigma_1 - \Pi^2 \sigma_2). \quad (4.3)$$

The polarization of electrons is a key to understanding the vacuum Hall effect. To clarify its role let us consider how a Dirac electron is coupled to an electric field $\mathbf{E} = -\dot{\mathbf{A}}(t)$. The Dirac Hamiltonian is almost diagonalized by use of a suitable Foldy-Wouthey transformation [9] $U = \exp(\gamma^k \Pi^k \xi)$ with $\xi = (1/2|\Pi|) \tan^{-1}(|\Pi|/m)$:

$$H^{\text{FW}} = \beta \epsilon + \Delta H, \quad (4.4)$$

where $\epsilon = \sqrt{m^2 + \Pi^2}$, the matrix Ξ thereby remains unchanged.

The extra term $\Delta H = -iU \partial_t U^{-1} \propto \mathbf{E}$ in general contains “odd” pieces (connecting positive- and negative-energy states). Their effect, however, becomes of $O(E^2)$ through further diagonalization, and is not of our present concern.

The “even” part given by

$$(\Delta H)^{\text{even}} = \frac{e}{2\epsilon(m + \epsilon)} \boldsymbol{\sigma} \cdot [\mathbf{E} \times (\mathbf{p} + e\mathbf{A})] \quad (4.5)$$

is a relativistic generalization of the well-known coupling $(e^2/4m^2) \boldsymbol{\sigma} \cdot (\mathbf{E} \times \mathbf{p})$ of a spin to an electric field. It is clear from the derivation that $-(\Delta H)^{\text{even}}$ is essentially the Berry phase; indeed, it recovers Eq. (2.5) for $\sigma_3 \rightarrow \pm 1$.

The physical picture becomes more transparent if one looks at the current carried by the negative-energy electron explicitly. Functionally varying the Hamil-

tionian $H^{(-)} = \epsilon + (\Delta H)^{\text{even}}$ with respect to $\mathbf{A}(t)$ yields the current:

$$\mathbf{j}^i = -e\dot{\mathbf{x}}^i + \partial P^i / \partial t. \quad (4.6)$$

Here the drift current $-e\dot{\mathbf{x}}^i$ follows from the equation of motion $\partial H^{(-)} / \partial \mathbf{A}^i = e\partial H^{(-)} / \partial \mathbf{p}^i = e\dot{\mathbf{x}}^i$:

$$-e\dot{\mathbf{x}}^i = (e/\epsilon)\Pi^i - (e^2 m / 4\epsilon^3) \epsilon_{ik} E^k + \dots \quad (4.7)$$

The second term $\dot{P}^i$ denotes the displacement current associated with the electric polarization of negative-energy electrons,

$$P^i \equiv \frac{\partial H^{(-)}}{\partial \mathbf{A}^i} = \frac{e}{2\epsilon(\epsilon + m)} \epsilon_{ik} \Pi^k \sim -\frac{e}{2(\epsilon + m)} \epsilon_{ik} \dot{\mathbf{x}}^k. \quad (4.8)$$

This electric polarization has its origin in moving magnetic moments. Thus the following picture emerges: As the electrons get accelerated ($\ddot{\mathbf{x}}^k \sim (e/\epsilon)E^k$) in the direction of an applied electric field, the displacement current

$$\dot{P}^i = -(e^2 m / 4\epsilon^3) \epsilon_{ik} E^k + \dots \quad (4.9)$$

begins to flow in the perpendicular direction. At the same time, the electrons drift in the same direction with velocity $(\dot{\mathbf{x}}^i)_\perp \sim (em/4\epsilon^3) \epsilon_{ik} E^k$. These drift current and displacement current contribute in equal amount to give

$$\mathbf{j}^i = (e/\epsilon)\Pi^i - (e^2 m / 2\epsilon^3) \epsilon_{ik} E^k. \quad (4.10)$$

[The omitted terms (...) in Eqs. (4.7) and (4.9) combine to cancel exactly.] Collecting $\mathbf{j}^i$ for the Dirac sea again reproduces the correct Hall conductance (2.8),

as seen from Eq. (2.7). This shows clearly that the Hall current is carried by the Dirac sea and that the polarization of the Dirac sea, both magnetic ($\Xi = 1$) and electric ($\tilde{\mathbf{P}}$), is responsible for the Hall effect in 1+2 dimensions.

The vacuum Hall current leads to another related phenomenon, induced charge, when there is a weak local magnetic field $B(\mathbf{x})$ normal to the plane. The current conservation law implies that charge is accumulated in response to B :

$$\rho^{\text{ind}} = -e\Delta n_e - \text{div} \mathbf{P} = -(e^2 m / 2\epsilon^3) B. \quad (4.11)$$

The drift-induced density change $\Delta n_e = -\int dt \text{div}(\dot{\mathbf{x}})$ and the polarization charge $-\text{div} \mathbf{P} = -(e^2 m / 4\epsilon^3) B + \dots$ turn out to contribute to ρ^{ind} almost equally. For the Dirac sea the induced charge per unit area is $\Delta Q = -e\Delta N_e$ with

$$\Delta N_e = eB / (4\pi). \quad (4.12)$$

Note that the Hall effect and induced charge become less prominent as the number of positive-energy electrons, polarized oppositely to the Dirac sea and thus flowing in the opposite direction, increases.

The $\Xi = -1$ electrons (ϕ_2, ϕ_3) are polarized in a way precisely opposite to the $\Xi = 1$ electrons (ϕ_1, ϕ_4). Correspondingly, for the Dirac electrons in (1+3) dimensions the Dirac sea is unpolarized, being filled with equal numbers of negative-energy electrons of opposite spin polarization, and no Hall effect takes place in the absence of a magnetic field.

V. Dirac electrons in a magnetic field

So far free electrons have been considered. In this section we consider Dirac electrons on a plane in a perpendicular magnetic field and uncover a connection between the Hall conductance and asymmetry in the spectrum.

A homogeneous magnetic field $B_z = B > 0$ is introduced through the replacement (the Landau gauge) $A^1(t) \rightarrow -yB + A^1(t)$ in the Hamiltonian (2.2). The relativistic cyclotron motion of the $\Xi = 1$ electron is readily determined for $A(t) = 0$: The positive- and negative-energy Landau levels $E_{\pm n} = \pm\sqrt{m^2 + 2neB}$ with $n = 1, 2, \dots$ are paired, except for a negative-energy $n = 0$ level $E_{-0} = -m$. Associated with the positive-energy levels E_n ($n = 1, 2, \dots$) are the wave functions:

$$\psi_{np}(x, y) = \frac{1}{\sqrt{2}} \left(\frac{\sqrt{1 + (m/E_n)} \chi_{n-1}(y - y_0)}{\sqrt{1 - (m/E_n)} \chi_n(y - y_0)} \right) e^{ip_x x}, \quad (5.1)$$

where $\chi_n(y)$ stand for the usual harmonic-oscillator wave functions; $y_0 \equiv p_x/(eB)$. For the wave functions associated with the negative-energy levels E_{-n} with $n = 0, 1, 2, \dots$, replace $m \rightarrow -m$ and $\chi_n \rightarrow -\chi_n$ in Eq. (5.1). As is well-known [10], each Landau level n has degeneracy in p_x of $eB/(2\pi)$ per unit area.

The eigenfunctions of $H[A]$ for $A(t) \neq 0$ are constructed from $\psi_{np}(x, y)$,

$$\psi_{np}(x, y|A) = \exp[-ie(y - y_0)A^2] \psi_{np}(x, y - e(A^1/B)), \quad (5.2)$$

without affecting the energy eigenvalues. As before, as $A(t)$ is adiabatically turned on and off, $A(t)$ varies along a closed path C in (A^1, A^2) space. The Berry phase associated with a Landau level n (for fixed p_x) is given by the general formula [8]:

$$\gamma_n[C] = \int F_n dA^1 dA^2, \quad (5.3)$$

$$F_n = \sum_k F_n^k, \quad (5.4)$$

$$F_n^k = \frac{i}{(E_n - E_k)^2} \left(\langle n | \delta_1 H | k \rangle \langle k | \delta_2 H | n \rangle - (n \leftrightarrow k) \right), \quad (5.5)$$

where $\delta_i H \equiv \partial H[A]/\partial A^i$ ($i = 1, 2$). Here $|n\rangle$ and $|k\rangle$ stand for the instantaneous eigenstates of $H[A]$. The Berry curvature F_n^k derives from the virtual transition $|n\rangle \rightarrow |k\rangle \rightarrow |n\rangle$ with $k \neq n$.

In the present model, $F_n^k \neq 0$ only for $k = \pm(n+1), \pm(n-1)$. For $n \geq 1$

$$\begin{aligned} F_n^{\pm(n+1)} &= \frac{e^2(E_n - m)(E_{n+1} \pm m)}{2E_n E_{n+1} (E_n \mp E_{n+1})^2}, \\ F_n^{\pm(n-1)} &= -\frac{e^2(E_n + m)(E_{n-1} \mp m)}{2E_n E_{n-1} (E_n \mp E_{n-1})^2}. \end{aligned} \quad (5.6)$$

For $F_{-n}^{-(n+1)}$ and $F_{-n}^{-(n-1)}$ replace m by $-m$ in F_n^{n+1} and F_n^{n-1} , respectively. A direct calculation shows that F_n is independent of Landau-level indices:

$$F_n = \sum_k F_n^k = e/B. \quad (5.7)$$

Actually there is a simpler way to derive this: Evaluate the time variation of the wave function (5.2),

$$\dot{\psi}_{np}(x, y|A) = -ie[(y - y_0)\dot{A}^2 - i(\dot{A}^1/B)\partial_y + (e/B)\dot{A}^1 A^2] \psi_{np}(x, y|A). \quad (5.8)$$

From the $\dot{A}^1 A^2$ term follows the result $F_n = e/B$. Thus, in view of the degeneracy $(eB/2\pi)L_x L_y$, each filled Landau level gives rise to the Hall conductance

$$\sigma_n = (eB/2\pi)F_n = e^2/(2\pi) = e^2/h. \quad (5.9)$$

The vacuum of the present system is the Dirac sea with all negative-energy Landau levels filled. At first sight, it appears that negative-energy levels combine

to yield an infinitely large Hall conductance for the vacuum, $\sum_{n=0}^{\infty} \sigma_{-n} \rightarrow \infty$, which is unacceptable. Actually, a physically sensible answer is obtained if one treats the bottom ($E \rightarrow -\infty$) of the Dirac sea carefully. For regularization let us cut off the energy spectrum and retain only a finite number $(2N+1)$ of Landau levels n with $-E_N \leq E_n \leq E_N$.

The regularized Dirac sea induces the CS term proportional to the sum:

$$\sum_{n=0}^N \sum_{k=-N}^N F_{-n}^k = \sum_{n=0}^N F_{-n} + D_{-N}, \quad (5.10)$$

where $F_{-n} = e/B$. The contribution from the bottom of the Dirac sea,

$$D_{-N} \equiv -(F_{-N}^{N+1} + F_{-N}^{-(N+1)}) = -\frac{e}{B} \left(N + \frac{1}{2} + \frac{m}{2E_N} \right), \quad (5.11)$$

turns out nonvanishing for $N \rightarrow \infty$, yielding the finite result

$$\sum_{n=0}^N \sum_{k=-N}^N F_{-n}^k = e/(2B) + O(1/\sqrt{N}). \quad (5.12)$$

What emerges through this regularization is the working of the Pauli exclusion principle or Fermi statistics: Note that cancellation has taken place among a majority of terms in Eq. (5.10) owing to the antisymmetry of F_n^k ,

$$F_n^k = -F_k^n, \quad (5.13)$$

which follows from Eq. (5.5). This structure implies that the virtual transitions $n \rightarrow k \rightarrow n$ and $k \rightarrow n \rightarrow k$ contribute to the Hall conductance in different sign; in particular, their contributions combine to vanish, as they should if both n and k were occupied levels. Thus this is interpreted as a manifestation of Fermi statistics.

Enforcing Fermi statistics provides a simple and yet enlightening representation for Eq. (5.12): Note the antisymmetry $F_n^k = -F_k^n$ and make the rearrangement,

$$\sum_{n=0}^N \sum_{k=-N}^N F_{-n}^k = \frac{1}{2} \left(\sum_{n=0}^N \sum_{k=-N}^N F_{-n}^k - \sum_{n=1}^N \sum_{k=-N}^N F_n^k \right) \quad (5.14)$$

$$= \frac{1}{2} \left(\sum_{n=0}^N F_{-n} - \sum_{n=1}^N F_n \right) + \frac{1}{2} \Delta_N, \quad (5.15)$$

where $F_{\pm n} = e/B$. In this case the boundary contribution (through $\pm E_N$)

$$\Delta_N \equiv F_N^{N+1} + F_N^{-(N+1)} - F_{-N}^{N+1} - F_{-N}^{-(N+1)} = -(e/B)(m/E_N) \quad (5.16)$$

vanishes as $N \rightarrow \infty$. The Hall conductance is now cast in the form of spectral asymmetry:

$$\sigma_{yx}^{\text{vac}} = \frac{e^2}{4\pi} \left\{ n(-\Lambda < E \leq -m) - n(m \leq E < \Lambda) \right\} \quad (\Lambda \rightarrow \infty), \quad (5.17)$$

where $n(\dots)$ stands for the number of Landau levels within the specified energy range. In the present case, there is only one unpaired level $E_{-0} = -m$ contributing to the asymmetry; the result $\sigma_{yx}^{\text{vac}} = e^2/(4\pi)$ turns out to be the same as in the absence of the magnetic field.

The connection between the Hall conductance and the spectral asymmetry, as given by Eq. (5.17), is not limited to the case of the vacuum. The basic identity (5.14) is promoted to a more general form with the replacement $\sum_{n=0}^N \rightarrow \sum_{n=-r}^N$ and $\sum_{n=1}^N \rightarrow \sum_{n=r+1}^N$ (r is an integer) made on both sides of it. Correspondingly, when physical electrons fully occupy levels up to E_F , the Hall conductance is again

expressed in terms of spectral asymmetry below and above E_F :

$$\sigma_{yx}(E_F) = \frac{e^2}{4\pi} \{ n(-\Lambda < E \leq E_F) - n(E_F < E < \Lambda) \}. \quad (5.18)$$

When the highest Landau level r is partially filled with filling fraction $\nu_r \leq 1$, Eq. (5.18) is replaced by a more general expression

$$\sigma_{yx}(E_F) = \frac{e^2}{4\pi} \left\{ \sum_{n=-N}^r \nu_n - \sum_{n=r}^N (1 - \nu_n) \right\}, \quad (5.19)$$

where, of course, we have to set $\nu_n = 0$ for $n \geq r + 1$ and $\nu_n = 1$ for $n \leq r - 1$. Here the Hall conductance is expressed in terms of an asymmetry in the number of occupied and unoccupied modes.

Formulas (5.17) and (5.19) hold equally for the $\Xi = \pm 1$ electrons. The spectrum of the $\Xi = -1$ electron is obtained by simply interchanging the positive- and negative-energy spectra of the $\Xi = 1$ electron, as seen from Eq. (4.3). Correspondingly, when there are (either $\Xi = 1$ or $\Xi = -1$) physical electrons with the total filling fraction ν^+ ,

$$\sigma_{yx}(E_F) = \frac{e^2}{2\pi} \left(\nu^+ + \frac{1}{2} \text{sgn}\{\Xi\} \right). \quad (5.20)$$

The vacuum Hall effect disappears for the unpolarized Dirac electron, whose spectrum is symmetric, $\pm E_n$ for $n = 0, 1, 2, \dots$. The Hall conductance is proportional to the filling fraction (per spin) so that $\sigma_{yx}(E_F) = (e^2/2\pi) \nu_D^+$.

VI. Spectral asymmetry and induced quantum numbers

There is a topological reason [3] for spectral asymmetry, which is traced back to the chiral anomaly in 1+1 dimensional quantum electrodynamics. Note that the $\Pi^1 \sigma_1 + \Pi^2 \sigma_2$ part of the Hamiltonian (2.2) is essentially the (Euclideanized) Dirac operator $D[A]_2$ in 1+1 dimensions, which is known to have nonzero index $\text{Ind}[A]_2 \propto F_{12}$ (or the chiral anomaly). This implies that $\Pi^1 \sigma_1 + \Pi^2 \sigma_2$ has chirally-unpaired zero-energy modes [12] for $F_{12} \sim B \neq 0$ and correctly accounts for the spectral asymmetry [3,4]. In this section we attempt to understand this spectral asymmetry and related phenomena from a physical point of view.

Let us first note that in the presence of a homogeneous magnetic field $B > 0$ the Hamiltonian $\beta \sqrt{m^2 + (\mathbf{p} + e\mathbf{A})^2}$, representing relativistic mass increase alone, has a symmetric energy spectrum $E_{\pm n}^{(0)} = \pm \sqrt{m^2 + 2eB(n + \frac{1}{2})}$ with $n = 0, 1, 2, \dots$. Each spinless Landau level $E_{\pm n}^{(0)}$ has degeneracy of $eB/(2\pi)$ per unit area, which is equal to the number of free electrons (per spin) with energy ϵ within the range $\sqrt{m^2 + 2eBn} < |\epsilon| < \sqrt{m^2 + 2eB(n+1)}$, as seen from the density of state $dN_e = \epsilon d\epsilon/(2\pi) = (eB/2\pi) dn$. See Fig. 1. Correspondingly, for the purpose of counting the number of modes one can associate the discrete Landau-level spectrum with an equivalent free-electron spectrum; such a connection is well-known [10] for non-relativistic electrons.

Next recall the non-relativistic Zeeman coupling $\Delta H = (e/2m)\beta \boldsymbol{\sigma} \cdot \mathbf{B}$, which is simply $eB\Xi/(2m)$ for the $\Xi = \pm 1$ electrons. It is now clear that it is the relativistic Zeeman effect that works to shift up the symmetric spectrum $E_{\pm n}^{(0)}$ by $\Delta n = 1/2$ to generate asymmetry in the spectrum $E_{\pm n} = \pm \sqrt{m^2 + 2eBn}$ of the $\Xi = 1$ electron, as illustrated in Fig. 1.

To draw some important consequences of spectral asymmetry compare the two free-electron spectra associated with the spinless spectrum and the Zeeman-shifted spectrum in Fig. 1. For the Dirac sea the net effect is the emergence of an extra number of spin-down negative-energy electrons (of energy $-m < \epsilon < -\sqrt{m^2 - eB}$),

$$\Delta N_e = (eB/2\pi)\Delta n = eB/(4\pi) \quad (6.1)$$

per unit area, relative to the vacuum of free electrons. These extra free-electron modes are responsible for the induced charge $\Delta Q = -e\Delta N_e = -e^2 B/4\pi$ and for $\Delta \Xi = \Delta N_e = eB/(4\pi)$. This charge is induced in association with the change of the vacuum as a homogeneous magnetic field is gradually turned on. Interestingly, the vacuum Hall current can be attributed to the drift of this induced charge ΔQ with velocity $v^j = \epsilon_{jk} E^k/B$ in response to a Hall field E^k :

$$\sigma_{yx}^{\text{vac}} = (e^2/2\pi)\Delta n = e^2/(4\pi). \quad (6.2)$$

These Hall conductance and induced charge are in agreement with Eqs. (2.8) and (4.12), and are correctly summarized in the relativistic CS term $\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$.

It is interesting to note that unlike the $B = 0$ case the Hall current in Eq. (6.2) is totally attributed to the drift current. No displacement current arises because the Dirac sea drifts homogeneously with constant velocity $\mathbf{E}/B$ (so that $\dot{\mathbf{P}} = 0$).

For the $\Xi = -1$ electron the spinless levels get Zeeman-shifted down by $\Delta n = 1/2$ and an unpaired $E_{+0} = m$ Landau level emerges. The spin-up negative-energy electrons thereby get reduced by the same number $\Delta N_e = eB/(4\pi)$. This is reinterpreted as the emergence of spin-down positron modes; hence the induced charge $\Delta Q = e^2 B/(4\pi)$ and $\Delta \Xi = eB/(4\pi)$. Both the induced charge and the

vacuum Hall conductance reverse sign here since the charge carriers have opposite charge.

For the vacuum of unpolarized Dirac electrons in a homogeneous magnetic field there is no net induced charge. Still the quantum number Ξ continues to be induced, $\Delta \Xi = eB/(2\pi)$. This is independently verified by a relativistic Feynman diagram calculation of the vacuum expectation value $\langle \psi^\dagger \Xi \psi \rangle = \langle \bar{\psi} \sigma^{12} \psi \rangle$ (with 4-component spinors but in 2 spatial dimensions):

$$\langle \bar{\psi} \sigma^{12} \psi \rangle = -4imB \int \frac{d^3 p}{(2\pi)^3} \frac{1}{(p^2 - m^2)^2} = \frac{eB}{2\pi}. \quad (6.3)$$

Actually, $\Delta \Xi$ by definition is equal to asymmetry in the numbers of the $\Xi = \pm 1$ modes residing in the Dirac sea. The induced asymmetry $\Delta \Xi = eB/(2\pi)$ therefore is attributed to the presence of an unpaired $\Xi = 1$ Landau level with $E_{-0} = -m$.

VII. Summary and discussion

In this paper we have studied physical mechanisms responsible for induced charge and current in the system of relativistic electrons in (1+2) dimensions. The induced charge and current are a manifestation of the quantum nature of the Dirac sea.

We have examined the case $E^2 > B^2$ first. There intrinsic spin polarization and electric polarization of the Dirac sea are responsible for the vacuum Hall current, which is composed of both the conduction current and the displacement current in the Dirac sea. These vacuum currents, when coupled to a weak local magnetic field, work to induce charge. We have shown that the Hall conductance is directly related to the geometry of the phase space of electrons.

In the presence of a strong magnetic field (or for $B^2 > E^2$) completely different physical mechanisms arise, yet leading to the same amount of induced charge and current. There induced charge is a consequence of the Zeeman shift of the spin-polarized Dirac sea and the vacuum Hall current is attributed to the drift of the induced charge; there is no displacement current in this case. We have shown that the Hall conductance is directly related to spectral asymmetry caused by the Zeeman shift.

Finally we wish to comment on the relevance of our analysis to the quantum Hall effect [6,13,14]. We have seen that Fermi statistics plays a crucial role in connecting the Hall conductance to spectral asymmetry. Since Fermi statistics is built in the general structure of the Berry phase, it is natural to expect that such a connection is quite general. In fact, it is extended to a realistic case of nonrelativistic electrons in the presence of impurities: Electrons are scattered by impurities and each Landau level is broadened into a subband. Suppose that a

subband contains N eigenmodes labeled in the order of increasing energy. The contribution of each eigenmode α with $\alpha = 1, 2, \dots, N$ to the Hall conductance is written as [15]

$$\sigma_\alpha \propto 1 + g_\alpha, \quad (7.1)$$

where 1 stands for the free-drift conduction which is universal for all α , as in Eq. (5.7), while

$$g_\alpha = \sum_\beta g_\alpha^\beta \quad (7.2)$$

stands for the effect of scattering of the eigenmode α by impurities (via virtual modes β). The g_α^β are derived from the Berry phase and enjoy the antisymmetry $g_\alpha^\beta = -g_\beta^\alpha$. As a result, a completely filled subband receives no net effect of scattering (because $\sum_\alpha g_\alpha = \sum_\alpha \sum_\beta g_\alpha^\beta = 0$). When the subband is filled up to the r -th mode, the associated Hall conductance is written in the form of generalized spectral asymmetry (weighted with σ_α) below and above the r -th mode:

$$\sum_{\alpha=1}^r (1 + g_\alpha) = \frac{1}{2}N + \frac{1}{2} \left(\sum_{\alpha=1}^r (1 + g_\alpha) - \sum_{\alpha=r+1}^N (1 + g_\alpha) \right). \quad (7.3)$$

Localization is another ingredient essential for quantization of the Hall conductance [13,14]: In the presence of impurities a considerable fraction of eigenmodes is considered to be localized in space and ceases to carry current; $\sigma_\alpha = 0$ for localized modes. In consequence, as long as the Fermi energy E_F lies within the spectrum of localized modes, the Hall conductance retains a plateau value. Note that the Hall conductance thereby is written in the form of asymmetry in the spectrum (weighted with σ_α) of extended modes below and above E_F .

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FIGURE CAPTION

FIG. 1. The symmetric spectrum of a spinless electron (on the left) gets Zeeman-shifted to yield asymmetry in the spectrum of the $\Xi = 1$ electron (on the right). The shaded spectra refer to the free-electron spectra associated each degenerate Landau level. The energy scale is labeled in terms of n according to the rule $E_n = \text{sgn}\{n\} \sqrt{m^2 + 2eB|n|}$.

Fig. 1