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Perturbative Approach to Quantum Gravity**

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Unreasonable Postulate in the Perturbative Approach to Quantum Gravity

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The conventional perturbative approach to quantum gravity is based on the expansion in powers of $\sqrt{\kappa}$, where κ denotes the Einstein gravitational constant. The introduction of a square root is due to the unreasonable postulate that the $\kappa \rightarrow 0$ limit of the gravitational field is a c -number. It is more natural that it is a q -number, which can be determined explicitly by the theory, and then the expansion becomes that in powers of κ but not of $\sqrt{\kappa}$. Thus the nonrenormalizability of Einstein gravity should be completely reconsidered in the light of the new expansion.

As is well known, quantum Einstein gravity, i.e., the quantum theory of general relativity, is not renormalizable in the conventional perturbation theory. It should be concluded, therefore, that *either* quantum Einstein gravity *or* the conventional perturbation theory to it is inadequate. Most physicists have taken the first alternative and indulged in seeking for other candidates of quantum gravity, such as supergravity, superstring, etc. It is quite curious that the second alternative has been forgotten without making any serious discussion. The reason for this seems to be the popularity of the path-integral approach: Since the concept of the sum over all possible metrics is difficult to be defined, the path-integral for gravity is usually defined by regarding the gravitational field as a Minkowskian tensor field in a flat space.

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Perturbation theory is based on the expansion in powers of $\sqrt{\kappa}$, where κ denotes the Einstein gravitational constant. Since $\sqrt{\kappa}$ has the dimension of length in natural units, the high-energy contributions from $E \gtrsim 1/\sqrt{\kappa}$ are treated very badly. This fact is quite likely to be the cause for the well-known nonrenormalizability of quantum Einstein gravity. But, of course, this situation is inevitable as long as one works in the small- κ approximation. Putting this point into pigeonhole, I wish to point out that the conventional perturbative approach to quantum gravity is still *guilty*: It is based on an unreasonable postulate! Hence any of its consequences, such as nonrenormalizability, should be distrusted.

The Lagrangian density of quantum Einstein gravity can be written as

$$\mathcal{L}_{\text{QEG}} = \frac{1}{2\kappa} \sqrt{-g} R + \mathcal{L}', \quad (1)$$

where g and R stand for the determinant of the gravitational field $g_{\mu\nu}$ and the scalar curvature, respectively, and $\mathcal{L}'$ consists of the matter-field Lagrangian density and the gauge-fixing plus Faddeev-Popov one. The precise expression for $\mathcal{L}'$ needs not be specified here; the important point is that $\mathcal{L}'$ does *not* involve κ .

The conventional postulate in the perturbative approach to quantum gravity is to set

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \sqrt{\kappa} h_{\mu\nu}(x), \quad (2)$$

where $\eta_{\mu\nu}$ denotes the Minkowski metric and $h_{\mu\nu}(x)$ is nonsingular as $\kappa \rightarrow 0$. That is, one assumes the following:

1. $g_{\mu\nu}^{(0)}(x) \equiv \lim_{\kappa \rightarrow 0} g_{\mu\nu}(x)$ is a *c*-number.
2. The expansion parameter is $\sqrt{\kappa}$ but not κ .

If the vacuum $|0\rangle$ is translationally invariant, $\langle 0|g_{\mu\nu}(x)|0\rangle$ is independent of x^μ , whence the first assumption implies that $g_{\mu\nu}^{(0)}(x)$ is essentially $\eta_{\mu\nu}$ (If $|0\rangle$ is not translationally invariant, $g_{\mu\nu}^{(0)}(x)$ gives an *a priori* background curved spacetime.).

Anyway, the reason for introducing the first assumption seems to be the fact that particle physics, which is constructed under the setting of $\kappa = 0$ from the outset, can be well described by the Minkowskian quantum field theory. One should note, however, that what can be inferred from the success of particle physics is that the commutator between

$g_{\mu\nu}(x)$ and any matter field should vanish as $\kappa \rightarrow 0$; this proposition does not necessarily imply that $g_{\mu\nu}^{(0)}(x)$ is a *c*-number because there are gravitational degrees of freedom.

The reason for introducing the second assumption is that if Eq. (2) is substituted into the Einstein-Hilbert Lagrangian density $(2\kappa)^{-1} \sqrt{-g} R$, then its zeroth-order part precisely becomes the Lagrangian density of a free massless tensor field. This is very good for the conventional perturbation theory. This “accidental” luck has dazzled physicists for so long time: They have believed, without examining its logical basis, that Eq. (2) would be the correct starting point of quantum gravity.

Thus, it may be concluded that the reason for postulating Eq. (2) is just that without it one cannot formulate the conventional perturbation theory for quantum gravity. Applicability, itself, of the conventional perturbative approach to quantum gravity has never come into question before encountering the divergence problem, as long as the small- κ expansion is supposed to be admitted. I wish to emphasize that the starting point of the small- κ expansion should be determined by the theory itself but not by any *ad hoc* assumption.

As is well known, since the Einstein-Hilbert action is general-coordinate invariant, the corresponding four degrees of freedom of $g_{\mu\nu}(x)$ are quantized by introducing a gauge-fixing term $\mathcal{L}_{\text{GF}}$ [included in $\mathcal{L}'$ of Eq. (1)]. That is, the equal-time commutation relations for those parts of $g_{\mu\nu}(x)$ are governed by $\mathcal{L}_{\text{GF}}$. But since $\mathcal{L}_{\text{GF}}$ is assumed to be independent of κ , all of those equal-time commutators cannot vanish as $\kappa \rightarrow 0$. Thus, $g_{\mu\nu}^{(0)}(x)$ is *not a c-number but a q-number*. This fact can be confirmed by calculating $g_{\mu\nu}^{(0)}(x)$ explicitly for a particular choice of $\mathcal{L}_{\text{GF}}$.

Since it is unnatural to introduce any particular *c*-number metric into the fundamental Lagrangian density, I set

$$\mathcal{L}_{\text{GF}} = -\tilde{g}^{\mu\nu} \partial_\mu b_\nu \quad (3)$$

with $\tilde{g}^{\mu\nu} \equiv \sqrt{-g} g^{\mu\nu}$, so as to be $GL(4)$ invariant. Here the B-field $b_\rho(x)$ is a bosonic $GL(4)$ -covariant vector field. The Faddeev-Popov ghost term $\mathcal{L}_{\text{FP}}$ is added so as to secure BRS invariance (i.e., quantum version of local symmetry). Then canonical quantization

can be carried out and all equal-time commutation relations are explicitly calculated [1]. Their $\kappa \rightarrow 0$ limits are as follows [2]:

$$R_{\mu\nu}^{(0)} = 0, \quad (4)$$

$$\partial_\mu \tilde{g}^{\mu\nu(0)} = 0, \quad (5)$$

etc. and

$$[g_{\mu\nu}^{(0)}(x), g_{\lambda\rho}^{(0)}(y)]_0 = 0, \quad (6)$$

$$[g_{\mu\nu}^{(0)}(x), \partial_0 g_{\lambda\rho}^{(0)}(y)]_0 = 0, \quad (7)$$

$$[g_{\mu\nu}^{(0)}(x), b_\rho^{(0)}(y)]_0 = -i(\tilde{g}^{00(0)})^{-1}(\delta_\mu^0 g_{\rho\nu}^{(0)} + \delta_\nu^0 g_{\mu\rho}^{(0)})\delta^3(x-y), \quad (8)$$

etc., where a subscript, 0, of a commutator indicates to set $x^0 = y^0$. While Eqs. (4)–(7) imply

$$[g_{\mu\nu}^{(0)}(x), g_{\lambda\rho}^{(0)}(y)] = 0 \quad (9)$$

four-dimensionally, Eq. (8) shows that $g_{\mu\nu}^{(0)}(x)$ is *not* a c-number.

Apart from the dimensionality, the above situation is completely the same as in the two-dimensional quantum gravity whose Lagrangian density is given by [3]

$$\mathcal{L}_{2\text{DQG}} = \sqrt{-g} R b + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}}, \quad (10)$$

where the first term is a gauge-fixing term for Weyl invariance, b being the Weyl B-field, and $\mathcal{L}_{\text{GF}}$ and $\mathcal{L}_{\text{FP}}$ are formally (i.e., apart from the dimensionality) the same as in the quantum Einstein gravity discussed above. Abe and I [4,5] have succeeded in solving this two-dimensional quantum gravity *exactly* in terms of a set of Wightman functions, which are given in closed form and are free of divergence. This solution, apart from the part related to the Weyl B-field, can be used also as the zeroth-order approximation to quantum Einstein gravity, because all relevant field equations and equal-time commutation relations are common to both apart from the dimensionality. I therefore claim that the q -number $g_{\mu\nu}^{(0)}(x)$ is explicitly known.

In proceeding to higher-order approximations to quantum Einstein gravity, there is no reason for introducing the quantities of order $\sqrt{\kappa}$. Since all field equations and commutation relations involve κ at most *linearly*, it is quite natural to expand $g_{\mu\nu}$ as

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \kappa g_{\mu\nu}^{(1)} + \kappa^2 g_{\mu\nu}^{(2)} + \dots \quad (11)$$

Then, of course, I have [2]

$$[g_{\mu\nu}^{(0)}(x), g_{\lambda\rho}^{(1)}(y)] \neq 0. \quad (12)$$

It gives the physically nontrivial lowest-order contributions. In the conventional approach, one had to introduce the quantity of order $\sqrt{\kappa}$ to describe the quantity corresponding to Eq. (12), because $g_{\mu\nu}^{(0)}(x)$ was a *c*-number. This is the origin of the unnatural apparition of a square root.

I add a remark on the perturbative expansion of gauge theory. Although the conventional expansion parameter is the coupling constant g , it is due to the requirement that the $g \rightarrow 0$ limit should be a free-field theory. If gA_μ^a is replaced by A_μ^a so as to remove the g dependence of covariant derivative, the Lagrangian density of gauge theory is written as

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4g^2} F^{\mu\nu a} F_{\mu\nu}^a + \mathcal{L}', \quad (13)$$

where g is involved explicitly neither in $F_{\mu\nu}^a$ nor in $\mathcal{L}'$. Thus the natural expansion parameter is g^2 but not g [6]. It has been shown [7] that the zeroth-order approximation to gauge theory is nothing but the BF theory [8], which is known to be exactly solvable.

In the expansion in powers of κ or of g^2 , proposed above, is BRS invariant in each order in sharp contrast to the conventional perturbation theory. That is, the Ward-Takahashi or Slavnov-Taylor identities are satisfied automatically in each order.

In conclusion, I summarize my assertion. In the conventional perturbative approach to quantum gravity, one assumes that $g_{\mu\nu}^{(0)}(x)$, the $\kappa \rightarrow 0$ limit of $g_{\mu\nu}(x)$, is a *c*-number *a priori*, but this postulate is unreasonable. If one starts with Eq. (1), $g_{\mu\nu}^{(0)}(x)$ is necessarily a *q*-number, which can be explicitly determined by the theory. Starting with it, one naturally obtains an expansion in powers of κ but not of $\sqrt{\kappa}$. The latter is a quantity

which is not involved in the fundamental action; its introduction is totally due to the artificiality of the conventional perturbation theory. Thus, I wish to claim that the well-known nonrenormalizability of quantum Einstein gravity must now be completely reconsidered in the light of the new expansion.

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