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QUARK MASS MATRICES AND GENERATION MIXING IN THE STANDARD MODEL WITH NON-COMMUTATIVE GEOMETRY*)

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Abstract

A version of the standard model based on non-commutative geometry allows to classify three quark generations in a twelve-dimensional, reducible, but indecomposable representation which provides a natural framework for inter-generational interactions. Assuming the quark generations to be initially degenerate (thus blaming electroweak interactions for the observed splittings) we find mass matrices and CKM mixing identical with, or very close to, published empirical analyses in agreement with experiment but derived from a very different physical picture. We present fully analytic expressions for the mass matrices and the mixing matrices which are functions of the mass eigenvalues and a small number of parameters. Our results confirm our previous experience: models based on non-commutative geometry explain some of the *qualitative* features of the standard model which previously were empirical inputs.

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1. Introduction

The pattern of the quark mass matrices and the structure of the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix that follows from these belong to the great challenges in our attempts to understand the detailed pattern of the standard model of strong and electroweak interactions of quarks and leptons. While purely phenomenological analyses of the CKM mixing are well developed and, in fact, are fairly successful in comparison with experiment, there is as yet little progress in the theoretical understanding, at a more fundamental level, of the matrices of quark masses.

On the other hand, application of ideas of non-commutative geometry [1] to particle physics has brought forth, over the last couple of years, novel extensions of Yang-Mills theories coupled to fermions which put some of the qualitative, but so far empirical, features of the standard model in a new perspective. The most ambitious, and mathematically most sophisticated, construction is the one proposed by Connes and Lott [2] which is based on a generalized Dirac(-Yukawa) operator over a manifold that consists of ordinary space-time endowed with an internal discrete space. In the simplest case, which is the one used in constructing the standard model, the internal space is a two-point space. Connes and Lott postulate that the odd pieces of that Dirac operator contain precisely the “physical” quark and lepton mass matrices, coming in three generations, the quark sector and the lepton sector being related by generalized Poincaré duality. In their framework this assumption seems to lead to the amazing prediction that, at the classical level, the Higgs potential is determined quantitatively by the quark and lepton masses. Although this is still being debated (see e.g. [3]), we do not believe that one can maintain such quantitative relations between the parameters of the model. No such relation would survive quantization. Furthermore, a closer analysis shows that even at the classical level relations between parameters are not canonical and can in fact be changed. Rather, as worked out in more detail in [4], we claim that the main virtue of this construction as well as of the other, alternative constructions that were proposed over the last years [5, 6, 7, 8, 9, 10] is to predict some of the *qualitative* features of the standard model which previously were ad-hoc inputs taken from phenomenology.

The Marseille-Mainz construction [6, 8] - very much like [5, 7] - is formulated at the level of the underlying algebra¹, by constructing first the superconnection and the bosonic sector of the model. The Dirac-Yukawa operator is a derived quantity whose detailed form, therefore, is not an input as in [2]. In the electroweak sector the Marseille-Mainz model leads rather naturally to the graded Lie algebra $su(2|1)$ as a classifying algebra for quarks and leptons, $su(2|1)$ acting on the fields of the standard model in the sense of what we called a weak symmetry in [4]. In particular, quarks coming in $N = 3$ generations fit very naturally into the reducible but indecomposable representations that this algebra is known to admit [12] and which are semi-direct sums of three copies of the fundamental representation reproducing the right $SU(2) \times U(1)$ quantum

¹In [11] we have shown that this is also possible in the case of Connes’ and Lott’s model. This reformulation allows to compare their construction most economically with the others.

numbers of right-handed and left-handed quark fields. As shown in [8] these “triangular” representations provide a natural framework for generation mixing. This is a novel feature of this non-commutative version of the standard model that does not occur in ordinary Yang-Mills theory.

The purpose of this paper is to study in more detail the quark mass matrices and the CKM mixing matrix in the framework of the reducible, but indecomposable representations of $su(2|1)$ (an earlier attempt was made in [13]). This analysis is based on the following assumption [8]: quark mass differences and state mixing are due to electroweak interactions only, in such a way that if these interactions were switched off, the quarks in each charge sector would be degenerate. We show that this assumption leads to a form of the mass matrices in the *up* and the *down* sector which is very close to the form postulated by Fritzsch and collaborators on more empirical grounds [14, 15]. This result is remarkable because this empirical form of the mass matrices is justified by an assumption very different from ours: Indeed, Fritzsch et al. assume that, due to some chiral symmetry at work in the model, the quarks of the first two generations initially are massless and that their masses are generated only through mixing with the massive, heavy, *t* and *b* quarks, respectively. Furthermore, since we derive mass matrices which, for the case of two generations, are identical with the ones postulated by Fritzsch et al., and, in the case of three generations, can come very close to theirs, the non-commutative standard model of the Marseille-Mainz group is able to describe quark mixing in fair agreement with experiment.

In sec. 2 we construct the relevant, classifying representation for three generations of quarks explicitly. We give the corresponding representation of the superconnection and identify the terms which give rise to quark mass terms. As charge conjugation in a model based on non-commutative geometry contains some subtleties [8] sec. 3 is devoted to this topic. After these preparations we write down the mass terms for *up* quarks and for *down* quarks in sec. 4. We establish the correspondence to more phenomenological forms of the mass matrices advocated in the literature [15, 17] and, thus, relate our results to standard, successful, phenomenology of CKM mixing. In sec. 5 we give an analytic expression for the full CKM mixing matrix for three generations which will serve as a basis for further analyses. The final sec. 6 gives some conclusions and an outlook on further applications of non-commutative models.

2. Classifying representations for three generations and superconnection

The graded Lie algebra $su(2|1)$ on which the model is built [8] is generated by four generators ($T_k \equiv I_1, I_2, I_3, Y$) for its *even* subalgebra (even with respect to the $\mathbb{Z}_2$ grading), and four generators ($\Omega_a \equiv \Omega_+, \Omega_-, \Omega'_+, \Omega'_-$) for the *odd* part. The graded commutators that define this algebra are repeated here for the sake of reference, viz

$$[I_k, I_l] = i\epsilon_{klm}I_m, \quad [Y, I_k] = 0, \quad (1)$$

$$\begin{aligned}
[I_1, \Omega_{\pm}] &= \frac{1}{2}\Omega_{\mp}, & [I_2, \Omega_{\pm}] &= \pm i\frac{1}{2}\Omega_{\mp}, & [I_3, \Omega_{\pm}] &= \pm\frac{1}{2}\Omega_{\pm}, \\
[I_1, \Omega'_{\pm}] &= -\frac{1}{2}\Omega'_{\mp}, & [I_2, \Omega'_{\pm}] &= \mp i\frac{1}{2}\Omega'_{\mp}, & [I_3, \Omega'_{\pm}] &= \pm\frac{1}{2}\Omega'_{\pm}, \\
[Y, \Omega_{\pm}] &= -\Omega_{\pm}, & [Y, \Omega'_{\pm}] &= \Omega'_{\pm}, \\
\{\Omega_{\pm}, \Omega'_{\mp}\} &= \pm I_3 + \frac{1}{2}Y, & \{\Omega_{\pm}, \Omega'_{\pm}\} &= I_1 \pm iI_2, & \{\Omega_{\pm}, \Omega_{\pm}\} &= 0, \\
\{\Omega_{\pm}, \Omega_{\mp}\} &= 0, & \{\Omega'_{\pm}, \Omega'_{\pm}\} &= 0, & \{\Omega'_{\pm}, \Omega'_{\mp}\} &= 0.
\end{aligned} \tag{2}$$

$$\begin{aligned}
\{\Omega_{\pm}, \Omega'_{\mp}\} &= \pm I_3 + \frac{1}{2}Y, & \{\Omega_{\pm}, \Omega'_{\pm}\} &= I_1 \pm iI_2, & \{\Omega_{\pm}, \Omega_{\pm}\} &= 0, \\
\{\Omega_{\pm}, \Omega_{\mp}\} &= 0, & \{\Omega'_{\pm}, \Omega'_{\pm}\} &= 0, & \{\Omega'_{\pm}, \Omega'_{\mp}\} &= 0.
\end{aligned} \tag{3}$$

Each individual quark generation is classified in the simplest typical representation $[Y_0 = 1/3, I_0 = 1/2]$ of $su(2|1)$ compatible with the requirement that chiral anomalies be absent [8, 16]. When decomposed into representations (I, Y) of $SU(2) \times U(1)$ this is

$$[Y_0 = \frac{1}{3}, I_0 = \frac{1}{2}] \rightarrow (I = \frac{1}{2}, Y = \frac{1}{3}) \oplus (I = 0, Y = \frac{4}{3}) \oplus (I = 0, Y = -\frac{2}{3}).$$

Three generations of quarks are classified in the reducible but indecomposable representation [12] that consists of the semi-direct sum of three copies of the above, viz

$$[Y_0 = \frac{1}{3}, I_0 = \frac{1}{2}] \oplus [Y_0 = \frac{1}{3}, I_0 = \frac{1}{2}] \oplus [Y_0 = \frac{1}{3}, I_0 = \frac{1}{2}]. \tag{4}$$

In this representation the generators have block triangular form as follows

$$\hat{T}_i = \begin{pmatrix} (T_i)^{11} & 0 & 0 \\ (T_i)^{21} & (T_i)^{22} & 0 \\ (T_i)^{31} & (T_i)^{32} & (T_i)^{33} \end{pmatrix}, \quad \hat{\Omega}_a = \begin{pmatrix} (\Omega_a)^{11} & 0 & 0 \\ (\Omega_a)^{21} & (\Omega_a)^{22} & 0 \\ (\Omega_a)^{31} & (\Omega_a)^{32} & (\Omega_a)^{33} \end{pmatrix}. \tag{5}$$

With the ordering $\Psi^T = (\psi^{(1)}, \psi^{(2)}, \psi^{(3)})$ and $(\psi^{(n)})^T = (u_L^{(n)}, d_L^{(n)}, u_R^{(n)}, d_R^{(n)})$, $n = 1, 2, 3$, of the basis states one identifies first those elements in the 4×4 block matrices which are or can be different from zero. For example, the blocks along the main diagonal of I_3 and of Y are given by, respectively, $(I_3)^{ii} = \text{diag}(1/2, -1/2, 0, 0)$ and $(Y)^{ii} = \text{diag}(1/3, 1/3, 4/3, -2/3)$. The nondiagonal blocks in I_3 are found to vanish, but those in Y and in $I_{1/2}$ initially do not vanish. For reasons to become clear below we do not write them out here. The odd generators, in turn, have the form (with $i \geq k$)

$$(\Omega_+)^{ik} = \left(\begin{array}{cc|cc} 0 & 0 & a_{ik} & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & \alpha_{ik} & 0 & 0 \end{array} \right), \quad (\Omega_-)^{ik} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & b_{ik} & 0 \\ \hline 0 & 0 & 0 & 0 \\ \beta_{ik} & 0 & 0 & 0 \end{array} \right). \tag{6}$$

$$(\Omega'_+)^{ik} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & c_{ik} \\ 0 & 0 & 0 & 0 \\ \hline 0 & \gamma_{ik} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right), \quad (\Omega'_-)^{ik} = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{ik} \\ \hline \delta_{ik} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right). \tag{7}$$

Clearly, the complex entries of these matrices and the elements of the even generators $\hat{Y}, \hat{I}_i$ are not all independent. The relations that exist between them are determined by the requirement that the block-triangular matrices (5) obey the graded commutators (1) – (3) of the algebra. For example, for the blocks along the main diagonal in (5) the first anticommutator (3) and the commutators with $\hat{I}_\pm$ give the conditions

$$a_{ii}\delta_{ii} = \frac{2}{3}, \quad d_{ii}\alpha_{ii} = -\frac{1}{3}, \quad b_{ii} = a_{ii}, \quad c_{ii} = -d_{ii}, \quad \beta_{ii} = -\alpha_{ii}, \quad \gamma_{ii} = \delta_{ii}.$$

Furthermore, there will be a few more conditions, to be discussed in sec. 4, for the even generators. We have obtained these conditions in part with the aid of the algebraic manipulation program MATHEMATICA but we do not write down all of them explicitly here. It suffices to notice that the parameters α_{ik} and δ_{ik} with $i \geq k$ that will appear in the mass matrices of *up* and *down* quarks, respectively, are independent. All relations there are can be satisfied by suitable choice of the other parameters not appearing in the mass matrices.

The interaction of the quarks with the vector bosons of $SU(2) \times U(1)$ and with the Higgs fields follows from the superconnection of the model,

$$\mathcal{A} = i \left\{ a I \cdot W + \frac{1}{2} b Y W^{(8)} + \frac{1}{\mu} [\Omega'_- \phi^{(0)} + \Omega'_+ \phi^{(+)} + \Omega_+ \overline{\phi^{(0)}} + \Omega_- \overline{\phi^{(+)}}] \right\}, \quad (8)$$

suitably expressed in the Clifford algebra describing the quark spinors [8]. The constants a and b are real numbers, μ is the mass scale of the theory. Via the construction of the corresponding supercurvature which yields the Lagrangian in the spontaneously broken phase [8, 9, 10], $\mathcal{A}$ gives rise to a mass term for the quarks of the form

$$\mu \bar{\Psi}(x) (\Omega_+ + \Omega'_-) \Psi(x) + \text{h.c.}.$$

As noted in [8], and in contrast to familiar Dirac mass terms, this expression is not invariant under charge conjugation and, therefore, the contribution of the antiparticles must be added to it explicitly. In other terms, we have to construct also the charge conjugate of this mass term, a topic to which we now turn. For later reference we note here the well-known transformation behaviour of the boson fields under charge conjugation:

$$\begin{aligned} {}^c W^{(\pm)} &= W^{(\mp)}, \quad {}^c W^{(3)} = -W^{(3)}, \quad {}^c W^{(8)} = -W^{(8)} \\ {}^c \phi^{(+)} &= \overline{\phi^{(+)}}, \quad {}^c \phi^{(0)} = -\overline{\phi^{(0)}}, \quad {}^c \overline{\phi^{(0)}} = \phi^{(0)}, \quad {}^c \overline{\phi^{(+)}} = -\phi^{(+)}. \end{aligned} \quad (9)$$

3. Charge conjugation in a graded Lie algebra

Consider a graded Lie algebra of the type $su(m|n)$ whose even part $su(m) \times su(n) \equiv Lie(G)$ generates the Lie group $G = SU(m) \times SU(n)$. Assume this group G to be a global symmetry of the Lagrangian of a theory containing fermions or, for the case of a

gauge theory coupled to fermionic matter, to be the structure group of that theory. Let the fermionic fields span an p -dimensional representation, reducible or irreducible, of G , $\Psi^T = (\psi_1, \dots, \psi_p)$. Charge conjugation will take the field ψ_k to ${}^c\psi_k = \eta_k i \gamma^2 \psi_k^*$, where η_k is a phase of square 1, i.e. $\eta_k = \pm 1$. Clearly, whether or not these phases are related among themselves depends on the type of representation chosen. But even in case they are related there will always be some convention to be made in choosing the phases. Of course, this convention will be reflected in the explicit form of the charge conjugates of the generators of $Lie(G)$.

In what follows we write η (without subscript) for the diagonal matrix of charge conjugation phases

$$\eta := \text{diag} (\eta_1, \dots, \eta_p). \quad (10)$$

The generators of the *even* part of $su(m|n)$, hence of $Lie(G)$, are denoted by T_l , while those of its *odd* part are denoted generically by Ω_a . If the group element $g \in G$ acts on the fermionic multiplet from the left, $\Psi' = U(g)\Psi$, then the same element acts on the charge conjugate multiplet ${}^c\Psi$ (from the left) by ${}^cU = \eta U^*(g)\eta$. The standard convention in expressing the group elements in terms of the generators is $U(g) = \exp\{i \sum \alpha_l U(T_l)\}$ for particle fields, and by ${}^cU(g) = \exp\{i \sum \alpha_l U({}^cT_l)\}$ for antiparticle fields, with $\alpha_l \in \mathbb{R}$ and T_l hermitean. With η as defined in eq. (10) and using $\eta^2 = \mathbb{1}$ one deduces the relationship

$$U({}^cT_l) = -\eta U(T_l^*)\eta. \quad (11)$$

Charge conjugation C is entirely defined by the input described above, i.e. by the global symmetry G of the theory, by the choice of the particle multiplet Ψ , and, in case there remains some freedom, by the convention chosen in defining the phase matrix (10). Eq. (11) gives the corresponding antiautomorphism of the Lie algebra $Lie(G)$, i.e. of the even part of the graded algebra.

If the behaviour of the Lagrangian under charge conjugation is known, it should be possible to deduce the action of this antiautomorphism on the generators Ω_a of the odd part of the algebra or, more precisely, on their representations $U(\Omega_a)$ in the space spanned by the fermion fields Ψ . This is what we work out next.

The graded Lie algebra contains, among others, commutation rules between even and odd generators which are of the type

$$[T_i, \Omega_a] = C_{iab} \Omega_b.$$

By taking the complex conjugate of this relation, multiplying from the left and from the right by η , making use of eq. (11), and imposing the graded commutators of the algebra for the charge conjugate generators it is an easy matter to identify terms of the type $\eta U(\Omega_a^*)\eta$ which are candidates for the charge conjugates $U({}^c\Omega_b)$. Such a set is not unique, however. Indeed, if h is an automorphism of the algebra which commutes with its *even* generators, then $h\eta U(\Omega_a^*)\eta h^{-1}$ is another such set. In fact, we need further information if we wish to fix the charge conjugates of the odd generators. We illustrate

these matters in the case of the algebra $su(2|1)$, for the example of one quark generation, described by an irreducible, typical, representation of $su(2|1)$, as well as for the example of three generations described by the reducible, but indecomposable, representation (4) described above.

For simplicity, we follow from here on the standard convention of writing the symbols of the generators T_k, Ω_a for their matrix representation in the space of the fermion fields. (We already did so in sec. 2.) In the case of $su(2|1)$ the commutators (2) show that (Ω_+, Ω_-) form an isospin doublet with $i_3 = +1/2, i_3 = -1/2$, respectively, and hypercharge -1 , while (Ω'_+, Ω'_-) form another such doublet with hypercharge $+1$. From this observation it is immediately clear that ${}^c\Omega_{\pm}$ have the same matrix structure as $\Omega'_{\mp}$, respectively, and that ${}^c\Omega'_{\pm}$ have the same structure as $\Omega_{\mp}$. Indeed, for this algebra the charge conjugates of the four odd generators are given by

$${}^c\Omega_{\pm} = \rho h \eta \Omega'_{\mp} \eta h^{-1}, \quad {}^c\Omega'_{\pm} = \rho' h \eta \Omega_{\mp} \eta h^{-1},$$

where ρ and ρ' are real numbers whose product must be $\rho\rho' = -1$. This is easily verified by checking the graded commutators (1) – (3) of $su(2|1)$ and making use of eq. (11) above. In the defining representation of $su(2|1)$ one has $\rho = 1, \rho' = -1$. Therefore the final answer is

$${}^c\Omega_{\pm} = h \eta \Omega'_{\mp} \eta h^{-1}, \quad {}^c\Omega'_{\pm} = -h \eta \Omega_{\mp} \eta h^{-1}. \quad (12)$$

One also verifies that taking the charge conjugates twice one obtains ${}^c({}^c\Omega_a) = -\Omega_a$ which is to be expected since the pairs (Ω_+, Ω_-) and (Ω'_+, Ω'_-) are $SU(2)$ doublets.

(i) For one quark generation we have $\Psi^T = (u_L, d_L, u_R, d_R)$. The charge conjugation phases are chosen as usual to be $\eta = \text{diag}(1, -1, 1, 1)$, so that ${}^c\Psi^T = ({}^cu_R, -{}^cd_R, {}^cu_L, {}^cd_L)$. From the explicit form of the even generators

$$I_i = \left(\begin{array}{c|c} \tau_i/2 & 0_{2 \times 2} \\ \hline 0_{2 \times 2} & 0_{2 \times 2} \end{array} \right) \quad Y = \text{diag} \left(\frac{1}{3}, \frac{1}{3}, \frac{4}{3}, -\frac{2}{3} \right)$$

we see that the automorphism h which commutes with all even generators must be

$$h = \text{diag}(1, 1, x, y) \quad \text{with } x, y \text{ nonzero} \in \mathbb{C}, \quad (13)$$

or a multiple thereof. In order to fix h we appeal to the following physical argument. The odd part of the superconnection (8) reads (except for the scale factor μ)

$$\mathcal{A}_{\text{odd}} = i \left\{ \Omega_+ \overline{\phi^{(0)}} + \Omega_- \overline{\phi^{(+)}} + \Omega'_- \phi^{(0)} + \Omega'_+ \phi^{(+)} \right\}.$$

It describes Yukawa couplings between the Higgs fields $\phi^{(0)}$ and $\phi^{(+)}$ and their hermitean conjugates and quark fields of definite handedness, i.e. matrix elements between a Higgs field and two quark fields, all three of which carry well-defined quantum numbers with respect to the global symmetry group $G = SU(2) \times U(1)$. These matrix elements from

which the operators Ω_a have disappeared, have a well-defined transformation behaviour under charge conjugation. The result that one derives by applying charge conjugation to these matrix elements must be obtainable also from the charge conjugate of $\mathcal{A}_{odd}$, which by eqs. (12) and (9) is given by

$${}^c\mathcal{A}_{odd} = -i h \eta \left\{ \Omega_-'^* \phi^{(0)} - \Omega_+'^* \phi^{(+)} + \Omega_+^* \overline{\phi^{(0)}} - \Omega_-^* \overline{\phi^{(+)}} \right\} \eta h^{-1}.$$

This requirement is met if ${}^c\mathcal{A}_{odd}$ is equal to

$${}^c\mathcal{A}_{odd} = -i \left\{ \Omega_-'^* \phi^{(0)} + \Omega_+'^* \phi^{(+)} + \Omega_+^* \overline{\phi^{(0)}} + \Omega_-^* \overline{\phi^{(+)}} \right\}.$$

The two forms of the odd part of the connection are equal if and only if $h\eta$ commutes with Ω_+ and Ω_-' , but anticommutes with Ω_- and $\Omega_+'.$ This condition fixes the automorphism h uniquely. Indeed, one finds $x = 1$ and $y = -1$, so that the product of η and h is $h\eta = \text{diag}(1, -1, 1, -1)$.

(ii) In the case of the twelve-dimensional indecomposable representation (4) there is more freedom in choosing the phase matrix η and, consequently, the charge conjugates (12) in so far as each quark generation (k) could carry its own overall phase $\exp(i n_k \pi)$, with n_k even or odd integer. As a consequence, the charge conjugates ${}^c\widehat{\Omega}_a$ will have the form

$$\begin{pmatrix} (\Omega_b^*)^{11} & 0 & 0 \\ (-)^{n_2+n_1} (\Omega_b^*)^{21} & (\Omega_a^*)^{22} & 0 \\ (-)^{n_3+n_1} (\Omega_b^*)^{31} & (-)^{n_3+n_2} (\Omega_b^*)^{32} & (\Omega_b^*)^{33} \end{pmatrix}.$$

Below we shall choose $n_1 = n_2 = n_3 = 0$ so that no additional signs will appear in this formula. Nevertheless, it is interesting to note that there are also other possibilities. For example, if n_1, n_2 and n_3 are chosen such that both $(n_1 + n_2)$ and $(n_3 + n_2)$ are odd integers, then $(n_1 + n_3)$ is an even integer. Therefore, the expression above with minus signs in the (21) and (32) positions, and a plus sign in the (31) position also belongs to a representation of $su(2|1)^2$.

4. General mass matrices for up and down quarks

With the results of the preceding section the mass terms for the quarks in the Lagrangian are $\mathcal{L}_{mass} = 1/4(C + {}^c C)$, where C is given by

$$C = \mu \overline{\Psi} (\Omega_+ + \Omega_-') \Psi + \text{h.c.}, \quad (14)$$

and its charge conjugate by

$${}^c C = \mu {}^c \overline{\Psi} ({}^c \Omega_+ + {}^c \Omega_-') {}^c \Psi + \text{h.c.}.$$

²The case with minus signs in all off-diagonal positions that was assumed in [8] is not admissible because it does not belong to a representation of $su(2|1)$.

The latter can easily be expressed in terms of the original fields by replacing ${}^c\Psi$ by its definition and by inserting the expressions (12) for the charge conjugate operators. One finds

$${}^cC = -\mu \bar{\Psi} h (\Omega_+ - \Omega'_-) h^{-1} \Psi + \text{h.c.} \quad (15)$$

Keeping in mind our earlier result $h = \text{diag}(1, 1, 1, -1)$ and the specific form of the matrices Ω_+ and Ω'_- , cf. eqs. (6) and (7), we see that the *up* part and the *down* part of the mass term are different: The former gets contributions only from Ω'_- , the latter only from Ω_+ , because the terms containing a_{ik} in Ω_+ and, similarly, the terms containing d_{ik} in Ω'_- cancel,

$$\mathcal{L}_{mass} = \frac{1}{4}(C + {}^cC) = \frac{1}{2} \left(\mu (\bar{\Psi} \Omega'_- \Psi)_{up} + \mu (\bar{\Psi} \Omega_+ \Psi)_{down} \right) + \text{h.c.} \quad (16)$$

When written in terms of individual quark fields this is seen to be

$$\mathcal{L}_{mass} = \frac{1}{2} \mu \left(\sum_{i,k=1}^3 \bar{u}_L^{(i)} T_{ik}^u u_R^{(k)} + \sum_{i,k=1}^3 \bar{d}_L^{(i)} T_{ik}^d d_R^{(k)} \right) + \text{h.c.}, \quad (17)$$

where the matrices T^u and T^d are triangular matrices, viz

$$T^u = \begin{pmatrix} \delta_{11}^* & 0 & \\ \delta_{21}^* & \delta_{22}^* & 0 \\ \delta_{31}^* & \delta_{32}^* & \delta_{33}^* \end{pmatrix}, \quad T^d = \begin{pmatrix} \alpha_{11}^* & 0 & \\ \alpha_{21}^* & \alpha_{22}^* & 0 \\ \alpha_{31}^* & \alpha_{32}^* & \alpha_{33}^* \end{pmatrix}, \quad (18)$$

the complex parameters α_{ik} and δ_{ik} being the same as in eqs. (6) and (7). In order to transform these matrices to a more customary, hermitean form we perform a unitary transformation U on the R-fields and make use of the following theorem [18]:

Theorem: Let T be a nonsingular, lower triangle matrix. Then there exists a unitary matrix U and a hermitean matrix H such that T admits the polar decomposition $T = HU$. If in addition T is positive this decomposition is unique.

In our application of this theorem we do not need the explicit form of the unitary U because this matrix is absorbed in the right-handed states. The hermitean H , on the other hand, is obtained as follows.

Let V be the unitary matrix which diagonalizes the hermitean product $TT^\dagger$,

$$V(TT^\dagger)V^\dagger = D_0. \quad (19)$$

As $TT^\dagger$ is positive, we can take the square roots of its eigenvalues but can freely decide on the signs of these roots. Below, we shall choose signs such that matrices of (linear) masses have the diagonal form $\text{diag}(m_1, -m_2, m_3)$ with all m_i positive. Let Δ_0 be a square root of D_0 , with a suitable choice of signs. The hermitean matrix H is then given by

$$H = V^\dagger \Delta_0 V. \quad (20)$$

Note that in case where T itself is not positive the decomposition as guaranteed by the theorem becomes unique as soon as one makes a choice on the signs in the square root Δ_0 of D_0 . Below we shall give explicit formulae for V^u, V^d , for *up* and *down* quarks, respectively, and shall calculate the hermitean mass matrices H^u, H^d in either case. This will allow for a direct comparison with other ansätze for these matrices. The CKM mixing matrix, finally, will be given by the simple expression

$$V_{CKM} = V^u V^{d\dagger}. \quad (21)$$

Before we turn to the physically motivated assumptions on the mass matrices (18) announced in the introduction, it is instructive to count parameters in the reducible but indecomposable representation of the generators. The four odd generators $\hat{\Omega}_a$ each contain 6×2 complex constants, i.e. 48 in total. The even generators $\hat{I}_3, \hat{I}_\pm$, and $\hat{Y}$ can be obtained from them by the first two anticommutators (3). While $\hat{I}_3$ is block-diagonal, $\hat{I}_\pm$, and $\hat{Y}$, in general, are found to have nonvanishing off diagonal blocks $(\)^{21}, (\)^{31}, (\)^{32}$. These depend on the same set of 48 constants. Working out the remaining graded commutators by means of MATHEMATICA we find 30 independent conditions, leaving 18 free parameters.

Let us return for a moment to the even subalgebra of $su(2|1)$ and the symmetry group $G = SU(2) \times U(1)$ that it generates. Our construction of the representation (4) has led us to a triangular, hence nonunitary representation of this group. On the other hand, we started from the assumption that each generation of quarks was already classified in an irreducible representation of G . Therefore, it seems natural to require all even generators to have block-diagonal form. Thus, we impose the conditions

$$(\hat{I}_\pm)^{ik} = 0 = (\hat{Y})^{ik}, \quad \text{for } (ik) = (21), (31), (32). \quad (22)$$

An alternative would be the one discussed in [8]. As noted there the off diagonal blocks, if they do not vanish, will lead to unwanted interaction terms such as flavor changing neutral currents, unless their contributions are cancelled by their charge conjugates. For this to happen, the charge conjugate off diagonal blocks must have negative signs. As shown in sec. 3 this is possible only for two, but not all three of them. There are then only two possibilities: Either we require all even generators to be block-diagonal, or else we allow two of the off diagonal blocks to be different from zero, take their charge conjugate with opposite sign, and require the third to vanish³. As this latter possibility seems rather artificial one is again led to (22).

Imposing the conditions (22) yields 6 more independent relations so that, eventually, we are left with 12 free complex parameters, in agreement with the result (18). Of course, not all of these parameters $(\alpha_{ik}, \delta_{ik})$ will be physically relevant. Indeed, by absorbing redundant phases into the base states one sees that each of the matrices (18) can be

³Note that this problem is typical for the case of three generations. It would not occur for two generations.

reexpressed in terms of 6 real numbers and a real phase. Thus, both T^u and T^d will have the generic form

$$T = \begin{pmatrix} \alpha & 0 & 0 \\ x_1 & \beta & 0 \\ x_3 e^{i\phi} & x_2 & \gamma \end{pmatrix} \quad (23)$$

with $\alpha, \beta, \gamma, x_1, x_2, x_3 \in \mathbb{R}$ and ϕ a real phase.

5. Mass matrices and CKM mixing matrix

We now turn to the assumption proposed in [8] and mentioned in the introduction, and work out its consequences. In the absence of electroweak interactions all off diagonal blocks in the representation (5) vanish. The blocks along the main diagonal of $\hat{\Omega}_a$ are identical for all three generations. In the representation (4) there is nothing that singles out any individual quark family and no ordering of the generations is preferred. As a consequence, in this situation the quark masses in every charge sector are degenerate, $T = \text{diag}(m_0, m_0, m_0)$. When the electroweak interaction is present it seems natural to assume that its effect *within* each generation is the same, i.e. that the diagonal blocks $(\Omega_a)^{ii}$ remain equal. The observed mass splittings are due to the nondiagonal blocks in $\hat{\Omega}_a$, eq. (5), which, by the nature of the reducible, indecomposable representation (4), connect different generations. With this assumption the generic form (23) of the mass matrix reduces to⁴

$$T = \begin{pmatrix} \alpha & 0 & 0 \\ x_1 & \alpha & 0 \\ x_3 e^{i\phi} & x_2 & \alpha \end{pmatrix} \quad (24)$$

with α and x_i real, ϕ a real phase. Clearly, the analysis will be the same in either charge sector, *up* or *down*, although the values of the parameters will be different. Therefore, we use the general form (24) for both the *up* and the *down* quarks. We denote its eigenvalues by (m_1, m_2, m_3) with $m_1 \equiv m_u, m_2 \equiv m_c, m_3 \equiv m_t$ in the case of T^u , and $m_1 \equiv m_d, m_2 \equiv m_s, m_3 \equiv m_b$ in the case of T^d .

The product $(TT^\dagger)$ is hermitean and has eigenvalues (m_1^2, m_2^2, m_3^2) . Its characteristic polynomial yields the following relations between the squared masses and the parameters

$$\begin{aligned} m_1^2 m_2^2 m_3^2 &= \alpha^6 \\ m_1^2 + m_2^2 + m_3^2 - 3\alpha^2 &= x_1^2 + x_2^2 + x_3^2 \\ m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2 - \alpha^2(m_1^2 + m_2^2 + m_3^2) &= x_1^2 x_2^2 - 2\alpha x_1 x_2 x_3 \cos \phi. \end{aligned} \quad (25)$$

From here on the principle of how to proceed is clear. First one derives the unitary matrix V that diagonalizes the positive matrix $(TT^\dagger)$, cf. eq. (19). Then, after having chosen the square root of D_0 such that, e.g., $\Delta_0 = \text{diag}(m_1, -m_2, m_3)$ one calculates the

⁴The results for the more general case $\alpha \neq \beta \neq \gamma$ was also derived analytically. We give the corresponding results for the matrices H and V in the appendix.

hermitean mass matrix (20). After having done this in either charge sector one obtains the CKM matrix from eq. (21). Before we present our results for the realistic case of three generations let us consider the simpler example of two generations.

5.1 Example: two generations of quarks

In this example T^u and T^d , eq. (24), have the form

$$T = \begin{pmatrix} \alpha & 0 \\ x_1 & \alpha \end{pmatrix}, \quad \alpha, x_1 \in \mathbb{R}. \quad (26)$$

The determinant and the trace of $(TT^\dagger)$ yield $\alpha = \sqrt{m_1 m_2}$ and $x_1 = m_1 - m_2$. Choosing $\Delta_0 = \text{diag}(m_1, -m_2)$ the matrix V is found to be

$$V = \frac{1}{\sqrt{m_1 + m_2}} \begin{pmatrix} \sqrt{m_2} & -\sqrt{m_1} \\ \sqrt{m_1} & \sqrt{m_2} \end{pmatrix}, \quad (27)$$

while the hermitean form of the mass matrix is

$$H = - \begin{pmatrix} 0 & \sqrt{m_1 m_2} \\ \sqrt{m_1 m_2} & m_2 - m_1 \end{pmatrix}. \quad (28)$$

We note that H , eq. (28), has exactly the same form as the mass matrix postulated by Fritzsch [15]

$$H_F = - \begin{pmatrix} 0 & Y \\ Y & M \end{pmatrix} \equiv - \begin{pmatrix} 0 & \alpha \\ \alpha & -x_1 \end{pmatrix}, \quad (29)$$

on the basis of an assumption which is opposite to ours: the idea behind the ansatz (29) is that, initially, quark 1 is massless while quark 2 is heavy and has the mass M . The effect of the mixing matrix element Y is to give quark 1 a small mass m_1 and to shift the mass of quark 2 from M to m_2 .

In contrast to this, our ansatz (26) which follows from $su(2|1)$ representation theory, assumes the diagonal matrix elements of the mass matrix to be equal, i.e. to be independent of the generation, and attributes the lifting of the initial mass degeneracy to the generation mixing contained in x_1 . Although the philosophy is very different in the two approaches, the result is exactly the same. In particular, we obtain the same formula for the Cabibbo angle [8] from eq. (21), viz

$$\cos \theta_c = \frac{\sqrt{m_u m_d} + \sqrt{m_c m_s}}{\sqrt{(m_u + m_c)(m_d + m_s)}}. \quad (30)$$

This simple example exhibits in a nutshell a general pattern that will also be found in the more realistic case of three generations to which we now turn⁵.

⁵We note in passing that the two-generation equation (30) already yields the correct order of magnitude of mixing angles in the CKM matrix. Indeed, when evaluated as written in (30) one finds an angle of about 9.2° . Applying it to (c, t, s, b) one finds 4.8° , while for (u, t, d, b) it is 2.0° .

5.2 The case of three generations

The matrices V can be calculated analytically also for the case of three generations of quarks. As the calculation is straightforward but rather lengthy we do not reproduce it here. The result is as follows

$$V = \begin{pmatrix} f(m_1)/N_1 & g(m_1)/N_1 & h(m_1)/N_1 \\ f(m_2)/N_2 & g(m_2)/N_2 & h(m_2)/N_2 \\ f(m_3)/N_3 & g(m_3)/N_3 & h(m_3)/N_3 \end{pmatrix}, \quad (31)$$

where the function symbols stand for

$$\begin{aligned} f(m_i) &= \alpha^2 x_1 x_2 - \alpha x_3 e^{i\phi} (\alpha^2 - m_i^2) \\ g(m_i) &= m_i^2 x_1 x_3 e^{i\phi} - \alpha x_2 (\alpha^2 - m_i^2) \\ h(m_i) &= (\alpha^2 - m_i^2)^2 - x_1^2 m_i^2, \end{aligned} \quad (32)$$

and where the denominators are given by

$$N_1 = \left\{ [(\alpha^2 - m_1^2)^2 - m_1^2 x_1^2] (m_3^2 - m_1^2) (m_2^2 - m_1^2) \right\}^{\frac{1}{2}} \quad (33)$$

$$N_2 = \left\{ [m_2^2 x_1^2 - (\alpha^2 - m_2^2)^2] (m_3^2 - m_2^2) (m_2^2 - m_1^2) \right\}^{\frac{1}{2}} \quad (34)$$

$$N_3 = \left\{ [(\alpha^2 - m_3^2)^2 - m_3^2 x_1^2] (m_3^2 - m_1^2) (m_3^2 - m_2^2) \right\}^{\frac{1}{2}}. \quad (35)$$

In performing this calculation we have made use of the relations (25) which follow from the characteristic polynomial of $(TT^\dagger)$. We have arranged matters such that m_1 is the smallest, m_3 the largest mass. Furthermore, the positivity of the radicants in eqs. (33) - (35) restricts the allowed range of x_1 , viz

$$\frac{|\alpha^2 - m_2^2|}{m_2} \leq x_1 \leq \min \left(\frac{|\alpha^2 - m_1^2|}{m_1}, \frac{|\alpha^2 - m_3^2|}{m_3} \right)$$

The hermitean mass matrix is obtained from V , eq. (31), as indicated in eq. (20). Again, the calculation is lengthy and was done in part by means of the program MATHEMATICA. The result is the following. Let

$$\begin{aligned} N &= (m_3 - m_2)(m_2 - m_1)(m_1 + m_3) \\ S &= m_1 - m_2 + m_3 \\ P &= m_1 m_2 + m_2 m_3 - m_1 m_3 \\ Q &= m_1^2 m_2^2 + m_2^2 m_3^2 + m_3^2 m_1^2 \\ R &= m_1(m_2^3 - m_3^3) + m_2(m_1^3 + m_3^3) + m_3(m_2^3 - m_1^3). \end{aligned}$$

We note that these quantities are related by $R - 2P^2 = NS$ and, with the first equation (25), by $Q - P^2 = 2\alpha^3 S$. The matrix elements of H are found to be

$$\begin{aligned}
H_{11} &= \alpha^2 \frac{P + \alpha S - 2\alpha^2 - x_2^2}{N} \\
H_{22} &= \frac{Px_1^2 - Q + \alpha^2(P + x_2^2) + \alpha^3 S + \alpha^4}{N} \\
H_{33} &= \frac{-Px_1^2 + R - Q - 2\alpha^2 P + 2\alpha^3 S + \alpha^4}{N} \\
H_{12} &= \alpha \frac{x_1(P - x_2^2) + \alpha x_2 x_3 e^{-i\phi} - \alpha^2 x_1}{N} \\
H_{13} &= \alpha \frac{x_3 P e^{-i\phi} + \alpha x_1 x_2 - \alpha^2 x_3 e^{-i\phi}}{N} \\
H_{23} &= \frac{(x_1 x_3 e^{-i\phi} + \alpha x_2)P - \alpha^3 x_2}{N}
\end{aligned} \tag{36}$$

It should be kept in mind that in these formulae x_3 and the angle ϕ , through eqs. (25), are given by x_1, x_2 and the squared masses.

Before we turn to phenomenology it is instructive to subject formulae (36) and (31) to some simple tests. By eqs. (36) one verifies that $\text{tr}\{H\} = S$. As to our expression for V , eq. (31), one verifies that by setting $x_2 = x_3 = 0$ the third generation decouples from the first two and one returns to the case discussed in subsec. 5.1. Similarly, setting $x_1 = x_3 = 0$ decouples the first generation, and setting $x_1 = x_2 = 0$ decouples the second generation. Finally, a particularly interesting test case consists in sending the mass of the lightest quark to zero. With $m_1 \rightarrow 0$ eqs. (25) gives $\alpha = 0$ and the matrix (24) becomes

$$T = \begin{pmatrix} 0 & 0 & 0 \\ x_1 & 0 & 0 \\ x_3 & x_2 & 0 \end{pmatrix} \tag{37}$$

which reduces to the case of two generations if we set $x_1 = x_2 = \sqrt{m_2 m_3}$, $x_3 = m_2 - m_3$. The massless quark 1 decouples, and in the (2,3) sector we recover the matrices V , eq. (27), and H , eq. (28), of the two generation case with m_1 replaced by m_2, m_2 by m_3 .

5.3 Contact with phenomenology

Both in the *up* sector and in the *down* sector the mass m_1 is much smaller than the other two. Therefore, in the realistic case where all three generations mix the matrix H is not far from the limit case (37) from which follows $H_{11} = H_{12} = H_{13} = 0$, and, with the above choice, also $H_{22} = 0$. The setting $H_{11} = H_{22} = H_{13} = 0$, on the other hand, is precisely the starting point of the analysis proposed by Fritzsch [15, 17] which is based on the idea that initially only one quark is massive, the other two picking up their masses through mixing between generations.

Our formulae suggest to expand the matrix elements of H and of V in terms of the quantity α , or, equivalently, in terms of $m_1^{1/3}$, keeping m_2 and m_3 fixed. We have done

so and we find, indeed, that H_{11} remains small, of the order of α^3 , H_{22} is of order α and sizeably smaller than H_{33} which starts with a term proportional to $m_3 - m_2$. Similarly, the off diagonal matrix elements are of the correct size, in agreement with what is expected empirically [15, 17]. We emphasize, in particular, that Fritzsch' ansatz fits correctly the hierarchy of the three CKM mixing angles [14]. Thus, we find mass matrices and, hence, the CKM matrix in accord with more phenomenological analyses with the bonus that all quantities are computable in terms of a few parameters. In a future publication we plan to perform a numerical fit of these parameters and a more detailed comparison with experiment. Also more specific questions such as dependence of mixing matrix elements on the top mass and maximal CP-violation [19] will be dealt with in a later publication.

6. Conclusions

In this work we have shown that the twelve-dimensional, reducible, but indecomposable representation (4) of $su(2|1)$ provides a canonical framework for nontrivial mass matrices of quarks and, hence, for Cabibbo-Kobayashi-Maskawa mixing of the quark generations. Adding to this the rather natural assumption that, initially, the three generations, in each charge sector, are degenerate, and that mass splittings and CKM mixing are caused by the off diagonal, inter-generational, pieces in the representation (4), we find hermitean mass matrices, eq. (20), which are very close to the ones proposed by Fritzsch [14, 15] on more phenomenological grounds. This result is surprising because Fritzsch' ansatz is based on a physical assumption that is very different from ours. In his ansatz, initially, only the heaviest quark in a given charge sector is massive while the other two are massless. Their masses, in turn, are then generated by the mixing with the massive, heavy partner. We note that the CKM mixing matrix that is derived from these *up* and *down* mass matrices is in good agreement with experiment.

Thus, to a certain extent, we provide a theoretical basis for the more phenomenological ansatz [14, 15], though the underlying physical picture is rather different. Our results, summarized in eqs. (31) and (36), are obtained in a fully analytic form. They have the distinctive property that they depend explicitly on the quark mass eigenvalues and four real parameters, x_1, x_2 in either charge sector. Furthermore, they can be expanded in terms of the lightest quark mass so that relative orders of magnitude become evident. Having made contact with the phenomenological analyses and having found mass matrices with the correct pattern, our CKM mixing matrix is in accord with experiment. A more quantitative fit to the data is deferred to a future publication. Also, a similar analysis of lepton masses is in progress.

Our encouraging results confirm our understanding [8, 4] of the extensions of Yang-Mills theory within non-commutative geometry: the graded Lie algebra $su(2|1)$ acts as a classifying algebra for quarks and leptons, the corresponding differential calculus yields the Yang-Mills-Higgs Lagrangian in a way which explains some of the *qualitative* features of the standard model which hitherto were ad hoc inputs. It is gratifying to

see that generation mixing appears naturally within that framework, without having to invoke new inputs, and that it can be described in a quantitatively correct manner.

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Appendix: The matrices V and H in the general case

For the sake of completeness we give here the expressions for V , eq. (19), and H , eq. (20), for the most general case (23) with $\alpha \neq \beta \neq \gamma$. The quantities N, S, P, Q, R are the same as in the main text. The relations (25) which follow from the characteristic polynomial of $(TT^\dagger)$, now read

$$\begin{aligned} m_1^2 m_2^2 m_3^2 &= \alpha^2 \beta^2 \gamma^2 \\ m_1^2 + m_2^2 + m_3^2 - (\alpha^2 + \beta^2 + \gamma^2) &= x_1^2 + x_2^2 + x_3^2 \\ Q - \alpha^2 \beta^2 - \beta^2 \gamma^2 - \gamma^2 \alpha^2 &= \alpha^2 x_2^2 + \beta^2 x_3^2 + \gamma^2 x_1^2 + x_1^2 x_2^2 - 2\beta x_1 x_2 x_3 \cos \phi. \end{aligned} \quad (38)$$

The unitary matrix V has the same form as in eq. (31), with the functions (f, g, h) and the denominators (N_1, N_2, N_3) replaced as follows

$$\begin{aligned} f(m_i) &= \alpha \beta x_1 x_2 - \alpha x_3 e^{i\phi} (\beta^2 - m_i^2) \\ g(m_i) &= m_i^2 x_1 x_3 e^{i\phi} - \beta x_2 (\alpha^2 - m_i^2) \\ h(m_i) &= (\alpha^2 - m_i^2)(\beta^2 - m_i^2) - x_1^2 m_i^2, \end{aligned} \quad (39)$$

$$N_1 = \left\{ [(\alpha^2 - m_1^2)(\beta^2 - m_1^2) - m_1^2 x_1^2] (m_3^2 - m_1^2)(m_2^2 - m_1^2) \right\}^{\frac{1}{2}} \quad (40)$$

$$N_2 = \left\{ [m_2^2 x_1^2 - (\alpha^2 - m_2^2)(\beta^2 - m_2^2)] (m_3^2 - m_2^2)(m_1^2 - m_2^2) \right\}^{\frac{1}{2}} \quad (41)$$

$$N_3 = \left\{ [(\alpha^2 - m_3^2)(\beta^2 - m_3^2) - m_3^2 x_1^2] (m_3^2 - m_1^2)(m_3^2 - m_2^2) \right\}^{\frac{1}{2}}. \quad (42)$$

The matrix H , finally, is given by

$$\begin{aligned} H_{11} &= \frac{\alpha^2 P + \alpha \beta \gamma S - \alpha^2 (\beta^2 + \gamma^2 + x_2^2)}{N} \\ H_{22} &= \frac{P(x_1^2 + \beta^2) - Q + \alpha^2 (\gamma^2 + x_2^2) + \alpha \beta \gamma S}{N} \\ H_{33} &= \frac{-P(x_1^2 + \alpha^2 + \beta^2) + R - Q + 2\alpha \beta \gamma S + \alpha^2 \beta^2}{N} \\ H_{12} &= \alpha \frac{x_1(P - x_2^2) + \beta x_2 x_3 e^{-i\phi} - \gamma^2 x_1}{N} \\ H_{13} &= \alpha \frac{x_3 P e^{-i\phi} + \beta x_1 x_2 - \beta^2 x_3 e^{-i\phi}}{N} \\ H_{23} &= \frac{(x_1 x_3 e^{-i\phi} + \beta x_2)P - \alpha^2 \beta x_2}{N}. \end{aligned} \quad (43)$$

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