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Hawking Radiation from (2+1)-Dimensional Black Hole

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Abstract

By using the frequency mixing method, we study the Hawking radiation of massless conformally coupled scalar field in the, recently discovered, 3 dimensional black hole geometry. We find the Hawking temperature which agrees with the previous results with appropriate normalization of timelike Killing vector. We also show that the angular velocity plays the role of chemical potential.

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1. Recently it is shown [1, 2] that there exists a black hole solution in the three-dimensional Einstein general relativity with a negative cosmological constant. Even though the curvature is constant, the solution has an event horizon and non-zero Hawking temperature.

There exist two parameters M and J which can be identified with the mass and angular momentum of the black hole. For non-vanishing J , it has an ergosphere and inner horizon, just like the 4-dimensional Kerr black hole solution.

In this paper we study the Hawking radiation of massless conformally coupled scalar field in this background black hole geometry. The Hawking temperature given in [1] is recovered. We also find that the angular momentum plays the role of chemical potential, which is analogous to the Kerr black hole case.

2. The Banados-Teitelboim-Zanelli(BTZ) metric [1] for three-dimensional black hole solution is

$$ds^2 = -N^2(r)dt^2 + N^{-2}(r)dr^2 + r^2(N^\varphi dt + d\varphi)^2, \quad (1)$$

where

$$N^2 = -M + \frac{r^2}{l^2} + \frac{J}{4r^2},$$

$$N_\varphi = -\frac{J}{2r^2}, \quad (2)$$

and l is related to the cosmological constant by $\lambda = -l^{-2}$.

We first consider the case when $M = 1$ and $J = 0$:

$$ds^2 = -\left(\frac{r^2}{l^2} - 1\right)dt^2 + \left(\frac{r^2}{l^2} - 1\right)^{-1}dr^2 + r^2 d\hat{\varphi}^2, \quad (3)$$

where $\hat{\varphi}$ is identified with a period $2\pi\nu$. This coordinate system covers the half of the anti-de Sitter space (AdS) which can be realized as the three-dimensional hyperboloid,

$$v^2 - w^2 - x^2 + y^2 = -l^2, \quad (4)$$

in a four-dimensional space with metric

$$ds^2 = dv^2 - dw^2 - dx^2 + dy^2. \quad (5)$$

The parametrization for the black hole geometry (3) is given by

$$\begin{aligned} v &= \sqrt{\hat{r}^2 - l^2} \cosh(\hat{t}/l), \\ w &= \sqrt{\hat{r}^2 - l^2} \sinh(\hat{t}/l), \\ x &= \hat{r} \cosh \hat{\varphi}, \\ y &= \hat{r} \sinh \hat{\varphi}. \end{aligned} \quad (6)$$

On the other hand, the parametrization which cover the whole AdS space is given by

$$\begin{aligned} v &= l \tan \chi \cos \theta, \\ w &= l \sin \tau \sec \chi, \\ x &= l \cos \tau \sec \chi, \\ y &= l \tan \chi \sin \theta, \end{aligned} \quad (7)$$

where $0 \leq \chi \leq \pi/2$, $0 \leq \theta < 2\pi$ and $0 \leq \tau \leq 2\pi$. The metric can be read as

$$ds^2 = l^2 \sec^2 \chi [-d\tau^2 + d\chi^2 + \sin^2 \chi d\theta^2], \quad (8)$$

which is conformal to the half of the Einstein static universe (ESU):

$$ds_E^2 = l^2 [-d\tau^2 + d\chi^2 + \sin^2 \chi d\theta^2], \quad (9)$$

where $0 \leq \chi \leq \pi$. These metrics are related by $g_{\mu\nu}^E = \Omega^2 g_{\mu\nu}$ where $\Omega = \cos \chi$, the conformal factor, is given by $\Omega = \cos \chi$.

AdS is well-known to have not only a closed time-like curve but also a timelike boundary at spatial infinity through which information can propagate.

To avoid the closed time-like curves, we consider the universal covering space of AdS (CAdS) in which τ takes any real value[3].

To resolve the problem of time-like boundary we follow the prescription of [4], i.e. take the Cauchy surface as a pair of surfaces $\{\tau, 0 \leq \chi \leq \pi/2\}$ and $\{\tau + \pi, 0 \leq \chi \leq \pi/2\}$ in CAdS, denoted by Σ_1 and Σ_2 , respectively. This will be shown to be equivalent to the surface $\{\tau, 0 \leq \chi < \pi\}$ in ESU. This choice of the Cauchy surface can be justified by the fact that Σ_1 and Σ_2 are causally disconnected.

3. The matter field we consider is the massless conformally coupled scalar field ϕ , which satisfies the wave equation

$$\square \phi - \frac{1}{8} R \phi = 0. \quad (10)$$

We consider the metric (8) as a spacetime in the far past $\mathcal{I}^-$ before the black hole formation and the BTZ metric (3) as a spacetime in the future $\mathcal{I}^+$. The mode solutions of the scalar field ϕ corresponding to each metric (3) and (8) define two different vacua, the in-vacuum $|0\rangle_{in}$ and the out vacuum $|0\rangle_{out}$, respectively. Then the in-vacuum $|0\rangle_{in}$, which can be regarded as initial configuration, is shown to have Plank distribution with non-zero temperature T , in the far future. This can be shown by Bogoliubov transformation between two modes corresponding (3) and (8).

In the metric (3), the mode solution of the equation (10) can be written in the form

$$\phi = e^{-i\omega t} e^{i\mu\varphi} \frac{R_{\omega\mu}(\hat{r})}{\sqrt{\hat{r}}}, \quad (11)$$

where $R_{\omega\mu}(\hat{r})$ satisfies the radial equation

$$\frac{d^2 R_{\omega\mu}}{d\hat{r}^{*2}} + \left\{ \omega^2 - N^2 \frac{m^2}{r^2} + \frac{R}{8} + \frac{1}{l^2 r^{\frac{1}{2}}} - \frac{1}{4} \frac{N^2}{r^{\frac{5}{2}}} \right\} R_{\omega\mu} = 0, \quad (12)$$

with

$$\hat{r}^* = \frac{l}{2} \ln\left(\frac{\hat{r}-l}{\hat{r}+l}\right), \quad (13)$$

and $\mu\nu \in Z$. Near the horizon, $\hat{r} \rightarrow l$ or $\hat{r}^* \rightarrow -\infty$, the effective potential in (12) vanishes and modes are given by

$$\phi_\omega = A_\omega e^{-i\omega(\hat{t}-\hat{r}^*)} e^{i\mu\hat{\phi}}. \quad (14)$$

This solution is consisted of positive frequencies with respect to the timelike Killing vector $\partial_{\hat{t}}$, satisfying

$$\mathcal{L}_{\partial_{\hat{t}}} \phi_\omega = -i\omega \phi_\omega. \quad (15)$$

We shall now consider the solution of (10) in the metric (8). We note that, for the massless conformally coupled scalar field ϕ , the conformally transformed field, $\phi_E = \Omega^{-\frac{1}{2}} \phi = (\cos\chi)^{-\frac{1}{2}} \phi$, satisfies the same wave equation (10) in the ESU (9):

$$\square \phi_E - \frac{1}{8} R_E \phi_E = 0, \quad (16)$$

with $R_E = 2/l^2$. In this space, the positive frequency mode solution is given by

$$\phi_E = \frac{1}{\sqrt{2\omega'}} e^{-i\omega'\tau} Y_{\omega'-\frac{1}{2},m'}(\chi, \theta), \quad (17)$$

where $Y_{\omega'-\frac{1}{2},m'}(\chi, \theta)$ are spherical harmonics with $\omega' - \frac{1}{2} \in Z$, $|m'| \leq \omega' - \frac{1}{2}$.

For the quantum field theory in CAdS, we follow the method of [4], in which the well-defined quantization in ESU has been used. The modes $\phi_{\omega',m'}^E$ in ESU form an orthonormal basis, along with the inner product defined by

$$(\alpha, \beta)_E = i \int_{\tau=\text{const}} \alpha \bar{\partial}_0 \beta^* g^{00} \sqrt{-g^E} d\chi d\theta. \quad (18)$$

The wave function ϕ^E satisfies the condition

$$\phi^E(\tau, \chi, \theta) = -i\phi^E(\tau - \pi, \pi - \chi, \theta). \quad (19)$$

This implies that for the massless field without any interaction, the wave-function $\phi^E(\phi, \chi, \theta)$ in $0 < \chi < \pi/2$ can be determined from $\phi^E(0, \chi, \theta)$ in $\pi/2 < \chi < \pi$. Therefore the Cauchy data on the complete surface at

$\tau = 0$ is equivalent to the Cauchy data on the pair of incomplete surfaces, $\{\tau = 0, 0 < \chi < \pi/2\}$ and $\{\tau = \pi, 0 < \chi < \pi/2\}$. This means that, when we use the orthonormal basis in ESU as the one in CAdS, we have to define the inner product as

$$(\alpha, \beta) = i \int_{\Sigma_1 \cup \Sigma_2} \alpha \bar{\partial}_0 \beta^* g^{00} \sqrt{-g} d\chi d\theta. \quad (20)$$

The mode solutions in CAdS become

$$\phi_{\omega',m'} = \frac{1}{\sqrt{2\omega'}} e^{-i\omega'\tau} \sqrt{\cos\chi} Y_{\omega'-\frac{1}{2},m'}(\chi, \theta), \quad (21)$$

where $\omega' - \frac{1}{2} \in Z$, $\omega' - \frac{1}{2} \geq 0$ and $|m'| \leq \omega' - \frac{1}{2}$. These are the positive frequency modes with respect to the timelike Killing vector $\partial_{\hat{t}}$. The spherical harmonics $Y_{\omega'-\frac{1}{2},m'}(\chi, \theta)$ can be expanded as

$$Y_{\omega'-\frac{1}{2},m'}(\chi, \theta) = e^{-im'\theta} \sum_{k=0}^{\omega'-\frac{1}{2}} (a_k e^{ik\chi} + b_k e^{-ik\chi}). \quad (22)$$

Since we are interested in the outgoing field, we take only $u_{\omega',m'}$ as our basis

$$u_{\omega',m'}(\chi, \theta) = e^{im'\theta} \sqrt{\cos\chi} \sum_{k=0}^{\omega'-\frac{1}{2}} a_k e^{ik\chi}. \quad (23)$$

We are now ready to calculate the Hawking radiation. The mode solution $v_{\omega,\mu}$, outside the black hole, can be expanded in terms of $u_{\omega',m'}$ as

$$v_{\omega,\mu} = \sum_{\omega',m'} (\alpha_{\omega\mu;\omega'm'} u_{\omega',m'} + \beta_{\omega\mu;\omega'm'} u_{\omega',m'}^*). \quad (24)$$

The $\alpha_{\omega\mu;\omega'm'}$ and $\beta_{\omega\mu;\omega'm'}$ are the Bogoliubov coefficients which are given by the inner products,

$$\begin{aligned} \alpha_{\omega\mu;\omega'm'}(\Sigma) &= \frac{1}{2\pi} (u, v)_\Sigma = \alpha_{\omega\mu;\omega'm'}(\Sigma_1) + \alpha_{\omega\mu;\omega'm'}(\Sigma_2) \\ \beta_{\omega\mu;\omega'm'}(\Sigma) &= \frac{1}{2\pi} (u, v^*)_\Sigma = \beta_{\omega\mu;\omega'm'}(\Sigma_1) + \beta_{\omega\mu;\omega'm'}(\Sigma_2). \end{aligned} \quad (25)$$

For our convenience, we choose Σ_1 and Σ_2 as the hypersurfaces at $\tau = 0$ and $\tau = \pi$, respectively. The observer outside the black hole, which is causally connected with Σ_1 , is in the region $\{\chi \neq 0, (0 \leq \theta < \pi/2, \text{ or } 3\pi/2 < \theta \leq 2\pi)\}$ and the horizon is located at $\theta = \pi/2$ and $\theta = 3\pi/2$. As is well-known, the Hawking radiation appears in this calculation because the positive frequency mode $v_{\omega\mu}$ in the outside the black hole includes the negative frequency mode $u_{\omega'\mu'}$ in the coordinates which cover the whole CAdS space. This effect is dominant near the horizon, hence we can compute the Bogoliubov coefficients by just taking the approximation in that region. $v_{\omega\mu}$ is approximated near the horizon as

$$v_{\omega\mu} = \begin{cases} A_{\omega\mu} e^{\frac{i\omega}{2} \ln[-\frac{1}{2}\chi(\frac{\pi}{2}-\theta) + \frac{\pi^2}{4}(\frac{\pi}{2}-\theta)^2]} e^{i\mu\chi}, & \theta \leq \frac{\pi}{2} \\ A_{\omega\mu} e^{\frac{i\omega}{2} \ln[-\frac{1}{2}\chi(\theta-\frac{3\pi}{2}) + \frac{\pi^2}{4}(\theta-\frac{3\pi}{2})^2]} e^{i\mu\chi}, & \frac{3\pi}{2} \leq \theta \end{cases} \quad (26)$$

with $\tau \approx 0$. $v_{\omega\mu}$ has very high frequency near the horizon, the dominant contribution to $\alpha_{\omega\mu;\omega'\mu'}$ comes from the modes with $e^{i(\omega'-1/2)\chi}$. For that mode, $\alpha_{\omega\mu;\omega'\mu'}$ is given by

$$\begin{aligned} \alpha_{\omega\mu;\omega'\mu'}(\Sigma_1) &\approx \frac{il}{2\pi} \frac{A}{\sqrt{2\omega'}} \int d\chi \frac{\sin\chi}{\sqrt{\cos\chi}} \\ &\times \left\{ \int_{\pi/2}^0 d\theta \exp[i\hat{\omega}l \ln(\frac{\chi}{2}\varepsilon_1) - im'\theta - i(\omega' - \frac{1}{2} - \mu)\chi](-i\omega' + \frac{i\hat{\omega}l}{\chi\varepsilon_1}) \right. \\ &+ \left. \int_{3\pi/2}^{2\pi} d\theta \exp[i\hat{\omega}l \ln(\frac{\chi}{2}\varepsilon_2) - im'\theta - i(\omega' - \frac{1}{2} - \mu)\chi](-i\omega' + \frac{i\hat{\omega}l}{\chi\varepsilon_2}) \right\} \\ &= -\frac{il(-i)^{m'}}{2\pi} \frac{A}{\sqrt{2\omega'}} \int d\chi \frac{\sin\chi}{\sqrt{\cos\chi}} \int_0^{\pi/2} d\varepsilon \left(\frac{\varepsilon}{2}\right)^{i\hat{\omega}l} (e^{im'\varepsilon} + (-1)^{m'} e^{-im'\varepsilon})(i\omega' - \frac{i\hat{\omega}l}{\chi\varepsilon}), \end{aligned} \quad (27)$$

where in the first line, $\varepsilon_1 = \pi/2 - \theta$, $\varepsilon_2 = \theta - 3\pi/2$. From this, one can see that

$$\beta_{\omega\mu;\omega'\mu'}(\Sigma_1) = \alpha_{\omega\mu;-\omega'-m'}(\Sigma_1) = \alpha_{\omega\mu;-\omega'\mu'}(\Sigma_1). \quad (28)$$

Using the fact that the dominant contribution comes from near $\chi = 0$ and large ω' , we have

$$\begin{aligned} \alpha_{\omega\mu;\omega'\mu'}(\Sigma_1) &\approx \frac{(-i)^{m'+1} A}{2\pi\sqrt{2\omega'}} (P_{\omega m'} \cdot (\frac{1+i\hat{\omega}l}{1-\frac{1}{2\omega'}-\frac{\mu}{\omega'}}) - Q_{\omega m'} \cdot i\hat{\omega}l) \\ &\times \Gamma(1+i\hat{\omega}l)(i(\omega' - \frac{1}{2} - \mu))^{-i\hat{\omega}l-1} \end{aligned} \quad (29)$$

where

$$\begin{aligned} P_{\omega m'} &= \int_0^{\frac{\pi}{2}} d\varepsilon \left(\frac{\varepsilon}{2}\right)^{i\hat{\omega}l} (e^{im'\varepsilon} + (-1)^{m'} e^{-im'\varepsilon}) \\ Q_{\omega m'} &= \int_0^{\frac{\pi}{2}} d\varepsilon \left(\frac{\varepsilon}{2}\right)^{i\hat{\omega}l} (e^{im'\varepsilon} + (-1)^{m'} e^{-im'\varepsilon}). \end{aligned}$$

There is no contribution from Σ_2 hypersurface for the observer outside the black hole, located in $-\pi/2 \leq \tau \leq \pi/2$. The integrand of (27) is analytic in the lower half ω' plane. For $\omega' > \mu$, we can use Eq.(29) to perform the analytic continuation of α to the lower half ω' plane and obtain the relation

$$|\beta|^2 = e^{-2\pi l\omega} |\alpha|^2. \quad (30)$$

This relation, along with the normalization condition of the wavefunction,

$$\sum_{\omega'm'} (|\alpha_{\omega\mu;\omega'm'}|^2 - |\beta_{\omega\mu;\omega'm'}|^2) = 1, \quad (31)$$

gives us the Plank distribution

$$<0|_m N_{\omega\mu}^{out} |0>_m = \sum_{\omega'm'} |\beta_{\omega\mu;\omega'm'}|^2 = \frac{1}{e^{2\pi l\omega} - 1}, \quad (32)$$

which shows that the non-rotating black hole radiates at the Hawking temperature $\hat{T} = (2\pi l)^{-1}$. In the next section we extend this result, by using the simple coordinate transformations, to the case of rotating black hole.

4. Our calculations for $M = 1$ and $J = 0$ case with metric (3) can easily be extended to the case of arbitrary M and J with metric (1).

For nonvanishing J , the black hole has ergosphere whose outer limit, the so-called 'static limit', is at $r_s = M\sqrt{l}$. In this case we can introduce the outgoing null coordinates $(u, r, \hat{\varphi})$,

$$\begin{aligned} du &= dt - dr^* = dt - \frac{dr}{N^2(r)}, \\ d\hat{\varphi} &= d\varphi - \frac{N_\varphi}{N^2} dr, \end{aligned} \quad (33)$$

for which the metric is given by

$$ds^2 = -(N^2 - r^2 N_\varphi^2) du^2 + r^2 d\hat{\varphi}^2 - 2 du dr - J du d\hat{\varphi}. \quad (34)$$

Alternatively the metric can be written as

$$ds^2 = -N^2 dt^2 + N^2 dr^{*2} + r^2 (N^\varphi dt + d\varphi)^2, \quad (35)$$

with

$$r^* = \frac{l^2}{2(r_+^2 - r_-^2)} \left\{ r_+ \ln\left(\frac{r - r_+}{r + r_+}\right) - r_- \ln\left(\frac{r - r_-}{r + r_-}\right) \right\}. \quad (36)$$

In this metric, the positive-frequency mode solutions are of the form

$$\phi = e^{-i\omega t} e^{im\varphi} \frac{R(r)}{\sqrt{r}}, \quad (37)$$

where $R(r)$ satisfies the radial equation,

$$\frac{d^2 R}{dr^{*2}} + [(\omega + m N_\varphi)^2 - N^2 \left\{ \frac{m^2}{r^2} + \frac{R}{8} + \frac{1}{2} \frac{d}{dr} \left(\frac{N^2}{r^{\frac{3}{2}}} \right) + \frac{N^2}{2r^{\frac{3}{2}}} \right\}] R = 0. \quad (38)$$

Near the horizon, $r \rightarrow r_+$, the solution is of the form

$$R = e^{\pm i(\omega - m\Omega_H)r^*}, \quad (39)$$

where $\Omega_H = J/(2r_+^2)$ is the angular velocity of the horizon. Therefore the positive-frequency outgoing modes which replace (11) in the general case are

$$\phi = A_\omega e^{-i\omega t} e^{im\varphi} e^{i(\omega - m\Omega_H)r^*}. \quad (40)$$

The metric (1) becomes that of (3) by the coordinate transformations,

$$\begin{aligned} \hat{t} &= \left(\frac{r_+}{l}\right) - r_- \varphi, \\ \hat{\varphi} &= \left(\frac{r_+ \varphi}{l}\right) - \left(\frac{r_-}{l^2}\right) t, \\ \hat{r}^2 &= l^2 \left(\frac{r_+^2 - r_-^2}{r_+^2 - r_-^2}\right). \end{aligned} \quad (41)$$

Under the transformations (41), the modes (40) turn into

$$\phi = A_\omega \exp[-i(\omega - m\Omega_H) \frac{lr_+}{r_+^2 - r_-^2} (\hat{t} - \hat{r}^*)] \exp[i \frac{lr_+}{r_+^2 - r_-^2} (m - \omega l^2 \Omega_H) \hat{\varphi}]. \quad (42)$$

This shows that

$$< 0 | {}_{in} N_\omega^{out} | 0 >_{in} = \frac{1}{e^{2\pi(\omega - m\Omega_H)\beta - 1}}, \quad (43)$$

where

$$T = \beta^{-1} (= \hat{T} \frac{r_+^2 - r_-^2}{lr_+}) = \frac{r_+^2 - r_-^2}{2\pi r_+ l^2}. \quad (44)$$

Note that $m\Omega_H$ plays the role of chemical potential [5].

The Hawking temperature we have found depends on the normalization of time-like Killing vector. Since the space-time is asymptotically anti-de Sitter, there is no natural normalization we can take. We have chosen the timelike Killing vector ∂_t as shown in (15).

Furthermore the temperature T' as seen by the observer located at r outside black hole is given by the Tolman relation [6],

$$T' = (g_{00})^{-\frac{1}{2}} T = \frac{T}{N(r)}, \quad (45)$$

which vanishes in the asymptotic region. This implies that there is an infinite redshift at infinity as the black hole solution is asymptotically AdS.

The back reaction of scalar field to the background geometry is an obvious next step to study, which is in progress.

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