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Quantum corrections to instanton calculations in the Abelian Higgs model

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Abstract

In the $(1+1)$ -dimensional Abelian Higgs model we calculate the final state quantum corrections to the fermion number violating cross-section in the background of the instanton. The arising determinants are treated in the Schwinger proper time formalism using some refined approximation techniques which are controlled in several ways and are also applied to the well known case of the quantum double well for comparison.

1 Introduction

The electroweak interactions violate baryon number conservation. This is known since the work of 't Hooft ([1]) but only much later it was realized by Ringwald ([2]) and Espinosa ([3]) that this may lead to sizable baryon number violating cross-sections in the TeV region. The $(1+1)$ -dimensional Abelian Higgs model ([4]–[9]) also allows for baryon number violation and hence is a convenient playground to discuss this question in a simpler context. Quantum corrections to the non-perturbative cross-section are essential in order to arrive at quantitative results. Here we calculate the cross-section in quasi-classical approximation around the instanton solution including the *final state* quantum corrections. The latter being given in terms of determinants we are faced with the more general problem of calculating determinants of functional operators with covariant derivatives and matrix potentials. Since there are no exact results known we will carry out an approximate calculation based on Schwinger's heat kernel representation ([10]). We follow the works of L. Carson et al. ([11], [12]) in the context of sphaleron fluctuations. We propose some additional techniques by which the convergence of the approximate results may be controlled and improved. The well known case of the double well determinant serves to compare our method with an exact result.

2 The model

The model considered here consists of an abelian gauge field, a Higgs field, and axially coupled massless fermions in one time and one space dimension. We start with the euclidean Lagrangian

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + (D_\mu \Phi)^\dagger D_\mu \Phi + \lambda \left(|\Phi|^2 - \frac{v^2}{2} \right)^2 + \sum_{j=1}^{n_f} \bar{\psi}^{(j)} i \gamma_\mu D_\mu^F \psi^{(j)} \quad , \quad (1)$$

with common conventions and

$$D_\mu^F = \partial_\mu - ig \gamma_5 A_\mu \quad ,$$

assuming weak coupling $v \gg 1$. The particle density current acquires an anomaly:

$$\langle 0 | \partial_\mu J_\mu^F(x) | 0 \rangle = n_f \frac{g}{2\pi} \epsilon_{\mu\nu} F_{\mu\nu}(x) \quad , \quad J_\mu^F = \sum_{j=1}^{n_f} \bar{\psi}^{(j)} \gamma_\mu \psi^{(j)} \quad . \quad (2)$$

The fermion number is therefore violated by $2n_f$ units of the topological charge:

$$\Delta N_F = 2n_f Q \quad , \quad Q = \frac{g}{4\pi} \int d^2x \epsilon_{\mu\nu} F_{\mu\nu} \quad . \quad (3)$$

As is well known the model has an instanton solution of the field equations

$$\Phi = \frac{v}{\sqrt{2}} e^{i(Q\vartheta + g\beta)} f(r) \quad , \quad gA_\mu = \epsilon_{\mu\nu} \partial_\nu (-\alpha) + \partial_\mu \beta \quad (4)$$

r and ϑ being the polar coordinates around the center of the instanton. The corresponding Dirac operator has one zero-mode $\not{D}^F \chi = 0$ with

$$\chi = \exp \left\{ -\frac{Q}{|Q|} \alpha + i\beta \gamma_5 \right\} n_\chi \xi \quad . \quad (5)$$

In the remainder of this article we assume the special ratio

$$\frac{m_H^2}{m_A^2} = 2\lambda/g^2 = 1 \quad . \quad (6)$$

In this case de Vega and Schaposnik ([4]) found an exact solution of the field equations in the $Q = 1$ sector which we will use in the following. We work in the singular gauge $\beta = -Q\vartheta$ so that the instanton fields take the following form:

$$A_\mu(x) = \frac{\epsilon_{\mu\nu}(x_\nu - x_{I\nu})}{gr^2} \{a(r) + Q\} \quad , \quad \Phi(x) = \frac{v}{\sqrt{2}} f(r) \quad , \quad r = |x - x_I|; \quad (7)$$

the physical Higgs excitation is

$$\sigma(x) = \frac{v}{\sqrt{2}} \{ f(r) - 1 \} \quad . \quad (8)$$

The profile functions $f(r)$ and $a(r) := -r\alpha'(r)$ approach the ordinary vacuum $a = 0$, $f = 1$ for $r \rightarrow \infty$:

$$a + 1 \sim Z g v r K_1(gvr) \quad , \quad f \sim 1 - Z K_0(gvr) \quad . \quad (9)$$

In case 6 the numerical constant reads

$$Z = 1.7079 \quad , \quad (10)$$

it will enter our formulas of the fermion number violating cross-section. Finally, the action of the instanton configuration turns out to be

$$S_0 = \pi v^2 \quad (11)$$

which is big because of the weak coupling limit.

3 Fermion number violating cross-section

We want to calculate the cross-section for the following process:

$$\bar{f} + \bar{f} \rightarrow (2n_f - 2) f + n_A A + n_H H \quad . \quad (12)$$

We know that according to (3) fermion number is violated by $2n_f$ units in a configuration with an instanton of topological charge $Q = 1$. We will therefore calculate the path integral

$$Z[J; \eta, \bar{\eta}] = \frac{1}{Z_0} \int \mathcal{D}A_\mu \mathcal{D}\Phi \mathcal{D}\Phi^\dagger e^{-S_{\text{boson}}[J]} Z_\psi \quad (13)$$

with

$$Z_\psi[\eta, \bar{\eta}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} e^{-S_F + \bar{\psi}\eta + \bar{\eta}\psi} \quad (14)$$

in saddle point approximation around the instanton. Integration over the fermions by standard methods gives the Green's function

$$\begin{aligned} G &= \int \mathcal{D}A_\mu \mathcal{D}\Phi \mathcal{D}\Phi^\dagger e^{-S_{\text{boson}}} \cdot \left(\frac{\det'(i\mathcal{D})}{\det'(i\mathcal{D})} \right)^{n_f} \\ &\times \prod_{i=1}^{2n_f} \chi^{(i)}(x_i - R) \cdot \prod_{k=1}^{n_A} A_{\mu_k}(y_k - R) \cdot \prod_{l=1}^{n_H} \Phi^{(\dagger)}(z_l - R) \end{aligned} \quad (15)$$

in which every fermion family is represented by one left and one right handed particle. The fermionic determinant (with the zero modes omitted, indicated by the prime) is given by

$$\frac{\det'(i\mathcal{D})}{\det'(i\mathcal{D})} = \exp \left\{ i \frac{g^2}{2\pi} \int d^2x \alpha(x) \epsilon_{\mu\nu} F_{\mu\nu}(x) \right\} = \exp \left\{ i \frac{g^2}{\pi} \int d^2x \alpha(x) \partial^2 \alpha(x) \right\} \quad (16)$$

and may be treated as a prefactor because of our assumption $S_0 = \pi v^2 \gg 1$. We want to calculate the remaining integral in Gaussian approximation, writing

$$A_\mu = A_\mu^0 + a_\mu \quad , \quad \Phi = \Phi^0 + \varphi \quad . \quad (17)$$

In every pre-exponential factor the full fields $\Psi = A_\mu$ or Φ , respectively, are replaced just by the background fields Ψ^0 which, finally, will be the instanton gauge and Higgs fields (the index 0 will be omitted in the following). In this procedure, however, we have to encounter zero modes of the exponential, i.e. directions in functional space in which integration is not Gaussian. The instanton

itself is invariant under spatial translations and rotations, and it is readily seen that because of gauge invariance the initial Lagrangian is invariant under the variation

$$a_\mu = \partial_\mu \alpha \quad , \quad \varphi = ig\alpha \Phi \quad . \quad (18)$$

For this gauge zero mode we choose to add to the Lagrangian a gauge fixing term of the t'Hooft Feynman type ([14])

$$\mathcal{L}_{gf} = \frac{1}{2} G^2 \quad , \quad G = -\partial_\mu a_\mu + ig (\varphi^\dagger \Phi - \Phi^\dagger \varphi) \quad . \quad (19)$$

The fields rescaled,

$$A_\mu \rightarrow v \bar{A}_\mu \quad , \quad a_\mu \rightarrow a_\mu \quad , \quad \Phi \rightarrow \frac{1}{\sqrt{2}} \bar{\Phi} \quad , \quad \varphi \rightarrow \frac{\bar{\varphi}}{\sqrt{2}} \quad , \quad z_\mu := gv x_\mu \quad , \quad D_\mu := \partial_\mu - i \bar{A}_\mu \quad , \quad (20)$$

according to Faddeev/Popov the path integral has to be multiplied with the corresponding determinant

$$\det_{\text{FP}} = \det \left(-\partial^2 + \bar{\Phi}^\dagger \bar{\Phi} \right) \quad (21)$$

and the volume of the gauge group 2π . We drop the bars in the following.

Finally, we split the Higgs fluctuation into real and imaginary part $\varphi = \varphi_1 + i\varphi_2$ and introduce the following matrix notation:

$$\psi = \begin{pmatrix} a_1 \\ a_2 \\ \varphi_1 \\ \varphi_2 \end{pmatrix} \quad , \quad \hat{\mathcal{D}}_\mu := \partial_\mu \mathbf{1}_{4 \times 4} - i A_\mu \hat{T} \quad , \quad \hat{T} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} \quad , \quad (22)$$

$$\hat{V} := \begin{pmatrix} \Phi^\dagger \Phi & 0 & i [D_1 \Phi - (D_1 \Phi)^\dagger] & D_1 \Phi + (D_1 \Phi)^\dagger \\ 0 & \Phi^\dagger \Phi & i [D_2 \Phi - (D_2 \Phi)^\dagger] & D_2 \Phi + (D_2 \Phi)^\dagger \\ i [D_1 \Phi - (D_1 \Phi)^\dagger] & i [D_2 \Phi - (D_2 \Phi)^\dagger] & * & \left(\frac{2\lambda}{g^2} - 1 \right) \frac{\Phi + \Phi^\dagger}{2} \frac{\Phi - \Phi^\dagger}{2i} \\ D_1 \Phi + (D_1 \Phi)^\dagger & D_2 \Phi + (D_2 \Phi)^\dagger & \left(\frac{2\lambda}{g^2} - 1 \right) \frac{\Phi + \Phi^\dagger}{2} \frac{\Phi - \Phi^\dagger}{2i} & ** \end{pmatrix} \quad (23)$$

with the abbreviations

$$\begin{aligned} * &= \Phi^\dagger \Phi + \frac{\lambda}{g^2} (\Phi^\dagger \Phi - 1) + \left(\frac{2\lambda}{g^2} - 1 \right) \left(\frac{\Phi + \Phi^\dagger}{2} \right)^2 \quad , \\ ** &= \Phi^\dagger \Phi + \frac{\lambda}{g^2} (\Phi^\dagger \Phi - 1) + \left(\frac{2\lambda}{g^2} - 1 \right) \left(\frac{\Phi - \Phi^\dagger}{2i} \right)^2 \end{aligned}$$

where we set $2\lambda = g^2$. We may then write

$$\frac{1}{2!} \delta^2 S = \int d^2 x (\delta^2 \mathcal{L} + \mathcal{L}_{gf}^1) = \frac{1}{2!} \int d^2 z \psi^T \hat{M} \psi \quad , \quad \hat{M} = -\hat{\mathcal{D}}^2 + \hat{V} \quad . \quad (24)$$

There are the spatial zero modes left which may be handled by the collective coordinate formalism ([15]). For the translations they read

$$a_\mu^R = \frac{1}{\sqrt{n_R}} R_\lambda \partial_\lambda A_\mu + \partial_\mu \Lambda \quad , \quad \varphi^R = \frac{1}{\sqrt{n_R}} R_\lambda \partial_\lambda \Phi + ig \Lambda \Phi$$

where to fulfill the gauge condition we have to set

$$\Lambda = -\frac{1}{\sqrt{n_R}} R_\lambda A_\lambda \quad ,$$

arriving at

$$a_\mu^R = \frac{1}{\sqrt{n_R}} R_\lambda F_{\lambda\mu} \quad , \quad \varphi^R = \frac{1}{\sqrt{n_R}} R_\lambda D_\lambda \Phi \quad . \quad (25)$$

The integral over the zero modes is replaced by the one over the collective coordinate

$$n_R \cdot \int d^2 R \quad (26)$$

where the normalization factor is

$$n_R = S_0 = \pi v^2 \quad . \quad (27)$$

The rotational zero modes are

$$a_\mu^\vartheta = -\frac{\vartheta}{\sqrt{n_\vartheta}} (z_\nu \epsilon_{\nu\rho} \partial_\rho A_\mu + \epsilon_{\mu\nu} A_\nu) + \partial_\mu \Lambda \quad , \quad \varphi^\vartheta = -\frac{\vartheta}{\sqrt{n_\vartheta}} z_\mu \epsilon_{\mu\nu} \partial_\nu \Phi + ig \Lambda \Phi$$

allowing for a gauge transformation

$$\Lambda = \frac{\vartheta}{\sqrt{n_\vartheta}} (x_\nu \epsilon_{\nu\rho} A_\rho + \Lambda')$$

where the gauge condition now implies

$$(\partial^2 - \Phi^\dagger \Phi) \Lambda' = \epsilon_{\mu\rho} F_{\mu\rho} \quad . \quad (28)$$

It follows

$$a_\mu^\vartheta = \frac{\vartheta}{\sqrt{n_\vartheta}} (x_\nu \epsilon_{\nu\rho} F_{\mu\rho} + \partial_\mu \Lambda') \quad , \quad \varphi^\vartheta = -\frac{\vartheta}{\sqrt{n_\vartheta}} (x_\nu \epsilon_{\nu\rho} D_\rho \Phi + ig \Lambda' \Phi) \quad , \quad (29)$$

and collective coordinate integration yields

$$\int d\zeta^{(\vartheta)} = \sqrt{n_\vartheta} \int_0^{2\pi} d\vartheta \quad .$$

We have solved eq.(28) numerically and found for the normalization factor

$$n_\vartheta = 2\pi v^2 \cdot 2.77 \quad . \quad (30)$$

The remaining eigenvalues are all positive so that the Gaussian integration formula is applicable provided that the (primed, i. e. with the zero modes omitted) determinant is finite. We regularize by referring to the ordinary vacuum, i. e. dividing by the determinant of

$$\hat{M}_0 = -\hat{\mathcal{D}}_0^2 + \hat{V}_0 \quad \text{with} \quad \hat{V}_0 := \hat{V}[A_\mu^0 \rightarrow 0, \Phi^0 \rightarrow 1] = 1_{4 \times 4} \quad , \quad -\hat{\mathcal{D}}_0^2 = -\partial^2 1_{4 \times 4} \quad . \quad (31)$$

Similarly we calculate the ratio of the Faddeev Popov determinant and the one in which the Higgs field is replaced by its vacuum expectation value. In the following, these ratios of determinants will be called fluctuation and Faddeev Popov determinant, respectively. We will calculate them approximately using Schwinger's proper time representation. To our knowledge they have never been calculated before and in particular there are no exact results to compare with. We therefore choose to treat also the determinant of the double well operator in the indicated formalism (see following section). Before going over to the determinants we will continue calculating the cross-section of the reaction (12). Collecting all results obtained thus far the fermion number violating Green's function in the instanton background (cf. eqs. 5, 7, and 8) reads

$$\begin{aligned} G(x, y, z) = & n_R \sqrt{n_\vartheta} g^2 v^2 e^{-S_0} (2\pi)^2 \int d^2 R \frac{\det_{\text{FP}}}{\sqrt{\det_{\text{F1}}}} \left(\frac{\det'(i\mathcal{D})}{\det'(i\vartheta)} \right)^{n_f} \\ & \times \prod_{i=1}^{2n_f} \chi^{(i)}(x_i - R) \cdot \prod_{k=1}^{n_A} A_{\mu_k}(y_k - R) \cdot \prod_{l=1}^{n_H} \sigma(z_l - R) \end{aligned} \quad (32)$$

from which we deduce, using Laplace transform methods ([16], [17]) and summing over the number of final bosons N , the total inclusive cross-section in the ultrarelativistic limit $m \rightarrow 0$

$$\sigma_{\text{inc}} = 2 m^{2n_f} B \left(\frac{E}{m\gamma} \right)^{2\gamma-2} \quad (33)$$

with

$$B = \left(\sqrt{2} \pi^2 n_R \sqrt{n_\vartheta} e^{-S_0} (2\pi)^2 \frac{\det_{\text{FP}}}{\sqrt{\det_{\text{F1}}}} \right)^2 \left(\frac{\det'(i\mathcal{D})}{\det'(i\partial)} \right)^{2n_f} \left(4\pi \frac{n_X^2}{\mu^2} \right)^{2n_f} \quad (34)$$

and

$$\gamma = \frac{3\pi v^2 Z^2 + 2n_f - 2}{2} . \quad (35)$$

We may compare the mass $m\gamma$ with the mass of the sphaleron which is given by ([18])

$$M_{\text{Sph}} = \frac{2}{3} m v^2 . \quad (36)$$

Note that the instanton calculation breaks down in this range of energy so that the question if fermion number violating processes become unsuppressed cannot be answered here.

4 Calculation of determinants

In this section we calculate three determinants of the form $-\hat{\mathcal{D}}^2 + \hat{V}$ the matrix structure being trivial in the double well and the Faddeev Popov case. The double well operator $M(2, 2)$ is a special case of the Pöschl Teller operator ([20])

$$M(s, a) = -\frac{d^2}{dz^2} + V(s, a) , \quad V(s, a) = a^2 - \frac{s(s+1)}{\cosh^2(z)} .$$

Formally it resembles the Faddeev Popov and the fluctuation operators. It has one (translational) zero mode, and the primed determinant is known exactly:

$$\frac{\det' \left(-\frac{d^2}{dz^2} + 4 - \frac{6}{\cosh^2(z)} \right)}{\det' \left(-\frac{d^2}{dz^2} + 4 \right)} = \frac{1}{12} \quad (37)$$

which we have to compare our approximate results with.

In Schwinger's proper time representation ([10]) the ratio of determinants is given by

$$\det \frac{\hat{M}}{\hat{M}_0} = \exp \left\{ - \int_0^\infty \frac{dt}{t} (J(t) - J_0(t)) \right\} , \quad J(t) = \text{Tr} \left(e^{-t\hat{M}} \right) . \quad (38)$$

Following the calculations by Carson in the sphaleron background ([11]) we perform the trace with plane waves:

$$\text{Tr} A = \text{tr} \int \frac{d^d z d^d p}{(2\pi)^d} e^{-ipz} A(z) e^{+ipz}$$

the “small” tr indicating the sum over matrix diagonal elements. The exponential under the t -integral will be expanded. But, for practical reasons the expansion has to be cut off at some finite value. To achieve nevertheless satisfying convergence of the t -integral we will factor out some arbitrary scale μ and investigate the dependence of the result on this scale. We arrive at the following central formula:

$$J(t) - J_0(t) = \frac{e^{-t\mu}}{(4\pi t)^{d/2}} \text{tr} \int d^d z \sum_{n=1}^\infty \frac{(-t)^n}{n!} O_n . \quad (39)$$

In fact, for practical reasons the sum has to be cut off at some finite value N . Letting grow N from 1 to a maximum value we are able to control the convergence of the sum.

The coefficients $O_n = O_n^{(1)} - O_n^{(0)}$ are given in the appendix for the cases of scalar and matrix operators, for $n = 1 \dots 5$ and $n = 1 \dots 3$, respectively.

4.1 Space integration

In the case of the double well determinant the integrals

$$A_n := \int_{-\infty}^{\infty} d\tau O_n^{\text{dw}} \quad (40)$$

may be calculated using the formula

$$\int_{-\infty}^{\infty} \frac{dz}{\cosh^{2n}(z)} = \frac{2^n (n-1)!}{(2n-1)!!} .$$

We find

$$\begin{aligned} A_1 &= -6 \\ A_2 &= 12\mu - 6 \\ A_3 &= -18\mu^2 + 18\mu - 33/5 \\ A_4 &= 24\mu^3 - 36\mu^2 + 132/5\mu - 246/35 \\ A_5 &= -30\mu^4 + 60\mu^3 - 66\mu^2 + 246/7\mu - 51/7 . \end{aligned}$$

The Faddeev Popov and fluctuation coefficients are integrated numerically using the instanton profile functions. In the first case

$$\overline{B}_n = \frac{1}{4\pi} \int d^2z O_n^{\text{FP}} \quad (41)$$

$$\begin{aligned} \overline{B}_1 &= -1.001 \\ \overline{B}_2 &= -1.587 + 2.002\mu \\ \overline{B}_3 &= -1.906 + 4.756\mu - 3.003\mu^2 \\ \overline{B}_4 &= -2.066 + 7.617\mu - 9.512\mu^2 + 4.004\mu^3 \\ \overline{B}_5 &= -2.141 + 10.32\mu - 19.04\mu^2 + 15.85\mu^3 - 5.005\mu^4 , \end{aligned}$$

in the latter case

$$\overline{C}_n = \frac{1}{4\pi} \int d^2z O_n^{\text{FI}} \quad (42)$$

$$\begin{aligned} \overline{C}_1 &= -5.01 \\ \overline{C}_2 &= -5.01 + 10.0\mu \\ \overline{C}_3 &= -5.93 + 15.0\mu - 15.0\mu^2 . \end{aligned}$$

4.2 Proper time integration

Up to now we have calculated the integrand on the right hand side of the first of eq.(38). Since the expansions in powers of t start with $t^{1/2}$ for the double well and with t^0 for the other two determinants the t -integration is obviously divergent at the lower limit. In the first case, as a kind of regularization it suffices to use the integral representation of the gamma function

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad (43)$$

also for $x = 1/2$. For the Faddeev Popov and fluctuation determinants we use the zeta function regularization ([19])

$$\ln \left(\det \frac{\hat{M}}{\hat{M}_0} \right)_{\text{Reg}_\zeta} = -\zeta'(0) , \quad (44)$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Sp} \left\{ e^{-t\hat{M}} - e^{-t\hat{M}_0} \right\} dt . \quad (45)$$

We thus arrive at

$$-\ln \det_{\text{dw}} = \sum_{n=1}^N \frac{(-1)^n}{n!} \frac{A_n}{\sqrt{4\pi}} \frac{\Gamma(n-1/2)}{\mu^{n-1/2}} , \quad (46)$$

$$-\ln \det_{\text{FP}} = \bar{B}_1 \ln(\mu) + \sum_{n=2}^N \frac{(-1)^n}{n(n-1)} \frac{\bar{B}_n}{\mu^{n-1}} , \quad (47)$$

$$-\ln \det_{\text{F1}} = \bar{C}_1 \ln(\mu) + \sum_{n=2}^N \frac{(-1)^n}{n(n-1)} \frac{\bar{C}_n}{\mu^{n-1}} . \quad (48)$$

4.3 Handling of discrete eigenvalues

At this point we remember that we have to calculate the “primed” determinants, i.e. those where the zero eigenmodes have been omitted. There is one (translational) zero mode in the case of the double well operator, and the two translations and the rotation of the Abelian Higgs model instanton yield three zero eigenvalues of the fluctuation operator. Since the Faddeev Popov operator has no such modes we have already obtained the desired result which setting μ equal to the natural scale $V_0 = 1$ turns out to be

$$(\det_{\text{FP}})_{\text{Reg}_c} = e^{-0.24} = 0.79 . \quad (49)$$

Nevertheless we intend to look at the μ -dependence of the determinants. In addition to that we investigate the question if the Schwinger method can be improved by treating the other discrete (but non-vanishing) eigenvalues separately. As is well known the double well operator has one such eigenvalue $\lambda_1 = 3/4$ ([20]). Applying the Ritz variation method to the Faddeev Popov operator we find a discrete eigenvalue below the continuous spectrum (which begins at the natural scale V_0):

$$\lambda_1 \simeq 0.783 . \quad (50)$$

We will handle these modes just like the vanishing ones subtracting them from the Schwinger result in a way given below and multiplying the “two-primed” determinants with the exact eigenvalues themselves. We will thus use Schwingers method only to calculate the continuous spectrum of the operators.

The second of eq.(38) suggests the following method to split up an exactly known eigenvalue:

$$J(t) - J_0(t) = \left(e^{-\lambda_1 t} - e^{-\lambda_{(0)} t} \right) + \left(J'(t) - J'_0(t) \right) \quad (51)$$

and for two or more such eigenvalues

$$\begin{aligned} J(t) - J_0(t) &= \left(e^{-\lambda_1 t} - e^{-\lambda_{(0)} t} \right) + \left(e^{-\lambda_2 t} - e^{-\lambda_{(0)} t} \right) \\ &+ \left(J''(t) - J''_0(t) \right) \end{aligned} \quad (52)$$

and so on.

where we have to split up also the lowest eigenvalue (equal to V_0) of the reference operator since the Schwinger method requires that original and reference operator have the same number of eigenvalues. We may then write

$$J'(t) - J'_0(t) = \left(J(t) - J_0(t) \right) - \left(J^1(t) - J^1_0(t) \right) , \quad (53)$$

$$J^1(t) - J^1_0(t) = e^{-\mu t} \sum_{n=1}^{\infty} \frac{(-t)^n}{n!} a_n , \quad (54)$$

$$a_n = (\lambda - \mu)^n - (\lambda_{(0)} - \mu)^n . \quad (55)$$

a_n has to be replaced by a sum $a_n^1 + a_n^2 + \dots$ if there is more than one eigenvalue to be subtracted. We investigate the behaviour of the result order by order, letting N grow from 1 to the maximum of calculated coefficients (5 or 3, respectively).

Subtracting only the zero mode,

$$\lambda_1 = 0 \quad , \quad \lambda_{(0)1} = 1 \quad , \quad (56)$$

we find in the double well case

$$J'(t) - J'_0(t) = e^{-\mu t} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left(\frac{A_n}{\sqrt{4\pi}} t^{n-1/2} - a_n^1 t^n \right) . \quad (57)$$

Where do we have to cut off the a_n -sum? Neither the A_n - nor the a_n -sum are convergent, but the difference should be. We propose two possibilities: first, we might take equal numbers of A_n 's and a_n 's the highest t -power of the latter exceeding the one of the former by $1/2$; second, we can stop the a_n -sum by one term earlier arriving at an inverse relation of the powers of t . Near the natural scale we find a systematical approach to the known exact result of $\ln \det$ from above (while N grows from 1 to 5) in the first case (figure 1, 1a) and from below (N again growing from 1 to 5) in the latter case (figure 1, 1b) the average of the two coming very close to $\ln(1/12)$ in each order (figure 1, 1c). (Here, as in the following figures, the graph diverging the fastest for $\mu \rightarrow \infty$ corresponds to $N = 1$, the flattest one to the maximum value of N .)

This picture does not change essentially when we subtract also the second eigenvalue (figure 1, 2a-c). With

$$\lambda_2 = 3/4 \quad , \quad \lambda_{(0)2} = 1 \quad (58)$$

we have now

$$-\ln \det'' \frac{\hat{M}}{\hat{M}_0} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n \mu^n} \left\{ \frac{\sqrt{\mu} A_n}{\sqrt{4\pi}} \frac{\Gamma(n-1/2)}{(n-1)!} - (a_n^1 + a_n^2) \right\} \quad (59)$$

and finally

$$\ln \det' \frac{\hat{M}}{\hat{M}_0} = \ln \det'' \frac{\hat{M}}{\hat{M}_0} + \ln \left(\frac{3}{4} \right) . \quad (60)$$

Again the average of the two proposed possibilities of subtracting the discrete mode with cut-off $N = 1 \dots 5$ lies very close to the exact result (figure 1, 2c).

We would like to mention that the authors of ref. [21] applied Schwinger's proper time representation to the double well determinant, but without investigating the dependence on a factored out scale μ nor the arbitrariness in subtracting the zero mode.

In the case of the Faddeev Popov determinant the Schwinger method seems to reproduce the lowest discrete eigenvalue very well: subtraction of

$$b_n = (\lambda_1 - \mu)^n - (1 - \mu)^n$$

(see eq. 50) yields

$$\ln (\det_{FP})_{\text{Reg}_c} = \ln (\lambda_1) - \bar{B}_1 \ln (\mu) - \frac{b_1}{\mu} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n \mu^n} \left(\frac{\mu \bar{B}_n}{n-1} - b_n \right) , \quad (61)$$

and the above result (eq.47; figure 2, 1) is scarcely changed, neither by the two subtraction methods (figure 2, 2a/b) nor by their average (figure 2, 2c). Again, the graphs flatten when the cut-off N grows from 1 to 5.

In the case of the fluctuation determinant, finally, we have to subtract three zero modes, i.e.

$$c_n = 3 \{ (-\mu)^n - (1 - \mu)^n \} ; \quad (62)$$

we arrive at

$$\ln (\det'_{\text{Fl}})_{\text{Reg}_\zeta} = - \left\{ \bar{C}_1 \ln(\mu) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)} \frac{\bar{C}_n}{\mu^{n-1}} \right\} + \left\{ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{c_n}{\mu^n} \right\} . \quad (63)$$

The two subtraction schemes and their average (figure 3, a-c) are again plotted the upper limit of the sum growing from 1 to 3.

Setting again μ equal to 1 (the natural scale) and taking into account all calculated orders our final result reads

$$(\det'_{\text{Fl}})_{\text{Reg}_\zeta} = 4.53 . \quad (64)$$

5 Conclusion

Collecting the results of the preceeding sections we find for the “holy grail” function of our problem

$$\sigma_{\text{inc}} = 4\pi^4 \left(4\pi m \frac{n_x^2}{\mu^2} \frac{\det'(i\mathcal{D})}{\det'(i\partial)} \right)^{2n_f} \exp(2\pi v^2 F) \quad (65)$$

$$F = -1 + \frac{\gamma-1}{\pi v^2} \ln \frac{E}{m\gamma} + \frac{1}{\pi v^2} \ln \left\{ n_R (2\pi \sqrt{n_\vartheta}) \frac{\det_{\text{FP}}}{\sqrt{\det'_{\text{Fl}}}} \right\} \quad (66)$$

where the first term reproduces the usual exponential tunnelling suppression, the second term shows the growing with phase space, and the third contains the quantum correction determinants including zero modes. Plugging in numbers we find in the limit $v \gg 1$

$$F \approx -1 + 4.4 \ln \frac{E}{21 M_{\text{Sph}}} + \frac{1}{\pi v^2} \ln \left\{ 34 (\pi v^2)^{3/2} \right\} . \quad (67)$$

We have used four techniques to control the approximate results for the determinants obtained by Schwinger’s proper time representation: (i) in eq.(39) we allow for a scale μ differing from 1; (ii) we plot the logarithms of every calculated determinant order by order in the expansion in n ; (iii) we subtract the vanishing eigenvalue(s) in two different ways, finally taking their average; (iv) we treat other discrete eigenvalues separately. We find that in higher orders the graphs flatten more and more as it should be reflecting the final independence of the determinant on the scale μ . We also conclude that we achieve good results setting this scale equal to the natural one. Non-vanishing discrete eigenvalues lying below the natural scale do not change the picture. The possible conjecture that because of the exponential factored out in eq.(39) one has to take for μ the lowest lying eigenvalue does not hold. Finally, taking the average of the two subtraction methods of zero modes yields satisfying results already in low orders.

We agree with Carson et al. ([11],[12]) in that the present method is preferable to the one of D’yakonov, Petrov, and Yung. Indeed, we cannot follow the reasoning behind their procedure. This question has been re-investigated recently ([23]) obtaining the same result.

We expect that the method will apply also to more complicated cases as for example more general instanton-antiinstanton configurations contributing to the cross-section via the optical theorem.

We would like to thank J. Kripfganz and A. Ringwald for helpful discussions.

A Appendix

The double well and the Faddeev Popov operators contain only scalar operators; there is no gauge field entering the calculation. Defining $v := V - \mu$ we find

$$O_1^{(1)}(v) = v \quad (68)$$

$$O_2^{(1)}(v) = v^2 \quad (69)$$

$$O_3^{(1)}(v) = v^3 - 1/2 v \partial^2 v \quad (70)$$

$$O_4^{(1)}(v) = v^4 - 3/5 v^2 \partial^2 v - 3/5 (\partial^2 v) v^2 - 2/5 (\partial_\mu v) v (\partial_\mu v) + 1/5 v \partial^4 v \quad (71)$$

$$O_5^{(1)}(v) = v^5 - v^3 \partial^2 v - v^2 (\partial^2 v) v - 4/3 v^2 (\partial_\mu v) (\partial_\mu v) - 1/3 v (\partial_\mu v) (\partial_\mu v) v \\ + 2/3 (\partial_\mu v) (\partial_\mu v) v^2 + 1/7 v^2 (\partial^4 v) + 1/21 v (\partial^4 v) v + 1/7 (\partial^4 v) v^2 \\ + 2/21 v (\partial^2 v) (\partial^2 v) + 1/7 (\partial^2 v) v (\partial^2 v) + 2/21 (\partial^2 v) (\partial^2 v) v - 1/14 v \partial^6 v \quad , \quad (72)$$

agreeing with the article of Carson ([11]). For the $O_n^{(0)}$ replace V by V_0 everywhere.

The calculation in the case of an additional gauge field turns out to be much more complicated. We give here only the result of $\text{tr} \left(O_n^{(1)} - O_n^{(0)} \right)$. In the general case of arbitrary background fields the first three coefficients read

$$\text{tr} O'_1 = (2x + 3) (\Phi^{0\dagger} \Phi^0 - 1) \quad (73)$$

$$\text{tr} O'_2 = (5/2 x^2 + x + 3) (\Phi^{0\dagger} \Phi^0 - 1)^2 + (3x^2 + x + 6) (\Phi^{0\dagger} \Phi^0 - 1) \\ + 8 (D_\mu \Phi^0)^\dagger D_\mu \Phi^0 - 1/3 F_{\mu\rho}^0 F_{\mu\rho}^0 \quad (74)$$

$$\text{tr} O'_3 = (7/2 x^3 + 3/4 x^2 + 3/2 x + 1) (\Phi^{0\dagger} \Phi^0 - 1)^3 \\ + (27/4 x^3 + 3/4 x^2 + 3x + 9) (\Phi^{0\dagger} \Phi^0 - 1)^2 + (9/2 x^3 + 3/2 x + 9) (\Phi^{0\dagger} \Phi^0 - 1) \\ + (x^2 + 4x + 25) (\Phi^{0\dagger} \Phi^0 - 1) (D_\mu \Phi^0)^\dagger D_\mu \Phi^0 + (x^2 - 2x + 25) (D_\mu \Phi^0)^\dagger D_\mu \Phi^0 \\ - 3 (x - 1) \left\{ \Phi^{0\dagger} (D_\mu \Phi^0) - (D_\mu \Phi^0)^\dagger \Phi^0 \right\} \left\{ \Phi^{0\dagger} (D_\mu \Phi^0) - (D_\mu \Phi^0)^\dagger \Phi^0 \right\} \\ + (x^2 + x + 5/4) \left\{ \partial_\mu (\Phi^{0\dagger} \Phi^0) \right\} \left\{ \partial_\mu (\Phi^{0\dagger} \Phi^0) \right\} + 4 (D_\mu D_\rho \Phi^0)^\dagger (D_\mu D_\rho \Phi^0) \\ - 1/2 F_{\mu\rho}^0 F_{\mu\rho}^0 \left\{ x + 1 + (2x + 1) (\Phi^{0\dagger} \Phi^0 - 1) \right\} - 1/5 (\partial_\mu F_{\mu\rho}^0) (\partial_\alpha F_{\alpha\rho}^0) \quad , \quad (75)$$

$$x := \frac{2\lambda}{g^2} = \frac{m_H^2}{m_A^2} \quad ,$$

the prime indicating that we have temporarily set μ equal to 0. To come to arbitrary μ we have derived the following property: let $O_n(v|v_0) := O_n^{(1)}(v) - O_n^{(1)}(v_0)$ ($v = V - \mu$). Then the following “binomial” formula holds:

$$O_n(v|v_0) = \sum_{k=0}^{n-1} \binom{n}{k} (\lambda - \mu)^k O_{n-k}(v - (\lambda - \mu)|v_0 - (\lambda - \mu)) \quad (76)$$

for arbitrary λ . Identifying the above O'_n with $O_n(V|V_0)$ we find the needed quantities setting $\lambda = 0$.

References

- [1] G. 't Hooft, *Phys. Rev. Lett.* **37** (1976) 8; *Phys. Rev.* **D14** (1976) 3432; (E) *Phys. Rev.* **D18** (1978) 2199.
- [2] A. Ringwald, *Nucl. Phys.* **B330** (1990) 1.
- [3] O. Espinosa, *Nucl. Phys.* **B343** (1990) 310.
- [4] H. de Vega and F. Schaposnik, *Phys. Rev.* **D14** (1976) 1100.
- [5] F. Schaposnik, *Phys. Rev.* **D18** (1978) 1183.
- [6] S. Raby and A. Ukawa, *Phys. Rev.* **D18** (1978) 1154.
- [7] L. McLerran, A. Vainshtein, and M. Voloshin, *Phys. Rev.* **D42** (1990) 171.
- [8] J. Kripfganz and A. Ringwald, *Mod. Phys. Lett.* **A5** (1990) 675.

- [9] A. Ringwald, *Phys. Lett.* **B285** (1992) 113.
- [10] J. Schwinger, *Phys. Rev.* **82** (1951) 664.
- [11] L. Carson, *Phys. Rev.* **D42** (1990) 2853.
- [12] L. Carson and L. McLerran, *Phys. Rev.* **D41** (1990) 647.
- [13] D. D'yakonov, V. Petrov, and A. Yung, *Sov. J. Nucl. Phys.* **39** (1984) 150.
- [14] G. 't Hooft, *Nucl. Phys.* **B33** (1971) 173; K. Fujikawa, B. Lee, and A. Sanda, *Phys. Rev.* **D6** (1972) 2923 and references therein.
- [15] J. Gervais and B. Sakita, *Phys. Rev.* **D11** (1975) 2943.
- [16] E. Byckling and K. Kajantie, *Particle Kinematics* (Kap. 6), Wiley, London 1973.
- [17] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Tables of Integral Transforms*, McGraw-Hill Book Company, Inc., New York 1954.
- [18] A. Bochkarev and M. Shaposhnikov, *Mod. Phys. Lett.* **A2** (1987) 991.
- [19] S. Hawking, *Commun. Math. Phys.* **55** (1977) 133.
- [20] S. Flügge, *Practical Quantum Mechanics 1* (Chap. 39), Springer Verlag, Berlin a. Heidelberg 1971.
- [21] J. Fuchs und A. Ringwald, CERN-TH.6668/92.
- [22] D. D'yakonov, V. Petrov, and A. Yung, *Sov. J. Nucl. Phys.* **39** (1984) 150.
- [23] D. Fliegner, diploma thesis, Heidelberg February 1994.

Figure 1: Logarithms of the double well determinant as a function of μ , subtracting only the zero (1) and both discrete modes (2) “naively” (a), “alternatively” (b), and averaging (c). The cut-off N growing from 1 to 5, the graphs flatten and approach the exact result (horizontal line).

Figure 2: Logarithms of the FP determinant without (1) and with subtraction of the discrete mode (2) as in figure 1.

Figure 3: Logarithms of the fluctuation determinant subtracting three zero modes as in figure 1.