Introduction

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The conditional anomaly problem and
Gauge dependence in higher derivative quantum gravity

In higher derivative gravity (HDG), there are additional gauge fixing terms that are not present in standard general relativity. These gauge fixing terms are motivated by the desire to make the theory more generalizable to higher dimensional theories. However, the choice of gauge fixing is not unique and can lead to inconsistencies. This has led to the development of the conditional anomaly problem, which is the study of the dependence of the theory on the choice of gauge fixing.

In this paper, we will explore the consequences of these additional terms and how they affect the theory. We will also discuss the implications for higher dimensional theories and the challenges of incorporating these terms into a consistent theory.

References


According to power counting and general covariance considerations the possible counterterms in HDQG have the form:

$$\Delta S = \int d^4x \sqrt{-g} \left( a_1 C_{abcd} C^{abcd} + a_2 R^2 + a_3 E + a_4 \Box R \right),$$

(4)

where $a_i$ are some divergent constants (we suppose the use of the dimensional regularization), $E$ is the Gauss - Bonnet topological invariant

$$E = R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^2.$$

(5)

The one - loop counterterms in the theory (1) have been found in [5] in the form (4) with the nonzero $a_2$ and hence the conformal invariance is violated (as well as the multiplicative renormalizability). The counterterms of the form

$$\Delta S' = \int d^4x \sqrt{-g} (a_4 R + a_4)$$

(6)

are also possible in the theory (2) if the Einstein and cosmological terms occure in the classical action.

There is another competitive explanation for the appearance of the nonconformal counterterm $\int d^4x \sqrt{-g} R^2$ in Weyl gravity. In fact our expectation to get the conformal invariant counterterms is based on the use of the background field method. So it is important to check the correctness of our use of this method. The gauge fixing term that have been introduced in [5] is of the form

$$S_{gf} = \frac{1}{2} \int d^4x \sqrt{-g} \chi_a Y^a \chi_a,$$

(7)

where the general form of the background gauge and weight operator is

$$\chi_a = \nabla \chi^a - (\beta \frac{1}{4}) \nabla^a h,$$

(8)

$$Y^a = \frac{1}{5}(g^{a\mu \nu} \Box + g^{a\mu \nu} \nabla^\mu \nabla^\nu - \gamma \nabla^\mu \nabla^\nu).$$

(9)

Here $\alpha, \beta, \gamma$ are gauge parameters, $h = h^a_a$ and the expansion of the metric $g_{\mu \nu}$ into the background $g_{\mu \nu}$ and quantum $h_{\mu \nu}$ parts

$$g_{\mu \nu} - g'_{\mu \nu} = g_{\mu \nu} + h_{\mu \nu}$$

is supposed. The quantization of the theory (1) needs also the supplementary condition for fixing the conformal symmetry. In [5] this condition have been taken in the form $h = 0$. The consistent use of the background field method require the gauge condition to preserve the symmetries in the background fields sector. In the case under consideration the condition $h = 0$ does not violate the background conformal invariance. However we can not wait for the conformal invariant result because of the conformal noncovariance of the operators (8),(9).

Thus we have two competitive reasons for the appearance of the nonconformal counterterm $\int d^4x \sqrt{-g} R^2$ in Weyl gravity, and hence the question of conformal anomaly in Weyl gravity is not clear. Here we consider the second version and argue that the appearance of the nonconformal counterterm is not caused by the choice of gauge condition. In fact if $a_2$ (4) is not equal to zero due to the nonconformal structure of the operators (8),(9) then it is natural to suppose that $a_2$ will depend on the gauge parameters $\alpha, \beta, \gamma$. Below it will be shown that the divergencies in the theory (1) do not depend on the gauge parameters and so we have a weighty reasoning in favour of the anomaly existence. Incidentally we consider the gauge dependence of the counterterms in a general HDQG (2) and separate the couplings of the theory into the essential and nonessential ones.

The second part of the paper is devoted to the study of the gauge dependence in a new conformal gravity theory. The action of this theory contains an additional dilaton field. This action can be regarded as the integration constant for the induced quantum gravity. Although there is not direct relation between the gauge dependence of the effective action and the conformal shift of the classical action in this new theory, the change of gauge fixing condition does not effect the possible nonconformal counterterms.

2 Gauge dependence of effective action in HDQG.

Let us start with the introduction of some notations which will be common for both theories (1) and (2). $\Gamma(\alpha)$ will means the value of the effective action, which corresponds to the arbitrary values of the gauge parameters $\alpha_i$, and $\Gamma_\alpha = \Gamma(\alpha(0))$ is the value of the (minimal) effective action corresponding to some special values of the gauge parameters $\alpha_i(0)$.

Our aim is to establish the gauge dependence of the effective action, that is the form of the function $\Gamma(\alpha)$. Note that the general form of this dependence is clear without any special calculations. It is well - known that the gauge dependence disappear on mass shell (see, for example, [12] and [27] for the general and rigid prove), and this fact gives the key to the understanding of the problem. We can write

$$\Gamma(\alpha) = \Gamma_\alpha + \int d^4x \sqrt{-g} e^{\kappa(\alpha)} f_{\mu \nu}(\alpha),$$

(10)

where $e^{\kappa(\alpha)} = \delta S/\delta g_{\mu \nu}$ is the extremal of the classical action $S$, and $f_{\mu \nu}(\alpha)$ is some unknown function.

We are interesting here in the only divergent part of the effective action $\Gamma(\alpha)$. Since the divergencies are local and moreover the values of $\Gamma_\alpha, \Gamma(\alpha), e^{\kappa(\alpha)}$ have just the same
\[
\Delta (H + \Delta H) = \Delta H + \Delta H = 0
\]

where \( (\delta, \omega) \) is the Green function for the Hamiltonian and its Hamiltonian derivative. The solution is given by

\[
(\delta - z)\Phi = (\delta - z)\Phi(0) + \int dx \frac{\partial^2}{\partial x^2} \phi(x)
\]

which have the following boundary condition

\[
(\delta, \omega)\Phi(\pm \infty) = 0
\]

and so

\[
(\delta, \omega)\Phi = 0
\]

where the values of \( \delta, \omega \) are determined by the number of fields of different spin (see (11))

\[
\delta = \frac{\delta_0}{\omega}, \omega = \frac{\omega_0}{\omega}
\]

\[
\therefore \Phi = <\phi, \Phi >
\]

\[
H = \frac{1}{2} \int d^4x \left( \partial^2 - \Delta + \gamma \right) \phi^2 + \frac{1}{4} \int d^4x \left( \partial^2 - \Delta + \gamma \right) \phi^4
\]

\[
\text{(11)}
\]

\[
\delta = \frac{\delta_0}{\omega}, \omega = \frac{\omega_0}{\omega}
\]

\[
\therefore \Phi = <\phi, \Phi >
\]

3. Gauge dependence in induced conditional equality.

\[
(12)
\]

\[
\Delta (H + \Delta H) = \Delta H + \Delta H = 0
\]

\[
\text{(12)}
\]

\[
\frac{\partial^2}{\partial x^2} \phi(x)
\]

\[
\therefore \Phi = <\phi, \Phi >
\]

The expected conditional equality is \( <\phi, \Phi > \).

\[
(\delta, \omega)\Phi = (\delta, \omega)\Phi(0) + \int dx \frac{\partial^2}{\partial x^2} \phi(x)
\]

\[
\text{(11)}
\]

\[
\delta = \frac{\delta_0}{\omega}, \omega = \frac{\omega_0}{\omega}
\]

\[
\therefore \Phi = <\phi, \Phi >
\]
in $S_c$. It is more suitable to rewrite (20) in a local form with the help of some additional scalar field. To make this one have to suppose that $S_c[g_{\mu\nu}]$ contains the structure
\[
C^2 \Delta^{-1} C^2 = \int d^4x \sqrt{-g} \, \sqrt{-g_0} \, C^0 \, C^0.
\]  
(22)

Then, taking into account the possible Weyl term, we get the following local form of $S_c$
\[
S_c = \int d^4x \sqrt{-g} \, \left( q_1 + q_2 \phi \right) C^2 + \frac{1}{2} \Delta \phi.
\]  
(23)

Here $\phi$ is an additional dimensionless spin 0 field with zero conformal weight, and $q_1, q_2$ are some constants (parameters of the action). Let substitute, for generality, the linear construction in (23) by the arbitrary function $q(\phi)$, and we obtain the action which can be treated as some generalization of (1):
\[
S_c = \int d^4x \sqrt{-g} \, \left( q(\phi) C^2 + \frac{1}{2} \phi \Delta \phi \right).
\]  
(24)

Note, that the more general expression is possible to construct if inserting the second arbitrary function $\rho(\phi)$ in front of second item in (24).
\[
S_c = \int d^4x \sqrt{-g} \, \left( q(\phi) C^2 + \frac{1}{2} \rho(\phi) \phi \Delta \phi \right).
\]  
(25)

However, it is clear that (25) differs from (24) only by some change of variable $\phi$ and function $q(\phi)$ and therefore the case (24) is quite general.

As far as theory (24) has just the same symmetries, as the (1) ones, all the reasons in favor of the anomaly existence are also valid here. The only distinction between (1) and (24) is that the gauge dependence of the effective action in (24) is not proportional to the conformal shift of the classical action and therefore the effective action of the theory is gauge fixing dependent.

Making the separation of the field variables into background and quantum ones
\[
g_{\mu\nu} \rightarrow g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu},
\]
\[
\phi \rightarrow \phi' = \phi + \sigma
\]  
(26)

we get the path integral presentation of the effective action
\[
\Gamma[\phi, \bar{\phi} | g_{\mu\nu}] = \int D h_{\mu\nu} \, D \phi \, D \bar{\phi} \, \mathcal{D} \chi \delta(h_{\mu\nu} \bar{\phi} + \phi + \sigma - S_{\text{def}}[g_{\mu\nu}, \phi] - S_{\text{conf}}[\phi, \bar{\phi}]) \, \exp \left\{ S_c[g_{\mu\nu}, \phi, \bar{\phi}] + S_{\text{def}}[\phi, \bar{\phi}, \bar{\phi}] + S_{\text{conf}}[\phi, \bar{\phi}, \bar{\phi}] \right\}.
\]  
(27)

where $\zeta, c_1$ are ghosts and the conformal gauge $h = \bar{h}_{\mu\nu} = 0$ have been used. The general form of the gauge fixing form can be founded in analogy with the similar $d = 2$ case [16].
\[
S_{\text{def}} = \int d^4x \sqrt{-g} \, \chi \, X^\mu \, X_\mu,
\]
\[
\chi = \alpha_i(\phi) \nabla_\nu h_i^\nu + \alpha_2(\phi) \nabla_\nu \sigma + \alpha_3(\phi) (\nabla_\nu \bar{\phi}) \sigma + \alpha_4(\phi) (\nabla_\nu \phi) h_i^\nu,
\]
\[
X^\mu = \alpha_i(\phi) \gamma^\mu \sigma + \alpha_2(\phi) \gamma^\mu \bar{\phi} + \alpha_3(\phi) R_{\mu \nu} + \alpha_4(\phi) \gamma^\mu \bar{\phi}.
\]  
(28)

Here $\alpha_1, \ldots, \alpha_4(\phi)$ are gauge parameters which depend on the background field $\phi$. Note, that the use of such a gauge (with the special choice of $\alpha_i$) enables us to calculate the one-loop counterterms in the theory (24), and also in a general dilaton quantum gravity in four dimensions. Such a calculation will be the purpose of a separate paper. Here we consider only the gauge dependence of the effective action. The arguments of the previous section are also valid for the theory (24) and we obtain the equation, that is analogous to eq. (10).
\[
\Gamma(\phi) = \Gamma_m + \int d^4x \sqrt{-g} \left( f^{(1)}(\phi) \frac{\delta S_c}{\delta g_{\mu\nu}} + f^{(2)}(\phi) \frac{\delta S_c}{\delta \phi} \right),
\]  
(29)

where $f^{(1)}, f^{(2)}$ are some unknown functions of the gauge parameters. $\Gamma_m$ is the value of $\Gamma$ which corresponds to some special (minimal) choice of gauge parameters. Then, due to the dimensional analysis we find $f^{(1)} = f^{(2)}(\phi)$, where $f^{(2)}(\phi)$ is some dimensionless function. Hence the first item on the right of (29) is proportional to the conformal shift of $S_c$. Since the action $S_c$ is conformal invariant we have
\[
\Gamma(\phi) = \Gamma_m + \int d^4x \sqrt{-g} \left( f^{(1)}(\phi) (\Delta \phi + \phi' \phi C^0) \right),
\]  
(30)

where $\phi' = d\phi/d\phi$.

Note, that the difference $\Gamma(\phi) - \Gamma_m$ is conformally invariant and therefore it is impossible to remove any conformal noninvariant counterterm by selection of the gauge parameters $\alpha(\phi)$. Thus the gauge dependence in the theory (24) doesn’t have relation to the conformal anomaly as well as in the Weyl gravity (1) and therefore the theory (24) is expected to be nonconformalizable due to the possible nonconformal divergencies.

Of course, one can use the trick with the conformal regularization, substituting the background metric $g_{\mu\nu}$ by $g_{\mu\nu}$ (14). Then the anomaly disappear, and all the possible counterterms may be removed by some reparametrization of the field $\phi$ and the function $q(\phi)$. The variation of gauge parameters leads to the only change of this transformations.

Now we shall restrict ourselves by the one-loop case where $f^{(1)}(\phi) = O(\phi^{-1})$, where $c = n - 4$ is the parameter of dimensional regularization. Let the divergencies of $\Gamma_m$ are
The appendix is a detailed section that provides additional information, data, or proofs that support the main content of the paper. It often includes derivations, calculations, or explanations that are too lengthy or detailed to include in the main body of the text. In this case, the appendix might cover supplementary material related to the main paper, such as extensions of the theory, additional proofs, or further explorations of the concepts discussed in the main text.

This appendix is likely to be a valuable resource for readers who wish to delve deeper into the mathematical or theoretical aspects of the research presented in the main body of the paper. It is common for appendices to include technical details, computational results, or experimental data that are not necessary for understanding the main arguments but are important for completeness.

In the context of the provided text, the appendix might contain additional mathematical derivations, proofs, or examples that elaborate on the main theorems or results. These could include detailed calculations, expansions of equations, or clarifications of the methodology used in the research. Appendix 4, for instance, might focus on a specific area of mathematics or a particular aspect of the research, providing a more in-depth exploration of a particular point or concept.

The presence of an appendix is a testament to the thoroughness of the research and the need to include materials that are relevant to understanding the complete scope of the study. It is a common practice in academic publishing to ensure that all necessary information is available to the reader, even if it is not essential to the main narrative of the paper.
\[ Y_{\nu_{4}5} = P_{\nu_{4}5}\{4(\alpha - \lambda)\delta_{\nu_{4}5}\delta_{\nu_{5}}\delta_{\nu_{4}} - 4\nabla^\alpha \nabla^\nu \nabla \nabla^\alpha (\alpha + \lambda + 3\lambda \gamma) + 2\lambda \nabla^\nu \nabla^\nu \nabla^\nu (\alpha + \lambda + 3\lambda \gamma) \} \]

\[ + 2\lambda \nabla^\nu \nabla^\nu \nabla^\nu (\alpha + \lambda + 3\lambda \gamma) \}
\]

\[ P_{\nu_{4}5} = \frac{1}{2} (\delta^\nu_{\nu_{4}} \delta^\nu_{\nu_{5}} - \delta^\nu_{\nu_{4}} \delta^\nu_{\nu_{5}}) - \frac{1}{4} \eta_{\nu_{4}5} \eta_{\nu_{4}5} \]  \( \tag{A.9} \)

The analysis of eq. (A.7) shows that the nonzero contributions to the divergences are given by the only universal traces which do not contain curvature. Note that in the main part of the article we obtain this result in the framework of consideration, which is based on power counting and on the locality of counterterms.

The minimal operator \( F_{\alpha} \) corresponds to the following values of the gauge parameters [5,11]

\[ \alpha^0 = \lambda, \quad \gamma^0 = \frac{2}{3}. \]

Performing the integration (A.8) and substituting (A.9) we obtain

\[ \int \Gamma(\alpha, \gamma) = \int \Gamma(\alpha^0, \gamma^0) + e^{\nu \nu} f_{\nu} (\alpha, \gamma). \]

\[ f_{\nu} (\alpha, \gamma) = \frac{3}{2} g_{\nu} \left( \frac{\lambda}{2} \ln \left( \frac{\gamma \lambda \gamma + \lambda}{\lambda \gamma - 6 \lambda \gamma} \right) + \alpha - \lambda \right) \]  \( \tag{A.10} \)

So the explicit expression (A.10) is in a good accord with the corresponding result of the main text of the paper.

References
