Abstract

Recently it was suggested that the problem of species doubling with Kogut-Susskind lattice fermions entails, at finite chemical potential, a confusion of particles with antiparticles. What happens instead is that the familiar correspondence of positive-energy spinors to particles, and of negative-energy spinors to antiparticles, ceases to hold for the Kogut-Susskind time derivative. To show this we highlight the role of the spinorial “energy” in the Osterwalder-Schrader reconstruction of the Fock space of non-interacting lattice fermions at zero temperature and nonzero chemical potential. We consider Kogut-Susskind fermions and, for comparison, fermions with an asymmetric one-step time derivative.
1 Introduction

The implementation of finite baryon density on the lattice [1, 2] has proven to be a hard task. Several proposals have been made to explain the unexpectedly low chemical potential \( \mu \) at which the baryon density and other quantities start to grow. The onset at a \( \mu \) much lower than the value \( m_N/3 \), as expected for a free nucleon gas, could be due to the enhanced number of flavours (when using Kogut-Susskind fermions) producing a much stronger nuclear binding than in nature [3, 4].

As the onset is so early that its energy can be almost confounded with the existence of a ‘baryonic’ Goldstone mode, it was also interesting to search for possible enhancement of mesonic modes due to some kind of lattice artifact. In [5] it was pointed out that by selecting only the positive-energy poles of the free Kogut-Susskind propagator one still obtains energies and baryon number densities composed of particles as well as antiparticles. This is in contrast to Dirac fermions in space-time continuum where positive-energy spinors always correspond to particles and negative-energy spinors to antiparticles. As also observed in [5], that feature of continuous fermions is reproduced on the lattice if an asymmetric, one-step time derivative is used. (In fact, such fermions have the same Hilbert space as Susskind’s fermions [6] on the Hamiltonian lattice.) It was then suggested in [5] that the Kogut-Susskind action does not allow a clear distinction of particles and antiparticles. That conclusion, however, is not justified. Rather, as we show below, it is the correspondence of positive/negative-energy spinors to particles/antiparticles which is destroyed by the Kogut-Susskind two-step time derivative.

The Hilbert spaces corresponding to space-time lattice fermions have all been reconstructed long ago as part of the issue of the positivity of their transfer matrices. This was done in [7, 8, 9] for Wilson fermions, and in [10] for staggered fermions. In all those references, a direct comparison was made between matrix elements in a standard fermion Fock space and the Grassmann functional integral. While this method enables one to deal with interacting fermions from the outset, it does not assign a Fock-space interpretation to the path-integral variables themselves. For Kogut-Susskind fermions, moreover, auxiliary Grassmann integrations had to be introduced. In the present context, where we want to identify the role of positive- and negative-energy spinors (the Fourier transforms of the path-integral variables) in the fermion Fock space, we therefore prefer to work our way through the Osterwalder-Schrader reconstruction scheme [11, 12] which does not anticipate any knowledge of the Hilbert
space. For an introduction to this formalism we refer the reader to [13]. Fortunately, the present issue allows us to restrict ourselves to non-interacting fermions at zero temperature, but at finite chemical potential \( \mu \).

Furthermore, we will restrict our examples to cases where the inverse fermion matrix can be given explicitly, in terms of spinorial energy eigenstates and eigenvalues. Thus we consider fermion actions of the general form

\[
S = \bar{\psi}_n S_{nn'} \psi_{n'}; \quad n = (n_1, n_2, n_3, n_4) = (\sigma, n_4)
\]  

where

\[
S_{nn'} = D(\mu)_{n_4 n_4'} \delta_{\sigma, \sigma'} + \delta_{n_4, n_4'} \frac{a_\sigma}{a_{\sigma'}} H_{\sigma, \sigma'}
\]

such that the time-changing (derivative) part \( D \) commutes with the equal-time (Hamiltonian) part \( H \). This form can be achieved, both for Kogut-Susskind and for time-asymmetric [5] fermions, by first absorbing a \( \gamma_4 \) factor into the \( \bar{\psi} \) integration variables and then diagonalizing in spin and restricting to one of the components by analogy to Kawamoto-Smit, but using \( \alpha \)'s as \( (i\gamma_4\gamma_1)^{n_1} (i\gamma_4\gamma_2)^{n_2} (i\gamma_4\gamma_3)^{n_3} \). For Wilson fermions, such a simplification does not seem to be possible.

For the asymmetric fermions it will be advantageous to allow for an anisotropy of the space-time lattice; hence the appearance in (1) of the spatial and temporal lattice spacings, \( a_\sigma \) and \( a_{\sigma'} \).

In section 2 we recall the basic steps in the procedure of Osterwalder-Schrader reconstruction, and the standard tools we have at hand for non-interacting fermions. In section 3 we apply this, in some detail, to fermions with a one-step time derivative. We thus recover the familiar zero-temperature scenario at finite chemical potential. In section 4 we provide the analogous expressions for Kogut-Susskind fermions. The spinorial “energy” here enters in two kinds of time dependency, a smooth one and an alternating one. A comparison of the operator expressions for the energy and for the baryon number in Fock space (as opposed to the linear space spanned by the Grassmann variables) will show that a chemical potential enhancing the baryons will deplete the antibaryons. In the concluding section 5 we discuss relations with the positivity of the transfer matrices and with finite temperature.

## 2 Basic ideas of Reconstruction

We wish to recover the vectors of the multi-fermion Hilbert space, their scalar product, and the operators of time evolution and baryon number, from the path-integral expectation values and correlations. The basic ideas are as follows [13]. Both bra
and ket vectors contribute to \( \langle \ldots \rangle_{\text{path integral}} \). The ket vectors are the functionals of Grassmann variables \( \psi_\tau \) and \( \tilde{\psi}_\tau \) with Euclidean times \( \tau > 0 \) (more precisely, they are equivalence classes of such functionals; cf. below). The bra vectors are the same with \( \tau < 0 \). Thus bra and ket functionals are related through a time-reversal operation \( \Theta \). On Grassmann variables it acts as

\[
\Theta \psi(\tau) = \tilde{\psi}(-\tau) \quad \Theta \tilde{\psi}(\tau) = \psi(-\tau)
\]

and is extended as an antilinear operation on functionals of \( \psi \) and \( \tilde{\psi} \). We shall adopt the rule that complex conjugation of a product of Grassmann variables implies reversed ordering of factors.

The scalar product of a bra and ket vector is given by a path-integral correlation: if \( F \) and \( G \) are ket functionals, and \( |F\rangle \) and \( |G\rangle \) their interpretations as Hilbert space vectors, their scalar product is

\[
\langle F|G \rangle = \langle \Theta F \ G \rangle_{\text{path integral}}
\]

However, for a quantum-statistical interpretation the scalar product must satisfy \( \langle F|F \rangle \geq 0 \). This translates into the criterion of reflection-positivity [11] for the path-integral expectations

\[
\langle \Theta F \ G \rangle_{\text{path integral}} \geq 0 \tag{2}
\]

If it holds there would still be, in general, null-norm functionals \( F \neq 0 \) with \( \langle F|F \rangle = 0 \) but those can be factored away. This apparent mathematical subtlety here turns out to comprise the phenomenon of species doubling due to the time discretization. In fact, fermions with a one-step time derivative [5], which have no extra species doubling in the time direction, have twice as many degrees of freedom to factor away than, for example, Kogut-Susskind fermions.

The transfer matrix derives from a unit shift of the time index on all the Grassmann variables in a ket functional.

Finally, a fermion field operator derives from the multiplication of ket functionals by a Grassmann variable at zero time. This will allow us to identify the baryon number operator from the corresponding path-integral observable.

It will be essential in the following to have an explicit representation of Grassmann-integral expectation values and correlations at hand. In the most general finite-dimensional case, if \( \psi_n \) and \( \tilde{\psi}_n \) are the integration variables and if \( O[\psi, \tilde{\psi}] \) is any observable or correlation in terms of them, then [14]

\[
\langle O \rangle_{\text{path integral}} = \frac{1}{\det S} \int \prod_n d\psi_n d\tilde{\psi}_n \exp \left( \sum_{n,m} \tilde{\psi}_n S_{nm} \psi_m \right) O[\psi, \tilde{\psi}]
\]

4
\[
= \exp \left( \sum_{\alpha, \beta} S^{-1}_{n\alpha \beta} \frac{\partial}{\partial \psi_{\alpha}} \frac{\partial}{\partial \psi_{\beta}} \right) O[\psi, \bar{\psi}] \bigg|_{\psi = \bar{\psi} = 0}
\] (3)

To keep the notation as simple as possible, we shall refer only to Grassmann derivatives “from left to right” (\(\partial/\partial \psi = \bar{\partial}\)). For an efficient calculation of signs it is convenient to note \(\bar{\partial} \bar{\partial} \bar{F} \bar{G} = -\bar{F} \bar{\partial} \bar{\partial} \bar{G}\) where \(\bar{\partial}\) is a Grassmann derivative “from right to left”.

We shall be only concerned here with the Euclidean time structure of the inverse action kernel \(S^{-1}_{nm}\), where \(n = (\sigma_1, \sigma_2, \sigma_3, n_4)\) etc. We shall use two different labelings of time slices: \(n_4 = 0, \pm 1, \pm 2, \ldots\) as appropriate for the space-time functional-integral approach, and \(\tau = \pm 1, \pm 2, \ldots\) which will emphasize later on the correspondence between bra and ket functionals. For later convenience we note that

\[\tau = n_4 \quad \text{for} \quad n_4 > 0 \quad \tau = n_4 - 1 \quad \text{for} \quad n_4 \leq 0\]

With respect to the spatial indices, we transform into a basis in which the Hamiltonian part of the action is diagonal. For Kogut-Susskind fermions this involves spatial Fourier transformation and linear combination of the momenta at the edges of the Brillouin zone. Let \(p\) denote a suitable (multi)index for the basis vectors. For example, \(p\) can be chosen as

\[p = (p_1, p_2, p_3, \lambda, s, f) \quad p_k \in \left[ -\frac{\pi}{2a_s}, \frac{\pi}{2a_s} \right], \quad \lambda = \pm 1, \quad s = \pm \frac{1}{2}, \quad f = u, \bar{d} \]

where \(p_k\) are the components of spatial momentum. The rest of the Brillouin zone, reachable by addition of “edge” momenta \((\pm \pi/a, \pm \pi/a, \pm \pi/a)\) corresponds to the degrees of freedom of spinorial “energy” sign \(\lambda\), spin \(s\), and flavour \(f\) [15]. In the following, \(\lambda\) will be of particular interest to us. Using

\[E(p) = \sqrt{\sum_k \sin^2 p_k + m^2}\]

the spinorial “energy” is \(\lambda E(p)\).

We can now rewrite the action kernel of equation (1) as

\[S_{n\alpha \beta} = \delta_{\alpha, \beta} \left( D(\mu)_{n_4, n_4} + \delta_{n_4, n_4} \frac{a_s}{a_s} \lambda E(p) \right)\]

In the following we reconstruct the fermion Fock space, and the operators of energy and baryon number in particular, for two versions of the time derivative.
We first analyse fermions with a one-step, asymmetric time derivative. This will lead to expectation values for the energy and baryon number resembling those familiar from space-time continuum [5]. In fact, for a one-step time derivative the Fock space is completely defined by the equal-time part of the action, $H$, which coincides with Susskind’s spatial fermions [6]. The Hamilton operator $\tilde{H}$ will be slightly different, though, due to finite temporal lattice spacing.

The time-derivative term is [5]

$$D = \sum_p \sum_{n_4} \tilde{\psi}_{p,n_4} \left( e^{\mu} \psi_{p,n_4+1} - \psi_{p,n_4} \right)$$

and thus the total action kernel is

$$S_{nn'} = \delta_{pp'} \left( (e^{\mu} \delta_{n_4+1,n_4'} - \delta_{n_4,n_4'}) + \lambda E(p) \frac{\partial}{\partial a_{\sigma}} \delta_{n_4,n_4'} \right)$$

Using $\epsilon = \ln(1 - \frac{a_{\sigma}}{a_{\sigma}} \lambda E(p))$ the inverse kernel reads

$$S^{-1}_{nn'} = \begin{cases} -e^{-\epsilon (r-\mu)(n_4-n_4')} & \text{if } n_4 \leq n_4' \\ 0 & \text{if } n_4 > n_4' \end{cases} \delta_{pp'} \text{ for } \epsilon > \mu \quad (4)$$

$$S^{-1}_{nn'} = \begin{cases} e^{-\epsilon (r-\mu)(n_4-n_4')} & \text{if } n_4 \leq n_4' \\ 0 & \text{if } n_4 > n_4' \end{cases} \delta_{pp'} \text{ for } \epsilon < \mu \quad (5)$$

It should be mentioned here that the zeros above will produce zero-norm states later on. Right at $\epsilon = \mu$ the inverse kernel is unambiguously determined by the antiperiodic boundary condition with respect to time. The appropriate limit is

$$S^{-1} = \frac{1}{2} \left( S^{-1}_{r-\mu-0} + S^{-1}_{r-\mu+0} \right)$$

To evaluate any scalar product or matrix element, it is convenient to adapt operation (3) to the special case of $O = \tilde{F} \cdot G$ with $\tilde{F}$ a negative-time and $G$ a positive-time functional. Sorting out the derivatives acting on $\tilde{F}$ or $G$ exclusively, we can write

$$\exp \sum_{nn'} S_{nn'}^{-1} \frac{\partial}{\partial \tilde{\psi}_n} \frac{\partial}{\partial \psi_{n'}} \tilde{F} G = G_0 \left( (G_\pm \tilde{F})(G_\pm G) \right) \quad (6)$$

where

$$G_\pm = \exp \sum_p \sum_{\tau, \tau'} \tilde{S}_{\tau,\tau'}^{-1}(p) \frac{\partial}{\partial \tilde{\psi}^\prime_\tau} \frac{\partial}{\partial \tilde{\psi}_{\tau'}}$$

for $\tau, \tau' > 0$

$$= \exp \sum_p \sum_{\tau, \tau'} \tilde{S}_{\tau,\tau'}^{-1}(p) \frac{\partial}{\partial \tilde{\psi}^\prime_\tau} \frac{\partial}{\partial \tilde{\psi}_{\tau'}}$$

for $\tau, \tau' < 0$
while the remaining “mixing” term is

\[ G_0 = \exp \left( \sum_{p: \epsilon > \mu} (-e^{-\mu}) \left( \sum_{\tau > 0} e^{(\mu-\epsilon)\tau} \frac{\partial}{\partial \psi_{p,-\tau}} \right) \left( \sum_{\tau' > 0} e^{(\mu-\epsilon)\tau'} \frac{\partial}{\partial \bar{\psi}_{p,-\tau'}^\dagger} \right) \right) \times \]

\[ \times \exp \left( \sum_{p: \epsilon < \mu} e^{\mu-2\epsilon} \left( \sum_{\tau > 0} e^{(\epsilon-\mu)\tau} \frac{\partial}{\partial \psi_{p,\tau}} \right) \left( \sum_{\tau' > 0} e^{(\epsilon-\mu)\tau'} \frac{\partial}{\partial \bar{\psi}_{p,\tau'}}^\dagger \right) \right) \]

(7)

The operations \( G_+ \) and \( G_- \) only modify the ket and bra functionals, respectively. The operation intertwining kets and bras is \( G_0 \), and the last step is to discard all remaining \( \psi \)s and \( \bar{\psi} \)s. It will be most convenient to formulate the fermion Hilbert space in terms of \( \tilde{\mathcal{F}} = G_- \mathcal{F} \) and \( \mathcal{G}' = G_+ \mathcal{G} \), i.e., at the \( G_0 \) stage after the \( G_+ \) and \( G_- \) operations.

We are aiming at operator expressions for the energy (transfer matrix \( \hat{T} \) or Hamiltonian \( \hat{H} \)) and for the baryon number. As \( G_+ \) and \( G_- \) depend on \( \tau - \tau' \) only, they commute with the time evolution. Thus \( \hat{T} \) can be determined at the \( G_0 \) stage immediately; see below. The baryon number operator is most easily identified from the path-integral observable “at \( \tau = 0 \)” (i.e., involving \( \tau = -1 \) and \( \tau = 1 \)) which is

\[ B = \frac{\partial S}{\partial \mu} = \sum_\sigma e^{\mu} \bar{\psi}_{\sigma,-1} \psi_{\sigma,1} = \sum_p e^{\mu} \bar{\psi}_{p,-1} \psi_{p,1} \]

(8)

In the second equality we changed over to the eigenbasis of the spinorial energy, and arranged Grassmann variables in time order. \( \bar{\psi}_{p,-1} \) and \( \psi_{p,1} \) can now be interpreted as multiplication operators acting on bra and ket functionals, respectively. Thus the matrix elements of the baryon number operator are

\[ \langle F | \hat{B} | G \rangle = e^\mu \langle \Theta(\psi_{p,1} F) \psi_{p,1} G \rangle_{\text{path integral}} = e^\mu G_0 \{ \Theta(\psi_{p,1} F) G_+ \psi_{p,1} G \} \]

To bring the field operators (multiplication operators) of the above equation to the \( G_0 \) stage, we apply the Baker-Hausdorff formula, obtaining for example the following primed form (with any ket functional \( G \))

\[ G_+ \psi_{p,1} G = \left( \psi_{p,1} + \begin{cases} 1 & \epsilon > \mu \\ 0 & \epsilon < \mu \end{cases} \right) \times e^{-\mu} \sum_{\tau > 0} e^{(\mu-\epsilon)\tau} \frac{\partial}{\partial \psi_{p,\tau}} \mathcal{G}' \]

(9)

On the bra side, we obtain the \( \Theta \)-reflected expression. In a similar way, by commuting \( \psi \)s and \( \bar{\psi} \)s through \( G_0 \) one obtains the adjoints of field operators. For example, we have

\[ G_0 \bar{\psi}_{p,-1} = \left( \bar{\psi}_{p,-1} + \begin{cases} 0 & \epsilon > \mu \\ 1 & \epsilon < \mu \end{cases} \right) \times e^{-\epsilon} \sum_{\tau > 0} e^{(\epsilon-\mu)\tau} \frac{\partial}{\partial \bar{\psi}_{p,\tau}} \mathcal{G}_0 \]

7
\[
= \tilde{\psi}_{p,-1} G_0 + G_0 \left( \begin{array}{cc}
0 & \epsilon > \mu \\
1 & \epsilon < \mu
\end{array} \right) \times e^{-\epsilon} \sum_{\tau > 0} e^{(\epsilon - \mu)\tau} \frac{\partial}{\partial \psi_{p,\tau}}
\]

The \( \tilde{\psi}_{p,-1} \) on the RHS is discarded when all \( \psi \) and \( \tilde{\psi} \) are set to zero in the final step of the scalar product. The bracketed expression on the RHS is the ket-adjoint to the bra-multiplication by \( \tilde{\psi}_{p,-1} \). To further develop our example, this can be used to rewrite \( G_0 \left\{ (\Theta \psi_{p,1} F') \psi_{p,1} G' \right\} \) as \( G_0 \left\{ (\Theta F') \psi_{p,1} \psi_{p,1} G' \right\} \) which is part of the baryon number operator we are interested in. From an expression such as the last one, in which all operator action is thrown on the kets, one finally abstracts away the functionals and the scalar product operation.

On the \( G_0 \) stage, as we see from (7) and (9), Grassmann derivatives occur only in specific combinations of time indices for each “momentum” index \( p \). The fermion ket Hilbert space takes its simplest form if we change variables such that

\[
\sum_{\tau > 0} e^{(\mu - \epsilon)\tau} \frac{\partial}{\partial \psi_{p,\tau}} =: \frac{\partial}{\partial \Psi_p} \quad \epsilon > \mu,
\]

\[
\sum_{\tau > 0} e^{(\epsilon - \mu)\tau} \frac{\partial}{\partial \psi_{p,\tau}} =: \frac{\partial}{\partial \Psi_p} \quad \epsilon < \mu.
\]

When the derivatives act on ket functionals we obtain the same values as before if we substitute\(^1\)

\[
\tilde{\psi}_{p,\tau} \rightarrow e^{(\mu - \epsilon)\tau} \tilde{\Psi}_p \quad \epsilon > \mu,
\]

\[
\psi_{p,\tau} \rightarrow e^{(\epsilon - \mu)\tau} \Psi_p \quad \epsilon < \mu.
\]

The substitution results in functionals of \( \Psi_p \) and \( \tilde{\Psi}_p \) only; this indicates that those are the only relevant (non-null) variables. Furthermore, the \( \Psi \) are non-null only for \( \epsilon < \mu \) (in ket-functionals) and the \( \tilde{\Psi} \) only for \( \epsilon > \mu \). This will be different for Kogut-Susskind fermions, leading to an additional doubling of species there.

In terms of the new variables, the baryon number (8) for time-asymmetric fermions takes the simple operator form

\[
\hat{B} = \sum_{p: \epsilon > \mu} \tilde{\Psi}_p \frac{\partial}{\partial \tilde{\Psi}_p} + \sum_{p: \epsilon < \mu} \left( 1 - \Psi_p \frac{\partial}{\partial \Psi_p} \right)
\]

At the \( G_0 \) stage, the transfer matrix is also easily identified. If all Grassmann variables in a ket functional are shifted by one time step, equation (10) shows that \( \tilde{\Psi}_p \) picks up a factor of \( e^{\mu - \epsilon} \),

\(^1\)In terms of Linear Algebra, we are considering the vector space spanned by the Grassmann variables and the dual space which is spanned by the Grassmann derivatives; in this sense we are changing from one pair of dual bases to another.
and $\Psi$ a factor of $e^{\gamma}$. The generator of such a transformation is

$$\hat{H} - \mu \hat{B} = -\log \hat{T} = \sum_{\epsilon > \mu} (\epsilon - \mu) \frac{\partial}{\partial \Psi^\dagger} + \sum_{\epsilon < \mu} (-\epsilon + \mu) \Psi \frac{\partial}{\partial \Psi}$$

(12)

We can rewrite this in the standard Fock-space form, using $\epsilon_m = \log (1 + \frac{a}{2\pi m})$:

$$\hat{H} - \mu \hat{B} = \sum_{\epsilon > \epsilon_m} (\epsilon - \mu) b^*_\epsilon b_\epsilon + \sum_{\epsilon > \epsilon_m} (\epsilon + \mu) \bar{b}^*_\epsilon \bar{b}_\epsilon + \sum_{\epsilon > \epsilon_m} (\epsilon - \mu)$$

$$\hat{B} = \sum_{\epsilon > \epsilon_m} \left( b^*_\epsilon b_\epsilon - \bar{b}^*_\epsilon \bar{b}_\epsilon \right) + \sum_{\epsilon < \epsilon_m} 1$$

(13) (14)

where we have defined creation and annihilation operators for antibaryons (displaying only the spinorial energy instead of the full momentum index) by

$$b^*_\epsilon = \Psi_{-\epsilon} \quad \bar{b}_\epsilon = \partial / \partial \Psi_{-\epsilon} \quad \epsilon > \epsilon_m$$

and for baryons by

$$b^*_\epsilon = \begin{cases} \Psi_{\epsilon} & \epsilon > \mu \\ \partial / \partial \Psi_{\epsilon} & \mu > \epsilon > \epsilon_m \end{cases} \quad b_\epsilon = \begin{cases} \partial / \partial \Psi_{\epsilon} & \epsilon > \mu \\ \Psi_{\epsilon} & \mu > \epsilon > \epsilon_m \end{cases}$$

The $b_\epsilon^*$ and $\bar{b}_\epsilon$ satisfy canonical anticommutation relations with $b_\epsilon$ and $\bar{b}_\epsilon$, but for $\epsilon$ or $\mu$ near the UV cutoff they are only proportional to the hermitian adjoints:

$$\bar{b}_\epsilon^\dagger = e^{-2\epsilon} b_\epsilon^* \quad \text{for all } \epsilon > \epsilon_m$$

$$b_\epsilon^\dagger = e^{\epsilon} b_\epsilon^* \quad \epsilon > \mu \quad b_\epsilon^\dagger = e^{\mu-2\epsilon} b_\epsilon^* \quad \mu > \epsilon > \epsilon_m$$

As we see from (12) the functional representing the ground state $|0;\mu\rangle$ at chemical potential $\mu$ is $G_0 = 1$. In terms of Fock space operators it is characterized by

$$\bar{b}_p |0;\mu\rangle = 0 \quad \text{for all relevant } p$$

$$b_\epsilon |0;\mu\rangle = 0 \quad \epsilon > \mu$$

$$\bar{b}_\epsilon^\dagger |0;\mu\rangle = 0 \quad \mu > \epsilon > \epsilon_m$$

(15)

Thus the ground state is filled with baryons of all momenta, spins and flavours with energies in the range $\mu > \epsilon(p) > \epsilon_m$.

The zero-point energy in (13) is due to the above definition of $\hat{H} - \mu \hat{B}$ as a plain generator of a temporal translation. Had we defined the baryon number operator as a plain generator of phase rotations, we would have avoided the divergent but $\mu$-independent contribution in (14) which we must now subtract by hand.
It should be noted that definition (13), based on a single time step, is valid only if \( \epsilon \) in the above definition \( \epsilon = \ln(1 - \frac{\omega_r^2}{\epsilon} \lambda E(p)) \) is a real number for all energy-momentum indices \( p \). Since the maximal eigenvalue of the free staggered Hamiltonian is \( \sqrt{3 + m^2} \), \( m \) being the fermion mass, the condition is \( a_x < a_x / \sqrt{3 + m^2} \). If the temporal spacing is too coarse to satisfy this, then one can still define a two-step Hamiltonian, \( \hat{H} = -\frac{1}{2} \log T^2 \) as customary for staggered fermions. This amounts to discarding any imaginary component that \( \epsilon \) might acquire.

### 4 Kogut-Susskind time derivative

We now turn to the case of the two-step time derivative as it is used with Kogut-Susskind fermions [1, 2]. The total action kernel is

\[
S_{n,n'} = \delta_{p,p'} S_{n,n'}(p) \text{ where } S_{n,n'}(p) = \frac{1}{2} \left( e^\mu \delta_{n,n_4} + e^{-\mu} \delta_{n,n_4'} \right) + \lambda E(p) \delta_{n,n_4}.
\]

Let us restrict to \( \mu \geq 0 \) and put \( \epsilon = \text{arsinh}(\frac{a_x}{m} \lambda E) \). The inverse kernel \( S_{n,n'}^{-1}(p) \) then reads

\[
\frac{e^{-\mu(n_4-n_4')} e^{\mu(n_4-n_4')}}{\cosh \epsilon} \times \begin{cases} 
1 & \text{for } n_4 \geq n_4' \\
(-1)^{n_4-n_4'} & \text{for } n_4 \leq n_4' 
\end{cases} \quad \epsilon > \mu
\]

\[
\frac{e^{-\mu(n_4-n_4')} e^{\mu(n_4-n_4')}}{\cosh \epsilon} \times \begin{cases} 
(-1)^{n_4-n_4'}+1 & \text{for } n_4 \geq n_4' \\
-1 & \text{for } n_4 \leq n_4' 
\end{cases} \quad \epsilon < -\mu
\]

\[
\frac{e^{-\mu(n_4-n_4')} e^{\mu(n_4-n_4')}}{\cosh \epsilon} \times \begin{cases} 
1 & \text{for } n_4 > n_4' \\
0 & \text{for } n_4 \leq n_4' 
\end{cases} \quad |\epsilon| < \mu
\]

Again, we change over to time labels \( \tau, \tau' \) and split the summation over time according to (6). In particular, the intertwining operation analogous to (7) now reads

\[
G_0 = \exp \sum_{p} \frac{e^{\epsilon + \mu}}{\cosh \epsilon} \left( \sum_{\tau \geq 0} e^{-(\epsilon + \mu) \tau} \frac{\partial}{\partial \psi_{p,\tau}} \right) \left( \sum_{\tau' \geq 0} e^{-(\epsilon + \mu) \tau'} \frac{\partial}{\partial \psi_{p,\tau'}} \right)
\]

\[
\times \exp \sum_{p} \frac{e^{\epsilon - \epsilon}}{\cosh \epsilon} \left( \sum_{\tau \geq 0} (-1)^\tau e^{(\epsilon - \mu) \tau} \frac{\partial}{\partial \psi_{p,\tau}} \right) \left( \sum_{\tau' \geq 0} (-1)^\tau' e^{(\epsilon - \mu) \tau'} \frac{\partial}{\partial \psi_{p,\tau'}} \right)
\]

\[
\times \exp \sum_{p} \frac{e^{-\epsilon - \mu}}{\cosh \epsilon} \left( \sum_{\tau \geq 0} (-1)^\tau e^{(\mu - \epsilon) \tau} \frac{\partial}{\partial \psi_{p,\tau}} \right) \left( \sum_{\tau' \geq 0} (-1)^\tau' e^{(\mu - \epsilon) \tau'} \frac{\partial}{\partial \psi_{p,\tau'}} \right)
\]

\[
\times \exp \sum_{p} \frac{e^{-\epsilon - \mu}}{\cosh \epsilon} \left( \sum_{\tau \geq 0} e^{(\epsilon + \mu) \tau} \frac{\partial}{\partial \psi_{p,\tau}} \right) \left( \sum_{\tau' \geq 0} e^{(\epsilon + \mu) \tau'} \frac{\partial}{\partial \psi_{p,\tau'}} \right)
\]

From this we can identify the relevant Grassmann variables at the \( G_0 \) stage of the scalar product. The procedure is largely the
same as for asymmetric fermions. The rules of substitution for differential operators are

\[
\sum_{\tau > 0} e^{-\nu \tau} \frac{\partial}{\partial \psi_{p,\tau}} \to \frac{\partial}{\partial \Psi^+_p} \quad \sum_{\tau > 0} e^{-\nu \tau} \frac{\partial}{\partial \psi_{p,\tau}} \to \frac{\partial}{\partial \Psi^-_p}
\]

\[
\sum_{\tau > 0} (-1)^\tau e^{-\nu \tau} \frac{\partial}{\partial \psi_{p,\tau}} \to \frac{\partial}{\partial \Psi^-_p} \quad \sum_{\tau > 0} (-1)^\tau e^{-\nu \tau} \frac{\partial}{\partial \psi_{p,\tau}} \to \frac{\partial}{\partial \Psi^+_p}
\]

In ket functionals, Grassmann variables are replaced as follows:

\[
\begin{align*}
\psi_{p,\tau} &\to (-1)^\tau e^{(\mu - \nu) \tau} \Psi^-_p & \psi_{p,\tau} &\to e^{-(\nu + \mu) \tau} \Psi^+_p \\
|\psi| < \mu & \quad (\tilde{\psi}_{p,\tau} \text{ irrelevant}) & \tilde{\psi}_{p,\tau} &\to e^{(\nu - \mu) \tau} \tilde{\psi}^+_p + (-1)^\tau e^{(\nu + \mu) \tau} \tilde{\psi}^-_p \\
\epsilon < -\mu & \quad \tilde{\psi}_{p,\tau} \to e^{(\nu + \mu) \tau} \tilde{\psi}^+_p & \tilde{\psi}_{p,\tau} &\to (-1)^\tau e^{(\nu + \mu) \tau} \tilde{\psi}^-_p
\end{align*}
\]

The baryon number as a path-integral observable \([1, 2]\) on the zero-time slice is

\[
B = \frac{\partial S}{\partial \mu} = \sum_p \left( e^\mu \tilde{\psi}_{p,-1} \psi_{p,1} + e^{-\mu} \tilde{\psi}_{p,1} \psi_{p,-1} \right)
\]

An intermediate result for the baryon number operator at \(G_0\) level is

\[
\hat{B} = \frac{1}{2} e^\mu \sum_{p: |\epsilon| > \mu} \left( \frac{1}{\cosh \epsilon} \frac{\partial}{\partial \Psi^-_p} \right) \left( e^{-\nu - \mu} \Psi^+_p + e^{\nu - \mu} \frac{\partial}{\partial \Psi^-_p} \right) - \frac{1}{2} e^{-\mu} \sum_{p: |\epsilon| > \mu} \left( - \frac{1}{\cosh \epsilon} \frac{\partial}{\partial \Psi^-_p} + \Psi^+_p \right) \left( - e^{\nu + \mu} \Psi^-_p + e^{\nu + \mu} \frac{\partial}{\partial \Psi^-_p} \right) + \frac{1}{2} e^\mu \sum_{p: |\epsilon| < \mu} \left( \frac{1}{\cosh \epsilon} \frac{\partial}{\partial \Psi^-_p} - \frac{1}{\cosh \epsilon} \frac{\partial}{\partial \Psi^-_p} \right) \left( e^{-\nu - \epsilon} \Psi^+_p - e^{\nu - \epsilon} \Psi^-_p \right) - \frac{1}{2} e^{-\mu} \sum_{p: |\epsilon| < \mu} \left( \Psi^+_p + \Psi^-_p \right) \left( e^{\nu + \epsilon} \frac{\partial}{\partial \Psi^+_p} + e^{\nu - \epsilon} \frac{\partial}{\partial \Psi^-_p} \right) + \frac{1}{2} e^\mu \sum_{p: |\epsilon| < \mu} \left( - \frac{1}{\cosh \epsilon} \frac{\partial}{\partial \Psi^-_p} + \Psi^+_p \right) \left( - e^{\nu - \epsilon} \Psi^-_p + e^{\nu - \epsilon} \frac{\partial}{\partial \Psi^-_p} \right) - \frac{1}{2} e^{-\mu} \sum_{p: |\epsilon| < \mu} \left( \frac{1}{\cosh \epsilon} \frac{\partial}{\partial \Psi^+_p} + \Psi^-_p \right) \left( e^{\nu + \epsilon} \Psi^+_p + e^{\nu - \epsilon} \frac{\partial}{\partial \Psi^-_p} \right)
\]

This actually simplifies to

\[
\hat{B} = \sum_{p: |\epsilon| > \mu} \left( \tilde{\psi}^-_p \frac{\partial}{\partial \Psi^-_p} - \psi^-_p \frac{\partial}{\partial \Psi^+_p} \right) + \sum_{p: |\epsilon| < \mu} \left( 1 - \psi^-_p \frac{\partial}{\partial \Psi^-_p} - \psi^+_p \frac{\partial}{\partial \Psi^+_p} \right) + \sum_{p: |\epsilon| < \mu} \left( \tilde{\psi}^+_p \frac{\partial}{\partial \Psi^+_p} - \tilde{\psi}^-_p \frac{\partial}{\partial \Psi^-_p} \right)
\]

\[
(16)
\]
The transfer matrix of Kogut-Susskind fermions is known to be not of the preferable form $e^{-\hat{H}}$ with $\hat{H}$ hermitian [10]. Nevertheless one can define a quantum field theory using the two-step Hamiltonian $\hat{H} = -\frac{1}{\hbar^2} \log T^2$. This is hermitian whenever $\hat{T}$ is. Thus we obtain, generalizing to $\mu > 0$,

$$\hat{H} - \mu \hat{B} = \sum_{p: \epsilon > \mu} \left( (\epsilon + \mu) \Psi^+_p \frac{\partial}{\partial \Psi^+_p} + (\epsilon - \mu) \Psi^-_p \frac{\partial}{\partial \Psi^-_p} \right)$$

$$+ \sum_{p: 0 < \epsilon < \mu} \left( (\epsilon + \mu) \Psi^+_p \frac{\partial}{\partial \Psi^+_p} + (\epsilon - \mu) \Psi^-_p \frac{\partial}{\partial \Psi^-_p} \right)$$

$$+ \sum_{p: \epsilon < -\mu} \left( (\epsilon + \mu) \Psi^-_p \frac{\partial}{\partial \Psi^-_p} + (\epsilon - \mu) \Psi^+_p \frac{\partial}{\partial \Psi^+_p} \right)$$

This can be rewritten in Fock form, using $i = \pm$ and $\epsilon_m = \text{arsinh} \left( \frac{\alpha_n}{\sigma_n} \right)$, by identifying antibaryon annihilation and creation operators as

$$\bar{b}^\pm = \partial/\partial \Psi^\pm \quad \bar{b}^\pm = \Psi^\pm \quad \epsilon > \epsilon_m$$

and the corresponding baryon operators as

$$b^\pm = \partial/\partial \Psi^\pm \quad b^\pm = \Psi^\pm \quad \epsilon > \epsilon_m$$

$$b^\pm = \Psi^\pm \quad b^\pm = \partial/\partial \Psi^\pm \quad \mu > \epsilon > \epsilon_m$$

We then have

$$\hat{H} - \mu \hat{B} = \sum_{p: \epsilon > \mu} \sum_{\epsilon} \left( (\epsilon - \mu) b_p^* \bar{b}^\pm_p + (\epsilon + \mu) \bar{b}_p^* \bar{b}^\pm_p \right) + \sum_{\epsilon > \epsilon_m} \sum_{\mu > \epsilon} \left( \epsilon - \mu \right)$$

We note that creation and annihilation operators are only defined for multi-indices $p$ with a positive energy, and that for the Grassmann variables this translates, depending on the baryon number and flavour, into either positive or negative spinorial “energy”.

As in the asymmetric case, the $b^\pm$ and $\bar{b}^\pm$ are only proportional to the hermitian adjoints of $b^\pm$ and $\bar{b}^\pm$:

$$\bar{b}^\pm = e^{-\epsilon - \mu} \cosh \epsilon \bar{b}^\pm \quad \epsilon > \epsilon_m$$

$$b^\pm = e^{-\epsilon + \mu} \cosh \epsilon b^\pm \quad \epsilon > \mu \quad b^\pm = e^{-\epsilon + \mu} \cosh \epsilon b^\pm \quad \mu > \epsilon > \epsilon_m$$

This does not affect the hermiticity of $\hat{H}$.

As with asymmetric fermions, the ground state functional is $G_{\mu}^{\Psi, \bar{\Psi}} = 1$. In terms of creation and annihilation operators this implies the same relations as in (15) for both of the time-like flavours. The ground state has a nonzero baryon density if $\sinh \mu > \frac{\alpha_n}{\sigma_n} \mu$. Thus, again, $\mu$ and $m$ are related in a nonlinear way, reminding us of a degree of ambiguity in the quantification of chemical potentials on the lattice [16, 17].
5 Conclusions

We have elaborated on how the sign of the "energy" of Grassmann functional integral variables is associated with the Fock-space degrees of freedom in the two cases of Kogut-Susskind and time-asymmetric lattice fermions. Our main conclusion is that the usual implementation of a chemical potential for the baryon number, namely through real-valued propagation factors in the discretized time derivative for any kind of fermions, will indeed enhance or deplete the baryons and antibaryons just as it is supposed to do.

It is quite plausible that the role of the spinorial "energy" is not altogether the same for Kogut-Susskind fermions as it is with continuous Euclidean time. For continuous time, the positivity of the transfer matrix is guaranteed already by the basic requirement (2) for the very existence of an underlying quantum-statistical model [11, 8]. By contrast, the Kogut-Susskind transfer matrix has both positive and negative eigenvalues at any time-like lattice spacing [10]. In fact, the hermiticity of the energy in this case is accomplished by simply discarding the sign of the negative eigenvalues of $\hat{T}$, namely in defining $\hat{H} = -\frac{1}{2} \log \hat{T}^2$ instead of $\hat{H} = -\log \hat{T}$. That redefinition of the energy is avoided with Wilson fermions [8]. It can also be avoided with the asymmetric time-derivative of section 2, provided that the temporal lattice spacing is sufficiently smaller than the spatial one. For non-interacting fermions the condition is $a_t < a_s / \sqrt{3} + m^2$; similar bounds exist for fermions interacting with a compact gauge field.

In computational practice, and related analytical study such as [2, 16, 5], the energy expectation value $\varepsilon$ of lattice fermions at finite temperature is not derived from the logarithm of the transfer matrix, $\varepsilon = \text{tr} \hat{H} e^{-\hat{N} s}$, but by differentiating the action with respect to the temporal lattice spacing. In terms of Fock space operators this can be obtained from $\varepsilon = \text{tr} \hat{E} T^{N_s} = \text{tr} \hat{E} e^{-\hat{H} N_s}$ where $\hat{E}$ is the operator reconstructed from the path-integral observable $\sum_p \bar{\psi}_{p,1} E(p) \psi_{p,1}$ for one-step fermions, or $\frac{1}{2} \sum_p (\bar{\psi}_{p,-1} E(p) \psi_{p,-1} + \bar{\psi}_{p,1} E(p) \psi_{p,1})$ for Kogut-Susskind fermions. In fact, $\hat{E}$ and $\hat{H}$ are different only for momenta comparable to the UV cutoff so that the use of $\hat{E}$ to obtain an energy density is certainly justified by its computational advantages.

It is reassuring to find that particles and antiparticles are safely recognized by their internal transformation properties (of which we have considered the baryon number) and not by any kind of spinorial energy. The latter would be an ill-defined concept, indeed, in the presence of dynamical gauge fields. The Hamiltonian of interacting fermions can only be reconstructed...
as part of the complete, translation-invariant fermion-boson field theory. Naturally, such a reconstruction had to emphasize local terms (as it did in [7, 8, 10]) rather than global diagonalizations.

As the time-asymmetric action could provide an exact algorithm with two flavours, it will be interesting to study it further in the SU(3) interactive case, in concordance with our view that the early onset problem could be due to the excessive proliferation of degenerate flavours.

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References


