Critical behavior in $c = 1$ matrix model with branching interactions

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Abstract

Motivated by understanding the phase structure of $d > 1$ strings we investigate the $c = 1$ matrix model with $g'(\text{tr} M(t))^2$ interaction which is the simplest approximation of the model expected to describe the critical phenomena of the large-$N$ reduced model of odd dimensional matrix field theory. We find three distinct phases: (i) ordinary $c = 1$ gravity phase, (ii) a branched polymer phase and (iii) an intermediate phase. Further we can also analyse the one with slightly generalized $g^{(2)}(\text{tr} M^2(t))^2 + g^{(3)}(\text{tr} M^2(t))^3 + \cdots + g^{(n)}(\text{tr} M^2(t))^n$ interaction. As a result the multi-critical versions of the phase (ii) and (iii) are found.

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1 Introduction

Matrix models are very powerful tools so far in the analysis of $d(\leq 1)$-dimensional noncritical strings or two-dimensional quantum gravity coupled to the conformal matter with the central charge $c \leq 1$. They have given us many informations about string susceptibility exponents, various correlation functions of scaling operators, non-perturbative structures obtained from double scaling limit and so on [1].

But in $d > 1$ case the corresponding matrix models cannot be exactly solvable at the present, thus it is necessary to exploit techniques for at least approximately solving [2] [3] [4]. Recently Alvarez-Gaumé et.al. tried the analysis of higher dimensional matrix models by using the large-$N$ reduced model [5] [6]. The $d$-dimensional (lattice) matrix field theory we consider is defined by in $d=\text{even}$ case

$$Z_{(d=\text{even})} = \int \prod_x dM(x) \exp \left[ -N \sum_x \frac{1}{2} \sum_{\mu=1}^d \left( M(x + \mu) - M(x) \right)^2 
+ \frac{m^2}{2} M(x)^2 + \frac{g}{4} M(x)^4 \right],$$

(1)

where $x$ is a site on $d$-dimensional hypercubic lattice and $M(x)$ is a $N \times N$ hermitian matrix. In $d=\text{odd}$ case, we decompose space-time as $\mathbb{R} \times (d-1)$-dim. lattice, and the theory is defined by

$$Z_{(d=\text{odd})} = \int \prod_x D M(t, x) \exp \left[ -N \sum_x \int dt \frac{1}{2} \left( \frac{\dot{M}(t, x)^2}{M(t, x)^2} + \frac{m^2}{2} M(t, x)^2 + \frac{g}{4} M(t, x)^4 \right) \right],$$

(2)

where $t$ is a continuous parameter and $x$ is a site on $(d-1)$-dimensional lattice. This theory is free from divergence because of an ultra-violet cutoff (lattice) and an infra-red cutoff (mass $m$). In large-$N$ limit eq.(1) reduced to a one-matrix model

$$Z_{(d=\text{even})}^r = \int dM \exp \left[ -N \text{tr} \left( \frac{1}{2} \sum_{\mu=1}^d (\Gamma_\mu M T^\dagger_\mu - M)^2 + \frac{m^2}{2} M^2 + \frac{g}{4} M^4 \right) \right]$$

(3)

and eq.(2) to a matrix quantum mechanics

$$Z_{(d=\text{odd})}^r = \int D M(t) \exp \left[ -N \int dt \frac{1}{2} \dot{M}(t)^2 + \frac{1}{2} \sum_{\mu=1}^{d-1} (\Gamma_\mu M(t) \Gamma^\dagger_\mu - M(t))^2 + \frac{m^2}{2} M(t)^2 + \frac{g}{4} M(t)^4 \right]$$

(4)
where \( \Gamma_\mu \)'s are traceless \( SU(N) \) matrices commuting only up to an element of the center of \( SU(N) \),
\[
\Gamma_\mu \Gamma_\nu = Z_{\nu \mu} \Gamma_\nu \Gamma_\mu, \quad Z_{\mu \nu} = e^{2\pi i n_{\mu \nu}/N},
\]
and the integers \( n_{\mu \nu} \) are defined mod \( N \) [7] [8]. The dimensionality of the lattice is completely reduced, but as the price of which the twist matrices \( \Gamma_\mu \) must be introduced. In spite of this simplification, due to the twist matrices an angular integration cannot be performed exactly, so we have to use some approximation. If we carry out the angle integral term by term in the expansion of a hopping term \( N \sum_{\mu=1}^d \text{tr}(\Gamma_\mu M \Gamma_\mu^\dagger M) \) in eq.(3), infinite terms of type \( (\sum_{i=1}^N \lambda_i^k)(\sum_{j=1}^N \lambda_j^k) \cdots \) are induced. \( (\lambda_i \)'s are eigenvalues of \( M \).) In order to get the knowledge about the theory (3) Alvarez-Gaume et.al. investigated the model restricting the induced terms to finite and suggested the rich phase structure.

In this paper we are interested in the theory (4). In order to get the hints for the phase structure we exactly solve \( c = 1 \) matrix model with \( (\text{tr} M(t)^2)^2 \) interaction, which is obtained as a simplest nontrivial approximation of the result of angular integration in (4). The model is described by the action
\[
S = \int_{-T/2}^{T/2} dt \left[ N \text{tr} \left( \frac{1}{2} \dot{M}(t)^2 + \frac{1}{2} M(t)^2 + g M(t)^4 \right) + g' (\text{tr} M(t)^2)^2 \right].
\]
We can take a continuum limit by remaining \( g' \) fixed and approaching \( g \) to \( g_c(g') \) a point on a critical line in \( g - g' \) plane. Then we find three distinct phases, which is characterized by the behavior of string susceptibility
\[
\chi = \lim_{T \to \infty} \frac{1}{TN^2} \frac{\partial^2 \ln Z}{\partial g^2} \Bigg|_{g' \text{fixed}},
\]
(i) a \( c = 1 \) gravity phase \( -\chi \sim 1/\ln(g - g_c(g')) \),
(ii) a branched polymer phase \( -\chi \sim (g - g_c(g'))^{-1/2} \), and
(iii) a phase in between (i) and (ii) \( -\chi \sim \ln(g - g_c) \). Especially the phase (iii) is interesting. It seems that it suggests the existence of a continuum theory of \( c > 1 \) matter coupled to gravity.

We can also analyze the model with slightly generic interaction
\[
g^{(2)} (\text{tr} M^2(t))^2 + g^{(3)} (\text{tr} M^2(t))^3 + \cdots + g^{(n)} (\text{tr} M^2(t))^n
\]
similarly.

2. **The reduced model for \( d = \text{odd} \)**

In eq.(4) the term-by-term angle integral with respect to a hopping term
\[
-N \int dt \text{tr} \left[ \frac{1}{2} \dot{M}(t)^2 - \sum_{\mu=1}^{d-1} \Gamma_\mu M(t) \Gamma_\mu^\dagger M(t) \right]
\]
induces derivative coupling terms, say
\[ \int dt (\sum_{i=1}^{N} \dot{\lambda}_i(t)^2)(\sum_{i=1}^{N} \lambda_i(t)^i) \cdots, \]
as well as the interactions with no derivatives. Since we are interested in critical (infra-red) properties of the system, we may expect that the derivative terms are irrelevant. Thus we will consider the $c = 1$ matrix model containing no derivative interactions of the form
\[ S = \int dt \left[ \text{tr} \left( \frac{1}{2} \dot{M}(t)^2 + \frac{1}{2} M(t)^2 + g M(t)^4 \right) \right. \]
\[ + N^2 \sum_{k_1, \ldots, k_n} g_{k_1, \ldots, k_n} \frac{1}{N} \text{tr} M(t)^k \frac{1}{N} \text{tr} M(t)^l \cdots \].

It is very interesting to understand the phase structure of this model. For the purpose of this paper we shall consider as a simple approximation the following system
\[ Z = \int DM(t) e^{-S} \]
\[ S = \int dt \left[ \text{tr} \left( \frac{1}{2} \dot{M}(t)^2 + \frac{1}{2} M(t)^2 + g M(t)^4 \right) + g'(\text{tr} M(t)^2)^2 \right] \quad (9) \]
and solve it exactly in $N \to \infty$ limit.

We can give a geometrical interpretation for the term $g'(\text{tr} M(t)^2)^2$, similar to the $c = 0$ case [6]. It provides a touching (or branching) point between two surfaces, where the height $t$ remains unchanged. In general, the term
\[ N^2 g_{k_1, \ldots, k_n} \frac{1}{N} \text{tr} M(t)^{k_1} \cdots \frac{1}{N} \text{tr} M(t)^{k_n} \]
represents a touching of $n$ surfaces at a common point unchanging the height. Thus the model (9) describes a interacting random surface in one-dimension rather than free surfaces in the case of ordinary matrix models (containing only a single trace in the action) [9].

### 3 The exact solution of the model in the large-$N$ limit

Now we obtain the exact solution of the model (9) in the large-$N$ limit. Introducing a collective field
\[ \phi(x, t) = \frac{1}{N} \text{tr} \delta(x - M(t)), \quad (10) \]
and its conjugate momentum $\pi(x,t)$ satisfying the commutator

$$[\phi(x,t), \pi(y,t)] = i\delta(x-y),$$

(11)

the leading order in $N$ of the collective Hamiltonian reads [10]

$$H = \frac{1}{2N^2} \int dx (\partial_x \pi(x)) \phi(x) (\partial_x \pi(x))
+ N^2 \int dx \left[ \frac{\pi^2}{6} \phi(x)^3 + \left( \frac{1}{2} x^2 + g x^4 \right) \phi(x) \right]
+ N^2 g' \left( \int dx x^2 \phi(x) \right)^2
+ N^2 \mu_F \left( 1 - \int dx \phi(x) \right).$$

(12)

In the last term $\mu_F$ is a lagrange multiplier respect to the constraint

$$\int dx \phi(x) = 1.$$  

(13)

The saddle point solution for $\phi(x)$ is given by

$$\phi_0(x) = \frac{1}{\pi} \sqrt{2(\mu_F - U(x))},$$

(14)

where using the second moment of $\phi$

$$c = \int dx x^2 \phi(x)$$

(15)

the effective potential $U(x)$ is written as

$$U(x) = \left( \frac{1}{2} + 2g'c \right) x^2 + g x^4.$$  

(16)

The range of $x$ is in the interval $(-x_-, x_+)$. The parameter $x_-$ ($x_+$) defined as the smaller (bigger) one of the positive solutions of an equation

$$\mu_F - U(x) = 0$$

(17)

with $g < 0$. For a while we should proceed the argument in the case of $g < 0$.

Substituting the saddle point value $\phi_0(x)$, using the elliptic integrals $E(k), K(k)$ with the modulus $k$ defined by

$$k^2 = \frac{x_-}{x_+} = \frac{1 + 4g'c - \sqrt{(1 + 4g'c)^2 + 16g\mu_F}}{1 + 4g'c + \sqrt{(1 + 4g'c)^2 + 16g\mu_F}}$$

(18)

and introducing the function

$$f(k) = \frac{2\sqrt{2}}{3\pi} \left[ K(k)(k^2 - 1) + E(k)(k^2 + 1) \right]$$

(19)
we obtain from (13)
\[ \mu_F = \left( \frac{1}{f(k)} \right)^{4/3} (-g)^{1/3} k^2 \]  
(20)
and from (15)
\[ c = \frac{2\sqrt{2}}{15\pi} (-2g)^{-1/3} \left( \frac{1}{f(k)} \right)^{5/3} \times \left( K(k)(-k^4 + 3k^2 - 2) + E(k)(-2k^4 + 2k^2 - 2) \right) . \]  
(21)
Of course the formula of the free energy also can be written, but the following analysis leads to very tedious calculations. Fortunately without it we can see the string susceptibility from eq.(21) as below. The string susceptibility \( \chi \) (7) is nothing but a two point connected correlator of \( trM^4 \) operators
\[ \chi = \lim_{T \to \infty} \frac{1}{T} < \int_{-T/2}^{T/2} dt trM(t)^4 \int_{-T/2}^{T/2} dt' trM(t')^4 > \text{conn.} \]  
(22)
on the other hand \( \frac{\partial c}{\partial g} \) is a connected two point function of \( trM^2 \) and \( trM^4 \) operators
\[ - \frac{\partial c}{\partial g} = \lim_{T \to \infty} \frac{1}{T} < \int_{-T/2}^{T/2} dt trM(t)^2 \int_{-T/2}^{T/2} dt' trM(t')^4 > \text{conn.} \]  
(23)
Thus due to universality, these two quantities must exhibit a same critical behavior as a connected correlator of two puncture operators.

From eqs.(18) and (21) we get the relation of \( g \) and \( g' \) with \( k \)
\[ \frac{2\sqrt{2}}{15\pi} g' = \frac{(-g)^{1/3} f(k)^{5/3} + 2(1 + k^2)(-g)f(k))}{2(k^4 - k^2 + 1) E(k) + (-k^4 + 3k^2 - 2) K(k)} . \]  
(24)
This equation determines a curve \( g = g(k, g') \) in \( g - g' \) plane for any fixed value of \( k \). In eq.(21) the singular behavior of \( c \) can come from the following two roots:

1. the singularity of \( K(k) \) when \( k \to 1 \)
2. the singularity of the \( g \)-dependence of \( k \) determined by eq.(24) when \( \frac{\partial g}{\partial k}|_{g' \text{fixed}} = 0 \).

The singularity of type (1) forms a critical line (i) \( g = g(1, g') \) in the phase diagram (fig.1). About that of type (2), its critical line (ii) is nothing but an envelope of family of the curves with parameter \( k \) \( \{ g = g(k, g') \}_{0 < k < 1} \). And boundary of the critical lines (i) and (ii) \( g = g' = 5\sqrt{5}/36\sqrt{3}\pi \) forms a critical point (iii) by itself.

We investigate the critical behavior when \( k \to 1 \). As a result of expansion of \( g \) and \( k \) about \( g_c(g') = g(1, g') \) and 1, eqs.(21) and (24) become respectively
\[ \frac{g - g_c}{-g_c} = \frac{15}{8} \frac{g' - g_c}{g' - 10g_c} k'^4 \ln \frac{4}{k'} - \frac{15}{32} \frac{5g' + g_c}{g' - 10g_c} k'^4 + O \left( k'^6 \ln \frac{4}{k'} \right) \]  
(25)
$$e = \frac{1}{5} \left( \frac{3\pi}{4\sqrt{2}} \right)^{2/3} (-g_c)^{-1/3} \times \left[ 1 + \frac{45}{8} \frac{g_c}{g' - 10g_c} k'^4 \ln \frac{4}{k'} \frac{255}{32} \frac{g_c}{g' - 10g_c} k'^4 + O \left( k'^6 \ln \frac{4}{k'} \right) \right]$$

(26)

where $k'^2 = 1 - k^2$.

Therefore the critical behavior of $\chi$ is

$$\chi \sim -\frac{\partial e}{\partial y} = \frac{\partial c}{\partial y} \begin{cases} \frac{\text{const}}{\ln \left| g - g_c \right|} & (g' > g_c(g')) \\ \text{const.} \ln (g - g_c) & (g' = g_c(g')) \end{cases} \ .$$

(27)

Here, the case of $g' > g_c(g')$ realizes the phase on the critical line (i) which is ordinary $c = 1$ gravity phase, and $g' = g_c(g')$ case corresponds to the critical point (iii). The susceptibility exponent $\gamma_{str}$ defined by $\chi \sim (g - g_c)^{-\gamma_{str}}$ is zero in the both phases, however the logarithmic corrections differently appear.

The behavior near the envelope (ii) is easily seen by expanding $g$ and $k$ about the values on the envelope $g_c(g'), k_c(g')$.

$$g(k, g') - g_c = \frac{\partial^2 g}{\partial (k^2)^2} \bigg|_{g' \text{ fixed}} \left( k^2 - k_c^2 \right)^2 + \cdots \ .$$

(28)

It can be shown to be

$$\chi \sim \frac{\text{const}}{\sqrt{g - g_c}} \ ,$$

(29)

thus we recognize that $\gamma_{str} = 1/2$ and the line (ii) exhibits branched polymer phase [11] [12].

The similar analysis can be done for positive $g$ and the branched polymer phase can be found on the line connected to (ii).

4 Discussion

With the view of understanding the phase structures of $d > 1$ strings we have investigated the $c = 1$ matrix model with $(\text{tr} M(t)^2)^2$ interaction and as a result the three phases are found. Further we can apply the above analysis to the slightly generalized model defined by the following action

$$S = N \int \left[ \frac{1}{2} \dot{M}(t)^2 + \frac{1}{2} M^2(t) + g M(t)^4 \right]$$

$$+ g^2 (\text{tr} M^2(t))^2 + g^3 (\text{tr} M^2(t))^3 + \cdots + g^{[n]} (\text{tr} M^2(t))^n \ ,$$

(30)

7
by replacing $2g^e$ in eqs.(16) and (18) to $2g^{(2)}c + 3g^{(3)}c^2 + \cdots + ng^{(n)}c^{n-1}$. If the couplings $g^{(1)}, \ldots, g^{(n)}$ are tuned as

$$\frac{\partial g}{\partial (k^2)} = \frac{\partial^2 g}{\partial (k^2)^2} = \cdots = \frac{\partial^n g}{\partial (k^2)^n} = 0 \quad (\text{at } k = k_c) \quad (31)$$

and $g$ is analytic with respect to $k^2$ near $k = k_c$, then $\partial g/\partial k^2 \sim (k^2 - k_c^2)^{j} \sim (g - g_c)^{j/(j+1)}$. For $k_c \neq 1$ (corresponding to the multi-critical version of the phase (ii) in the previous model ) since $c$ is nonsingular, $\chi$ behaves as

$$\chi \sim (g - g_c)^{-j/(j+1)}. \quad (32)$$

Also for $k_c = 1$ (corresponding to the multi-critical version of the phase (iii) )

$$\frac{\partial c}{\partial k^2} = (\text{const})k^2 \ln \frac{4}{k^2} = (\text{const})(g - g_c)^{1/(j+1)} \ln(g - g_c), \quad (33)$$

therefore the susceptibility is

$$\chi \sim (g - g_c)^{(1-j)/(j+1)} \ln(g - g_c). \quad (34)$$

Here $j$ is an integer which takes $1, \ldots, n$.

The intermediate phase (iii) is expected to have a possibility realizing well-defined $d > 1$ continuum string theory. In order to have any insights about it, it would be necessary to show that there is a phase with respect to some $j$ where macroscopic loops have non-zero expected value, as done in ref. [5].

References


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**Figure Caption**

Fig.1. Phase diagram of the theory defined by eq.(9). The curve is the critical 
line of the theory. The piece of the curve (i) belonging to $g' > 5\sqrt{5}/36\sqrt{3}\pi$ comes 
from the singularity of type (1) and corresponds to the ordinary $c = 1$ gravity phase. 
The $g' < 5\sqrt{5}/36\sqrt{3}\pi$ piece (ii), from the singularity of the type (2), corresponds to 
the branched polymer phase. The point between (i) and (ii) $g = g' = 5\sqrt{5}/36\sqrt{3}\pi$ 
is a critical point with respect to an intermediate phase.