Harvard-Smithsonian Center for Astrophysics

Preprint Series

No. 3782

(Received January 18, 1994)

A SIMPLE DERIVATION OF THE ASYMPTOTIC CASIMIR INTERACTION OF A PAIR OF FINITE SYSTEMS

Larry Spruch

Department of Physics, New York University
and

Harvard-Smithsonian Center for Astrophysics

James F. Babb

Harvard-Smithsonian Center for Astrophysics

and

Fei Zhou

Department of Physics, New York University

To appear in
Physical Review A

HARVARD COLLEGE OBSERVATORY

SMITHSONIAN ASTROPHYSICAL OBSERVATORY

60 Garden Street, Cambridge, Massachusetts 02138

SMITHSONIAN ASTROPHYSICAL OBSERVATORY
Centennial Year 1990

Center for Astrophysics
Preprint Series No. 3782

**A SIMPLE DERIVATION OF THE ASYMPTOTIC CASIMIR
INTERACTION OF A PAIR OF FINITE SYSTEMS**

Larry Spruch
Department of Physics, New York University
and
Harvard-Smithsonian Center for Astrophysics

James F. Babb
Harvard-Smithsonian Center for Astrophysics

and

Fei Zhou
Department of Physics, New York University

A simple derivation of the asymptotic Casimir interaction of a pair of finite systems

Larry Spruch^{††}, James F. Babb[†] and Fei Zhou[†]

[†]*Department of Physics, New York University, New York, New York 10003*

^{††}*Institute for Theoretical Atomic and Molecular Physics, Harvard-Smithsonian Center for Astrophysics, 60 Garden Street, Cambridge, MA 02138*

Even if all constituents move slowly, the interaction between a pair of systems predicted by non-relativistic theory can differ drastically at large separations of the systems from the true interaction, that which takes into account the finite speed of the virtual electromagnetic waves (generated by vacuum fluctuations) responsible for the interaction at large separations. We here adapt, improve, and generalize a rough physical picture presented previously, and arrive at exact results for the asymptotic interaction, $V(r)$. One need use neither the Schrödinger equation nor the Dirac equation; *all* that need be known of either system is its electric dipole polarizability. [For a neutral system, one needs the static value $\alpha(\omega = 0)$ for a spherically symmetric system and the static tensor $\alpha_{ij}(\omega = 0)$ for a non-spherically symmetric system; the $\alpha(\omega)$ for an electron is $-e^2/m\omega^2$]. Apart from its simplicity and the insights it leads to, the approach provides a uniform derivation for a number of known results, and one result not previously given, the asymptotic interaction of a distant electron and a cylindrically symmetric core, a Rydberg H_2 molecule and a Rydberg carbon atom, with their cores in their ground states, being prototypes.

PACS numbers: 31.30.Jv, 31.50.+w, 31.90.+s

I. INTRODUCTION

The retarded (long-range, Casimir) interactions between various pairs of systems, for pairs chosen from atoms, molecules, ions, and electrons, have been obtained by a variety of methods. These include unitary transformations of the Hamiltonian [1], relativistic dispersion theory [2], time-ordered Feynman diagrammatic techniques through fourth order [3, 4] a Lagrangian formulation [5] and stochastic electrodynamics [6], all based on energy studies. There is also a study based on momentum considerations [7]. The electrons are studied using the Schrödinger or Dirac equations. Finite systems have also been studied by obtaining interactions involving dielectric walls, and then taking the walls to consist of widely separated atoms [8, 9, 10]. A recent and very nice review which includes, among other subjects, the surface mode approach, has been given by Milonni [11]. See also Levin and Micha [12].

The various pair interactions are known as a function of the separation of the systems, for separations anywhere from a few a_0 to ∞ . We will limit our considerations to asymptotic separations, and to pairs chosen from among electrons, atoms, ions, and molecules, and we will denote the Casimir interaction by $V(r)$.

Our purpose is two-fold. First, we note that most studies have considered a particular pair, while we would like to obtain a result—for the restricted conditions noted above—applicable to any of the various pairs. Second, we would like the approach to be conceptually simple, with the attendant physical insights, *and* with the extension to include non-spherical systems made obvious.

The method we will use is an adaptation and extension of one used earlier [13, 14, 15, 16]. The earlier version provided insights, and led to a single expression applicable to the various pair interactions; however, it relied very heavily, with incomplete justification, on a result obtained by other means [6]. We make three improvements on that earlier approach. First, that approach ignored the difference, for a given value of the momentum $\hbar k$ of a given mode, of the two states of polarization; we do not do so here. Second, since the significant contributions come from modes with λ (the wavelength divided by 2π) satisfying $\lambda \gtrsim r$, the field for each mode was taken to be the same at both systems; we do not now make that approximation. Third, there is no reliance—let alone incompletely justified reliance—on the results of others.

It may be instructive to observe that even though relativistic quantum-field-theoretic methods are used to obtain the asymptotic forms of the interactions, while non-relativistic methods using Coulomb potentials are all that are necessary to obtain short-range interactions, the former results have much simpler forms. The origin of the simplicity for large separations, large compared to the characteristic dimensions of either system, is the fact

that the dominant contributions come from $\lambda \gtrsim r$ —the ever-present factor $\exp i\mathbf{k} \cdot \mathbf{r}$ tends to average out for $kr = r/\lambda \gtrsim 1$ —and therefore $\lambda \gg D$, the larger of the two characteristic dimensions. We are therefore concerned with very low-frequency (virtual) photons, and it is well known that the interactions of such photons with a system are describable in terms of simple properties of the systems [17]. In our case, the relevant property of either system, *all* that need be known of it, is its electric dipole polarizability, the static scalar $\alpha(\omega = 0)$ for a neutral spherically symmetric atom or molecule, the static tensor $\alpha_{ij}(0)$ for a non-spherically symmetric system, and $\alpha(\omega) = -e^2/m\omega^2$ for a free or weakly bound electron. Compare, for example, the short-range and long-range interaction between two spherically symmetric atoms. In the course of the evaluation of the shorter-range r^{-6} van der Waals interaction there appear denominators which contain excitation energies of *both* atoms simultaneously; in the derivation of the long-range Casimir-Polder interaction [18], on the other hand, the denominators which appear contain excitation energies of one or the other atom, but not both, and the Casimir-Polder interaction for $r \sim \infty$ has the remarkably simple form given in Eq. (2.24) below. It is perhaps not then surprising that it is possible to give a relatively simple derivation of the asymptotic form of the interaction of a pair of finite systems.

II. THE ASYMPTOTIC INTERACTION OF TWO SPHERICALLY SYMMETRIC SYSTEMS

We begin by considering two spherically symmetric systems, labeled 1 and 2, centered at $\mathbf{r}_1$ and $\mathbf{r}_2$ with respect to an arbitrary origin, with dynamic polarizabilities $\alpha_1(\omega)$ and $\alpha_2(\omega)$, defined by Eq. (2.2) below. (See Fig. 1) The two systems are in a cube L on a side; L , the separation $r = |\mathbf{r}|$ (where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$) of the systems, and D , the larger of the characteristic dimensions of the systems, satisfy $L \gg r \gg D$, and the systems are far from any wall. The vacuum fluctuation electric field at an arbitrary point $\mathbf{X}$ has the standard form,

$$\begin{aligned} \mathbf{E}(\mathbf{X}) &= i \sum_q (2\pi\hbar\omega/\tau)^{1/2} \hat{\mathbf{e}}_q [a_q \exp(i\mathbf{k} \cdot \mathbf{X}) - a_q^\dagger \exp(-i\mathbf{k} \cdot \mathbf{X})] \\ &= \sum_q \mathbf{E}_q(\mathbf{X}), \end{aligned} \tag{2.1}$$

where $\tau = L^3$ is the volume of the box, $q \equiv (\mathbf{k}, \lambda)$ denotes a mode of the field, $\mathbf{k}$ is the wave vector, $\hat{\mathbf{k}} = \mathbf{k}/k$, $k = \omega/c$, and λ defines the state of polarization. $\hat{\mathbf{e}}_q$ is a real linear

polarization vector, and $a_q^\dagger$ and a_q are the standard creation and destruction operators. The $\alpha(\omega)$'s are defined by

$$\mathbf{p}_{iq} = \alpha_i(\omega)\mathbf{E}_q(\mathbf{r}_i), \quad i = 1 \text{ or } 2, \quad (2.2)$$

where $\mathbf{p}_{iq}$ is the electric dipole moment induced in the i -th system by $\mathbf{E}_q(\mathbf{r}_i)$ [Below, the notation $\mathbf{E}_{iq}$ will sometimes be used interchangeably with $\mathbf{E}_q(\mathbf{r}_i)$]. The $\alpha(\omega)$'s can be taken to be real. [This is often a good approximation. In the present context it is an excellent approximation, since the significant values of ω , $0 \lesssim \omega \lesssim c/r$, are arbitrarily small for ω arbitrarily large, and, finally, $\alpha(\omega \approx 0)$ is real.] Since $\mathbf{E}_q$ is real, it follows that the $\mathbf{p}_{iq}$'s are real.

The effect of the systems—in the presence of $\mathbf{E}$ —on one another, the interaction we seek, can be treated perturbatively. The perturbation interaction of the two systems generated by the mode q is

$$V_q(\mathbf{r}_1, \mathbf{r}_2) = \langle 0 | -\mathbf{p}_{1q} \cdot \mathbf{E}_q(2 \rightarrow 1) | 0 \rangle = \langle 0 | -\mathbf{p}_{2q} \cdot \mathbf{E}_q(1 \rightarrow 2) | 0 \rangle, \quad (2.3)$$

where $|0\rangle$ represents the photon vacuum state. $\mathbf{E}_q(i \rightarrow j)$ is the field at $\mathbf{r}_j$ generated by the induced dipole moment $\mathbf{p}_{iq}$ at $\mathbf{r}_i$. $\mathbf{E}_q(i \rightarrow j)$ represents a scattered wave. (Multiply scattered waves lead to higher order effects, as do induced multipole moments higher than dipole.) In Eq. (2.3), we can choose either form; we do not add the two forms, for to do so would be to double count. This is perhaps most easily seen by considering the classical interaction of the two systems induced by a uniform time-independent external field. The point warrants a further comment. Note the absence of a factor of $\frac{1}{2}$ in either of the two dipole-field interaction forms in Eq. (2.3); there is no factor of $\frac{1}{2}$ because, to leading order, the dipole moments $\mathbf{p}_{1q}$ and $\mathbf{p}_{2q}$ are independent of one another, having been induced by $\mathbf{E}_{1q}$ and $\mathbf{E}_{2q}$, respectively. Consider, however, an alternative derivation of Eq. (2.3) based on the energy of interaction of one of the system, say system 2, with the mode q and with system 1. That energy is

$$-\frac{1}{2}\alpha_2(\omega)[\mathbf{E}_{2q} + \mathbf{E}_q(1 \rightarrow 2)]^2, \quad (2.4)$$

where we have inserted a factor $\frac{1}{2}$ because we are here viewing the problem as the interaction of a dipole induced by a field with the field. In Eq. (2.4), the $\mathbf{E}_{2q}^2$ term is independent of $\mathbf{r}_1$ and does not contribute to $V_q(\mathbf{r}_1, \mathbf{r}_2)$, while the $\mathbf{E}_q^2(1 \rightarrow 2)$ term is of higher order. The relevant terms are the cross terms, which give

$$V_q(\mathbf{r}_1, \mathbf{r}_2) = \langle 0 | -\alpha_2(\omega)\mathbf{E}_{2q} \cdot \mathbf{E}_q(1 \rightarrow 2) | 0 \rangle, \quad (2.5)$$

which is equivalent to Eq. (2.3).

It follows from the properties of the creation and destruction operators that the systems must each interact with the same mode if they are to interact with one another. The interaction is therefore simply the sum over q ,

$$V(\mathbf{r}_1, \mathbf{r}_2) = \sum_q V_q(\mathbf{r}_1, \mathbf{r}_2) = \langle 0 | - \sum_q \alpha_2(\omega) \mathbf{E}_{2q} \cdot \mathbf{E}_q (1 \rightarrow 2) | 0 \rangle. \quad (2.6)$$

With $x \equiv kr$ and $\hat{\mathbf{n}} \equiv \mathbf{r}/r$, $\mathbf{E}_q(1 \rightarrow 2)$ is given by (see, for example, Eq. (9.18) of [19])

$$\mathbf{E}_q(1 \rightarrow 2) = \text{Re} \left(k^3 e^{ix} \left\{ [-(\mathbf{p}_1 \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}} + \mathbf{p}_1] \frac{1}{x} + [3(\mathbf{p}_1 \cdot \hat{\mathbf{n}}) \hat{\mathbf{n}} - \mathbf{p}_1] \left(\frac{1}{x^3} - \frac{i}{x^2} \right) \right\} \right). \quad (2.7)$$

Using Eqs. (2.1), (2.2), and (2.6), we arrive straightforwardly at

$$\begin{aligned} V(\mathbf{r}_1, \mathbf{r}_2) = V(\mathbf{r}) = \text{Re} \sum_{\mathbf{k}} \alpha_2(\omega) k^3 (2\pi\hbar\omega/\tau) e^{-i\mathbf{k} \cdot \mathbf{r}} e^{ix} \alpha_1(\omega) \\ \times \left[(-S + T) \frac{1}{x} + (3S - T) \left(\frac{1}{x^3} - \frac{i}{x^2} \right) \right] (-1). \end{aligned} \quad (2.8)$$

The (-1) is the value of

$$\langle 0 | -a_q a_q^\dagger | 0 \rangle, \quad (2.9)$$

the a_q^2 , $(a_q^\dagger)^2$, and $a_q^\dagger a_q$ terms vanishing. S and T are given by

$$S \equiv \sum_{\lambda} (\hat{\mathbf{e}}_q \cdot \hat{\mathbf{n}})^2 = 1 - (\hat{\mathbf{k}} \cdot \hat{\mathbf{n}})^2 = 1 - \cos^2 \theta = 1 - \mu^2, \quad (2.10)$$

and

$$T \equiv \sum_{\lambda} (\hat{\mathbf{e}}_q \cdot \hat{\mathbf{e}}_q) = 2, \quad (2.11)$$

where we let $\hat{\mathbf{n}}$ define the z axis, expressed $\mathbf{k}$ in terms of its spherical coordinates k, θ , and ϕ and set $\mu = \cos \theta$. We now make the replacements

$$\sum_{\mathbf{k}} \rightarrow \tau \int \frac{d^3 k}{(2\pi)^3} \rightarrow \frac{\tau}{(2\pi)^3} \int_0^\infty dk k^2 \int_{-1}^1 d\mu \times 2\pi, \quad (2.12)$$

using cylindrical symmetry in the second step. Since the factor multiplying $\exp(i\mathbf{k} \cdot \mathbf{r})$ is even in μ , we make the further replacement

$$e^{i\mathbf{k} \cdot \mathbf{r}} = e^{ix\mu} \rightarrow \cos x\mu. \quad (2.13)$$

Taking the real part leads to

$$V(r) = -\frac{\hbar c}{2\pi r^7} \lim_{\gamma \rightarrow 0} \int_0^\infty dx e^{-\gamma x} x^6 \alpha_1(xc/r) \alpha_2(xc/r) \int_{-1}^1 d\mu \cos(x\mu) [(1 - \mu^2)f(x) + 2g(x)], \quad (2.14)$$

where

$$f(x) = \frac{3}{x^2} \sin x + \left(\frac{3}{x^3} - \frac{1}{x} \right) \cos x, \quad (2.15)$$

$$g(x) = -\frac{1}{x^2} \sin x - \left(\frac{1}{x^3} - \frac{1}{x} \right) \cos x. \quad (2.16)$$

(To avoid a divergence, we introduced a convergence factor in Eq. (2.14). The introduction is justified by the fact that a proper formulation of $V(r)$ does not lead to a divergence; the divergence, at large ω , arises because the dipole approximation which we made fails for very high ω . The logical procedure is therefore to introduce a convergence factor before making the approximation, at which time it has no effect, and *then* to make the dipole approximation.)

The integration over μ is trivial, and gives the sought for result,

$$V(r) = -\frac{\hbar c}{\pi r^7} \lim_{\gamma \rightarrow 0} \int_0^\infty dx e^{-\gamma x} x^6 \alpha_1(xc/r) \alpha_2(xc/r) I(x), \quad (2.17)$$

where

$$I(x) \equiv \left(\frac{3}{x^6} - \frac{5}{x^4} + \frac{1}{x^2} \right) \sin 2x - \left(\frac{6}{x^5} - \frac{2}{x^3} \right) \cos 2x. \quad (2.18)$$

This form was first obtained [13] by recognizing, by a simple argument, that $V(r)$ was proportional to an integral which contained $\alpha_1(\omega)$ and $\alpha_2(\omega)$, and generalizing the form of the atom-atom potential—which contained $\alpha_1(0)$ and $\alpha_2(0)$ —obtained by Boyer [6] using a rather different approach; the static atomic polarizabilities $\alpha_1(0)$ and $\alpha_2(0)$ were simply taken inside the integral as dynamic polarizabilities, with the resultant $V(r)$ then applicable to electron-electron, electron-ion, and electron-atom pairs, as well as to atom-atom pairs. The replacement of $\alpha(0)$'s by $\alpha(\omega)$'s was a reasonable step, one which has been justified by the present analysis—which does not rely on Boyer's work—but the step must be exercised with care. There are forms of $V(r)$ for $r \sim \infty$ for an atom and a dielectric wall that are proportional to $\alpha(0)$ of the atom that do *not* give $V(r)$ for $r \sim \infty$ for an electron and a dielectric wall on taking $\alpha(0)$ inside the integral over ω as $\alpha(\omega) = -e^2/m\omega^2$ [10].

The various pair potentials now follow immediately. For two atoms we set

$$\alpha_1(\omega) = \alpha_1(0), \quad \alpha_2(\omega) = \alpha_2(0), \quad (2.19)$$

for an atom or ion (assumed to have infinite mass) and an electron we set

$$\alpha_1(\omega) = \alpha_1(0), \quad \alpha_2(\omega) = -e^2/m\omega^2, \quad (2.20)$$

and for two electrons we use

$$\alpha_1(\omega) = \alpha_2(\omega) = -e^2/m\omega^2. \quad (2.21)$$

(The interaction in the electron-electron case is much more complicated than for the first two cases. A proper analysis of the interaction of two charged particles was begun by Feinberg and Sucher, and has been continued by Sucher [20].) The various integrals can all be evaluated analytically. In particular, $\omega^6 I(\omega r/c) \sim \omega^5$ as $\omega \sim 0$, so that the integral in Eq. (2.17) is well behaved as $\omega \sim 0$ even for the electron-electron case, though some of the individual terms would diverge and must be combined before performing the integrations. The integration of the individual terms which do not diverge is given by

$$\begin{aligned} \lim_{\gamma \rightarrow 0} \int_0^\infty dx x^n e^{-\gamma x} \cos 2x &= \frac{n!}{2^{n+1}} (-1)^{(n+1)/2}, \quad n \text{ odd} \\ &= 0, \quad n \text{ even} \end{aligned} \quad (2.22)$$

and

$$\begin{aligned} \lim_{\gamma \rightarrow 0} \int_0^\infty dx x^n e^{-\gamma x} \sin 2x &= \frac{n!}{2^{n+1}} (-1)^{n/2}, \quad n \text{ even} \\ &= 0, \quad n \text{ odd.} \end{aligned} \quad (2.23)$$

In particular, one finds the Casimir-Polder interaction [18]

$$V_{\text{atom-atom}}(r) = -\frac{23}{4\pi} \frac{\alpha_1(0)\alpha_2(0)\hbar c}{r^7} \quad (2.24)$$

and the ion-electron interaction [3]

$$V_{\text{ion-el}}(r) = \frac{11}{4\pi} \frac{\alpha_2(0)\hbar e^2}{mcr^5}. \quad (2.25)$$

The result given in Eq. (2.25) is also valid for the interaction of an electron and a neutral atom [21]. Lundeen and co-workers have studied the Rydberg He atom intensively [22], but the n and l of the outer electron were not large enough for Eq. (2.25) to be valid. Analytic results have been obtained for $V_{\text{ion-el}}(r)$ for smaller values of r , and therefore of

n and l , (but for which there are still significant differences from the non-relativistic van der Waals results), and these analytic results [23, 24] have been confirmed to rather good accuracy [22].

In the above analysis we neglected the magnetic dipole moment induced by vacuum fluctuation magnetic fields, and the electric and magnetic fields generated by those magnetic moments. We also neglected the magnetic fields radiated by the induced electric dipole moments. The analysis could easily be adapted to include those effects; those effects are generally much smaller than the effect considered, that of the electric field generated by the induced electric dipole moment.

III. THE ASYMPTOTIC INTERACTION OF SPHERICALLY SYMMETRIC AND CYLINDRICALLY SYMMETRIC SYSTEMS

System 1 is assumed to be spherically symmetric, but system 2 is assumed to have cylindrical symmetry only. System 1 is at a very large distance r from system 2, at an angle θ_0 with respect to the symmetry axis of system 2, which is chosen to be the z axis. The anisotropic interaction between the two systems in the presence of the fluctuating electromagnetic field will be denoted by $V^{an}(r, \theta_0)$. It will often suffice to know only the average over θ_0 of $V^{an}(r, \theta_0)$, namely,

$$\begin{aligned}\bar{V}_{an}(r) &\equiv \int_0^{2\pi} V^{an}(r, \theta_0) \sin \theta_0 d\theta_0 / \int_0^{2\pi} \sin \theta_0 d\theta_0 \\ &= \frac{1}{2} \int_{-1}^1 V^{an}(r, \theta_0) d(\cos \theta_0).\end{aligned}\tag{3.1}$$

Ignoring all magnetic effects, it may well be expected on intuitive ground that $\bar{V}_{an}(r)$ can be obtained from the $V(r)$ for two spherically symmetric systems, given by Eq. (2.17), by the simple replacement

$$\alpha_2(\omega) \rightarrow \frac{\alpha_2^{\parallel}(\omega) + 2\alpha_2^{\perp}(\omega)}{3}\tag{3.2}$$

where $\alpha_2^{\perp}(\omega)$ and $\alpha_2^{\parallel}(\omega)$ are the dynamic electric dipole polarizabilities perpendicular and parallel, respectively, to the axis of symmetry of system 2; the factor 2 in front of $\alpha_2^{\perp}(\omega)$ is reasonable since there are two modes of vibration in the x, y plane to each mode of vibration parallel to the z axis. Since a symmetric system can be an atom, ion, or molecule

but not an electron, and since we are concerned with $r \sim \infty$ and therefore $\omega \sim 0$, we can replace $\alpha_2^\perp(\omega)$ by $\alpha_2^\perp(0)$ and $\alpha_2^\parallel(\omega)$ by $\alpha_2^\parallel(0)$. We may therefore expect to find

$$\bar{V}^{an}(r) = -\frac{\hbar c}{\pi r^7} \frac{\alpha_2^\parallel(0) + 2\alpha_2^\perp(0)}{3} \lim_{\gamma \rightarrow 0} \int_0^\infty dx e^{-\gamma x} x^6 \alpha_1\left(\frac{xc}{r}\right) I(x). \quad (3.3)$$

For system 1 a spherically symmetric atom or molecule, $\alpha_1(xc/r) = \alpha_1(\omega)$ can be taken outside the integral as $\alpha_1(0)$. The resultant interaction is then a special case of a more general result obtained by Craig and Power [25] in studies of the interaction of two neutral systems, neither of which need be spherically symmetric. [If both have cylindrical symmetry, one could make the same replacement for $\alpha_1(\omega)$ as given by Eq. (3.2) for $\alpha_2(\omega)$, and one could replace $\alpha_1^\perp(\omega)$ by $\alpha_1^\perp(0)$ and $\alpha_1^\parallel(\omega)$ by $\alpha_1^\parallel(0)$.] We will nevertheless give a formal proof of Eq. (3.3), for we wish to extend the result for $\bar{V}^{an}(r)$ to include the case in which system 1 is an electron, in which case $\alpha_1(xc/r)$ cannot be taken outside the integral. We will at the same time determine $V(r, \theta_0)$, a more detailed result which can be of interest.

We begin by noting that it is convenient to work with three sets of orthogonal axis. When both system were spherically symmetric, there was only one fixed axis, $\hat{n} = \mathbf{r}/r$, and we let $\hat{n}$ define the z axis. In the present context there are two fixed axes, $\hat{n} = \mathbf{r}/r$ and $\hat{e}_z$, where $\hat{e}_z$ is the symmetry axis of system 2. With $\hat{n}$ and $\hat{e}_z$ defining the xy plane, one set of axes is simply $\hat{e}_x$, $\hat{e}_y$ and $\hat{e}_z$. (See Fig. 2.) The second set is $\hat{u} = \hat{e}_y \times \hat{n}$, $\hat{e}_y$, and $\hat{n}$, and the third set is $\hat{k}$, $\hat{e}_k^{(1)}$, and $\hat{e}_k^{(2)}$, where $\mathbf{k}$ is the wave vector of the mode under consideration and the $\hat{e}_k^{(1)}$ and $\hat{e}_k^{(2)}$ are real linear polarization vectors. As for two spherically symmetric systems, there will appear the vector

$$e^{i\mathbf{k} \cdot (\mathbf{r}_2 - \mathbf{r}_1)} = e^{i\mathbf{k} \cdot \mathbf{r}} = e^{ix\hat{k} \cdot \hat{n}}, \quad (3.4)$$

where, as before, $x \equiv kr$. With $\mathbf{k} = k\hat{k} = (k, \theta, \phi)$, where θ and ϕ are defined relative to $\hat{u}$, $\hat{e}_y$ and $\hat{n}$, with $\hat{n}$ the polar axis, the exponent $ix\hat{k} \cdot \hat{n}$ is then $ix \cos \theta$. We do not initially demand that system 2 be cylindrically symmetric. We next note that, in the presence of an external field $\mathbf{E}_q(\mathbf{r}_2) \equiv \mathbf{E}_{2q}$, system 2 at $\mathbf{r}_2$ with dynamic electric dipole tensor polarizability coefficients $\alpha_{2ij}(\omega) = \alpha_{2ji}(\omega)$, where i and j run from 1 to 3, has an induced dipole moment $\mathbf{p}_{2q}$ with components

$$p_{2qi} = \sum_j \alpha_{2ij}(\omega) E_{2qj}. \quad (3.5)$$

The analog of Eq. (2.3) or Eq. (2.5) is

$$V_q(\mathbf{r}_1, \mathbf{r}_2) = \langle 0 | - \sum_i p_{2qi} E_{qi}(1 \rightarrow 2) | 0 \rangle$$

$$= \langle 0 | - \sum_{i,j} \alpha_{2ij}(\omega) E_{2qj} E_{qi} (1 \rightarrow 2) | 0 \rangle; \quad (3.6)$$

$E_{qi}(1 \rightarrow 2)$ is the i -th component of the field generated at $\mathbf{r}_2$ by the dipole moment $\mathbf{p}_1$ induced in system 1 by $\mathbf{E}_q(\mathbf{r}_1) \equiv \mathbf{E}_{1q}$. Straightforward but tedious algebra leads to

$$V_q(\mathbf{r}_1, \mathbf{r}_2) = \frac{2\pi\hbar\omega}{\tau} \alpha_1(\omega) e^{i\mathbf{k}\cdot\mathbf{r}} Q_q, \quad (3.7)$$

where

$$Q_q \equiv \sum_{i,j} \alpha_{2ij}(\omega) [\hat{n}_j \hat{\epsilon}_{qi} (\hat{\mathbf{n}} \cdot \hat{\epsilon}_q) f(x) + \hat{\epsilon}_{qi} \hat{\epsilon}_{qj} g(x)], \quad (3.8)$$

where $f(x)$ and $g(x)$ are defined by Eqs. (2.15) and (2.16). The indices i and j are with respect to the $\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z$ axes; $\hat{\epsilon}_{q2}$, for example, is the y component of $\hat{\epsilon}_q$. We use the notation $\hat{\epsilon}_{\mathbf{k}}^{(1)}$ for $\hat{\epsilon}_q$ with $q = (\mathbf{k}, \lambda = 1)$ and $\hat{\epsilon}_{\mathbf{k}}^{(2)}$ for $\hat{\epsilon}_q$ with $q = (\mathbf{k}, \lambda = 2)$.

For system 2 cylindrically symmetric, we have

$$\alpha_2(\omega) = \begin{pmatrix} \alpha_2^\perp(\omega) & 0 & 0 \\ 0 & \alpha_2^\perp(\omega) & 0 \\ 0 & 0 & \alpha_2^\parallel(\omega) \end{pmatrix}$$

and Q_q reduces to

$$Q_q = \alpha_2^\perp(\omega) \sum_{i=1,2} [\hat{n}_i \hat{\epsilon}_{qi} (\hat{\mathbf{n}} \cdot \hat{\epsilon}_q) f(x) + \hat{\epsilon}_{qi}^2 g(x)] \\ + \alpha_2^\parallel(\omega) [\hat{n}_3 \hat{\epsilon}_{q3} (\hat{\mathbf{n}} \cdot \hat{\epsilon}_q) f(x) + \hat{\epsilon}_{q3}^2 g(x)]. \quad (3.9)$$

Effectively, $\alpha_2^\perp(\omega)$ and $\alpha_2^\parallel(\omega)$ can be replaced by their static values $\alpha_2^\perp(0)$ and $\alpha_2^\parallel(0)$, respectively. It is convenient to rewrite Q_q as

$$Q_q = \frac{\alpha_2^\parallel(0) + 2\alpha_2^\perp(0)}{3} A_q - \frac{\alpha_2^\parallel(0) - \alpha_2^\perp(0)}{3} B_q. \quad (3.10)$$

We see by inspection that

$$A_q = (\hat{\mathbf{n}} \cdot \hat{\epsilon}_q)^2 f(x) + \hat{\epsilon}_q^2 g(x), \quad (3.11)$$

and

$$B_q = [(\hat{\mathbf{n}} \cdot \hat{\epsilon}_q)^2 - 3\hat{n}_3 \hat{\epsilon}_{q3} (\hat{\mathbf{n}} \cdot \hat{\epsilon}_q)] f(x) + [\hat{\epsilon}_q^2 - 3\hat{\epsilon}_{q3}^2] g(x). \quad (3.12)$$

Correspondingly, we have

$$V^{an}(r, \theta_0) = \bar{V}^{an}(r) + \Delta V^{an}(r, \theta_0), \quad (3.13)$$

where $\bar{V}^{an}(r)$, associated with A_q , is given by Eq. (3.1), and $\Delta V^{an}(r, \theta_0)$, associated with B_q , is

$$\begin{aligned} \Delta V^{an}(r, \theta_0) = & \frac{\alpha_2^{\parallel}(0) - \alpha_2^{\perp}(0)}{3} \frac{\hbar c}{2\pi r^7} \lim_{\gamma \rightarrow 0} \int_0^\infty dx \int_{-1}^1 d\mu \\ & \times \cos(x\mu) e^{-\gamma x} x^6 \alpha_1\left(\frac{xc}{r}\right) \int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{\lambda} B_q. \end{aligned} \quad (3.14)$$

Using the results given in the Appendix, we find

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{\lambda} B_q = (1 - 3\mu_0^2) \left[(1 - \mu^2) f(x) + \frac{1}{2} (1 - 3\mu^2) g(x) \right], \quad (3.15)$$

where $\mu_0 = \cos \theta_0$. We see immediately that, as expected, $\Delta V^{an}(r, \theta_0)$ vanishes if $\alpha_2^{\perp}(0) = \alpha_2^{\parallel}(0)$, and that, by virtue of that presence of the factor $1 - 3\mu_0^2$, the average of $\Delta V^{an}(r, \theta_0)$ over θ_0 vanishes even if $\alpha_2^{\perp}(0) \neq \alpha_2^{\parallel}(0)$. With Eq. (3.15), the integration over μ in Eq. (3.14) leads to

$$\Delta V^{an}(r, \theta_0) = (1 - 3\mu_0^2) \frac{\alpha_2^{\parallel}(0) - \alpha_2^{\perp}(0)}{3} \frac{\hbar c}{2\pi r^7} \lim_{\gamma \rightarrow 0} \int_0^\infty dx e^{-\gamma x} x^6 \alpha_1\left(\frac{xc}{r}\right) I'(x), \quad (3.16)$$

where

$$I'(x) = \left(\frac{3}{x^6} - \frac{1}{x^4} - \frac{1}{x^2} \right) \sin 2x - \left(\frac{6}{x^5} + \frac{2}{x^3} \right) \cos 2x. \quad (3.17)$$

As a check on Eq. (3.16) we take system 1 to be a spherically symmetric atom (or molecule). We can then replace $\alpha_1(xc/r)$ by $\alpha_1(0)$, and we obtain

$$\Delta V_{atom-atom}^{an}(r, \theta_0) = \frac{7\hbar c}{8\pi r^7} (1 - 3\mu_0^2) \alpha_1(0) \frac{\alpha_2^{\parallel}(0) - \alpha_2^{\perp}(0)}{3}. \quad (3.18)$$

The evaluation of $\bar{V}_{atom-atom}^{an}(r)$ is similar to that of $V_{atom-atom}(r)$ in Eq. (2.24), and the result, together with Eq. (3.18), can be written as

$$V_{atom-atom}^{an}(r, \theta_0) = -\frac{\hbar c \alpha_1(0)}{8\pi r^7} \left[46 \frac{\alpha_2^{\parallel}(0) + 2\alpha_2^{\perp}(0)}{3} - 7 \frac{\alpha_2^{\parallel}(0) - \alpha_2^{\perp}(0)}{3} (1 - 3\mu_0^2) \right], \quad (3.19)$$

which is consistent with the result by Craig and Power [25].

For a system of an electron and a cylindrically symmetric atom, we have

$$\alpha_1 \left(\frac{xc}{r} \right) = -\frac{e^2 r^2}{m x^2 c^2}, \quad (3.20)$$

and Eq. (3.16) reduces to

$$\Delta V_{atom-el}^{an}(r, \theta_0) = -(1 - 3\mu_0^2) \frac{\alpha_2^{\parallel}(0) - \alpha_2^{\perp}(0)}{3} \frac{25\hbar e^2}{8\pi m c r^5}. \quad (3.21)$$

The corresponding $\bar{V}_{atom-el}(r)$ can be determined in a similar way to that leading to Eq. (2.25). Finally, we find the interaction between a cylindrically symmetric atom and an electron to be

$$V_{atom-el}^{an}(r, \theta_0) = \frac{\hbar e^2}{8\pi m c r^5} \left[22 \frac{\alpha_2^{\parallel}(0) + 2\alpha_2^{\perp}(0)}{3} - 25 \frac{\alpha_2^{\parallel}(0) - \alpha_2^{\perp}(0)}{3} (1 - 3\mu_0^2) \right]. \quad (3.22)$$

Since this result is valid only asymptotically, it may well not be valid for experiments on the Rydberg H_2 molecule which have been made [26] and those which are now in progress. A more complete analysis, which reduces asymptotically to the result given in Eq. (3.22) but which is valid down to much smaller separations of the electron and the H_2^+ core, has been given [27].

ACKNOWLEDGMENT

The work was begun while one of us (LS) was visiting the Institute for Theoretical Atomic and Molecular Physics, and was completed while L.S. was at the Aspen Center for Physics. The hospitality of both organizations is gratefully acknowledged. The desirability of having available for confrontation with the experiments data a theoretical expression for the long-range Casimir interaction in the Rydberg H_2 molecule was pointed out to us by S. R. Lundeen; we are indebted to him for his having done so. We would like to thank Dr. P. W. Milonni for a useful remark. This work was partially supported by the National Science Foundation, under Grant No. PHY-9019745, and a grant to the Institute of Theoretical Atomic and Molecular Physics.

Footnote added in proof: Eq. (2.17) was derived under the assumption that the systems are in the asymptotic regime. In fact, in the approximation in which only dipole moments are considered and only two photons are exchanged—and magnetic effects, exchange effects, ..., are neglected—the atom-atom $V(r)$ of Eq. (2.17) is valid for *all* r . That is *not* the case, for example, for the electron-ion $V(r)$ of Eq. (2.17). We hope to discuss the origin of the difference at a later date.

APPENDIX EVALUATION OF SOME SUMS

We see from Fig. 2 that

$$\hat{\mathbf{u}} = \cos \theta_0 \hat{\mathbf{e}}_x - \sin \theta_0 \hat{\mathbf{e}}_z, \quad (\text{A1})$$

$$\hat{\mathbf{n}} = \sin \theta_0 \hat{\mathbf{e}}_x + \cos \theta_0 \hat{\mathbf{e}}_z \quad (\text{A2})$$

and

$$\hat{\mathbf{k}} = \sin \theta \cos \phi \hat{\mathbf{u}} + \sin \theta \sin \phi \hat{\mathbf{e}}_y + \cos \theta \hat{\mathbf{n}}. \quad (\text{A3})$$

Reexpressing $\hat{\mathbf{k}}$ in terms of $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$ gives

$$\begin{aligned} \hat{\mathbf{k}} &= (\sin \theta \cos \phi \cos \theta_0 + \cos \theta \sin \theta_0) \hat{\mathbf{e}}_x + \sin \theta \sin \phi \hat{\mathbf{e}}_y \\ &\quad + (\cos \theta \cos \theta_0 - \sin \theta \cos \phi \sin \theta_0) \hat{\mathbf{e}}_z. \end{aligned} \quad (\text{A4})$$

We choose the first polarization vector $\hat{\mathbf{e}}_{\mathbf{k}}^{(1)}$ to lie in the plane defined by $\hat{\mathbf{k}}$ and $\hat{\mathbf{n}}$, with angles $\theta + \pi/2$ and ϕ relative to the set of axes $\hat{\mathbf{u}}$, $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{n}}$. We then have

$$\begin{aligned} \hat{\mathbf{e}}_{\mathbf{k}}^{(1)} &= \cos \theta \cos \phi \hat{\mathbf{u}} + \cos \theta \sin \phi \hat{\mathbf{e}}_y - \sin \theta \hat{\mathbf{n}} \\ &= (\cos \theta \cos \phi \cos \theta_0 - \sin \theta \sin \theta_0) \hat{\mathbf{e}}_x + \cos \theta \sin \phi \hat{\mathbf{e}}_y \\ &\quad - (\cos \theta \cos \phi \sin \theta_0 + \sin \theta \cos \theta_0) \hat{\mathbf{e}}_z. \end{aligned} \quad (\text{A5})$$

The second polarization vector is chosen to be

$$\begin{aligned} \hat{\mathbf{e}}_{\mathbf{k}}^{(2)} &= \hat{\mathbf{k}} \times \hat{\mathbf{e}}_{\mathbf{k}}^{(1)} = -\sin \phi \hat{\mathbf{u}} + \cos \phi \hat{\mathbf{e}}_y \\ &= -\sin \phi \cos \theta_0 \hat{\mathbf{e}}_x + \cos \phi \hat{\mathbf{e}}_y + \sin \phi \sin \theta_0 \hat{\mathbf{e}}_z. \end{aligned} \quad (\text{A6})$$

The angles of $\hat{\mathbf{e}}_{\mathbf{k}}^{(2)}$ with respect to $\hat{\mathbf{u}}$, $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{n}}$ are $\pi/2$ and $\phi + \pi/2$. We note that

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_{\mathbf{k}}^{(1)} = \cos(\theta + \pi/2) = -\sin \theta. \quad (\text{A7})$$

and

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_{\mathbf{k}}^{(2)} = \cos(\pi/2) = 0. \quad (\text{A8})$$

Then we find

$$\sum_{\lambda} \hat{n}_3 \hat{e}_{q3} (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_q) = \hat{n}_3 \hat{e}_{\mathbf{k}3}^{(1)} (\hat{\mathbf{n}} \cdot \hat{\mathbf{e}}_{\mathbf{k}}^{(1)}) = \sin^2 \theta \cos^2 \theta_0, \quad (\text{A9})$$

and

$$\int_0^{2\pi} \frac{d\phi}{2\pi} \sum_{\lambda} \hat{e}_{q3}^2 = \frac{1}{2} \cos^2 \theta \sin^2 \theta_0 + \sin^2 \theta \cos^2 \theta_0 + \frac{1}{2} \sin^2 \theta_0. \quad (\text{A10})$$

References

- [1] E. A. Power, *Introductory Quantum Electrodynamics* (Elsevier, Amsterdam, 1964).
- [2] G. Feinberg and J. Sucher, *Phys. Rev. A* **2**, 2395 (1970).
- [3] E. J. Kelsey and L. Spruch, *Phys. Rev. A* **18**, 15 (1978).
- [4] J. F. Babb and L. Spruch, *Phys. Rev. A* **36**, 456 (1987).
- [5] K. Milton, L. L. DeRaad, Jr., and J. Schwinger, *Ann Phys. (N.Y.)* **115**, 388 (1978).
- [6] T. H. Boyer, *Phys. Rev.* **180**, 19 (1969).
- [7] P. W. Milonni, R. J. Cook, and M. E. Goggin, *Phys. Rev. A* **38**, 1621 (1988).
- [8] E. M. Lifshitz, *Zh. Eksp. Teor. Fiz.* **29**, 94(1954) [*Sov. Phys. JETP* **2**, 73(1956)]. See also L. D. Landau and E. M. Lifschitz, *Course of Theoretical Physics* (Pergamon, New York, 1960).
- [9] I. E. Dzyaloshinskii, E. M. Lifshitz and L. Pitaevskii, *Adv. Phys.* **10**, 165 (1961).
- [10] Y. Tikochinsky and L. Spruch, *Phys. Rev. A* ... (1993).
- [11] P. W. Milonni, *The Quantum Vacuum* (Academic Press, 1993).
- [12] *Long Range Forces: Theory and Recent Experiments in Atomic Systems*, edited by F. S. Levin and D. Micha, (Plenum Press, New York, 1992).
- [13] L. Spruch and E. J. Kelsey, *Phys. Rev. A* **18**, 845 (1978).
- [14] L. Spruch, *Phys. Today* **39**, 37 (1986).
- [15] L. Spruch, in *Long Range Forces: Theory and Recent Experiments in Atomic Systems*, edited by F. S. Levin and D. Micha (Plenum Press, New York, 1992).
- [16] Some ideas along the same lines were very briefly noted in [1].
- [17] F. E. Low, *Phys. Rev.* **96**, 1428 (1954); M. Gell-Mann and M. L. Goldberger, *Phys. Rev.* **96**, 1433 (1954).
- [18] H. B. G. Casimir and D. Polder, *Phys. Rev.* **73**, 360 (1948).

- [19] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975), Eq. (9.18).
- [20] J. Sucher, in *Long Range Forces: Theory and Recent Experiments in Atomic Systems*, edited by F. S. Levin and D. Micha (Plenum Press, New York, 1992), and references therein. See also, J. Sucher, unpublished.
- [21] J. Bernabéu and R. Tarrach, *Ann. Phys. (N.Y.)* **102**, 323 (1976).
- [22] E. A. Hessels, F. J. Deck, P. W. Arcuni, and S. R. Lundeen, *Phys. Rev. Lett.* **65**, 2765 (1990); **66**, 2544E (1991). See also Lundeen, in Ref. 11.
- [23] C. K. Au, G. Feinberg, and J. Sucher, *Phys. Rev. Lett.* **53**, 1145 (1984).
- [24] J. F. Babb and L. Spruch, *Phys. Rev. A* **38**, 13 (1988).
- [25] D. P. Craig and E. A. Power, *Internal Journal of Quantum Chemistry*, vol. **III**, 903 (1969).
- [26] W. G. Sturnus, E. A. Hessels, P. W. Arcuni, and S. R. Lundeen, *Phys. Rev. A* **44**, 3023 (1991), and references therein.
- [27] J. F. Babb and L. Spruch, unpublished.

Caption for Fig. 1

$\mathbf{r}$ defines the z axis, while $\mathbf{k}$ and $\mathbf{r}$ define the xz plane. The polarization vector $\hat{\mathbf{e}}_{\mathbf{k}}^{(2)}$ lies in the xz plane, at an angle $\theta + \pi/2$ to $\hat{\mathbf{e}}_z$, while $\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} = \hat{\mathbf{e}}_y$. The unit vector $\hat{\mathbf{k}}$, $\hat{\mathbf{e}}_{\mathbf{k}}^{(1)}$ and $\hat{\mathbf{e}}_{\mathbf{k}}^{(2)}$ define an orthogonal set of axes.

Caption for Fig. 2

System 1, indicated by a solid circle, is spherically symmetric. System 2 is cylindrically symmetric; its axis, indicated by the heavy horizontal line, defines the z axis. $\hat{\mathbf{n}}$ and $\hat{\mathbf{e}}_z$ define the xz plane. $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = r\hat{\mathbf{n}}$, $\hat{\mathbf{e}}_x$ and $\hat{\mathbf{e}}_y$ lie in the page. $\mathbf{k}$ is the wave vector, one polarization vector $\hat{\mathbf{e}}_{\mathbf{k}}^{(1)}$ lies in the plane defined by $\mathbf{k}$ and $\hat{\mathbf{n}}$, and the second polarization vector is given by $\hat{\mathbf{e}}_{\mathbf{k}}^{(2)} \equiv \hat{\mathbf{k}} \times \hat{\mathbf{e}}_{\mathbf{k}}^{(1)}$. With angles defined by the right-handed orthogonal set of axes $\hat{\mathbf{u}}$, $\hat{\mathbf{e}}_y$, and $\hat{\mathbf{n}}$, we have $\hat{\mathbf{k}} = (\theta, \phi)$, $\hat{\mathbf{e}}_{\mathbf{k}}^{(1)} = (\theta + \pi/2, \phi)$, and $\hat{\mathbf{e}}_{\mathbf{k}}^{(2)} = (\pi/2, \phi + \pi/2)$.

Figure 1

Figure 2