RECENT DEVELOPMENTS IN D=2 STRING FIELD THEORY

MICHIO KAKU

Physics Dept., City College of the City University of New York
New York, N.Y., 10031, USA

Received (received date)
Revised (revised date)

ABSTRACT
We review the recent developments in constructing string field theory in two-dimensions. We analyze the bewildering number of string field theories that have been proposed, all of which correctly reproduce the correlation functions of two-dimensional string theory. These include:

- free fermion field theory
- collective string field theory
- temporal gauge string field theory
- non-polynomial string field theory

We will analyze discrete states, the \( w(\infty) \) symmetry, and correlation functions in terms of these different string field theories. We will also comment on the relationship between these various field theories, which is still not well understood.

1. Introduction

At present, string theory\(^1\) gives us the best hope for a unified theory of all known interactions, including gravity. However, the frustrating aspect of string theory is that millions of classical solutions are now known of the string equations, perhaps corresponding to the set of all possible conformal field theories\(^2\). Which one, if any, describes the physical world?

There are indications that the theory is not Borel summable, meaning that these perturbative vacua are probably unstable, and that the true vacuum of the theory must be found non-perturbatively. Thus, the principle problem of string theory is to determine its true, non-perturbative vacuum. And the most conservative method of formulating a non-perturbative framework for string theory is through a second quantized string field theory\(^3\).

A first quantized theory is essentially a single-particle formalism, where the coordinates of a single particle \( x_\mu \) are quantized, according to the relations:

\[
[x_\mu, p_\nu] = i\delta_{\mu\nu}
\]

Because the first quantized theory describes the coordinates of a single particle \( x_\mu(t) \), it cannot easily describe multi-particle states. In particular, it is difficult to
describe the true vacuum of the theory.

In a first quantized point-particle formalism, interactions can be introduced, but only in a rather awkward fashion, by specifying by hand the set of all possible interacting graphs. We define the scattering amplitude via:

$$A(k_1, k_2, ...) \sim \sum_{\text{topologies}} \int Dx_i(t) \exp \left( iS + \sum_j i k_j x_j \right)$$

(2)

where we must arbitrarily impose the set of topologies (corresponding to the complete set of Feynman diagrams) over which we perform the functional integration. Thus, the topologies of the Feynman diagrams, the various relative weights of the Feynman diagrams, the particular choice of which Feynman diagrams to include, etc. are all rather arbitrary. In particular, it is difficult to prove the unitarity of the theory.

A second quantized theory, by contrast, is necessarily a multi-particle formalism. We begin not with the co-ordinates of a single particle, but with the field variables $\phi$ which can describe multi-particle states, and define the commutation relations to be:

$$[\phi(x), \pi(y)] = i\delta(x - y)$$

(3)

for equal times. In contrast to the first quantized theory, where the set of Feynman diagrams must be imposed as an additional, cumbersome constraint, in a second quantized theory everything is fixed once we specify the invariant Lagrangian. This in turn automatically yields the complete topology of Feynman diagrams, their relative weights, etc. Unitarity is proven simply by showing that the Hamiltonian is hermitian.

In principle, a second quantized string field theory should yield the true vacuum of string theory. Unfortunately, in practice string field theory has fallen far short of this ambitious goal, in part because it is defined in a specific background. At present, string field theory has only been able to reproduce the known results of perturbative string theory, and has provided little useful non-perturbative information.

In recent years, the most promising avenue to approach non-perturbative string theory has been to explore two-dimensional string theory, which is, remarkably enough, exactly soluble. Thus, two-dimensional string theory serves as a toy model or a theoretical laboratory in which one can obtain valuable intuitive insight into the full theory.

In two-dimensions, in turn, there are two ways in which to approach string theory:

1) We can use “matrix models,” in which we approximate the Riemann surfaces found in string theory via a discrete triangulation and then take the continuum limit. $c = 1$ matrix models begin with the one dimensional action:

$$L = \frac{1}{2} \text{Tr} \left( \hat{M}^2 - U(M) \right)$$

(4)
where $U(M)$ is the potential, and where $M(t)$ is a $N \times N$ hermitian matrix which is a function of time $t$. (The eigenvalues of $M$ will eventually become a continuous variable, which corresponds to the second dimension of space-time.) The great advantage of using matrix models is that they are exactly soluble and simple to use.

However, the disadvantage of matrix models is that almost all string degrees of freedom have been eliminated, and hence the intuitive concept of the string, in some sense, has disappeared. All we have left is the massless tachyon. This means that the rather mysterious “discrete states” and the $u(\infty)$ algebra appear in a rather subtle and obscure fashion.

More seriously, matrix models are only defined for $c \leq 1$, meaning that they cannot yield any realistic information about our physical world. This is a severe drawback, which has retarded the development of this approach.

2) We can formulate two-dimensional string theory via Liouville theory.

Liouville theory begins with the first quantized action:

$$S = \frac{1}{8\pi} \int d^2 \sigma \sqrt{g} \left[ g^{ab} \left( \partial_a \phi \partial_b \phi - \frac{3}{8} \partial_a \partial_b \phi \right) - Q R \phi + \mu e^{-\sqrt{2} \phi} \right]. \quad (5)$$

where $\phi$ is the Liouville field, a remnant of the original world-sheet metric $g_{\alpha\beta}$, and $\mu$ is the cosmological constant on the world-sheet. (This conformal action, where $\phi$ is treated as time, is written in the Euclidean metric. We can make the transition to the Minkowski metric with the substitution $\phi \rightarrow it$.)

The advantage of the Liouville approach is that it is transparently a string theory. All string degrees of freedom, including the familiar ghosts, are present. The tachyon, for example, appears with mass $m^2 = \frac{1}{16}(2-D)$, which vanishes if $D = 2$.

When re-expressed as a string field theory, this formalism also enjoys the advantage that the discrete states and $u(\infty)$ algebra appear naturally. As in gauge theory, we find that the three-string coupling constants are exactly the structure constants of $u(\infty)$, and that the discrete states emerge as solutions of the gauge constraints. In other words, the discrete states and $u(\infty)$ emerge as by-products of the gauge symmetry of string field theory, giving us an intuitive understanding of why they exist.

The Liouville action has some rather peculiar features. Because the action is not translation invariant in $\phi$, we will find that the light-cone gauge cannot be naively imposed on the theory. This means that we cannot use the gauge conditions to eliminate all higher string degrees of freedom. (This, in turn, helps to explain why there are discrete states which cannot be eliminated in the light-cone gauge.) Because translation invariance is broken, we will also find that the usual conservation of energy is violated.

However, the most serious complication of this action is that it is highly non-linear and, even at the free level, almost intractable. This is because the classical equation of motion for the Liouville field is $\partial_a \partial^a \phi \sim e^{\phi}$, which is highly non-linear.
Most results for Liouville theory are formulated in the approximation where the cosmological constant is taken to be zero: $\mu = 0$. In this limit, the theory becomes a free theory, and all correlation functions are easily found. These scattering amplitudes correspond to what is called “bulk” scattering. (For non-zero cosmological constant, there is an impenetrable barrier when the $e^{a\phi}$ term becomes large, so we have scattering or reflection off a wall. These scattering amplitudes for non-zero $\mu$ correspond to what is called “wall” scattering, and are much more difficult to solve.)

Progress in this area has been painstakingly slow. In fact, the explosion of research in matrix models helped to re-stimulate research in Liouville theory, as surprisingly simple results were found for correlation functions which were previously thought, using Liouville theory, to be intractable.

In this paper, we will study $D = 2$ string field theory from both perspectives. We will first develop matrix models via the theory of collective string field theory, and then we will discuss the string field theory of Liouville theory.

Our task is complicated by the fact that there are a bewildering variety of string field theories in two-dimensions, including:

(a) free fermion field theory\(^6\)
(b) collective field theory of Das and Jevicki\(^7\)
(c) temporal gauge field theory of Ishibashi and Kawai\(^8\)
(d) the non-polynomial BRST string field theory\(^9\) of the author.

(The 2D non-polynomial string field theory is a modification of the non-polynomial string field found in 26 dimensions by the author\(^10\) and the MIT\(^11\) and Kyoto groups\(^12\).)

Eventually, we hope that all of them can be shown to be nothing but different gauge choices of the same string field theory. For example, it is expected that the BRST Liouville string field theory can be reduced to the collective string field theory when all the BRST trivial states are eliminated. Because reparametrization invariance allows us to cancel two degrees of freedom, and since there are only two degrees of freedom in $D = 2$, this means that collective field theory is defined only in terms of a single tachyon, while BRST string field theory includes all redundant string states as well as BRST trivial states.

However, at present the complete relationship between all of these formalisms is largely unknown.

2. Matrix Models and Free Fermion Theory

In this section, we will first show that matrix models can be reduced to string theory in the double scaling limit, and that the theory is soluble because it is equivalent to a free fermion field theory.

Let us perform a simple counting of the various $N$ factors found in the Feynman
Recent Developments in String Field Theory in D=2

rules generated by the action for matrix models. We find:

\[
\begin{align*}
V &= \text{vertices} & \rightarrow & N \\
P &= \text{propagators} & \rightarrow & N^{-1} \\
L &= \text{loops} & \rightarrow & N
\end{align*}
\]  

(6)

The partition function therefore has the form:

\[
Z(g) = \sum_{h} N^{V-P+L} Z_h(g) = \sum_{h} N^{2-2h} Z_h(g)
\]  

(7)

This means that each Feynman diagram contributes an overall factor of \(N^{2-2h}\), where \(h\) is the number of handles or closed loops of the diagram, and where \(\chi = 2 - 2h\) is the Euler number. Thus, a \(1/N\) expansion decomposes the perturbation theory in terms of topologically equivalent diagrams. In the limit \(N \to \infty\), only the planar diagram, with no loops, survives. This limit is not that interesting. More instructive is taking the limit as the coupling \(g\) approaches some critical value, which alters the nature of the limit process. Then the combination of \(N \to \infty\) with \(g \to g_c\) gives us a non-trivial limit.

For example, in string theory it can be shown that \(Z_h\) obeys the property:

\[
Z_h(g) \sim f_h(g_c - g)^{2-\Gamma \chi/2}
\]  

(8)

where \(g_c\) represents a critical value for the coupling constant, \(\chi\) is the Euler number of the surface, and \(\Gamma\) is the critical exponent, given by \(\Gamma = (1/12)(D - 1 - \sqrt{(D-1)(D-25)})\).

Let us define \(\kappa^{-1} = (g_c - g)^{(2-\Gamma \chi)/2}\). Thus, we can define a “double scaling limit,” where we take the limit \(N \to \infty\), \(g \to g_c\), and:

\[
\text{double scaling limit} = \begin{cases} 
N & \to \infty \\
g & \to g_c \\
\kappa & \to \text{constant}
\end{cases}
\]  

(9)

Then the partition function becomes:

\[
Z(g) \to \sum_{h} \kappa^{2h-2} f_h
\]  

(10)

Thus, in the double scaling limit, we expect that matrix models provide a good description of string theory in two-dimensions. We should also point out that, in the double scaling limit, the precise nature of the potential \(U(M)\) washes out, and so the final results are largely independent of the nature of \(U(N)\).

The next step is to write matrix models as a field theory. To accomplish this, we will first write out the Hamiltonian and then make a change of variables:

\[
H = -\frac{1}{2} \Delta + U
\]
\[ \Delta = \sum_i \frac{\partial^2}{\partial M_i^2} + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial \text{Re} M_{ij}^2} + \frac{\partial^2}{\partial \text{Im} M_{ij}^2} \]

\[ U = \frac{1}{2} \text{Tr} M^2 + \frac{g}{N} \text{Tr} M^4 \] (11)

Likewise, the functional measure is given by:

\[ \prod_{ij} dM_{ij} = \prod_i dM_{ii} \prod_{i<j} d(\text{Re} M_{ij}) d(\text{Im} M_{ij}) \] (12)

Our strategy is to rewrite this Hamiltonian and measure in terms of the eigenvalues \( \lambda_i \) of the matrix \( M \), as well as the angular variables.

We can always decompose the matrix \( M(t) \) into:

\[ M(t) = \Omega^\dagger(t) \Lambda(t) \Omega(t) \] (13)

where \( \Omega \in SU(N) \) and \( \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N) \). With this decomposition, the Lagrangian now depends on the angular part \( \Omega \) via the term:

\[ \text{Tr} \dot{M}^2 = \text{Tr} \dot{\Lambda}^2 + \text{Tr} [\Lambda(\dot{\Omega} \Omega^\dagger)]^2 \] (14)

To eliminate this last term, it will be helpful to decompose it in terms of the generators of \( SU(N) \):

\[ \dot{\Omega} \Omega^\dagger = \frac{i}{\sqrt{2}} \sum_{i<j} \alpha_{ij} T_{ij} + \beta_{ij} \tilde{T}_{ij} + \sum_{i=1}^{N-1} \dot{\alpha}_i H_i \] (15)

where \( H_i \) are the diagonal generators of the Cartan subalgebra. \( T_{ij} \) is defined to be the matrix \( M \) such that \( M_{ij} = M_{ji} = 1 \), and zero elsewhere; \( \tilde{T}_{ij} \) is the matrix \( M \) such that \( M_{ij} = -M_{ji} = -i \), and all other entries are 0.

The Lagrangian is now written as:

\[ L = \sum_i \left( \dot{\lambda}_i^2 + U(\lambda_i) \right) + \sum_{i<j} (\lambda_i - \lambda_j)^2 (\dot{\alpha}_{ij}^2 + \dot{\beta}_{ij}^2) \] (16)

We also note that the measure of integration, in these new co-ordinates, looks like

\[ \mathcal{D} \Phi \equiv \mathcal{D} \Omega \prod_i d\lambda_i \Delta^2(\lambda), \]

where \( \Delta(\lambda) \) is the Vandermonde determinant \( \prod_{i<j} (\lambda_i - \lambda_j) \).

Because of the non-trivial Jacobian, the kinetic term for the eigenvalues becomes

\[ -\frac{1}{2\beta^2} \sum_{i=1}^N \frac{1}{\Delta^2(\lambda)} \frac{d}{d\lambda_i} \Delta^2(\lambda) \frac{d}{d\lambda_i} \] (17)

(where \( \beta \) corresponds to an effective inverse Planck’s constant, which goes as \( N \)).

This can also be rewritten as:

\[ -\frac{1}{2\beta^2} \Delta(\lambda) \sum_i \frac{d^2}{d\lambda_i^2} \Delta(\lambda). \] (18)
Then the total Hamiltonian can be expressed as:

\[ H = -\frac{1}{2\beta^2} \sum_i \frac{d^2}{d\lambda_i^2} \Delta(\lambda) + \sum_i U(\lambda_i) + \sum_{i<j} \left( \Pi_{ij}^2 + \tilde{\Pi}_{ij}^2 \right) \]

(19)

where \(\Pi_{ij}\) and \(\tilde{\Pi}_{ij}\) are conjugate to \(\alpha_{ij}\) and \(\beta_{ij}\), that is, they are the generators of left rotations on \(\Omega, \Omega^+\rightarrow A\Omega\).

Fortunately, we only need concern ourselves with the \(SU(N)\) singlet sector of the theory. This is because the angular terms in the Hamiltonian are all positive definite. To find the ground state of the system, we therefore need only examine the singlet sector. Our problem is thus reduced to examining the system defined by:

\[
\left( \sum_{i=1}^N h_i \right) \Psi(\lambda) = E \Psi(\lambda)
\]

\[ h_i = -\frac{1}{2\beta^2} \frac{d^2}{d\lambda_i^2} + U(\lambda_i) \]

(20)

where \(\Psi(\lambda) = \Delta(\lambda) \chi_{\text{sym}}(\lambda)\) is by construction antisymmetric because of the presence of the Vandermonde determinant \(\Delta\).

Several key points must be made. First, the wave function \(\Psi\), because of the presence of the Vandermonde determinant, is now fermionic. Second, the theory has now reduced to a system of uncoupled non-relativistic fermions acting under a potential. This is one way to see why the theory is exactly soluble. Although the boundary conditions may be a bit involved, the Hamiltonian is that of a free fermion theory.

The fermionic string field \(\Psi\) can be written as a linear superposition of an infinite number of states \(\psi_i\) with energy \(\epsilon_i\):

\[
\Psi(\lambda, t) = \sum_i \alpha_i \psi_i(\lambda)e^{-i\epsilon_i t}
\]

(21)

In this representation, we can easily re-write our original Hamiltonian in terms of this string field:

\[
H = \int d\lambda \left[ \frac{1}{2\beta^2} \frac{\partial \Psi^\dagger \partial \Psi}{\partial \lambda} + U(\lambda) \Psi^\dagger \Psi - \mu_F (\Psi^\dagger \Psi - N) \right]
\]

(22)

(we adjust the Lagrange multiplier \(\mu_F\) to equal the Fermi level).

This Hamiltonian, in turn, can be derived from the following action:

\[
S = \int dt d\lambda \left[ i \Psi^\dagger \dot{\Psi} - \frac{1}{2\beta^2} \frac{\partial \Psi^\dagger \partial \Psi}{\partial \lambda} - U(\lambda) \Psi^\dagger \Psi + \mu_F (\Psi^\dagger \Psi - N) \right]
\]

(23)

This string field theory action contains all the information of the \(c = 1\) matrix model for the \(SU(N)\) singlet sector. Notice that the action appears to be defined
in two dimensions if we treat $\lambda$ and $t$ as two space-time co-ordinates. This is how the Liouville mode appears in our formulation.

We should point out that, in this form, the free fermion action can yield many interesting features of two-dimensional string theory, such as the energy states, the scattering amplitudes, etc. However, instead of developing this formalism any further, let us now discuss a new formalism, collective string field theory, and then show the equivalence of free fermion field theory with collective string field theory.

3. Collective Field Theory

Next, we wish to develop the formalism of collective field theory of Jevicki and Sakita\textsuperscript{14} and derive the collective string field theory action. Collective field theory involves a change of variables, from an original set of fields to a set of invariant ones. Originally, the goal of collective field theory was to replace the variable $A_{\mu}^a$ appearing in gauge theory with the Wilson loop, involving an invariant trace of a closed loop. Although this has not been particular successful, it has enjoyed considerable success for two-dimensional string theory.

We start with a standard Hamiltonian defined with variables $q_i$:

$$H = -\frac{1}{2} \sum_i^N \frac{\partial^2}{\partial q_i^2} + V(q_i)$$

(24)

Next, we change variables from the $q_i$ to a field variable $\phi(x)$ as follows:

$$\phi(x) = f(x, q_1, q_2, ..., q_N)$$

(25)

where the mapping, we notice, is now overcomplete.

If we insert this transformation into the Hamiltonian, we naturally find, after a straightforward application of the chain rule:

$$H = \frac{1}{2} \int dx \omega(x, \phi) \frac{\delta}{\delta \phi(x)}$$

$$- \frac{1}{2} \int dx dy \Omega(x, y, \phi) \frac{\delta}{\delta \phi(x) \delta \phi(y)}$$

(26)

where:

$$\omega(x, \phi) = -\sum_i \partial_i^2 f(x, q_i)$$

$$\Omega(x, y, \phi) = \sum_i \partial_i f(x, q) \partial_i f(y, q)$$

(27)

Let us write $\pi(x) = -i(\delta / \delta \phi(x))$. Then, symbolically, the free Hamiltonian can be written as:
Recent Developments in String Field Theory in D=2 . . . 9

\[ H = \frac{1}{2} \pi \Omega \pi + i \frac{1}{2} \omega \pi \] (28)

where we have dropped all integral signs for the sake of clarity.

Unfortunately, this Hamiltonian actually yields incorrect answers for even simple systems. The reason is that we have not included the Jacobian of the transformation, and also because this new Hamiltonian is not hermitian on scalar products defined in this new space. The term which violates hermiticity is \( \omega \pi \).

In the old co-ordinates, scalar products are defined via functionals \( \Phi[\phi] \). In the new co-ordinates, scalar products are defined via \( \Psi[\phi] \), where we have the rescaling \( \Psi[\phi] = J^{1/2} \Phi[\phi] \), where \( J[\phi] \) is the Jacobian of the transformation. This means that expressions like \( \pi \Phi[\phi] \) have to be rescaled via factors of \( J^{-1/2} \):

\[
\pi(x)\Phi[\phi] = \pi J^{-1/2} \Psi([\phi]) = J^{-1/2} (\pi - iC) \Psi[\phi]
\]

\[ C = -\frac{1}{2} \frac{\delta \ln J[\phi]}{\delta \phi(x)} \] (29)

Inserting this new value of \( \pi \), shifted by the quantity \(-iC\), we now have, symbolically:

\[
H = \frac{i}{2} (\omega + i(\pi \Omega) - 2C\Omega) \pi
+ \frac{1}{2} (\pi \Omega \pi - C\Omega C) + \frac{i}{2} \omega C - \frac{i}{2} \Omega(\pi C)
\] (30)

It is now a simple matter to make the Hamiltonian hermitian. We will simply set the first line of the equation to zero. This not only restores the hermitian nature of the Hamiltonian, as desired, it also determines \( J \):

\[
\omega + \frac{\delta \Omega}{\delta \phi} - 2\Omega C = 0
\] (31)

whose solution is given by:

\[ C = \frac{1}{2} \Omega^{-1} [\omega + i(\pi \Omega)] \] (32)

With this new restriction, our Hamiltonian now becomes:

\[
H = \frac{1}{2} \pi \Omega \pi + \frac{1}{8} \left( \omega + \frac{\partial \Omega}{\partial \phi} \right) \Omega^{-1} \left( \omega + \frac{\partial \Omega}{\partial \phi} \right)
+ V(\phi) - \frac{1}{4} \frac{\delta \omega}{\delta \phi} - \frac{1}{4} \frac{\partial^2 \Omega}{\partial \phi \partial \phi}
\] (33)

Although this expression appears to be quite formidable, it simplifies considerably for concrete systems. Now, let us adapt this formalism to matrix models.
We will make a change of variables from $M(t)$ to a new variable $\phi_1(t)$, or its Fourier transform $\phi_1(t)$, which is given in terms of an invariant trace:

$$\phi_1(t) = \text{Tr} \left( e^{i M(t) t} \right) = \sum_{i=1}^{N} e^{i \lambda_i(t)} , \quad (34)$$

$$\phi(x,t) = \int \frac{dk}{2\pi} e^{-ikx} \phi_1(t) = \sum_{i=1}^{N} \delta(x-\lambda_i(t)) , \quad (35)$$

We see that $\phi(x,t)$ is nothing but a sum over delta functions of the eigenvalues of $M(t)$. There is, therefore, an additional constraint $\int dx \phi(x,t) = N$, which we can impose by adding a Lagrange multiplier term $\mu_F \left( \int dx \phi(x,t) - N \right)$. This Lagrange multiplier will correspond to the Fermi level.

Let us now insert the value of $\phi$ into $\omega$ and $\Omega$. A simple calculation yields:

$$\omega(k, \phi) = -\frac{\partial^2}{\partial M^2} \phi_1 = k^2 \int_0^1 d\alpha \phi_1 \phi_1^{(1-\alpha)}$$
$$\Omega(k', \phi) = \frac{\partial \phi_1}{\partial M} \frac{\partial \phi_1^{(1)}}{\partial M} = kk' \phi_1^{(1)} \phi_1^{(1)} \quad (36)$$

Written in $x$ space, these expressions become:

$$\omega(x, \phi) = 2 \frac{\partial}{\partial x} \int dy \frac{\phi(x) \phi(y)}{x-y}$$
$$\Omega(x, x', \phi) = \frac{\partial}{\partial x} \frac{\partial}{\partial x'} \left[ \delta(x-x') \phi(x) \right] \quad (37)$$

Now let us insert the value of $\omega$ and $\Omega$ into the Hamiltonian. Our final expression now becomes:

$$H = \int dx \left\{ \frac{1}{2} \Pi_x \phi \Pi_x \phi + \frac{\pi^2}{6} \phi^3 + [U(x) - \mu_F] \phi \right\} , \quad (38)$$

with the commutation relations:

$$[\phi(x), \Pi(y)] = \delta(x-y) \quad (39)$$

As before, we have restricted our system to the singlet sector of the full theory.

Starting with this Hamiltonian, it is not difficult to make the transition to the Lagrangian formalism and write down the action of the system:

$$S = \int dt \int dx \left[ \frac{1}{2} \partial_x^{-1} \phi \partial_x^{-1} \phi \right. \\
- \left. \frac{1}{6} \pi^2 \phi^3 + (\mu_F - U(x)) \phi(x,t) \right] \quad (40)$$
What is rather unusual about this system of equations is that the Hamiltonian is cubic, but the action is actually non-polynomial. This means that there are two distinct ways in which to derive the S-matrix of the theory, requiring quite different Feynman rules.

In practice, it will be more convenient, of course, to calculate Feynman diagrams with the cubic theory, rather than the non-polynomial one. To extract out the Feynman rules, we will find it useful to expand around the classical solution of the theory:

\[
\frac{1}{2} (\pi \phi_0(x))^2 + U(x) = \mu_F, \tag{41}
\]

\[
\pi \phi_0 = p_0(x) = \sqrt{2(\mu_F - U(x))}. \tag{42}
\]

Then, by power expanding around \(\phi_0\):

\[
\phi(x, t) = \phi_0(x) + \frac{1}{\sqrt{\pi}} \partial_x \eta(x, t) \tag{43}
\]

we find:

\[
H = \int dx \left\{ (\pi \phi_0) \left( \frac{1}{2} \Pi^2 + \frac{1}{2} \eta_x^2 \right) + \frac{\pi^2}{6} (\eta_x)^3 + \frac{\tau}{2} \Pi^2 \eta_x \right\}. \tag{44}
\]

In order to eliminate the \(\phi_0\) in the free part of the Hamiltonian, we will change variables from \(x\) to \(\tau\):

\[
\tau = \int^x dx \frac{d\tau}{\pi \phi_0}; \quad \frac{d\tau}{d\tau} = p_0 \tag{45}
\]

With this new variable, the Hamiltonian becomes, for the quadratic and cubic pieces:

\[
H_2 + H_3 = \int d\tau \left\{ \frac{1}{2} \left( \Pi^2 + (\partial_\tau \eta)^2 \right) + \frac{1}{6} \left[ (\partial_\tau \eta)^3 + 3 \Pi^2 (\partial_\tau \eta) \right] \right\}. \tag{46}
\]

There is also the linear term, given by:

\[
H_1 = \frac{1}{12} \int d\tau \left[ \frac{\phi'^4}{\phi_0} - \frac{1}{2} \left( \frac{\phi'}{\phi_0} \right)^2 \right] - \frac{1}{24} \int dx \left[ \frac{(\phi'^4)}{\phi_0} - 2 \phi'^4 \right]. \tag{47}
\]

From this, we can derive Feynman rules and calculate scattering amplitudes. Before we do this, however, we can use the form of the Hamiltonian found here to reveal some of the hidden symmetries of this system.

4. Discrete States and \(w(\infty)\)

To see that matrix models possess hidden symmetries, it is useful to write it in Hamiltonian form:
where $P$ is a matrix which is conjugate to $M$, and we take $U(M) = -\frac{1}{2} M^2$.

Now write down the following operators, which correspond to creation–annihilation operators:

$$B_n^\pm = \text{Tr} \left( P \pm M \right)^n , \quad n = 0, 1, 2, \ldots$$

What is remarkable is that these states have the following commutation relation with the Hamiltonian:

$$[H, B_n^\pm] = \mp im B_n^\pm ,$$

This also indicates that these states $B_n^\pm$ correspond to physical states with energy $\epsilon_n = \pm \text{im}$. (We can analytically continue this discrete imaginary momenta to real values, so that this operator corresponds to a physical state.)

We can also generalize this operator and write down another series of operators:

$$B_{n, \nu} = \text{Tr} \left( (P + M)^\nu (P - M)^\nu \right) ,$$

which has the commutation relations:

$$[H, B_{j, m}] = -2im B_{j, m} .$$

where:

$$m = \frac{n - \bar{n}}{2} , \quad j = \frac{n + \bar{n}}{2} .$$

These states have energy given by $-2im$.

We would now like to reveal the physical meaning of these operators. The easiest way to do this is to carry over these expressions to the collective field theory. In this way, we will be able to establish the field-theoretic meaning of these operators.

To do this, the simplest method is to construct a “dictionary” which will give us a quick, intuitive way in which to make the transition between matrix model quantities and collective field theory. To construct this dictionary, we will introduce the operator:

$$\alpha_\pm(x, t) = \Pi_\nu = \pm \pi \delta(x, t)$$

which has the commutation relations:

$$[\alpha_\pm(x), \alpha_\pm(y)] = \pm 2\pi \delta(x - y)$$

Written in this form, the Hamiltonian can be written as:

$$H = \frac{1}{2} \int \frac{dx}{2\pi} \left\{ \frac{1}{3} \left( \alpha_+^3 - \alpha_-^3 \right) - \left( x^2 - \mu \right) (\alpha_+ - \alpha_-) \right\} .$$
To make the transition between matrix models and collective field theory, the matrix $M$ is eventually replaced by its eigenvalues $\lambda$, which eventually become the variable $x$. Also, the conjugate matrix $P$ is replaced by its eigenvalues $p$, which becomes the operator $\alpha(x,t)$.

\[ M \rightarrow \lambda \rightarrow x, \quad P \rightarrow p \rightarrow \alpha(x,t). \] (57)

Furthermore, one can show that the trace operation is replaced by:

\[ \text{Tr} \{ \} \rightarrow \int \frac{dx}{2\pi} \int_{\alpha_-(x,t)}^{\alpha_+(x,t)} \frac{d\alpha}{2\pi} \{ \}, \] (58)

where $\alpha_\pm(x,t)$ are the chiral components of the scalar field density. For example, one can show that the matrix model Hamiltonian is easily transformed into the collective field theory Hamiltonian by these substitutions:

\[ \frac{1}{2} \text{Tr} \left( P^2 - M^2 \right) - \frac{1}{2} \left( p_0^2 - x^2 \right) \rightarrow \int \frac{dx}{2\pi} \int \frac{d\alpha}{2\pi} \frac{1}{2} \left( \alpha_+^2 - x^2 \right) = \frac{1}{2} \int \frac{dx}{2\pi} \frac{1}{3} \left( \alpha_+^3 - x^3 \right)^+ \ . \] (59)

With this dictionary, we can now find the meaning of the operator $B_n$ and $B_{i,m}$. The $B_n$ operators simply correspond to the tachyons of the theory.

Also, consider the operator:

\[ H_n = \frac{1}{2\pi} \int dx \int_{\alpha_-(x,t)}^{\alpha_+(x,t)} d\alpha \left( \alpha_+^2 - x^2 \right)^n, \] (60)

which possess the commutation relations:

\[ [H_n, H_m] = 0, \] (61)

which are conserved:

\[ \frac{d}{dt} H_n = \int dx \partial_x \left( \alpha_+^2 - x^2 \right) \left( \alpha_+^2 - x^2 \right)^n = 0 . \] (62)

This shows that the Hamiltonian is nothing but a member of a spectrum-generating algebra.

These elements, in turn, can be generalized to the larger set:

\[ O_{JM} = \int \frac{dx}{2\pi} \int_{\alpha_-}^{\alpha_+} d\alpha \left( \alpha + x \right)^J M^{+1} \left( \alpha - x \right)^J \! M^+ \! M^{+1}, \] (63)

which obeys the $w(\infty)$ commutation relations:

\[ [O_{J_1 M_1}, O_{J_2 M_2}] = 4i \left( (J_2 + 1)M_1 - (J_1 + 1)M_2 \right) O_{J_1 + J_2, M_1 + M_2} . \] (64)
Recent Developments in String Field Theory in $D=2$ \\

The states $B_{jm}$ therefore correspond, in collective field theory language, to the discrete states. They are physical states of the theory, but they differ physically from the tachyon states $B_n$ because they are defined only at discrete momenta.

In summary, we found that the states $B_n$ and $B_{jm}$ correspond, respectively, to the tachyon and the discrete states once we translate our operators into collective field theory language via this “dictionary.” We also found that the Hamiltonian is nothing but a single member of a spectrum-generating algebra, given by $w(\infty)$. The fact that we have an infinite number of conserved currents is yet another way to see that the model is soluble.

5. Equivalence with Free Fermion Model

At this point, it may seem strange that we have two entirely different formalisms in which to describe $D = 2$ string theory, the free fermion theory and the collective field theory. Actually, they are equivalent. To demonstrate this fact, we will find it useful to bosonize the free fermion fields in eq. (22) in terms of bosonic variables $P$ and $X$ as follows:

$$
\Psi_L = \frac{1}{\sqrt{2\pi}} : \exp \left[ i \sqrt{\pi} \int (P - X') d\tau \right] : ,
$$

$$
\Psi_R = \frac{1}{\sqrt{2\pi}} : \exp \left[ i \sqrt{\pi} \int (P + X') d\tau \right] :
$$

where $X$ is a massless two-dimensional periodic scalar field, and $P$ is its canonically conjugate momentum.

It is now a simple matter to insert these bosonized expressions into the action of the free fermion theory. We find, for example, expressions like:

$$
: \Psi_L^\dagger \partial_\tau \Psi_L - \Psi_R^\dagger \partial_\tau \Psi_R : = \frac{i}{2} : P^2 + (X')^2 :
$$

$$
: \Psi_L^\dagger \Psi_L + \Psi_R^\dagger \Psi_R : = -\frac{X'}{\sqrt{\pi}}
$$

$$
: \partial_\tau \Psi_L^\dagger \partial_\tau \Psi_L + \partial_\tau \Psi_R^\dagger \partial_\tau \Psi_R : = -\sqrt{\pi} : PX'P + \frac{1}{3}(X')^3 + \frac{1}{6\pi} X'''' : .
$$

It is now straightforward to substitute these expressions directly into the Hamiltonian:

$$
: H : = \frac{1}{2} \int_0^{T/2} d\tau \left[ P^2 + (X')^2 - \frac{\sqrt{\pi}}{\beta v^3} (PX'P + \frac{1}{3}(X')^3 + \frac{1}{6\pi} X''') - \frac{1}{2\beta \sqrt{\pi}} \frac{v''}{v^3} - \frac{5(v')^2}{2v^4} \right] : ,
$$

where $v = \sqrt{2(\mu F - U)}$. If we integrate by parts and discard the boundary terms, this reduces to:
: \[ H := \frac{1}{2} \int_{0}^{T/2} d\tau : \left[ P^2 + (X')^2 - \frac{\sqrt{\pi}}{2\beta \mu \sinh^2 \tau} (P X' P + \frac{1}{3} (X')^3) - 1 - \frac{3}{2} \coth^2 \tau \right] X' \right]^2 : \]

(68)

In this way, we can now directly compare the free fermion action with the action of collective field theory in eqs. (46) and (47), and we find that they are identical. Physically, the massless field \( \Phi \) describes small fluctuations of the Fermi surface. In the Hamiltonian, this reduces to (for \( \mu > 0 \) and \( v = \sqrt{2\mu} \sinh (\tau) \)):

: \[ H := \frac{1}{2} \int_{0}^{\infty} d\tau : \left[ P^2 + (X')^2 - \frac{\sqrt{\pi}}{2\beta \mu \sinh^2 \tau} (P X' P + \frac{1}{3} (X')^3) - 1 - \frac{3}{2} \coth^2 \tau \right] X' \right]^2 : \]

(69)

There is one subtle point. Like the free fermion action, this Hamiltonian is actually divergent at \( \tau = 0 \). This divergence can be regularized with zeta-function techniques. Alternatively, we may take this with \( \mu < 0 \). Then

: \[ H := \frac{1}{2} \int_{0}^{\infty} d\tau : \left[ P^2 + (X')^2 - \frac{\sqrt{\pi}}{2\beta \mu \cosh^2 \frac{\tau}{2}} (P X' P + \frac{1}{3} (X')^3) - 1 - \frac{3}{2} \coth^2 \tau \right] X' \right]^2 : \]

(70)

If we compare this result with the previous one derived from collective field theory, then we see that they are the same. Now that we have established the equivalence of the two theories, we can now use this expression to derive the Feynman rules and then the amplitudes.

6. S-matrices

We decompose the \( X \) and \( P \) fields into oscillators as follows:

\[ X(t, \tau) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{4\pi |k|}} \left( a(k) e^{i(k t - |k| \tau)} + a^\dagger(k) e^{-i(k t - |k| \tau)} \right), \]

\[ P(t, \tau) = -i \int_{-\infty}^{\infty} \frac{dk}{\sqrt{4\pi |k|}} \left( a(k) e^{i(k t - |k| \tau)} - a^\dagger(k) e^{-i(k t - |k| \tau)} \right), \]

(71)

such that \([a(k), a^\dagger(k')] = \delta(k - k')\). The Hamiltonian now decomposes into three pieces:

\[ H_2 = \int_{0}^{\infty} dk k a^\dagger(k) a(k), \]

\[ H_3 = \frac{i}{24 \pi^3 |\mu|} \int_{0}^{\infty} dk_1 dk_2 dk_3 \sqrt{k_1 k_2 k_3} \left( f(k_1 + k_2 + k_3) a(k_1) a(k_2) a(k_3) \right) \]
Recent Developments in String Field Theory in D=2 ...

\[ H_1 = -\frac{i}{48\pi^2|\mu|} \int_0^\infty dk \sqrt{k^2} \langle k | (a(k) - a^\dagger(k)) \rangle, \]

where \( H_2 \) is the free Hamiltonian, \( H_3 \) is the cubic interaction, and \( H_1 \) is the tadpole term, and:

\[
\begin{align*}
  f(k) &= \int_{-\infty}^\infty d\tau \frac{1}{\cosh^2 \tau} e^{ik\tau} = \frac{\pi k}{2\sinh(\pi k/2)}, \\
g(k) &= \int_{-\infty}^\infty \frac{1}{\cosh^2 \tau} \tau e^{ik\tau} = \frac{\pi(k^2 + 2k)}{4\sinh(\pi k/2)}.
\end{align*}
\]

One novel feature is the appearance of the functions \( f \) and \( g \), which appear in the three-tachyon and one-tachyon Hamiltonian. When calculating scattering amplitudes, they give rise to an infinite tower of poles at discrete momenta, i.e. they will represent the effect of the discrete states on the scattering amplitudes.

Now we calculate the S-matrix:

\[ S = 1 - 2\pi i\delta(E_i - E_f)T. \]

Each right-moving massless particle has energy equal to momentum, \( E = k \). Let us describe the case of non-forward scattering of particles of momenta \( k_1 \) and \( k_2 \) into particles of momenta \( k_3 \) and \( k_4 \). Second-order perturbation theory gives us:

\[ T(k_1, k_2; k_3, k_4) = \sqrt{E_1 E_2 E_3 E_4} \sum_i \frac{<k_3 k_4 | H_{int} | i> <i | H_{int} | k_1 k_2>}{E_1 + E_2 - E_i}, \]

where we sum over all intermediate states \( i \). The s-channel contribution is then given by:

\[
S^{(s)} (k_1, k_2; k_3, k_4) = -\frac{\alpha_s^2}{8\pi} \prod_{j=1}^4 E_j \int_0^\infty dk \left( \frac{k f^2(k_1 + k_2 - k)}{k_1 + k_2 - k + i\epsilon} - \frac{k' f^2(k_1 + k_2 + k)}{k_1 + k_2 + k - i\epsilon} \right),
\]

The first term is due to one-particle intermediate exchange. The second one is due to five-particle intermediate states (found by slicing the four-particle scattering amplitude in a particular fashion). Likewise, the t- and u-channel contributions can also be calculated. By adding all three channels together, we find:

\[ S(k_1, k_2; k_3, k_4) = -\frac{\pi \alpha_s^2}{8} \delta(E_1 + E_2 - E_3 - E_4) \prod_{j=1}^4 E_j \left( F(p_4) + F(p_5) + F(p_6) \right). \]
where \( p_s = k_1 + k_2, p_t = |k_1 - k_3|, p_u = |k_1 - k_4| \), and
\[
F(p) = \int_{-\infty}^{\infty} dk \left[ \frac{(p-k)^2}{\sinh^2(\pi(p-k)/2)} \right] \frac{k}{p - k + i \varepsilon \text{sgn}(k)}
\]
(78)

Using:
\[
\frac{1}{x - i \varepsilon} = \mathcal{P} \frac{1}{x} + \pi i \delta(x)
\]
(79)
we find \( F(p) = -i(4p/\pi) - (8/3\pi) \). The total amplitude thus becomes:
\[
S(E_{1,2}; E_{3,4}) = -\frac{g_s^2}{2} \delta(E_1 + E_2 - E_3 - E_4) \prod_{j=1}^{4} E_j (E_1 + E_2 + |E_1 - E_3| + |E_1 - E_4| - 2i) .
\]
(80)

Upon the Euclidean continuation \( E_j \rightarrow i|q_j| \), and inclusion of the external leg factors, this agrees with the calculation found in matrix models.

This amplitude is rather revealing for several reasons. The amplitude seems to consist of two parts, the momentum-dependent parts, given by \( |k_1 + k_2| \), etc., and a contact term, given by a constant. The momentum-dependent terms, in turn, correspond to the exchange of the tachyon. The contact term, on the other hand, emerges because we must sum over an infinite number of poles in the \( F(p) \) integration. These poles, we see, correspond to the discrete states which are defined at discrete momenta.

Thus, it is possible to define a second set of Feynman rules, seemingly different from the previous set, in which we explicitly reveal the presence of the tachyon exchange and the presence of an infinite tower of discrete states. In contrast to the previous set of Feynman rules, which are based on cubic interactions, the second set of Feynman rules are actually non-polynomial, since the contact interactions can be defined over \( N \) particle states.

After a simple rescaling, we can define this new set of Feynman rules as the tachyon propagator and the various contact vertex functions \( V_N \) arising from the discrete states:

\[
\begin{align*}
tachyon \text{ propagator} & \rightarrow \frac{|k|}{\mu} \\
V_3 & \rightarrow \frac{1}{\mu} \\
V_4 & \rightarrow \frac{4}{\mu^2} \\
V_5 & \rightarrow \frac{1}{\mu^3} \left( 32 + \sum_{i=1}^{5} k_i^2 \right)
\end{align*}
\]
(81)

The existence of two sets of Feynman rules, one based on cubic vertices, and the other being non-polynomial, is apparently a universal feature of string theory.
On one level, this is due to the fact that the collective field theory has an non-poly-nomial Lagrangian formulation as well as a cubic Hamiltonian formulation. Similarly, ordinary string field theory in 26 dimensions also has the cubic light-cone vertex function as well as the covariant non-polynomial form, both of which yield the same on-shell S-matrix.

As an added check, one can show that these Feynman rules, both cubic and non-polynomial, can be shown to agree with the usual value of the $N$ point amplitudes, which equals, after a rescaling in a special kinematical region: \[ T_N \sim \left( \frac{d}{d\mu} \right)^{N-3} \mu^{N-3} \] (82)

where:

\[ s = \frac{1}{\sqrt{2}} \sum_{i=1}^{N} |q_i| - (N - 2); \quad \sum_{i=1}^{N} q_i = 0 \] (83)

where $q_i = -k_i/2$.

7. Temporal Gauge String Field Theory

In the usual 26 dimensional critical string field theory, there are both cubic and non-polynomial actions. The cubic theory is associated with the light-cone gauge, where the cubic interaction corresponds to the splitting or fissioning of a closed string into two smaller pieces, conserving string length in the process, i.e., $l_1 + l_2 = l_3$. The 26 dimensional non-polynomial theory, by contrast, preserves the string length of each string, and hence the non-polynomial interactions do not conserve string length.

Similarly, collective field theory in two-dimensions has both a cubic and a non-polynomial form. One puzzling aspect of the collective field theory approach, however, is that the cubic version does not have any remnant of the string length. In fact, no where in the collective field theory does the string length enter.

Some light on this peculiar situation is shed by the temporal gauge string field theory, where the world-sheet graviton is quantized in the temporal gauge. It was found by Ishibashi and Kawai that the quantization of a pure $c = 0$ two-dimensional gravity theory (in zero space-time dimensions) yielded a purely cubic action, where the string length was preserved. They showed that the correlation functions of the temporal gauge gravity theory were identical to the one-matrix model theory by demonstrating that the $\tau$ functions of the theory obeyed the usual Virasoro conditions.

In two-dimensional gravity in this gauge, the physical states are labeled by the invariant string length, which is a generally covariant quantity, and hence interactions preserve string length. They then showed that this formalism generalized to the multi-matrix approach, showing that the $c \leq 1$ case also preserved string length.
in its cubic interactions. In the limit as $c \rightarrow 1$, one presumably approaches the two-dimensional string theory.

Perhaps the simplest way in which to derive this new string field theory is to apply the apparatus of collective field theory to the case of the one-matrix model, and then let the number of matrices go to infinity.

The multi-matrix approach begins with a finite sequence of hermitian matrices $M_i$, with an action:

$$S = -c \sum_i Tr(M_i M_{i+1}) + \sum_i V_i(M_i)$$  \hspace{1cm} (84)

Correlation functions can be computed from the functional integral:

$$Z = \int dM_1 \cdots dM_k e^{-S}$$  \hspace{1cm} (85)

with hermitian $N \times N$ matrices $M_i$, $i = 1, \cdots, k$.

It can be shown that this generates a sequence of matrix models such that $c$ is given by:

$$c = 1 - \frac{6}{(k+1)(k+2)}, \quad k = 1 \cdots n$$  \hspace{1cm} (86)

For $k = 1$, we have the one-matrix model with $c = 0$. For $k \rightarrow \infty$, we presumably have the $c = 1$ model. In the double scaling limit, these theories correspond to two-dimensional gravity coupled to conformal matter with various values of $c$.

The $U(N)$ invariant observables can be given by the traces

$$\phi C = Tr(M_1^{n_1} M_2^{n_2} \cdots)$$  \hspace{1cm} (87)

where the loop space index $C$ denotes the sequence of matrices $C = \{n_1, n_2, \cdots\}$.

There is one important complication, however. There are no dimensions of space-time present. This means that the Hamiltonian makes no sense, because there is no time parameter. To remedy this situation, we will use the stochastic quantization approach\(^{26}\) and introduce a fictitious time variable $t$ into the theory and take the limit as $t \rightarrow \infty$.

In the stochastic quantization approach, we begin with an action $S$ defined with a field $\varphi(x)$ and introduce a fictitious time variable $t$. The field $\varphi(x,t)$ obeys the time-dependent Langevin equation:

$$\frac{\partial}{\partial t} \varphi(x,t) = -\frac{\partial S}{\partial \varphi} + \eta$$  \hspace{1cm} (88)

where $\eta$ is the random variable.

The correlation functions are then obtained in the limit $t \rightarrow \infty$

$$\langle F(\varphi) \rangle = \lim_{t \rightarrow \infty} \int [d\varphi(x)] F(\varphi) P_t$$  \hspace{1cm} (89)
where the time evolution (of the probability distribution $P_t$) is given by

$$\frac{\partial}{\partial t} P_t = -H_{FP} P_t \quad (90)$$

with the Fokker-Planck Hamiltonian

$$H_{FP} = \frac{1}{2} \int \left( \frac{d}{d\varphi(x)} - \delta S \right) \frac{d}{d\varphi(x)} \quad (91)$$

In the limit $t \to \infty$, one can show that these stochastic correlation functions defined over a fictitious time approach the correct correlation functions.

Our task is now to re-write these equations in terms of the matrix model approach. To begin, let us start with the $c = 0$ one-matrix model. As in the $c = 1$ case, we will make the usual change of variables from $M$ to $\phi_n$, such that $\omega$ and $\Omega$ become, by the chain rule:

$$\phi_n = \text{Tr}(M^n)$$
$$\Omega(n, n') = n n' \phi_{n+n'-2}$$
$$\omega(n) = -n \sum_{n'=0}^{n-2} \phi_n \phi_{n-n-2} \quad (92)$$

The stochastic Hamiltonian and action are given by:

$$S = \text{Tr} \left( \frac{1}{2} M^2 - \frac{1}{3} M^3 + \cdots \right) \quad (93)$$

and

$$H = -\text{Tr} \left( \frac{\partial}{\partial M} - \frac{\partial S}{\partial \Omega} \right) \frac{\partial}{\partial M} \quad (94)$$

Making the standard change of variables, the Hamiltonian becomes

$$- \sum_n \left( \sum_{n=0}^{\infty} \phi_{n+m+2} \frac{\partial}{\partial \phi_m} + \sum_{r=0}^{n-2} \phi_r \phi_{n-r} - \Omega(S, \phi_n) \right) n \frac{\partial}{\partial \phi_n} \quad (95)$$

with

$$\Omega(S, \phi_n) = n (\mu \phi_{n+1} - \phi_{n+2} + \cdots) \quad (96)$$

(This Hamiltonian has an elegant structure. If we drop the $S$ dependent term, then it can be written as

$$H_3 = - \sum_n O_n n \Pi_n \quad (97)$$

with

$$O_{n+2} = \sum_{n=0}^{\infty} \phi_{n+m+2} \Pi_m + \sum_r \phi_r \phi_{m+r} \quad (98)$$
where we have a Virasoro algebra generated by $O_{n+2} = L_n$:

$$[L_n, L_m] = (n - m) L_{n+m} \quad (99)$$

The Hamiltonian is therefore of the form $H = -O_n n(d/d\phi_n)$.

Let us now take the continuum limit. We will take the $z$ representation, given by:

$$\phi(z) = \text{Tr} \frac{1}{z - M} = \int_0^\infty dLe^{-L^2} \phi_L = \int_0^\infty dLe^{-L^2} \text{Tr}(e^{LM}) \quad (100)$$

with:

$$\phi(z) = \sum_{n \geq 0} z^{-n-1} \phi_n$$

$$\partial \Pi(z) = \sum_{n \geq 0} z^{n-1} n \Pi_n \quad (101)$$

with:

$$O(z) = \sum_n z^{-n-2} O_n \quad (102)$$

This gives:

$$O(z) = \int dz' \phi(z') \frac{\phi(z)}{z - z'} \partial_z \Pi + \phi^2(z) \equiv (\phi \partial_z \Pi)(z) + \phi^2(z) \quad (103)$$

(The bracket notation means that there are only $z^{-n}$ components.) The Hamiltonian is now

$$H = -\int dz \left[ \phi(z) \partial_z \Pi(z) + \phi^2(z) + (z^2 - \mu z) \phi + (\mu - z) \phi + c \right] \partial_z \Pi(z) \quad (104)$$

The scaling limit is defined by $\mu = \mu + a^2 \Lambda$ and $z = z + a\zeta$. We will also make a shift:

$$\phi(z) = \frac{1}{2} (z\mu - z^2) + a^{3/2} \Phi(\zeta) \quad (105)$$

where $a$ represents a basic dimensional length. After a rescaling and a shift in the field, the Hamiltonian now becomes:

$$H = -a^{1/2} \int d\zeta \left[ \frac{1}{N^2 a^4} \Phi \partial_\zeta \Pi + \Phi(\zeta)^2 - \Phi^2 \right] \partial \Pi(\zeta) \quad (106)$$

So far, the Hamiltonian resembles the one found for $c = 1$ matrix models. But the next crucial step is to convert variables from $\zeta$ to $l$ (which will correspond to string length):

$$\Phi(\zeta) = \int_0^\infty dl \ e^{-l/2} \Phi(l) \quad (107)$$
Inserting this new field definition in terms of string length, we now find:\(^{21}\)

\[
H = - \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \Phi (l_1 + l_2) l_1 \Pi(l_1) l_2 \Pi(l_2) \\
+ (l_1 + l_2) \phi(l_1) \phi(l_2) \Pi(l_1 + l_2) \\
- \int_0^\infty dl \rho(l) \Pi(l)
\]

(108)

If we examine this Hamiltonian, we find that the string length \(l\) is preserved by the cubic interaction, i.e. closed strings simply break or fission. This is the action found by Ishibashi and Kawai. (In some sense, the fact that the action is now written in terms of fields which conserve string length is not surprising. If we have the product of three functions defined at the same point \(\zeta\) and then re-express each function in terms of its Fourier transform with variable \(l\), then the new expression will conserve \(l\) among the three transformed functions.)

The next step is to generalize these arguments to the \(c \leq 1\) case. In principle, all steps can be carried out as before, although there is some difficulty reproducing the constraints on the \(\tau\)-functions, which should obey the \(w(n)\) algebra constraints.

For the general case, we add an index \(i\) to the field \(\Phi\), which corresponds to the group theoretical height of the representation, where the height variable takes on its value on the nodes of an \(ADE\) or \(ADE\) Dynkin diagram. The world-sheet is divided into different neighborhoods, where the height variable takes on the same value in the same neighborhood.

The Hamiltonian is then given by:\(^{8}\)

\[
H = \sum_i \int_0^\infty dl_1 \int_0^\infty dl_2 \Phi_i^1(l_1) \Phi_i^1(l_2) \Phi_i^1(l_1 + l_2) \\
+ \sum_{i,j} C_{ij} \int_0^\infty dl_1 \int_0^\infty dl_2 \Phi_i^1(l_1 + l_2) \Phi_j^1(l_2) \Phi_j^1(l_1 + l_2) \\
+ g \sum_i \int_0^\infty dl_1 \int_0^\infty dl_2 \Phi_i^1(l_1 + l_2) \Phi_i^2(l_1) \Phi_i^2(l_2) l_1 l_2
\]

(109)

where \(C_{ij}\) is the connectivity matrix \((C_{ij} = 1\) when \(i\) and \(j\) are linked on the Dynkin diagram; otherwise, it vanishes).

In principle, this new Hamiltonian in the \(c \rightarrow \infty\) limit should approach the collective string field theory Hamiltonian when expressed in terms of the variable \(l\), although this final step has not yet been completed.

8. Liouville Theory

Several features of collective field theory are still rather mysterious. The algebra \(w(\infty)\) does not emerge as part of a gauge symmetry, but rather as a spectrum
Recent Developments in String Field Theory in $D=2$ . . .

generating algebra. Also, the meaning of the discrete states is rather obscure. The reason why collective field theory does not resemble a standard gauge theory is because all the string degrees of freedom have been eliminated, leaving only the tachyon. It is thus instructive to re-write all our results in a standard Liouville framework.

Our starting point is the Liouville action in eq. (5), with zero cosmological constant $\mu = 0$, which gives us the energy-momentum tensor:

$$
T^{(X, \phi)}_{zz} = -\frac{1}{2} (\partial_z X^i)^2 - \frac{1}{2} (\partial_z \phi)^2 + \frac{1}{2} Q \partial\bar{z}^2 \phi
$$

$$
T^{(b, c)}_{zz} = -2b_{zz} \partial_z c^2 + c^2 \partial_z b_{zz} .
$$

The Fourier components are given by:

$$
L_n = \frac{1}{2\pi i} \oint dz z^n T^{(X, \phi)}_{zz} + \frac{c + 1 + 3Q^2 - 26}{12} n(n^2 - 1) \delta_{n+m,0} .
$$

To cancel the ghost contribution to the conformal anomaly one must set $c + 1 + 3Q^2 - 26 = 0$. Therefore, we have:

$$
Q = \sqrt{\frac{25 - c}{3}} .
$$

For $c = 1$, we have $Q = 2\sqrt{2}$.

Let us analyze these states from the perspective of BRST string field theory. We start with the BRST operator:

$$
Q_{BRST} = \frac{1}{2\pi i} \oint dz c(z) \left( T^{(X, \phi)}(z) + \frac{1}{2} T^{(b, c)}(z) \right) .
$$

Then we introduce the field functional $\Phi(X, \phi, b, c)$, where $X$ is the string variable, $\phi$ is the Liouville field coming from the metric, and $b$ and $c$ are the Faddeev-Popov ghosts.

Then the free field action is given by:

$$
\langle \Phi | Q(b_0 - \bar{b}_0) | \Phi \rangle
$$

which is invariant under: $\delta |\Phi\rangle = Q|\Lambda\rangle$. The equations of motion are then given by:

$$
Q(b_0 - \bar{b}_0)|\Phi\rangle = 0 .
$$

The physical, on-shell spectrum is found by solving the usual constraints and equations of motion. We first note that we can use the gauge degree of freedom to eliminate all the $b$ and $c$ modes in the field functional $|\Phi\rangle$. Then the ghost-free field functional obeys the usual Virasoro conditions:
Recent Developments in String Field Theory in D=2

\begin{align}
L_n|\psi\rangle &= \overline{L_n}|\psi\rangle = 0 \quad \text{for } n > 0 \\
L_0|\psi\rangle &= \overline{L_0}|\psi\rangle = 1 \cdot |\psi\rangle.
\end{align}

(115)

Now let us find the lowest physical state. The ground state of the theory, given by a tachyon, is:

\[ |p, \epsilon\rangle = e^{i p X + \epsilon \phi(0)}|0\rangle. \tag{116} \]

The tachyon state satisfies

\[ L_n|p, \epsilon\rangle = \overline{L_n}|p, \epsilon\rangle = 0, \quad n > 0, \text{ and} \]

\[ L_0|p, \epsilon\rangle = \overline{L_0}|p, \epsilon\rangle = \left(1/2p^2 - 1/2 \epsilon(\epsilon + 2\sqrt{2})\right)|p, \epsilon\rangle. \tag{117} \]

In order to maintain the on-shell condition, this means that \( p^2 - \epsilon(\epsilon + 2\sqrt{2}) = 0 \) or, defining \( E = \epsilon + \sqrt{2} \),

\[ p^2 - E^2 = 0. \tag{118} \]

meaning that the tachyon is massless.

If we solve for \( \epsilon \), we find the rather unusual solution:

\[ \epsilon = -\sqrt{2} \pm p = -\sqrt{2} + \chi p. \tag{119} \]

For the case of zero cosmological constant, \( \mu = 0 \), this means that on-shell tachyons occur in two chiralities, \( \chi = \pm 1 \). When we calculate scattering amplitudes in the next section, we must keep track of these chiralities. (For the case of non-vanishing cosmological constant, the situation is much more involved. Although the Virasoro generators are quite non-linear, one can still show that the Virasoro algebra is preserved and that the tachyon state is a solution of the Virasoro constraints, but only for one value of the chirality.)

9. Tachyon Correlation Functions

Next, we would like to calculate the scattering amplitudes for tachyons. The calculation using string field theory is rather involved and will be presented later. We will thus first evaluate the tachyon scattering amplitudes using conformal field theory. The tachyon vertex is given by:

\[ T_\chi(p) = e^{ipX + (-\sqrt{2}\chi)p}\phi \tag{120} \]

so the correlation function becomes:

\[ \langle \prod_{i=1}^{N} T_\chi(p_i) \rangle. \tag{121} \]
When performing the functional integral in this correlation function, there is an important complication. Since the theory is translationally invariant in $X$, the correlator is non-vanishing only if $\sum_i p_i = 0$. However, the correlation function is not translation invariant in $\phi$, which means that $\epsilon$ is not conserved, even for vanishing cosmological constant. In particular, this means that we have difficulty performing the functional integration over the zero mode $\phi_0$, where $\phi = \phi_0 + \tilde{\phi}$.

In order to integrate over $\phi_0$, we must impose yet another condition on the momenta. We will demand that all terms proportional to $\phi_0$ in the integrand vanish. If we set $p = 0$, then the $\phi_0$ dependence of the integrand is given by the vertices $e^{i\epsilon\phi_0}$ as well as the curvature term in the action in eq. (5):

$$\exp \left[ \sum_{i=1}^N \epsilon_i \phi_0 + \frac{1}{8\pi} \int d^2\sigma \sqrt{g} R \phi_0 \right] = \exp \left[ \left( \sum_{i=1}^N \epsilon_i + 2\sqrt{2} \right) \phi_0 \right].$$

(122)

In order to make this vanish, we will demand that:

$$\sum_{i=1}^N \epsilon_i = -2\sqrt{2}$$

(123)

By imposing this constraint by hand, we can now do the $\phi_0$ integration, which reduces to $\int d\phi_0$, which is divergent but can be factored out. (If we go to the Minkowski metric, then the $\phi_0$ integral becomes a Fourier integral, so we have a delta function which enforces the previous condition.)

The functional integration over $X$ and $\tilde{\phi}$ is now trivial, and in fact reproduces the original Shapiro-Virasoro amplitude, except that the energy factors are shifted by the various constraints on energy. The shifted Shapiro-Virasoro amplitude for the scattering of $N+1$ tachyons, in turn, can be written as:

$$A_{X,\ldots,X_N}(p_1, \ldots, p_N) = \int \prod_{i=1}^{N-3} d^2z_i \prod_{i<j} \left| z_i - z_j \right|^{2q_i q_j}$$

(124)

which is the familiar amplitude, except that $q_i q_j = p_i p_j - \epsilon_i \epsilon_j$ and $\epsilon_i = -\sqrt{2} + \chi_i p_i$.

These integrals can be evaluated, yielding:

$$A_{X,\ldots,X_N}(p_1, \ldots, p_N, p_{N+1}) = \frac{\pi^{N-2}}{(N-2)!} \prod_{i=1}^N \frac{\Gamma(1 - \sqrt{2} p_i)}{\Gamma(\sqrt{2} p_i)}$$

(125)

The sum rules $\sum_i p_i = 0$ and $\sum_i \epsilon_i = -2\sqrt{2}$ must be satisfied. This, in turn, fixes the value of $p_{N+1}$ to be:

$$\sum_{i=1}^N p_i = -p_{N+1} = \frac{1}{\sqrt{2}}(N - 1).$$

(126)

The amplitudes of type $(1,N)$ are obtained by a parity flip.
These amplitudes exhibit a remarkable structure. They vanish unless there is exactly one particle with $\chi = -1$, with all other particles having positive chirality. This amplitude is also unusual because of the presence of poles at $p_i = n / \sqrt{2}$ on each external leg, which does not appear in ordinary field theory. These “leg poles,” which occur only at discrete values of momenta, correspond to the discrete states. (These leg poles, from a field theory point of view, can be dropped if we rescale our operators.)

Unfortunately, we have glossed over several difficult questions in hastily arriving at this amplitude. The problem with this amplitude is that we have taken a vanishing cosmological constant $\mu = 0$, as well as imposing by hand an additional constraint on the momenta.

It is possible to remove these constraints by using a clever trick, which is to “analytically continue in integers.” This may sound mathematically strange, but can probably be rigorously justified. To see how this is done, let us first perform the $\phi_0$ integration without imposing these constraints:

$$\langle \prod_i T_{X_i} \rangle = \mu^s \Gamma(-s) \langle \prod_i T_{X_i} \left( \int d^3 z \, e^{-\sqrt{2} \phi} \right)^s \rangle$$

where we have used the integral:

$$\mu^s \Gamma(-s) = \int_0^\infty dA \, A^{-s-1} \, e^{-\mu A}$$

with the definitions $A = e^{-\sqrt{2} \phi}$ and:

$$-\sqrt{2} s = \sum_i \epsilon_i + Q$$

In general, the matrix element over the tachyon vertex functions can easily be calculated for positive integers $s$, but is problematic for non-positive integers. Let us, for example, take the case of four-tachyon scattering, where $k_1, k_2, k_3$ are positive and $k_4$ is negative. Then the scattering amplitude, for positive integer $s$, can be evaluated as in the shifted Shapiro-Virasoro amplitude. The matrix element becomes:

$$A(k_i) \sim \mu^s \Gamma(-s) \int d^2 z \, \left[ 2^{(k_1+k_2-k_3-k_4)}/2^{(k_1+k_2-k_3-k_4)} \right] \left| 1 - z^{2(k_1+k_2-k_3-k_4)} \right|^s \left| 1 - w_i z \right|^{s-2}$$

$$\times \prod_{i=1}^4 \int d^2 w_i \, \left| w_i z \right|^{s-2} \left| 1 - w_i z \right|^{s-2}$$

$$\times \prod_{1 \leq i < j \leq 4} \left| w_i - w_j \right|^{-4}$$

Evaluating these integrals for positive integer $s$, we have:

$$A(k_i) \sim (s+1) \mu^s \Delta^s (1) \prod_{i=1}^4 (-\pi) \Delta \left[ \frac{1}{2} \epsilon_i^2 - k_i^2 \right]$$
where $\Delta(x) = \Gamma(x)/\Gamma(1-x)$.

Now let us rescale and eliminate the leg poles. We then have:

$$A(k_1, k_2, k_3, k_4) \sim (1 + s)\mu^s$$  \hspace{1cm} (132)$$

In the same way, we can now evaluate the $N$ point scattering amplitude for arbitrary $\mu$. For positive integer $s$, we now have:

$$\langle \prod_{i=1}^{N+1} T_{\chi_i} \rangle \sim \left( \frac{\partial}{\partial \mu} \right)^{N-2} \mu^{s+N-2}$$  \hspace{1cm} (133)$$

We now perform the “analytic continuation in integers,” and claim that this formula actually works for arbitrary $s$, not just positive integers. In this way, we re-derive the formula found in matrix models in eq. (82).

We should mention that the expressions for the $N$-point tachyon scattering amplitudes were calculated using conformal field theory, not string field theory. We will return to this calculation of the scattering amplitudes when we introduce the full non-polynomial string field action.

10. Discrete States in Liouville Theory

Let us look for discrete states more systematically by constructing the physical states within the string field $|\Phi\rangle$ using the oscillators of the $X$ and $\phi$ fields. Introducing the oscillators $\alpha_n$ and $\beta_n$ through:

$$\partial_z X = -i \sum_n \alpha_n z^{-n-1}, \quad \partial_z \phi = -i \sum_n \beta_n z^{-n-1}$$  \hspace{1cm} (134)$$

We define:

$$\alpha_{0}^{\mu} = (i\beta_{n}, \alpha_{n}), \quad q^{\mu} = (\epsilon, \, p), \quad Q^{\mu} = (Q, \, 0).$$  \hspace{1cm} (135)$$

The Virasoro generators take the form

$$L_n = (q^{\mu} + \frac{n + 1}{2}Q^{\mu})\alpha_{\mu,n} + \sum_k : \alpha_{n+k}^{\mu} \alpha_{\mu,-k} : \quad n \neq 0$$

$$L_0 = q_{\mu}(q^{\mu} + Q^{\mu}) + \sum_k : \alpha_{n+k}^{\mu} \alpha_{\mu,-k} :$$  \hspace{1cm} (136)$$

where $[\alpha_{n}^{\mu}, \alpha_{m}^{\nu}] = n \delta_{n+m,0} \eta^{\mu\nu}, \eta_{\mu\nu} = \text{diag}(-1,1)$ and the indices are raised and lowered with the $\eta$ metric.

Now let us construct some of the higher excited states which are solutions of the gauge constraints. The first excited state (for open strings) is given by:

$$|\psi_E\rangle = \epsilon_{\mu} \alpha_{0}^{\mu} |q\rangle.$$  \hspace{1cm} (137)$$
We can always generalize this to the closed string case by doubling the number of states, so that we use $\epsilon_{\mu} = \epsilon_{\bar{\mu}}$. The Virasoro conditions now give

$$(q_{\mu} + Q_\mu) \epsilon^\mu = 0 , \quad (q_{\bar{\mu}} + Q_{\bar{\mu}}) q^\mu = 0 .$$

(138)

Notice also that the state $q_\mu \sigma_{-1}^\mu |q\rangle$ (with polarization $\epsilon^\mu = q^\mu$) is a pure gauge state, $L_{-1} |q\rangle$.

For general $q_\mu$, the unique solution of the Virasoro conditions is $\epsilon_{\mu} \sim q_\mu$, which corresponds to the pure gauge state. This is in accord with the naive light-cone argument that there are no physical oscillator states in two-dimensional string theory.

There are two important exceptional momenta, however.

First, we can set $q^\mu = 0$. Then the gauge symmetry becomes trivial, and the polarization $\epsilon_{\mu} = (0, 1)$ yields a non-trivial physical state, whose vertex operator is $\partial X \bar{\partial} \phi$.

Second, we can choose $q^\mu = -Q_\mu$. In this case the constraints are trivially satisfied, and the polarization $\epsilon_{\mu} = (0, 1)$ again gives a physical state. The corresponding operator is $\partial X \bar{\partial} \phi e^{-2\sqrt{2}}$.

The reason why Liouville theory possesses these strange discrete states, as we mentioned earlier, is because the Liouville action is not translationally invariant in $\phi$. When imposing the light-cone gauge, we find that we cannot eliminate all higher string degrees of freedom. Thus, the existence of these discrete states is a direct consequence of the breakdown of translation invariance.

In this way, we can show that these discrete states persist for all higher levels. At first, it may seem to be an impossibly tedious task to explicitly calculate operator expressions for these higher discrete states. However, using the representations of the conformal group, this is actually not a difficult task.

To do this, we must examine representations of the $c = 1$ algebra created by compactifying on a circle. It is known that we can construct generators of the $SU(2)$ algebra defined on the circle, given by $\partial X$ and $e^{\pm iX}$. These operators, in turn, can be written as Kac-Moody operators $H_\Delta$ and ladder operators $H_{\pm}$:

$$H_{\pm}(z) = \oint \frac{du}{2\pi i} : e^{\pm iX(u+z)} : , \quad H_\Delta(z) = \oint \frac{du}{4\pi} \partial X (u+z) .$$

(139)

The allowed values of the chiral (rescaled) momenta are $p = n/2$, and the simplest primary fields are the $SU(2)$ highest weight states:

$$\psi_{j,j}(z) = e^{iJX}(z)$$

(140)

where $J = 0, 1/2, 1, 3/2, \ldots$.

Given the fact that $\phi_{j,j}$ transforms as the highest weight state, we can compute the other members of the representation by hitting this state with lowering ladder operators $H_{-}$. By repeatedly acting with the lowering operator $H_{-}$, one builds up a spin-$J$ $SU(2)$ multiplet of primary fields $\psi_{j,m}$, $m \in \{J, J-1, \ldots, -J\}$, with conformal dimension $\Delta = J^2$.
Recent Developments in String Field Theory in D=2

\[
\psi_{J,m}(z) = \left[ \frac{(J + m)!}{(J - m)! (2J)!} \right]^{1/2} \left[ \oint \frac{du}{2\pi i} : e^{-iX(z+u)} : \right]^{J-m} \psi_{J,J}(z) .
\] (141)

So far, these states are only functions of the string variable \(X\). In order to couple it to two-dimensional gravity, we must "dress" the state by inserting a vertex operator for the \(\phi\) field, in order to obtain physical operators with dimension one:

\[
\Phi_{J,m}(z) = \left[ (J + m)! (J - m)! (2J)! \right]^{1/2} \psi_{J,m}(z) : e^{\delta}(z) : .
\] (142)

where \(\delta = -1 \pm J\).

One can verify that the operators satisfy the Virasoro conditions. These states are not pure gauge states, because the corresponding states have non-vanishing norms.

Now that we have explicitly written down operator expressions for the infinite number of discrete states, we will examine the algebra created by these discrete states.

11. \(w(\infty)\) in Liouville Theory

An important property of the chiral vertex operators \(\Phi_{J,m}\) is that they form an interesting algebra under the O.P.E.:

\[
\Phi_{J_1,m_1}(z) \Phi_{J_2,m_2}(0) = \ldots + \frac{1}{z} (J_1 m_2 - J_2 m_1) \Phi_{J_1+J_2-1,m_1+m_2}(0) + \ldots .
\] (143)

where we have shown the only physical operator appearing on the right-hand side. This is isomorphic to a wedge sub-algebra of \(w(\infty)\).

So far, we have only analyzed the states within \(|\Phi\rangle\) which are ghost-free. Now let us analyze the states with ghosts in them and arrange them according to BRST cohomologies. We recall that the physical states of string theory are given by the cohomology classes of the BRST operator at ghost number one. This means that the physical operators are the ghost number one vertex operators which commute with \(Q_{BRST}\), modulo commutators involving the \(Q_{BRST}\) with other operators. In the BRST formalism, the tachyon vertex operators (for open strings) is created by multiplying the usual vertex operator by \(c\):

\[
V_k^{\pm} (z) = c(z) T_{\pm}(z)
\] (144)

and the discrete vertex operators are given by:

\[
Y_{J,m}^{\pm}(z) = c(z) \Phi_{J+1,m}(z) .
\] (145)

These operators are non-trivial only at the discrete values of the momenta. It can be shown that each \(Y\) operator has a partner of adjacent ghost number. For
example, the $Y_{J,m}^+$ operators have “partners” at ghost number zero, while the $Y_{J,m}^-$ have “partners” at ghost number two.

Now let us examine the fusion rules for these ghost number zero operators, usually denoted by $O_{J,m}$. Modulo BRST commutators, the fusion rules for these operators form what is called the “ground ring”:

$$O_{J,m} O_{J',m'} = O_{J+J',m+m'}$$  \hspace{1cm} (146)

Explicitly, their form is given, for the lowest states, by:

\begin{align*}
O_{0,0} & = \ 1 \\
O_{1,1} & = (cb + \frac{i}{2} \partial X - \frac{1}{2} \partial \phi) e^{\frac{1}{2} i(X + \phi)} \\
O_{1,-1} & = (cb - \frac{i}{2} \partial X - \frac{1}{2} \partial \phi) e^{\frac{1}{2} i(-iX + \phi)}
\end{align*}  \hspace{1cm} (147)

The entire ring, via the fusion rules, can be generated by the last two operators. One can think of the ring as a set of analogues of the identity operator, which occur at the discrete momenta.

For open strings, the BRST cohomology is complicated by the fact that there exists a non-trivial operator $a$ which is also a BRST commutator:

$$a = [Q, \phi] = c\partial \phi + 2\partial c$$  \hspace{1cm} (148)

(Normally, this would imply that $a$, a BRST commutator, is also BRST trivial. Therefore, it would be uninteresting. However, $\phi$ does not transform as a true conformal field, so $a$ is actually not BRST trivial, although it is BRST invariant. $a$ is therefore BRST non-trivial.) Thus, multiplication by $a$ increases the ghost number by one and helps to generate a new BRST cohomology.

To display this cohomology,\textsuperscript{26} it will be useful to use the language of string field theory. For open string field theory, we can fix the gauge by choosing $b_0 \Phi = 0$. The states which satisfy both $Q \Phi = 0$ and $b_0 \Phi = 0$ are grouped into the “relative cohomology” and are given by $O_{J,m}$ and $Y_{J,m}^+$. If we relax the $b_0 \Phi = 0$ constraint, then we have a larger set of states, called the “absolute cohomology,” given by:

\begin{align*}
G & = 0: \quad O_{J,m} \\
G & = 1: \quad Y_{J,m}^+, \quad aO_{J,m} \\
G & = 2: \quad aY_{J,m}^+ .
\end{align*}  \hspace{1cm} (149)

which exhausts the BRST cohomology for $\epsilon > -1$.

Let $P_{J,m}$ represent the conjugate to $Y_{J,m}^+$. Then there exists another cohomology set, similar to the above, for $\epsilon < -1$, given by:
This completes the open string BRST cohomology.

For closed strings, the situation is a bit more complicated. We can impose the condition \((b_0 - \tilde{b}_0)\Phi = 0\) as well as the \(b_0\) and \(\tilde{b}_0\) constraints. Thus, there are three sets of cohomologies. There is the relative cohomology, which satisfies the \((L_0 - \tilde{L}_0)\Phi = 0\) constraint as well as the \(b_0\) and \(\tilde{b}_0\) constraints. Then there is the semi-relative cohomology, which satisfies the \(b_0 - \tilde{b}_0\) constraint but not the \(b_0\) and \(\tilde{b}_0\) constraint. The semi-relative cohomology is given explicitly by:

\[
\begin{align*}
G &= 0 : \mathcal{O}_{j,m} \overline{\mathcal{O}}_{j,m'}^+ \\
G &= 1 : Y^+_{j,m} \overline{Y}^+_{j,m'}, \mathcal{O}_{j,m} \overline{Y}^+_{j,m'}, (a + \overline{a}) \cdot (\mathcal{O}_{j,m} \overline{\mathcal{O}}_{j,m'}) \\
G &= 2 : Y^+_{j,m} \overline{Y}^+_{j,m'}, (a + \overline{a}) \cdot (Y^+_{j,m} \overline{\mathcal{O}}_{j,m'}), (a + \overline{a}) \cdot (\mathcal{O}_{j,m} \overline{Y}^+_{j,m'}) \\
G &= 3 : (a + \overline{a}) \cdot (Y^+_{j,m} \overline{Y}^+_{j,m'}). 
\end{align*}
\tag{151}
\]

for \(\epsilon > -1\).

There is another BRST cohomology set for \(\epsilon < -1\), given by:

\[
\begin{align*}
G &= 0 : \mathcal{O}_{j,m} \overline{\mathcal{O}}_{j,m}^+ \\
G &= 1 : Y^-_{j,m} \overline{Y}^-_{j,m'}, \mathcal{O}_{j,m} \overline{Y}^-_{j,m'}, (a + \overline{a}) \cdot (\mathcal{O}_{j,m} \overline{\mathcal{O}}_{j,m'}) \\
G &= 2 : Y^-_{j,m} \overline{Y}^-_{j,m'}, (a + \overline{a}) \cdot (Y^-_{j,m} \overline{\mathcal{O}}_{j,m'}), (a + \overline{a}) \cdot (\mathcal{O}_{j,m} \overline{Y}^-_{j,m'}) \\
G &= 3 : (a + \overline{a}) \cdot (Y^-_{j,m} \overline{Y}^-_{j,m'}) 
\end{align*}
\tag{152}
\]

And lastly, there is the absolute cohomology, which satisfies none of the \(b\) constraints.

12. Non-polynomial String Field Theory

Now that we have discussed the preliminaries of the first quantized Liouville theory, we would like to reanalyze this information within the context of a second quantized field theory.

One advantage of introducing the string field \(\Phi\) is that we can now assemble the tachyon, discrete states, and higher BRST trivial states into one field.

Symbolically, we have:

\[
|\Phi(X, b, c, \phi)\rangle = |\text{tachyon}\rangle + |\text{discretestates}\rangle + |\text{BRST trivial states}\rangle 
\tag{153}
\]
Another advantage of introducing string field theory is that the mysterious $w(\infty)$ symmetry emerges in much the same way that symmetries emerge in ordinary gauge theory. For example, in gauge theory, the structure constants of $SU(N)$ emerge as the coupling of three gauge fields $A_a^\mu$. Similarly, we can show that the coupling of three string fields is proportional to the structure constants of $w(\infty)$. Thus, $w(\infty)$ is a subalgebra of the full gauge algebra of string field theory.

If $|jm\rangle$ represents a discrete state, then one can show that the three-string vertex function of string field theory, taken on discrete states, yields the $w(\infty)$ structure constants:

$$
\langle j_1, m_1 | j_2, m_2 | j_3, m_3 | V_3 \rangle \sim \langle \Psi_{j_1, m_1}(0) \Psi_{j_2, m_2}(1) \Psi_{j_3, m_3}(\infty) \rangle \\
\sim (j_1 m_2 - j_2 m_1) \delta_{j_3, j_1 + j_2 + 1} \delta_{m_2, m_1 + m_2}
$$

Now, let us discuss the action for $D = 2$ non-polynomial string field theory. The basic structure of the $D = 2$ action must be identical to the action for the 26 dimensional case. This is because the triangulation of moduli space with cylinders of equal circumference (but arbitrary length) is independent of the dimension of space-time. The action is therefore:

$$
L = \langle \Phi | Q | \Phi \rangle + \sum_{n=3}^{\infty} \alpha_n \langle \Phi^n \rangle
$$

where $Q = Q_0(b_1 - \bar{b}_1)$, $Q_0$ is the usual BRST operator, and $\langle \Phi^n \rangle$ represents a $n$ string vertex function, such that the $n$ strings meet to form the topology of a polyhedra. Note that there are more than one distinct polyhedra at each level. For example, there are 2 polyhedra at $N = 6$, 5 polyhedra at $N = 7$, and 14 polyhedra at $N = 8$.

The fact that closed string field theory (defined with strings of equal string length) requires an additional four-string graph to reproduce the Shapiro-Virasoro amplitude was first pointed out in ref. 10, where it was shown that the “missing region” of moduli space could be filled exactly by a four-string diagram with the topology of a tetrahedron.

For example, let $a_{ij}$ represent the distance that the $i$th and $j$th string share in common. Then $a_{ij} = 0$ if they share no common boundary. Then we have $\sum_i a_{ij} = 2\pi$ for fixed $j$, which simply says that the total circumference of the $j$th closed string is $2\pi$.

The lengths of the sides of the tetrahedron is therefore governed by four constraints, representing the four faces of a tetrahedron:

$$
a_{12} + a_{13} + a_{14} = 2\pi \\
a_{21} + a_{23} + a_{24} = 2\pi \\
a_{31} + a_{32} + a_{34} = 2\pi \\
a_{41} + a_{42} + a_{43} = 2\pi
$$
There are six unknowns $a_{ij}$ for a tetrahedron. There are four constraints on them, given above. So we have a net number of two degrees of freedom, which is the correct number of moduli needed to parametrize the Shapiro-Virasoro amplitude.

Similarly, for an $N$ sided polyhedra, the number of edges or sides is $3N - 6$, and the number of constraints is $N$ for $N$ faces, so that the total number of degrees of freedom is $2N - 6$, which is the correct number of Koba-Nielsen variables or moduli necessary to describe the $N$ string amplitude. In other words:

$$3N - 6 \text{ Edges} - N \text{ Faces} = 2N - 6 \text{ Moduli}$$

For the tetrahedron graph, we can choose $a_{12}$ and $a_{13}$ as independent variables. Then solving the constraints, we find that the missing region is parametrized by:

$$\text{Missing region} = \left\{ \begin{array}{l}
a_{12}, a_{13} \leq \pi \\
a_{12} + a_{13} \geq \pi
\end{array} \right. \quad (158)$$

This missing region is filled precisely by the four-string interaction with the topology of a tetrahedron. Similarly, it is not hard to generalize this counting for higher polyhedra.

The non-polynomial action, in turn, is invariant under the following gauge transformation:

$$\delta \Phi = [Q\lambda] + \sum_{n=1}^{\infty} \beta_n \Phi^n \lambda$$

If we insert $\delta \Phi$ into the action, we find that the result does not vanish, unless:

$$(-1)^n \langle \Phi | Q\lambda \rangle + n \langle Q\Phi | \Phi^{n-1} \lambda \rangle + \sum_{p=1}^{n-2} C_p^n \langle \Phi^{n-p} | \Phi^p \lambda \rangle = 0 \quad (160)$$

where the double bars mean that when we join two polyhedra, the common boundary has circumference $2\pi$.

(We should point out that the gauge transformation outlined above is actually anomalous, i.e. the measure of integration $D\Phi$ is not invariant under this gauge transformation. This means that the transition from the classical action to the quantum one requires additional non-polynomial terms at the higher loop level. There are two ways in which one can calculate these higher loop corrections. First, one can use the Fujikawa method and calculate a recursion relation which generates the complete quantum action.27 Or, one can use BV quantization.28)

The problem facing us, however, is that the vertices are anomalous unless we take into account the proper insertion factors. The Liouville action possesses terms like $R\phi$, which introduce curvature singularities on the world-sheet where the strings join. Thus, we must insert the proper insertion factors at these singular points in order to maintain gauge invariance.
To do this, we will find it convenient to bosonize the $b, c$ ghosts, such that $c = e^\sigma$ and $b = e^{-\sigma}$. This will allow us to treat $X, \phi$ and $\sigma$ on almost the same footing. Then let us introduce the notation:

\begin{align*}
Q^M &= \{0, Q, -3\} \\
\phi^M &= \{X^i, \phi, \sigma\}
\end{align*}

The insertion factor, placed at the points where strings join, can now be expressed as:

\begin{equation}
\prod_{j=1}^{2(N-2)} \left( e^{-(Q^M \phi^M j)^2} \right)_j
\end{equation}

where $j$ labels the $2(N-2)$ sites where we have curvature singularities on the string world sheet.

The $N$-string vertex function $|V_N\rangle$ can now be written down as $|V_N\rangle = \int B_N|V_N^0\rangle$, where $B_N$ contains line integrals over $b$ fields described by Beltrami differentials. The $|V_N^0\rangle$ is the usual vertex function, given by a series of delta functions representing the overlap of $N$ strings. Written out explicitly, $|V_N^0\rangle$ is:

\begin{equation}
|V_N^0\rangle = \left( \prod_{j=1}^{2(N-2)} e^{-(Q^M \phi^M j)^2} \right) \int \delta(\sum_{i=1}^{N} p_i^M + Q^M) \prod_{i=1}^{N} P_i
\end{equation}

\begin{equation}
\times \exp \left\{ \sum_{r, \delta, n, m=0}^{N} \sum_{n, m=0}^{\infty} \frac{1}{2} N^M_{r, \delta}^{M, \sigma} M^r_{n, \sigma} M^m_{n, \sigma} \right\}
\times \exp \left\{ \sum_{r, \delta, n, m=0}^{N} \sum_{n, m=0}^{\infty} \frac{1}{2} N^M_{r, \delta}^{M, \sigma} M^r_{n, \sigma} M^m_{n, \sigma} \right\} \left( \prod_{i=1}^{N} dP_i |P_i^M\rangle \right)
\end{equation}

where $P_i$ represents the operator which rotates the string field by $2\pi$, where $j$ labels the insertion points, where we have deliberately dropped an uninteresting constant, and where the state vector $|P_i^M\rangle$ and the Neumann functions, which describe the world-sheet, are defined in the usual way.

Naively, we expect that

\begin{equation}
\sum_{i=1}^{N} Q_i |V_N^0\rangle = 0
\end{equation}

The proof of this statement is far from obvious. The calculation of the anomalous term is quite non-trivial, involving subtle point-splitting methods at the point where strings join.

13. Proof of Gauge Invariance
Recent Developments in String Field Theory in $D=2$ . . .

The proof of gauge invariance is complicated by the fact that there are subtle but important anomalies in the calculation of BRST invariance. For three strings, the calculation is performed by writing the BRST operator as a line integral over the three strings:

$$\sum_{i=1}^{3} Q_{i}|V_{3}\rangle_{0} = \oint_{C_{1} + C_{2} + C_{3}} \frac{d\rho}{2\pi} e(\rho) \times \left\{ -\frac{1}{2} (\partial_{z} \phi)^{2} + \frac{d\epsilon}{d\rho} h(\rho) + \frac{Q}{2} (\partial_{z} \phi)^{2} \right\} |V_{3}\rangle_{0} \quad (165)$$

where $C_{i}$ are infinitesimal curves which together comprise circles which go around $\rho(z_{i})$ and $\rho(\bar{z}_{i})$.

The calculation is rather tedious, so we will only quote the final result, which is given by $^{3}$:

$$\left\{ pc(z_{0}) \left[ \frac{D}{24} - \frac{13}{12} + \frac{1}{24} + \frac{1}{8} Q^{2} \right] + \frac{d\epsilon(z_{0})}{dz} \left[ \frac{5D}{96} - \frac{65}{48} + \frac{5}{96} + \frac{5}{32} Q^{2} \right] \right\} |V_{3}\rangle_{0} \quad (166)$$

which cancels if:

$$D - 26 + 1 + 3Q^{2} = 0 \quad (167)$$

which is precisely the consistency equation for Liouville theory in $D$ dimensions. To check the accuracy of this result, we can also perform the calculation using bosonized co-ordinates, which yields:

$$\left\{ pe^{\sigma(z_{0})} \left[ \frac{D + 2}{24} + \frac{1}{8} (Q^{2} - 3^{2}) \right] + \frac{d\sigma(z_{0})}{dz} \left[ \frac{5(D + 2)}{96} + \frac{5}{32} (Q^{2} - 3^{2}) \right] \right\} |V_{3}\rangle_{0} \quad (168)$$

Once again, we find that the anomaly cancels if we set:

$$D + 2 + 3(Q^{2} - 3^{2}) = 0 \quad (169)$$

as desired. Since the point-splitting method isolates the anomalous contribution of each vertex of the polygon, it is then trivial to prove that all $N$-point vertices are also BRST covariant.

14. Derivation of the Shifted Shapiro-Virasoro Amplitude

To show that the non-polynomial string field theory is correct, we must also show that it reproduces the correlation functions of Liouville theory. This is non-trivial, because the conformal maps used in open string field theory cannot be used for the closed string case.

To perform this calculation, we first need the conformal map between the complex $z$-plane and the multi-sheeted $\rho$ plane, which describes the collision of several
strings of equal perimeter. This is not easy, because the conformal map used in the covariant open string field theory calculation cannot be used here.

By analyzing the zeros and singularities of the conformal map, we find that it is given by:

\[
\frac{d\rho(z)}{dz} = C \prod_{i=1}^{N-2} \frac{(z - z_i)(z - \bar{z}_i)}{\prod_{k=1}^{N}(z - \gamma_k)}
\]

(170)

where the \(N\) variables \(\gamma_i\) map to points at infinity (the external lines in the \(\rho\) plane) and the \(N-2\) pair of variables \((z_i, \bar{z}_i)\) map to the points where two strings collide, creating the \(i\)th vertex (which are interior points in the \(\rho\) plane).

The number of unknowns in this map minus the number of moduli equals the number of constraints:

\[
\text{Unknowns} - \text{Moduli} = \text{Constraints}
\]

(171)

More explicitly, we have:

\[
\{C, z_i, \bar{z}_i, \gamma_i\} \rightarrow 6N - 6 \text{ unknowns}
\]

\[
\{\rho'(\gamma_i), \rho(z_i) - \rho(\bar{z}_i)\} \rightarrow 4N \text{ constraints}
\]

\[
\{\gamma_{ij} + i\theta_{ij}\} \rightarrow 2N - 6 \text{ moduli}
\]

(172)

where \(\gamma_{ij}\) is the distance separating the \(i\)th and \(j\)th vertex functions, and \(\theta_{ij}\) is the relative angle separating these two vertices.

For the four point function, the map can be integrated using ordinary analytical functions. We use the identity:

\[
\frac{(z - z_1)(z - \bar{z}_1)(z - z_2)(z - \bar{z}_2)}{\prod_{i=1}^{N}(z - \gamma_i)} = 1 + \sum_{i=1}^{4} \frac{A_i}{z - \gamma_i}
\]

(173)

where we define \(z_i = ia_i + b_i\) and \(\bar{z}_i = -ia_i + b_i\) for complex \(a_i\) and \(b_i\), and:

\[
A_i = \frac{[(\gamma_i - b_1)^2 + a_1^2]}{\prod_{j=1,j\neq i}^{4}(\gamma_i - \gamma_j)}
\]

(174)

Then we can split the integral into two parts, with the result:

\[
\rho(z) = \rho_1(z) + \rho_2(z)
\]

\[
\rho_1(z) = \int_{y_1}^{y_2} \frac{Ndz}{\sqrt{(z - z_1)(z - \bar{z}_1)(z - z_2)(z - \bar{z}_2)}}
\]

\[
\rho_2(z) = \sum_{i=1}^{4} \int_{y_1}^{y_2} \frac{NA_i dz}{(z - \gamma_i)\sqrt{(z - z_1)(z - \bar{z}_1)(z - z_2)(z - \bar{z}_2)}}
\]

(175)

Written in this form, we can now perform all integrals exactly, using third elliptic integrals. It is then easy to show:
\[ \rho_1(z) = N g F(\phi, k') = N g t n^{-1}[\tan \phi, k'] \]

\[ \rho_2(z) = \sum_{i=1}^{4} \frac{g N A_i}{a_1 + b_i g_1 - g_i g_1} \left( g_1 F(\phi, k') + \frac{\omega_i - g_1}{1 + \omega_i^2} \left[ F(\phi, k') + \omega_i^2 \Pi(\phi, 1 + \omega_i^2, k') + \omega_i(\omega_i^2 + 1)f_i \right] \right) \] (176)

where:

\[ \omega_i = \frac{a_1 + b_i g_1 - \gamma_i g_1}{b_i - a_1 g_1 - \gamma_i} \]

\[ f_i = \frac{1}{2} (1 + \omega_i^2)^{-1} (k^2 + \omega_i^2)^{-1} \times \ln \frac{(k^2 + \omega_i^2)^{1/2} - (1 - \omega_i^2)^{1/2} \text{dn} u}{(k^2 + \omega_i^2)^{1/2} + (1 - \omega_i^2)^{1/2} \text{dn} u} \]

\[ \phi = \arctan \left( \frac{y - b_1 g_1}{a_1 + b_1 g_1 - g_1 y} \right) \] (177)

and where:

\[ A^2 = (b_1 + b_2)^2 + (a_1 + a_2)^2, \quad B^2 = (b_1 - b_2)^2 + (a_1 - a_2)^2 \]

\[ g_i^2 = [4a_i^2 - (A - B)^2] / [(A + B)^2 - 4a_i^2], \quad g = 2(A + B) \]

\[ g_1 = b_1 - a_1 g_1, \quad k^2 = 1 - k^2 = 4AB / (A + B)^2 \]

\[ u = \text{dn}^{-1}(1 - k^2 \sin^2 \phi) \] (178)

After a considerable amount of algebra, this expression simplifies to:

\[ \rho(z) = \sum_{i=1}^{4} \frac{g N A_i}{a_1 + b_i g_1 - g_i g_1} \frac{\omega_i - g_1}{1 + \omega_i^2} \left[ \omega_i^2 \Pi(\phi, 1 + \omega_i^2, k') + \omega_i(\omega_i^2 + 1)f_i \right] \] (179)

Now that we have the explicit conformal map for four-string scattering, it is straightforward to find the Jacobian which takes us from the moduli describing string scattering to the Koba-Nielsen variables. The string scattering is described by \( \tau = \tau + i\theta \), where \( \tau \) is the length of the intermediate string, and \( \theta \) is the relative angle that strings one and two are rotated with respect to string three and four.

Similarly, with a bit of work one can calculate expressions for \( d\tau \) and \( dx \). Putting everything together, we find:

\[ \frac{d\tau}{dx} = -\frac{\pi N}{2K(k)g_1 (1 - x)(\gamma_1 - \gamma_3)(\gamma_2 - \gamma_4)} \] (180)

If we take only the tachyon component of \( \Phi \), then the four point amplitude can be written as:
where we must sum over all permutations so that we integrate over the entire complex plane; where $b_\varnothing$ defined in the $\rho$ plane transforms into $\int_C dz (dz / dw) b_{zz}$ in the $z$-plane, where $C$ is the image in the $z$-plane of a circle in the $\rho$ plane which slices the intermediate closed string, where $V(z) = c(z) \bar{c}(z)V_0(z)$, where $V_0$ is the tachyon vertex without ghosts, and where the ghost part $A_G$ equals:

$$A_4 = \langle V_0 | \frac{b_\varnothing \bar{b}_0}{L_1 \pm L_2} | V_0 \rangle$$

$$= \int d\tau d\theta \left\{ \left\langle V(\infty) V(1) \left( \int_C dz (dz / dw) b_{zz} \right) \left( \int_C dz dz b_{\sigma\sigma} \right) V(\bar{z}) V(0) \right\rangle \right\}\right|^2$$

$$= \int d^2 \tau \exp \left[ \sum_k (i n_k \cdot \phi(i) + \epsilon_k \phi(i)) \right] A_G$$

(181)

(Notice that we have made a conformal transformation from the $\rho$ world sheet to the $z$ complex plane. In general, we pick up a determinant factor, proportional to the determinant of the Laplacian defined on the world sheet. However, after making the conformal transformation, we find that the determinant of the Laplacian on the flat $z$-plane reduces to a constant. Thus, we can in general ignore this determinant factor.)

Putting the Jacobian, the ghost integrand, and the string integrand together, we finally find:

$$A_4 = \int d^2 \tau \left| x^{2p_1 p_2 (1 - \bar{x})^{2p_2 p_3}} \right|^2$$

(184)

In two-dimensions, we have $p_1 \cdot p_2 = p_3 - \epsilon_i \epsilon_j$ where $\epsilon_i = -\sqrt{2} + \chi_i p_i$, where $\chi$ is the “chirality” of the tachyon state, so we reproduce the integral found in matrix models and Liouville theory. (The amplitude is non-zero only if the chiralities are all the same except for one external line.)

We must say, however, that our results are only good for $\mu = 0$. We saw earlier that, by “analytically continuing in integers,” we could use conformal field theory and manipulate the shifted Shapiro-Virasoro amplitude, thereby formally deriving the matrix model result. Unfortunately, we cannot use a similar trick for the non-polynomial calculation. Because of the highly non-linear nature of the
\( \mu \neq 0 \) Liouville theory, it is not known whether we can derive the \( \mu \neq 0 \) amplitudes using some generalization of the \( \mu = 0 \) non-polynomial string field theory. This is an open question.

15. Conclusion

In summary, we have been able to formulate two-dimensional string field theory in at least four different ways. In each method, we have a different way of viewing the origin of the discrete states and the \( w(\infty) \).

The first three methods (free fermion theory, collective field theory, and temporal gauge field theory) are easy to work with, because all gauge degrees of freedom have been explicitly eliminated. However, this obscures the string picture underlying the theory. Thus, discrete states and \( w(\infty) \) are easy to display in this formalism, but rather difficult to explain intuitively.

By contrast, the Liouville string field theory has all degrees of freedom intact. Calculationally, it is more difficult to use than the other string field theories. But since the string picture is left untouched, the discrete states and \( w(\infty) \) appear in much the same way as in ordinary Yang-Mills theory, as byproducts of the full gauge invariance of the theory.

In principle, all four field theories should emerge as gauge fixed versions of the same theory. However, because of the complicated nature of the interactions (both cubic and non-polynomial), this is still an open question.

We should also point out that string field theory still faces many problems. Most of the results of two-dimensional gravity and strings were first found in the first quantized formalism, not the second. String field theory still rarely yields new insights beyond the perturbative, first quantized approach. Also, string field theory is still formulated in a background dependent fashion, which is one reason why it is not clear how to accommodate black hole solutions in this formalism. All these problems, we are confident, will be solved with time.

Acknowledgements

We would like to acknowledge partial support from CUNY-FRAP 6-64435 and NSF PHY-9020495.

References

Recent Developments in String Field Theory in $D=2$.


For more complete references, see the reviews:

“Recent Developments in 2D String Theory,” by A. Jevicki, BROWN-HET-918.

For reviews, see:


8. N. Ishibashi, H. Kawai; KEK preprint KEK-Th 378 “String field theory of $c \leq 1$ non-critical strings.” (December 1993).
N. Ishibashi, H. Kawai; KEK Preprint KEK-Th 364 (July 1993).


Recent Developments in String Field Theory in $D=2$ . . .


