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Elastic waves in discontinuous media: three dimensional scattering

F.R. Molino

and

P.C. Sabatier

Laboratoire de physique mathématique *
Université Montpellier II, Place E. Bataillon
34095 Montpellier Cx 05, France

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Abstract

This report contains an exact study of elastic waves propagation and of their scattering in discontinuous media where hard reflectors are onion-like sets of surfaces. In order to reformulate the problem as a finite set of boundary integral equations, the wave motion between reflectors is represented by means of elastic potentials which involve vectorial densities on the surfaces. In the external medium, an outgoing asymptotic condition generalizes the Silver Müller (and the Sommerfeld) condition to the case of coupled waves (S and P waves) moving with different velocities. The uniqueness of the Green function, which guarantees the uniqueness of the direct problem solution, is proved. For any incident wave, and arbitrary number of surfaces, the transmission and scattering problems are studied, with and without the simplification obtained by assuming constant Poisson ratios. According to the parameters range, the equations which are obtained are well-posed, either as second kind Fredholm equations, or because they reduce to inverting the sum of the identity operator and a "small norm" bounded operator. The results can be used to describe rigorously the three dimensional scattering of elastic waves in the frequency domain for any kind of incident wave function (P, S, ..) as well as the response to a localized source.

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1 Introduction

1.1 Introduction

In two previous papers, ([Sabatier, Dolveck-Guilpart, 1988], [Sabatier, 1989]), we gave a complete theory of scalar waves through media showing both inhomogeneities and strong discontinuities (reflectors). Our aim was to give a rigorous and complete framework to any scattering study, in a form which generalise and be similar to that of potential scattering in quantum mechanics, or to acoustic and electromagnetic scattering by continuous media or homogeneous obstacles. Hence, the S-matrix should be obtained, and its dependence on medium characteristics be derived by constructive methods. Elastic waves are the object of the present paper, but not quite in the most general case as we did it for acoustic waves ([Sabatier, 1989]). Here we limit our exact study to disconnected discontinuity surfaces of arbitrary shape and number, separating homogeneous domains with arbitrary Lamé parameters. Inhomogeneities could be treated in addition, but except in the acoustic case ([Sabatier, 1989]), or unless approximations are introduced, they lead to so heavy mathematics that they shadow the physical meaning without real need for it. On the other hand, studies of acoustic wave scattering themselves need the present elastic wave formulation as soon as solid medias are involved.

Needless to say, we found in the literature several rigorous partial studies of elastic wave scattering in discontinuous media, and many more approximate ones, because of their relevance in seismic problems for instance. We also found extensions of our previous studies to the electromagnetic case([de Fazio, 1990]). Finally, we were able to get the book of Kupraze (1963), who gathered the knowledge on exact methods for elastic wave propagation through discontinuous surfaces, and is surprisingly not often cited in most recent rigorous partial studies. In fact the theorems given in our section 3 and 4 as tools for our study were originally derived by Betti, Liapounov, Kupraze himself, and others, and gathered in his book. However, since his notations and developments in functional analysis are often very difficult to read, we give here the theorems in the form we need them, with a sketch of their proofs, and refer occasionnaly the reader to Kupraze or to the doctoral dissertation of one of us (M) for more details. Kupraze gave also a treatment of the scattering problem (his problem A) but only with the ad hoc assumptions that the Poisson ratio $\frac{\lambda}{\mu}$ is an invariant. In the true general case, he treated a two values configuration by introducing an infinite series, in a way which cannot be generalized and used as a basis for computations. In fact, the computers are now able to deal with these huge problems, whose physical interest is obvious, and this is also why we think the present study has become necessary.

The difficulties of this general case arise not only from the presence of vector waves, as in the electromagnetic case, but mainly from the interference of two types of waves, S and P (for Shear and Pressure) moving *with different velocities*. In contrast, the electromagnetic case [Colton, Kress, 1992] only involves waves of fixed velocities, and its analysis is simpler.

One must use here the general *elastic* Green tensor (see [Morse, Feshbach 1953]) $G(x, y)$:

$$\left(G(x, y)\right)_{ij} = \frac{1}{\omega^2} \left(\partial_{x_i} \partial_{y_j} \left(\Phi^{k^p}(x, y) - \Phi^{k^s}(x, y) \right) + (k^s)^2 \Phi^{k^s}(x, y) \delta_{ij} \right) \quad (1.1)$$

$$= \frac{1}{\omega^2} \left(\partial_{x_i} \partial_{y_j} \frac{e^{ik_p|x-y|}}{|x-y|} + \epsilon_{irq} \partial_{x_q} \epsilon_{rut} \partial_{y_u} \frac{e^{ik_s|x-y|}}{|x-y|} \delta_{tj} \right) \quad (1.2)$$

where ∂_{x_i} denotes the partial derivative $\frac{\partial}{\partial x_i}$, ϵ_{ijk} is the absolutely antisymmetric tensor:

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ is an even permutation of } 1, 2, 3 \\ -1 & \text{if } i, j, k \text{ is an odd permutation of } 1, 2, 3 \\ 0 & \text{else} \end{cases} \quad (1.3)$$

and δ_{ij} is the Kronecker symbol. The Einstein summation convention is used throughout the paper; ω is the frequency, c^p and c^s the velocities (see (1.11) below), $k^p = \frac{\omega}{c^p}$, $k^s = \frac{\omega}{c^s}$ the wave numbers.

$G(x, y)$ is solution of

$$\begin{aligned} \text{NS}[G(x, y)] &= (c^p)^2 \text{grad}_x \text{div}_x G(x, y) - (c^s)^2 \text{rot}_x \text{rot}_x G(x, y) + \omega^2 G(x, y) \\ &= -\delta(x - y) \mathbf{I}, \end{aligned} \quad (1.4)$$

where $\mathbf{I}$ is the identity term. $G(x, y)$ will be used in the following instead of the classical Helmholtz-Green function

$$\Phi^{k^p}(x, y) = \frac{e^{ik_p|x-y|}}{|x-y|} \quad (1.5)$$

which would fit if $c^p = c^s$. Here we consider the case of constant parameters inside each surface ('hard scatterers problem'). More realistic cases will be treated later.

So as to solve the problem, a separation of the S and P-waves should be done, but they are physically coupled on the discontinuity surfaces of the parameters, and one must beware the loss of physically significant information. Thus our specific tool to solve the scattering problem will be the boundary integral equations method, which transforms the three-dimensional transmission problem into a two-dimensional integral equation. This requires representation theorems of the solution in terms of surface densities. In the first part of this paper we establish such theorems. Then we use them to prove the uniqueness of our Green function in this layered case. Thanks to this result and a suitable asymptotic condition, which generalizes the Silver-Müller condition, the problem is well-posed and we are able to write down the scattering theory.

1.2 Setting the problem

Let $\{S_\alpha\}, \alpha = 0, \dots, N-1$ be a set of "onion-like" non intersecting surfaces in $\mathbb{R}^3$ of class C^2 , enclosing the layers $\{\Omega_\alpha\}$, ($\alpha = 0, 1, 2, \dots, N-1$). Let Ω_N be the remaining open space

(See [Sabatier, 1989]). The space is filled by **elastic** media characterized by their Lamé parameters inside each layer, $\lambda_\alpha, \mu_\alpha$, and **density** ρ_α . The basic elements for modelling are

- a) the propagation equation (here Navier-Stokes) at each point inside layers,
- b) the boundary conditions on the **discontinuity** surfaces,
- c) an asymptotic condition for the **physical** solution, expressing clearly the character of the incident wave.

The boundary conditions between welded layers are

1. continuity of the displacement $u_i(x) (i = 1, 2, 3)$.
2. continuity of the normal traction on the surface $t_\nu[u](x) (i = 1, 2, 3)$, where

$$t_{ij}[u](x) = \lambda \delta_{ij} \partial_{x_p} u_p(x) + \mu \epsilon_{ijp} \epsilon_{prs} \partial_{x_r} u_s(x) + 2\mu \partial_{x_j} u_i(x) \quad (1.6)$$

and ν_i is the vector normal to S_i at x , pointing to Ω_{i+1} :

$$t_\nu[u](x) = \{t_{ij}[u]\nu_j(x)\} = 2\mu \frac{\partial u}{\partial \nu} + \lambda \nu \operatorname{div} u + \mu(\nu \wedge \operatorname{rot} u) \quad (1.7)$$

These conditions simply mean that the **layers** remain welded. Other conditions (see [Kupradze, 1963]) could be studied by a **similar** analysis, but we shall not do it here. Hence $u_i(x)$ will satisfy, for $i = 1, 2, 3$, and in each layer, e.g. that labelled by α :

$$\left\{ \begin{array}{l} (\lambda_\alpha + 2\mu_\alpha) \operatorname{grad}_x \operatorname{div}_x u(x, t) - \mu_\alpha \operatorname{rot}_x \operatorname{rot}_x u(x, t) = \rho_\alpha \frac{\partial^2 u}{\partial t^2}(x, t) \\ u_i(x) \text{ continuous across the surface } S_\alpha, \\ t_\nu[u] \text{ continuous across the surface } S_\alpha, \\ \text{asymptotic condition.} \end{array} \right. \quad (1.8)$$

For a harmonic wave

$$u(x, t) = u(x) e^{-i\omega t}, \quad (1.9)$$

the equation 1.8 will reduce in each layer Ω_α to:

$$\left\{ \begin{array}{l} (c_\alpha^p)^2 \operatorname{grad}_x \operatorname{div}_x u(x, t) - (c_\alpha^s)^2 \operatorname{rot}_x \operatorname{rot}_x u(x, t) + \omega^2 u(x, t) = 0 \\ u_i(x) \text{ continuous across the surface } S_\alpha, \\ t_\nu[u] \text{ continuous across the surface } S_\alpha, \\ \text{asymptotic condition.} \end{array} \right. \quad (1.10)$$

where

$$\left\{ \begin{array}{l} (c_\alpha^p)^2 = \frac{\lambda_\alpha + 2\mu_\alpha}{\rho_\alpha}, \\ (c_\alpha^s)^2 = \frac{\mu_\alpha}{\rho_\alpha}. \end{array} \right. \quad (1.11)$$

It follows from 1.10 that in each domain, $u(x)$ can be decomposed into the sum of a gradient and a rotational, at least in **one** obvious way, since the rotational and the divergence of $u(x)$ must exist everywhere **but** on the boundaries.

1.3 Contents

This article is organised as follows. In section 1 it is shown that the solution of Navier-Stokes equation inside an homogeneous domain Ω can be written in terms of potentials which only involve vectorial density functions supported by the surfaces bounding Ω . In section 2 we prove that a solution of a homogeneous transmission problem which in addition obeys the purely outgoing condition at ∞ must vanish. This ensures the uniqueness of Green's functions and the uniqueness of scattering or transmission problems solutions. The algebraic and analytic properties of elastic single and doublelayer potentials are demonstrated in section 4. They are applied in section 5, where the transmission and scattering problems are reduced to determining the vectorial density functions on each discontinuity surface. Finally, it is shown that the response to an elastic source can be treated by the same token. Some lengthy proofs are treated in appendices.

2 Representation theorems

2.1 Acoustic case (recall)

Modelling acoustic media layered with similar topology led us (See [Sabatier, 1989]) to seek a scalar field $\phi(x)$ solution of the following system:

$$\left\{ \begin{array}{l} \Delta\phi + k_\alpha^2\phi = 0; \quad k_\alpha^2 = \frac{\rho_\alpha\omega_\alpha^2}{\lambda_\alpha} \text{ inside layer } \Omega_\alpha, \\ \frac{\phi(x)}{a(x)} \text{ continuous across surfaces,} \\ \phi(x)\frac{\partial a}{\partial\nu}(x) - a(x)\frac{\partial\phi}{\partial\nu}(x) \text{ continuous across surfaces,} \\ \text{Sommerfeld outgoing asymptotic condition.} \end{array} \right. \quad (2.1)$$

where $a(x)$ is a function with given jumps across S_α . We recall the Sommerfeld scalar asymptotic condition

$$\left(\text{grad}_x \phi, \frac{x}{|x|} \right) - ik\phi(x) = o\left(\frac{1}{|x|} \right) \quad (2.2)$$

In this case the representation theorem is a consequence of the following Green theorem (See [Colton, Kress, 1983 p 68]) :

Theorem 1 *If $\phi(x)$ is a solution of the Helmholtz equation in Ω , then*

$$\phi(x) = \int_{\partial\Omega} \frac{\partial\Phi_k(x,y)}{\partial\mathbf{n}(y)} \phi(y) ds(y) - \int_{\partial\Omega} \Phi_k(x,y) \frac{\partial\phi(y)}{\partial\mathbf{n}(y)} ds(y),$$

where $\mathbf{n}$ is the outgoing normal vector on $\partial\Omega$. The same is true in $\mathbb{R}^3 \setminus \Omega$ where ϕ ensures the Sommerfeld asymptotic condition (2.2).

We note that $\phi(x)$ and $\frac{\partial\phi(y)}{\partial\mathbf{n}(y)}$, the surface densities used in this representation theorem, are the building blocks of the conserved physical quantities on the surfaces of discontinuity of the parameters. We shall obtain the elastic generalization of this result.

2.2 Representation theorem inside Ω

Let us introduce the following operator:

$$\mathbf{P}_{ij}^\alpha[\mathbf{u}](x)\mathbf{n}_j(x) =: \mathbf{P}_\mathbf{n}^\alpha[\mathbf{u}](x) \quad (2.3)$$

$$= \frac{\alpha + \mu}{\rho} \frac{\partial}{\partial\mathbf{n}} \mathbf{u}(x) + \frac{\lambda - \alpha + \mu}{\rho} \mathbf{n} \text{div} \mathbf{u} + \frac{\alpha}{\rho} (\nu \wedge \text{rot} \mathbf{u}). \quad (2.4)$$

This operator, for $\alpha = \mu$, gives

$$\frac{1}{\rho} \mathbf{t}_\mathbf{n}[\mathbf{u}] = \mathbf{T}_\mathbf{n}[\mathbf{u}] = \frac{2\mu}{\rho} \frac{\partial}{\partial\mathbf{n}} \mathbf{u}(x) + \frac{\lambda}{\rho} \mathbf{n} \text{div} \mathbf{u} + \frac{\mu}{\rho} (\mathbf{n} \wedge \text{rot} \mathbf{u}), \quad (2.5)$$

and for $\alpha = \mu(\lambda + \mu)/(\lambda + 3\mu)$ gives the so-called "pseudostress":

$$N_n^\alpha[u](x) = \frac{2\mu(\lambda + 2\mu)}{\rho(\lambda + 3\mu)} \frac{\partial}{\partial n} u(x) + \frac{(\lambda + \mu)(\lambda + 2\mu)}{\rho(\lambda + 3\mu)} n \operatorname{div} u + \frac{\mu(\lambda + \mu)}{\rho(\lambda + 3\mu)} (n \wedge \operatorname{rot} u). \quad (2.6)$$

We begin by proving the equivalent of the Helmholtz-Green theorem for the Navier-Stokes equations:

Theorem 2 (Betti-Green theorem) *Let Ω be an elastic domain characterised by the parameters λ, ρ, μ , and let u and v be two vector fields inside Ω . If u and v are two times differentiable in Ω , and the Navier-Stokes differential operator is:*

$$NS[u](x) = \frac{\lambda + 2\mu}{\rho} \operatorname{grad}_x \operatorname{div}_x u(x) - \frac{\mu}{\rho} \operatorname{rot}_x \operatorname{rot}_x u(x) \quad (2.7)$$

then :

$$\begin{aligned} \int_{\Omega} \{v(x) \cdot NS[u](x) - u(x) \cdot NS[v](x)\} d^3x = \\ \int_{\partial\Omega} \left\{ \frac{1}{\rho} v(x) \cdot t_n[u](x) - u(x) \cdot \frac{1}{\rho} t_n[v](x) \right\} ds(x) \end{aligned} \quad (2.8)$$

where t_n is defined in 1.6 and $n(x)$ is the vector normal to $\partial\Omega$, pointing outwards of Ω .

[Proof: It is a consequence of the formula

$$\int_{\Omega} \operatorname{div}_x u(x) d^3x = \int_{\partial\Omega} u(x) \cdot n(x) ds(x) \quad (2.9)$$

used here in the tensor form:

$$\int_{\Omega} \partial x_{i_p} A_{i_1 i_2 \dots i_p}(x) d^3x = \int_{\partial\Omega} A_{i_1 i_2 \dots i_p} n_{i_p}(x) ds(x), \quad (2.10)$$

assuming that ∂x_{i_p} means $\frac{\partial}{\partial x_{i_p}}$. We apply it to the Navier-Stokes operator, which is linked to P by:

$$\partial x_j \frac{1}{\rho} t_{jk}[u] = NS_k[u]. \quad (2.11)$$

The equation 2.8 follows. $\square$]

An important generalisation can be proved, using :

Theorem 3 *With the same assumptions and notations as in the Betti-Green theorem, we can write:*

$$\begin{aligned} \int_{\Omega} \{NS_j[u](x) v_j(x) - u_j(x) NS_j[v](x)\} d^3x = \\ \int_{\partial\Omega} \{v(x) \cdot P_n^\alpha[u](x) v(x) - u(x) \cdot P_{jn}^\alpha[v](x)\} ds(x) \end{aligned} \quad (2.12)$$

where $P_n^\alpha[v]$ is defined in 2.4 and $n(x)$ is the vector normal to $\partial\Omega$, pointing outwards of Ω ;

(See [Kupradze, 1963] for proof). A third useful vector operator can be constructed from the same building blocks. We call it the ghost operator:

$$\mathbf{Q}_n[\mathbf{u}] = \frac{\partial}{\partial \mathbf{n}} \mathbf{u}(x) - n \operatorname{div} \mathbf{u} + (\mathbf{n} \wedge \operatorname{rot} \mathbf{u}). \quad (2.13)$$

It is such that

$$\mathbf{T}_n[\mathbf{u}](x) - \mathbf{N}_n[\mathbf{u}](x) = \frac{2\mu^2}{\rho(\lambda + 3\mu)} \mathbf{Q}_n[\mathbf{u}](x). \quad (2.14)$$

The consistence of the two theorems Th2 and Th3 is expressed by

$$\int_{\partial\Omega} \{ \mathbf{Q}_n[\mathbf{u}](x) \cdot \mathbf{v}(x) - \mathbf{u}(x) \mathbf{Q}_n[\mathbf{v}](x) \} ds(x) = 0, \quad (2.15)$$

a direct consequence of (2.14).

As a corollary we have the:

Theorem 4 (Interior representation theorem) *If $\mathbf{u}(x)$ and $\mathbf{G}(x, y)$ are respectively two vector and tensor fields solutions of*

$$(c^p)^2 \operatorname{grad}_x \operatorname{div}_x \mathbf{u}(x) - (c^s)^2 \operatorname{rot}_x \operatorname{rot}_x \mathbf{u}(x) + \omega^2 \mathbf{u}(x) = 0 \quad (2.16)$$

$$(c^p)^2 \operatorname{grad}_x \operatorname{div}_x \mathbf{G}(x, y) - (c^s)^2 \operatorname{rot}_x \operatorname{rot}_x \mathbf{G}(x, y) + \omega^2 \mathbf{G}(x, y) = -\delta(x - y) \mathbf{I} \quad (2.17)$$

in Ω , with the definitions 1.11, we have

$$4\pi \mathbf{u}(x) = \int_{\partial\Omega} ds(y) \mathbf{P}_n^\alpha[\mathbf{u}](y) \cdot \mathbf{G}(x, y) - \int_{\partial\Omega} ds(y) \mathbf{P}_n^\alpha[\mathbf{G}(x, y)] \mathbf{u}(y), \quad (2.18)$$

where $\mathbf{P}_n^\alpha[\mathbf{G}(x, y)]$ is the second order tensor obtained by generalizing $\mathbf{P}_n^\alpha[\mathbf{u}]$ to tensorial arguments, and is written from :

$$\mathbf{P}_{ijk}^\alpha[\mathbf{G}(x, y)] = \frac{\lambda - \alpha + \mu}{\rho} \delta_{ik} \partial_{x_p} \mathbf{G}_{pj}(x, y) + \frac{\alpha}{\rho} \epsilon_{ikp} \epsilon_{prs} \partial_{x_r} \mathbf{G}_{sj}(x, y) + \frac{\alpha + \mu}{\rho} \partial_{x_k} \mathbf{G}_{ij}(x, y). \quad (2.19)$$

[**Proof:** $\mathbf{u}(x)$ and $\mathbf{G}(x, y)$ are solutions of:

$$\begin{cases} \partial_{x_j} \mathbf{P}_{ij}[\mathbf{u}(x)] = \mathbf{NS}_x[\mathbf{u}](x) = -\omega^2 \mathbf{u}_i(x) \\ \partial_{x_k} \mathbf{P}_{ijk}[\mathbf{G}(x, y)] = \mathbf{NS}_x[\mathbf{G}](x, y) = -\omega^2 \mathbf{G}_{ij}(x, y) - \delta_{ij} \delta(x - y), \end{cases} \quad (2.20)$$

these relations exhibiting the "duality" relation existing between the stress tensor and the Navier-Stokes equations. With the help of 2.20, applying the Betti-Green theorem to $\mathbf{u}(y)$ and the column vectors of $\mathbf{G}(x, y)$ give the result. $\square$]

2.3 Representation theorem outside Ω

Let us introduce the P- and S-components for the Green tensor $\mathbf{G}$:

$$\mathbf{G}(x, y) = \frac{1}{(c^p)^2} \mathbf{G}^p(x, y) + \frac{1}{(c^s)^2} \mathbf{G}^s(x, y), \quad (2.21)$$

with

$$\begin{cases} \mathbf{G}_{ij}^p(x, y) = \frac{1}{(k^p)^2} \partial_{x_i} \partial_{y_j} \Phi_{kp}(x, y) \\ \mathbf{G}_{ij}^s(x, y) = \frac{1}{(k^s)^2} \epsilon_{irq} \partial_{x_q} \epsilon_{rlj} \partial_{y_l} \Phi_{ks}(x, y) \end{cases} \quad (2.22)$$

satisfying

$$\begin{cases} \left(\text{div}_x \mathbf{G}^s \right)_j = \partial_{x_i} \mathbf{G}_{ij}^s(x, y) = -\partial_{y_i} \mathbf{G}_{ij}^s(x, y) = 0 \\ \left(\text{rot}_x \mathbf{G}^p \right)_{ij} = \epsilon_{ikl} \partial_{x_l} \mathbf{G}_{kj}^p(x, y) = -\epsilon_{ikl} \partial_{y_l} \mathbf{G}_{kj}^p(x, y) = 0 \end{cases}$$

At this point it must be emphasized that a P and S decomposition of $\mathbf{G}$ is in no way unique: the difference between two such decompositions would be a harmonic tensor, but since we are considering a transmission problem across boundaries, even if this tensor is vanishing at infinity - which is the case - it will not necessarily be zero everywhere, for its values on the boundaries are not known. However this absence of uniqueness does not weaken the asymptotic results given below.

The Green tensor obeys the following asymptotic condition:

Lemma 5 *If, with the notations above, x is fixed, and $|y| \rightarrow \infty$, then:*

$$\mathbf{P}^\alpha[\mathbf{G}](x, y) \cdot \hat{\mathbf{x}} - ik^p \mathbf{G}^p - ik^s \mathbf{G}^s = O\left(\frac{1}{|y|^2}\right) \quad (2.23)$$

where $\hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$.

[Proof: We can separate $\mathbf{P}^\alpha[\mathbf{G}](x, y)$ into P- and S-parts:

$$\begin{aligned} \mathbf{A}_{ij}(x, y) &= \frac{1}{\rho} (\mathbf{P}^\alpha[\mathbf{G}](x, y) \cdot \hat{\mathbf{x}} - ik^p \mathbf{G}^p - ik^s \mathbf{G}^s)_{ij} \\ &= \frac{\lambda - \alpha + \mu}{\rho} \hat{x}_i \partial_{x_r} \left(\frac{1}{(c^p)^2} \mathbf{G}^p \right)_{rj}(x, y) + \frac{\alpha + \mu}{\rho} \partial_{x_k} \left(\frac{1}{(c^p)^2} \mathbf{G}^p \right)_{ij}(x, y) \hat{x}_k \\ &\quad + \frac{\alpha}{\rho} \epsilon_{ipk} \epsilon_{prs} \partial_{x_r} \left(\frac{1}{(c^s)^2} \mathbf{G}^s \right)_{sj}(x, y) \hat{x}_k \\ &\quad + \frac{\alpha + \mu}{\rho} \partial_{x_k} \left(\frac{1}{(c^s)^2} \mathbf{G}^s \right)_{ij}(x, y) \hat{x}_k \\ &\quad - ik^p (\mathbf{G}^p)_{ij} - ik^s (\mathbf{G}^s)_{ij}. \end{aligned}$$

Lets us denote A^p and A^p the components of A depending respectively on G^p and G^s .
Introducing the definition for G^p and G^s given above, one obtains for A^p

$$\begin{aligned} A_{ij}^p(x, y) &= \frac{1}{(c^p)^2} \left\{ \frac{\lambda - \alpha + \mu}{\rho} \hat{x}_i \partial_{x_r} \left[\frac{1}{(k^p)^2} \partial_{x_r} \partial_{y_j} \frac{e^{ik^p|x-y|}}{|x-y|} \right] \right. \\ &\quad + \frac{\alpha + \mu}{\rho} \hat{x}_k \partial_{x_k} \left[\frac{1}{(k^p)^2} \partial_{x_i} \partial_{y_j} \frac{e^{ik^p|x-y|}}{|x-y|} \right] \\ &\quad \left. - ik^p (G_{ij}^p) \right\} \end{aligned}$$

Then using the following expansions of the partial derivatives of the Green function:

$$\begin{aligned} \partial_{x_a} \Phi_{kp}(x, y) &= -\partial_{y_a} \Phi_{kp}(x, y) \\ &= \frac{x_a - y_a}{|x-y|} (ik^p) \Phi_{kp}(x, y) + O\left(\frac{1}{|y|^2}\right), \end{aligned}$$

$$\partial_{x_a} \partial_{x_b} \Phi_{kp}(x, y) = \frac{x_a - y_a}{|x-y|} \frac{x_b - y_b}{|x-y|} (ik^p)^2 \Phi_{kp}(x, y) + O\left(\frac{1}{|y|^2}\right),$$

$$\partial_{x_a} \partial_{x_b} \partial_{x_c} \Phi_{kp}(x, y) = \frac{x_a - y_a}{|x-y|} \frac{x_b - y_b}{|x-y|} \frac{x_c - y_c}{|x-y|} (ik^p)^3 \Phi_{kp}(x, y) + O\left(\frac{1}{|y|^2}\right),$$

one obtains

$$\begin{aligned} A_{ij}^p(x, y) &= \frac{1}{\omega^2} \left[-(k^p)^2 \frac{\lambda - \alpha + \mu}{\rho} \hat{x}_i \frac{x_j - y_j}{|x-y|} (ik^p) \Phi_{kp}(x, y) \right. \\ &\quad + \frac{\alpha + \mu}{\rho} \frac{(x-y) \cdot \hat{x}}{|x-y|} \frac{x_i - y_i}{|x-y|} \frac{y_j - x_j}{|x-y|} (ik^p)^3 \Phi_{kp}(x, y) \\ &\quad - ik^p \left[\frac{1}{(k^p)^2} \partial_{x_i} \partial_{y_j} \Phi_{kp}(x, y) \right] \\ &\quad + O\left(\frac{1}{|y|^2}\right) \\ &= \frac{1}{(c^p)^2} (-ik^p) \left[\frac{y_j - x_j}{|x-y|} \right] \left[\frac{\lambda - \alpha + \mu}{\rho} \hat{x}_i + \frac{\alpha + \mu}{\rho} \frac{(x-y) \cdot \hat{x}}{|x-y|} \frac{x_i - y_i}{|x-y|} \right. \\ &\quad \left. - \frac{(\lambda + 2\mu)}{\rho} \frac{x_i - y_i}{|x-y|} \right] \Phi_{kp}(x, y) \\ &\quad + O\left(\frac{1}{|y|^2}\right) \end{aligned}$$

Using for $|y| \rightarrow \infty$:

$$\hat{x}_i - \frac{x_i - y_i}{|x-y|} = O\left(\frac{1}{|y|^2}\right),$$

the $O\left(\frac{1}{|y|^2}\right)$ term vanishes in the above expression and

$$A_{ij}^p(x, y) = O\left(\frac{1}{|y|^2}\right)$$

The S-term can be treated in the same way. $\square$

To proceed to the exterior representation theorem we must introduce an asymptotic condition for $u(x)$ similar to 2.23. Again, there is no "natural" way of decomposing the field into P and S parts, for the same reasons of non uniqueness sketched in the case of G . However, we know at least one decomposition in each domain, as it follows from 1.10. Let us write it as:

$$u(x) = \frac{1}{(c^p)^2} u^p(x) + \frac{1}{(c^s)^2} u^s(x), \quad (2.24)$$

$$rot_x u^p(x) = 0, \quad (2.25)$$

$$div_x u^s(x) = 0 \quad (2.26)$$

We can now write down the

Theorem 6 (Exterior representation theorem) *If $u(x)$ and $G(x, y)$ are two vector and tensor fields solutions of*

$$c_p^2 grad_x div_x u(x) - c_t^2 rot_x rot_x u(x) + \omega^2 u(x) = 0$$

$$c_p^2 grad_x div_x G(x, y) - c_t^2 rot_x rot_x G(x, y) + \omega^2 G(x, y) = -\delta(x - y)I$$

in $\mathbb{R}^3 \setminus \Omega$, with the "outgoing asymptotic condition":

$$P_{\hat{x}}^\alpha[u](y) - ik^p u^p(y) - ik^s u^s(y) = O\left(\frac{1}{|y|^2}\right), \quad (2.27)$$

we have

$$u_i(x) = - \int_{\partial\Omega} ds(y) G(x, y) \cdot P_n^\alpha[u](y) + \int_{\partial\Omega} ds(y) u(y) \cdot P_n^\alpha[G(x, y)](y) \quad (2.28)$$

$n(y)$ being always the outgoing normal vector on $\partial\Omega$.

[Proof : Let us write down the interior representation formula in full cartesian coordinates for $x \in (\mathbb{R}^3 \setminus \Omega) \cap B(0, R)$, where $B(0, R)$ is the 0-centered ball of radius R such that $\Omega \subset B(0, R)$:

$$\begin{aligned} u_i(x) = & - \int_{\partial\Omega} (G_{ji} P_{jk}^\alpha[u] n_k - P_{jik}^\alpha[G] n_k) u_j ds \\ & + \int_{B(0, R)} (G_{ji} P_{jk}^\alpha[u] \hat{x}_k - P_{jik}^\alpha[G] \hat{x}_k) u_j ds, \end{aligned} \quad (2.29)$$

with $\hat{x}_j = \frac{y_j}{|y|}$ as the normal outgoing vector on the R -radius sphere. Now in order to make the $B(0, R)$ integral vanish as $R \rightarrow \infty$, we should introduce the asymptotic conditions (2.23), (2.27) in 2.29. We write:

$$\begin{aligned} & \int_{B(0, R)} (G_{ji} P_{jk}^\alpha[u] \hat{x}_k - P_{jik}^\alpha[G] \hat{x}_k) u_j ds \\ & = \int_{B(0, R)} G_{ji} (P_{jk}^\alpha[u] \hat{x}_k - ik^p u_j^p - ik^s u_j^s) ds \\ & - \int_{B(0, R)} (P_{jik}^\alpha[G] \hat{x}_k - ik^p G_{ji}^p - ik^s G_{ji}^s) u_j ds \\ & + \Lambda(R) \end{aligned} \quad (2.30)$$

If we have $\lim_{R \rightarrow \infty} \Lambda(R) = 0$, then by the Schwarz theorem the integral (2.30) will vanish asymptotically, since the asymptotic condition 2.27 implies that

$$|u(x)| = O\left(\frac{1}{|x|}\right). \quad (2.31)$$

and letting R go to infinity in 2.29 our theorem will be proved.
Now $\Lambda(R)$ contains two terms of the form:

$$\int_{B(0,R)} G_{ji}^s(x,y) u_j^p(y) ds, \quad (2.32)$$

$$\int_{B(0,R)} G_{ji}^p(x,y) u_j^s(y) ds. \quad (2.33)$$

But all the columns of G^s and G^p , due to the definition of the tensorial divergence and rotational, are divergence free and rotational free vectors respectively. Thus for a given i , the contractions $G_{ji}^s(x,y) u_j^p(y)$ and $G_{ji}^p(x,y) u_j^s(y)$ can be considered as scalar products of a divergence free vector and a rotational free vector, which by the lemma 11, to be proved in the next section, give:

$$\left| G_{ji}^s(x,y) u_j^p(y) \right| = O\left(\frac{1}{|y|^3}\right), \quad (2.34)$$

$$\left| G_{ji}^p(x,y) u_j^s(y) \right| = O\left(\frac{1}{|y|^3}\right). \quad (2.35)$$

As an immediate consequence Λ goes to zero as R goes to infinity. $\square$]

3 Uniqueness results

In this part of the work, we shall prove the following theorem:

Theorem 7 *Let $S_\alpha, \alpha = 0, 2, \dots, N-1$ be a set of C^2 surfaces in $\mathbb{R}^3$ and $\Omega_\alpha, \alpha = 0, 1, \dots, N$ the corresponding layers with the topology described in section 1.2 .*

Then if $\mathbf{u}(x)$ is a vector solution of

$$\left\{ \begin{array}{l} (c_\alpha^p)^2 \text{grad}_x \text{div}_x \mathbf{u}(x) - (c_\alpha^s)^2 \text{rot}_x \text{rot}_x \mathbf{u}(x) + \omega^2 \mathbf{u}(x) = 0 \text{ in } \Omega_\alpha, \alpha \in 0, 1, \dots, N, \\ \mathbf{u}(x) \text{ continuous in } \mathbb{R}^3 \\ \mathbf{t}_\nu \text{ continuous through the surfaces,} \\ \left| \mathbf{t}[\mathbf{u}] \cdot \frac{x}{|x|} - ik^p \mathbf{u}^p - ik^s \mathbf{u}^s \right| = O\left(\frac{1}{|x|^2}\right) \end{array} \right. \quad (3.1)$$

Then $\mathbf{u}=0$ in $\mathbb{R}^3$.

This result will enforce the validity of our model assumptions, since it shows the uniqueness of the Green function associated to the problem we shall study. The theorem 6 can be derived from Kupradze's arguments in the case of one layer (1963, chap. 3). We give here a derivation which is closer to the arguments of [Colton, Kress, 1983].

3.1 Acoustic case

It is a common feature of the acoustic and elastic cases that the uniqueness demonstration is based essentially (1) upon the existence of a conserved current through the surface of discontinuity $\partial\Omega$ and (2) upon a Lemma concerning the asymptotic behavior of the solution, which is due to Rellich [Rellich, 1943] in the scalar case:

Lemma 8 *let k be a positive number and $\phi \in C^2(\mathbb{R}^3 \setminus \overline{\Omega})$ a solution of the Helmholtz equation satisfying the asymptotic Sommerfeld condition:*

$$\left(\frac{x}{|x|}, \text{grad}_x \phi(x) \right) - ik\phi(x) = O\left(\frac{1}{|x|^2}\right)$$

and with moreover

$$\int_{|x|=R} |\phi|^2 ds = o(1),$$

then $\mathbf{u}=0$ in $\mathbb{R} \setminus \overline{\Omega}$.

In the acoustic case the theorem to be proved was:

Theorem 9 *Let S_α and Ω_α be defined as in section (1.2); then if $\phi(x)$ is a solution of*

$$\left\{ \begin{array}{l} \Delta \phi(x) + k_N^2 \phi(x) = 0 \text{ in } \mathbb{R}^3 \setminus S = \bigcup_\alpha S_\alpha. \\ \frac{\phi(x)}{a(x)} \text{ continuous across surfaces,} \\ \phi(x) \frac{\partial a}{\partial \nu}(x) - a(x) \frac{\partial \phi}{\partial \nu}(x) \text{ continuous across surfaces,} \\ \text{Sommerfeld outgoing asymptotic condition.} \end{array} \right. \quad (3.2)$$

k_N being the wave number in the exterior domain Ω_N , and $a(x)$ containing the information on the discontinuities. Then $\phi = 0$ in $\mathbb{R}^3$.

We recall here a sketch of the proof [Sabatier, 1989]. From the existence of the conserved current

$$\bar{\phi} \frac{\partial \phi}{\partial \nu(x)} - \frac{\partial \bar{\phi}}{\partial \nu(x)} \phi$$

one can prove that

$$\text{Im} \int_{|y|=R} \left(\phi \frac{\partial \bar{\phi}}{\partial \nu(x)} \right) ds = 0.$$

Moreover it follows, from the assumptions of the theorem that:

$$\int_{|y|=R} \left(\left| \frac{\partial \phi}{\partial \nu(y)} \right|^2 + k^2 |\phi|^2 + 2 \text{Im} \left(\phi \frac{\partial \bar{\phi}}{\partial \nu(x)} \right) \right) ds = o(1),$$

so that immediately

$$\int_{|y|=R} |\phi|^2 = o(1),$$

which yields the result by direct application of the Rellich lemma. We could try to tie up the elastic and scalar case, using for example the equivalence between our asymptotic condition written separately for $u(x)$ and the classical Silver-Müller condition for $u^p(x)$ and $u^s(x)$. However, the elastic conserved current

$$\bar{u} \cdot t_\nu[u](x) - u \cdot \overline{t_\nu[u]}(x) \quad (3.3)$$

is not related in a simple way to the scalar current (although there exists a formal analogy between them). Hence we have to modify the arguments, with a special care of the different form of the conserved current.

3.2 Conserved current

Defining the *elastic current* as 3.3, the continuity conditions 3.1 imply that

$$\int_{S_i^+} (\bar{u} \cdot t_\nu(x) - u \cdot \overline{t_\nu(x)}) ds = \int_{S_i^-} (\bar{u} \cdot t_\nu(x) - u \cdot \overline{t_\nu(x)}) ds$$

so that the current is conserved across the discontinuity. Moreover it is also conserved across a homogeneous layer (i.e. from one discontinuity to a neighbouring one), by using the Green theorem. If the onion-like layers Ω_i are welded,

$$\begin{aligned} \int_{S_i^+} (\bar{u} \cdot t_\nu(x) - u \cdot \overline{t_\nu(x)}) ds - \int_{S_{i+1}^-} (\bar{u} \cdot t_\nu(x) - u \cdot \overline{t_\nu(x)}) ds \\ = \int_{\Omega_i} (\bar{u} \cdot \text{NS}(u) - u \cdot \overline{\text{NS}(\bar{u})}) d^3 y \\ = 0 \end{aligned}$$

where NS stands for the Navier-Stokes operator. By the same argument the current flux outwards the first layer Ω_0 (the scatterer Ω in the one-layered case) is obviously equal to zero:

$$\int_{S_1} \bar{\mathbf{u}} \cdot \mathbf{t}_\nu(x) - \mathbf{u} \cdot \overline{\mathbf{t}_\nu(x)} ds = 0. \quad (3.4)$$

3.3 Asymptotic behavior of the displacement

Theorem 10 *Let $\mathbf{u}(x)$ be a solution of the Navier-Stokes equations outside S_N , satisfying the asymptotic condition 2.27. Then we can write down*

$$\mathbf{u}(x) = \frac{e^{ik_N r}}{r} \sum_{n=0}^{+\infty} \frac{\mathbf{u}_{np}(\theta, \phi)}{r^n} + \frac{e^{ik_N^* r}}{r} \sum_{n=0}^{+\infty} \frac{\mathbf{u}_{ns}(\theta, \phi)}{r^n}, \quad (3.5)$$

the expansion being absolutely and uniformly converging outside of a ball including S_N . Moreover one can differentiate the expansion as many times as necessary, and obtain a uniformly and absolutely converging sum.

This result is a straightforward consequence of the representation theorem outside Ω , where we use the following expansion formula, due to Atkinson and Wilcox [Colton, Kress 1983]:

Lemma 11 *Let $\phi(x) \in C^2(\mathbb{R}^3 \setminus \bar{\Omega})$ be a solution of the Helmholtz equation outside some bounded set Ω , satisfying the Sommerfeld radiation condition. Let R_0 be such that:*

$$\partial\Omega_{R_0} = \{x \in \mathbb{R}^3 \mid |x| = R_0\} \subset \mathbb{R}^3 \setminus \bar{\Omega}.$$

Let r, θ, ϕ be the spherical coordinates associated to the axis $\hat{x}$.

Then $\mathbf{u}(x)$ can be expanded as:

$$\frac{e^{ikr}}{r} \sum_{n=0}^{+\infty} \frac{\mathbf{u}_n(\theta, \phi)}{r^n},$$

the expansion and its derivatives in r, θ, ϕ being absolutely and uniformly converging for all the variables.

We shall derive also two physically meaningful properties concerning the behavior of the first terms in the expansions of $\mathbf{u}^p$ and $\mathbf{u}^s$: the non-radial components of $\mathbf{u}^p$ behave as $O\left(\frac{1}{r^2}\right)$, and the radial one as $O\left(\frac{1}{r}\right)$, showing thus at infinity the dominant compressive character of the wave. The exact opposite result holds for the expansion of $\mathbf{u}^s$, for which the non-radial elements are dominant far from the scatterer.

Theorem 12 *If, in general, $\mathbf{u}^p$ and $\mathbf{u}^s$ are rotational-free and divergence-free respectively, we can write asymptotically:*

$$\mathbf{u}^p(x) = \frac{\mathbf{u}_{0,p}^r}{r} e^{ik^p r} \left(\frac{\vec{r}}{r} \right) + O\left(\frac{1}{r^2}\right) \quad (3.6)$$

$$\mathbf{u}^s(x) = \left(\frac{\mathbf{u}_{0,s}^\theta}{r} \vec{\theta} + \frac{\mathbf{u}_{0,s}^\phi}{r} \vec{\phi} \right) e^{ik^s r} + O\left(\frac{1}{r^2}\right). \quad (3.7)$$

[Proof: See appendix C]

3.4 Asymptotic behavior of the normal stress

Let us now put the normal stress $t_\nu[u](x)$ in a form which shows the leading asymptotic terms

Theorem 13 *If $|x| \rightarrow \infty$, and $\frac{x}{|x|} = \hat{x}$, the leading term of $t_\nu[u](x)$ is:*

$$\frac{1}{\rho_N} t[u]_{\hat{x}} = \frac{(\lambda_N + 2\mu_N)}{\rho_N} \hat{x} \operatorname{div}_x u + \frac{\mu_N}{\rho_N} [\operatorname{rot}_x u, \hat{x}] + O\left(\frac{1}{r^2}\right) \quad (3.8)$$

[**Proof:** See appendix C $\square$] Now, remember the Silver Müller asymptotic condition (cf [Colton, Kress, 1983])

$$[\operatorname{rot}_x u, \frac{x}{|x|}] + \operatorname{div}_x u \frac{x}{|x|} - iku = O\left(\frac{1}{r^2}\right). \quad (3.9)$$

We see that it would yield for u^p and u^s :

$$\begin{aligned} [\operatorname{rot}_x u^s, \frac{x}{|x|}] - ik^s u^s &= O\left(\frac{1}{r^2}\right) \\ \operatorname{div}_x u^p \frac{x}{|x|} - ik^p u^p &= O\left(\frac{1}{r^2}\right) \end{aligned}$$

which appear exactly with our outgoing asymptotic condition 2.27, and 3.8, taking into account that:

$$\frac{1}{\rho} t[u]_{\hat{x}} - ik^p u^p - ik^s u^s = \operatorname{div}_x u^p \frac{x}{|x|} - ik^p u^p + [\operatorname{rot}_x u^s, \frac{x}{|x|}] - ik^s u^s + O\left(\frac{1}{r^2}\right).$$

Hence we can write down:

Theorem 14 *Let*

$$u(x) = \frac{1}{(c^p)^2} u^p(x) + \frac{1}{(c^s)^2} u^s(x) \quad (3.10)$$

be a solution of the Navier-Stokes equations with

$$\operatorname{rot}_x u^p(x) = 0, \quad (3.11)$$

$$\operatorname{div}_x u^s(x) = 0, \quad (3.12)$$

Then the outgoing asymptotic condition

$$\frac{1}{\rho_N} t[u]_{\hat{x}} - ik^p u^p(x) - ik^s u^s(x) = O\left(\frac{1}{r^2}\right) \quad (3.13)$$

is equivalent to the Silver-Müller condition for u^p and u^s separately and therefore to the Sommerfeld asymptotic condition for their components.

3.5 Uniqueness proof

A different uniqueness proof can be found in [Kupradze, 1963], for a one layer structure only. Our uniqueness theorem can be obtained with the help of two lemmas analogous to the scalar case.

Lemma 15 *Let $\mathbf{u}$ be a solution of the Navier-Stokes equations, satisfying the asymptotic condition (2.27); then asymptotically:*

$$\operatorname{Im} \left[\overline{(k^p \mathbf{u}^p + k^s \mathbf{u}^s)} \cdot \frac{1}{\rho} \mathbf{t}[\mathbf{u}]_{\hat{\mathbf{x}}} \right] = \left((k^p)^2 |\mathbf{u}_{0,p}^r|^2 + (k^s)^2 (|\mathbf{u}_{0,s}^\theta|^2 + |\mathbf{u}_{0,s}^\phi|^2) \right) \frac{1}{r^2} + O\left(\frac{1}{r^3}\right) \quad (3.14)$$

with $\hat{\mathbf{x}} = \frac{\mathbf{x}}{|\mathbf{x}|}$.

[Proof: See appendix C $\square$]

The second lemma concerns the recurrence relation existing between the terms of the expansion of $\mathbf{u}(x)$. Again, an analogous result holds in the scalar case [Colton, Kress, 1983, page 74]:

Lemma 16 *Let*

$$\mathbf{u}(x) = \frac{e^{ik_N r}}{r} \sum_{n=0}^{+\infty} \frac{\mathbf{u}_{n,p}(\theta, \phi)}{r^n} + \frac{e^{ik_N^* r}}{r} \sum_{n=0}^{+\infty} \frac{u_{n,s}(\theta, \phi)}{r^n},$$

be a solution of the Navier-Stokes equations

$$c_p^2 \operatorname{grad}_x \operatorname{div}_x \mathbf{u}(x) - c_t^2 \operatorname{rot}_x \operatorname{rot}_x \mathbf{u}(x) + \omega^2 \mathbf{u}(x) = 0$$

Then the following relations holds:

$$\begin{aligned} -2ik^p n \mathbf{u}_{n,p} + (n)(n-1) \mathbf{u}_{n-1,p} + \mathbf{B}[\mathbf{u}_{n-1,p}] &= 0 \\ -2ik^s n u_{n,s} + (n)(n-1) u_{n-1,s} + \mathbf{B}[u_{n-1,s}] &= 0 \end{aligned}$$

where $\mathbf{B}$ is the Beltrami operator for the sphere

$$\mathbf{B} = \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

[Proof: Using the decomposition 3.5, one can write the Navier-Stokes equations in the following form:

$$\left(\Delta \mathbf{u}^p + (k^p)^2 \mathbf{u}^p \right) + \left(\Delta \mathbf{u}^s + (k^s)^2 \mathbf{u}^s \right) = 0. \quad (3.15)$$

Writing in spherical coordinates the laplacian operator, one obtains, (as in the corollary 3.8 of [Colton, Kress, 1983]) :

$$\begin{aligned} & \frac{e^{ik^p r}}{r} \sum_{n=0}^{+\infty} \left(-2ik^p n \mathbf{u}_{n,p} + (n)(n-1) \mathbf{u}_{n-1,p} + \mathbf{B} \cdot \mathbf{u}_{n-1,p} \right) \frac{1}{r^{n+1}} \\ & + \frac{e^{ik^s r}}{r} \sum_{n=0}^{+\infty} \left(-2ik^s n u_{n,s} + (n)(n-1) u_{n-1,s} + \mathbf{B} \cdot u_{n-1,s} \right) \frac{1}{r^{n+1}} \\ & = 0 \end{aligned}$$

□]

These two lemmas are similar to the corresponding ones in the scalar case ([Colton, Kress, 1983], p. 74) and follow from the equivalence of the condition (2.27) for u to the Sommerfeld one for its components.

We can now prove the main theorem:

Let R be such that

$$\Omega \subset B(0, R)$$

The asymptotic condition (2.27) implies that:

$$\begin{aligned} 0 &= \lim_{R \rightarrow \infty} \int_{|y|=R} \left| \frac{1}{\rho_N} P[u] \cdot \frac{x}{|x|} - ik^p u^p - ik^s u^s \right|^2 ds \\ &= \lim_{R \rightarrow \infty} \int_{|y|=R} \left| \frac{1}{\rho_N} P[u] \right|^2 ds + \int_{|y|=R} \left| k^p u^p + k^s u^s \right|^2 ds \\ &\quad + \lim_{R \rightarrow \infty} 2Im \int_{|y|=R} \overline{\left(k^p u^p + k^s u^s \right)} \cdot \frac{1}{\rho_N} t[u]_{\hat{x}} ds \end{aligned}$$

Then by the theorem 13 and since the lemma 14 implies that the third term is asymptotic to a positive term, we get:

$$\begin{aligned} u_{0,p}^r &= 0 \\ u_{o,s}^\theta &= u_{o,s}^\phi = 0 \end{aligned}$$

The cancelling of u outside of Ω follows from the lemma 14, and the recurrence relations, which kill all the terms in the expansion of u outside Ω . Now let us use the continuity conditions:

$$\begin{aligned} u &= 0 \text{ and } t[u]_{\hat{x}} = 0 \text{ on } S_{N-1}^+, \\ u &= 0 \text{ and } t[u]_{\hat{x}} = 0 \text{ on } S_{N-1}^-. \end{aligned}$$

Since u is equal to zero in Ω_N , it is therefore solution of

$$\left\{ \begin{array}{l} (c_\alpha^p)^2 \text{grad}_x \text{div}_x u(x) - (c_\alpha^s)^2 \text{rot}_x \text{rot}_x u(x) + \omega^2 u(x) = 0 \text{ in } \Omega_\alpha, \alpha \in 0, 1, \dots, N-2, \\ (c_{N-1}^p)^2 \text{grad}_x \text{div}_x u(x) - (c_{N-1}^s)^2 \text{rot}_x \text{rot}_x u(x) + \omega^2 u(x) = 0 \text{ in } \Omega_{N-1} \cup \Omega_N \cup S_{N-1} \\ u(x) \text{ continuous in } \mathbb{R}^3 \\ t_\nu[u](x) \text{ continuous through the surfaces,} \\ \left| \frac{1}{\rho} P[u] \cdot \frac{x}{|x|} - ik^p u^p - ik^s u^s \right| = O\left(\frac{1}{|x|^2}\right) \end{array} \right. \quad (3.16)$$

which is again the original problem , with the $(N-1)^{th}$ layer extending now to infinity. Applying N times the argument proves that u is zero all through $\mathbb{R}^3$. Hence the theorem 6 is proved.

4 Elastic single and double layer potentials

Our goal is to reduce the elastic scattering problem to a set of N integral equations on the surfaces S_i . This will be done in part 5 by means of an ansatz which is closer to that used in the acoustic transmission problem ([Colton, Kress, 1983]) than to that used in the acoustic scattering problem ([Sabatier, 1989]). It proceeds through representing $\mathbf{u}(\mathbf{x})$ inside each layer as a combination of elastic single and double layer vector potentials.

4.1 Single and double layer vectorial potentials

Let $\mathbf{G}_l$ be the Green tensor (1.2) defined with the parameters λ_l, μ_l, ρ_l of the domain Ω_l and let S_m be a discontinuity surface. We define the (elastic) single layer vectorial potential as

$$\mathcal{S}_{lm}[\mathbf{v}](\mathbf{x}) = 2 \int_{S_m} \mathbf{v}(\mathbf{y}) \cdot \mathbf{G}_l(\mathbf{x}, \mathbf{y}) d\mathbf{s}(\mathbf{y}). \quad (4.1)$$

It is well-known (Kupradze 1963, p. 28) that $\mathcal{S}(\mathbf{x})$ is a solution of the Navier-Stokes equations (1.10a) defined with λ_l, μ_l, ρ_l . Let ν_m be the unitary normal vector to the surface S_m at $\mathbf{x}$, pointing towards S_{m+1} , and let $\mathbf{P}_{l\nu_m}^\alpha$ be the generalized operator (see (2.4) at $\mathbf{x}$:

$$\mathbf{P}_{l\nu_m}^\alpha = \frac{\alpha + \mu_l}{\rho_l} \frac{\partial}{\partial \nu_m} + \frac{\lambda_l + \mu_l - \alpha}{\rho_l} \nu_m \operatorname{div} + \frac{\alpha}{\rho_l} \nu_m \wedge \operatorname{rot}. \quad (4.2)$$

The vectorial operator $\mathbf{P}_{l\nu_m}^\alpha$ can be applied to the symmetric tensor $\mathbf{G}$, giving a tensor. The result is readily derived by means of the tensor formulas (1.2) and (2.4). But one can also do it in a transparent way: so as to obtain the component $(\mathbf{P}[\mathbf{G}](\mathbf{x}))_{jk}$, fix j , consider the vector $(\mathbf{G})_j$ with components $(\mathbf{G})_{1,j}, (\mathbf{G})_{2,j}, (\mathbf{G})_{3,j}$, and derive $(\mathbf{P}[(\mathbf{G})_j](\mathbf{x}))_k$ as it would be done for $(\mathbf{P}[\mathbf{u}](\mathbf{x}))_k$.

Using $\mathbf{P}_{l\nu_m}^\alpha$ enables one to define a class of double layer potentials labelled by α . Kupradze (1963) calls "first kind" the one related to $\mathbf{T}_{l\nu_m}$, and "second kind" that related to $\mathbf{N}_{l\nu_m}$, where $\mathbf{N}_{l\nu_m}$ and $\mathbf{T}_{l\nu_m}$ are respectively defined from $\mathbf{T}_n$ and $\mathbf{N}_n$ (see (2.5) and (2.6)). Because of its smoother analytic properties, we use here only the latter one and call it the (elastic) double layer (vectorial) potential:

$$\mathcal{K}_{lm}^N[\mathbf{v}](\mathbf{x}) = 2 \int_{S_m} \mathbf{v}(\mathbf{y}) \cdot \mathbf{N}_{l\nu_m}(\mathbf{y}) [\mathbf{G}_l(\mathbf{x}, \mathbf{y})] d\mathbf{s}(\mathbf{y}), \quad (4.3)$$

where

$$\mathbf{N}_{l\nu_m}(\mathbf{y}) = \frac{2\mu_l(\lambda_l + 2\mu_l)}{\rho_l(\lambda_l + 3\mu_l)} \frac{\partial}{\partial \nu_m} + \frac{(\lambda_l + \mu_l)(\lambda_l + 2\mu_l)}{\rho_l(\lambda_l + 3\mu_l)} \nu_m \operatorname{div} + \frac{(\lambda_l + \mu_l)\mu_l}{\rho_l(\lambda_l + 3\mu_l)} \nu_m \wedge \operatorname{rot}. \quad (4.4)$$

Again, it is well known ([Kupradze, 1963] p. 28) that $\mathcal{K}_{lm}^N[\mathbf{v}](\mathbf{x})$ is a solution of the Navier-Stokes equations defined with λ_l, μ_l, ρ_l . We introduce also what can be called the

generalised derivatives of these potentials:

$$\mathcal{K}_{lmn}'^N[\mathbf{v}](x) = 2N_{l\nu_n} \left[\int_{S_m} \mathbf{v}(y) \cdot \mathbf{G}_l(x, y) ds(y) \right], \quad (4.5)$$

$$\mathcal{T}_{lmn}'^{TN}[\mathbf{v}](x) = 2T_{l\nu_n} \left[\int_{S_m} \mathbf{v}(y) \cdot N_{l\nu_m}(y) [\mathbf{G}_l(x, y)] ds(y) \right], \quad (4.6)$$

where l, n, m may be equal or different. Using $\mathbf{T}$ or $\mathbf{Q}$ instead of $\mathbf{N}$ would introduce $\mathcal{K}_{lmn}'^T$, $\mathcal{K}_{lmn}'^Q$, $\mathcal{T}_{lmn}'^{TT}$, $\mathcal{T}_{lmn}'^{NT}$ in an obvious way. $\mathcal{K}^T$ and $\mathcal{K}^Q$ solve (1.10a), as $\mathcal{K}^N$ and $\mathcal{S}$.

4.2 Continuity and jumps of potentials and generalised derivatives

The following results are well-known ([Kupradze, 1963] p. 36 ff):

If $v \in C^{0,\alpha}(S_m)$, the single layer potential (4.1) belongs to $C^{0,\alpha}$ throughout $\mathbb{R}^3$. With the same assumption, the double layer potential $\mathcal{K}_{lm}^N$ or $\mathcal{K}_{lm}^T$ belong to $C^{0,\alpha}$ inside and outside the domain bordered by S_m . For $x \in S_m$, let x^- be tending to x from inside Ω_m , x^+ tending to x from inside Ω_{m+1} . The jumps are, at $x \in S_m$:

$$\begin{aligned} \mathcal{K}_{lm}^N(x^-) &= 2 \int_{S_m} \mathbf{v}(y) \cdot N_{l\nu_m}(y) [\mathbf{G}_l(x, y)] ds(y) - 4\pi \mathbf{v}(x) \\ \mathcal{K}_{lm}^N(x^+) &= 2 \int_{S_m} \mathbf{v}(y) \cdot N_{l\nu_m}(y) [\mathbf{G}_l(x, y)] ds(y) + 4\pi \mathbf{v}(x) \end{aligned} \quad (4.7)$$

The same property holds for $\mathcal{K}_{lm}^T$. With the same assumption, the generalized derivative $\mathcal{K}_{lmn}'$ of $\mathcal{S}_{lm}$ belongs to $C^{0,\alpha}$ everywhere if $n \neq m$. If $n = m$, it shows jumps across S_m :

$$\begin{aligned} \mathcal{K}_{lmn}'^N(x^-) &= 2N_{l\nu_m}(x) \left[\int_{S_m} \mathbf{v}(y) \cdot \mathbf{G}_l(x, y) ds(y) \right] + 4\pi \mathbf{v}(x) \\ \mathcal{K}_{lmn}'^N(x^+) &= 2N_{l\nu_m}(x) \left[\int_{S_m} \mathbf{v}(y) \cdot \mathbf{G}_l(x, y) ds(y) \right] - 4\pi \mathbf{v}(x), \end{aligned} \quad (4.8)$$

where we wrote $\mathcal{K}_{lmn}'^N$ for $\mathcal{K}_{lmn}'^T$. The same property holds for $\mathcal{K}_{lmn}'^T$.

The operators $\mathcal{T}$ are defined and have their values in $C^{0,\alpha}$ if $\mathbf{v} \in (C^{1,\alpha}(S_m))^3$ and $x \in \mathbb{R}^3 \setminus S_m$, or any x if $n \neq m$. For $n = m$, the limit values of $\mathcal{T}_{lm}^{TT} =: \mathcal{T}_{lmm}^{TT}$ on either side of S_m are equal and define the value on S_m (Liapounov-Tauber theorem). We prove in the appendix B that the same holds for $\mathcal{T}_{lm}^{NN}$, but that $\mathcal{T}_{lm}^{TN}$ shows jumps as $x \rightarrow S_m$:

$$\begin{aligned} \left[\mathcal{T}_{lm}^{TN}(x^\pm) \right]_i &= \left[\mathcal{T}_{lm}^{TN}(x^\pm) \right]_i \\ &\quad \pm \frac{\mu}{\rho} \left(\frac{\mu}{\lambda + 3\mu} \right) \left\{ \nu_j [\text{Grad} \mathbf{v}_j]_i - \nu_i [\text{Grad} \mathbf{v}_j]_j \right\} \end{aligned} \quad (4.9)$$

(where the Einstein convention was used and $x \in S_m$). We shall use in the following

$$[\Theta(\mathbf{v})]_i = \nu_j [\text{Grad} \mathbf{v}_j]_i - \nu_i [\text{Grad} \mathbf{v}_j]_j \quad (4.10)$$

4.3 Analytic properties of the surface operators

For $S = \bigcup_{i=0}^{N-1} S_i$, we shall try solving the scattering problem by deriving density functions in $C^{0,\alpha}(S)$ and $C^{1,\alpha}(S)$. The potential will give the solution of the Navier-Stokes equations inside the regularity domains Ω_i , and the densities will be determined by managing the corresponding "surface operators" $\mathcal{S}_{lm}, \mathcal{K}_{lm}, \mathcal{K}'_{lmn}, \mathcal{T}_{lmn}(x \in S_m)$. Since the S_i are unconnected, there is a qualitative difference between the "diagonal" operators $\mathcal{T}_{lm}, \mathcal{K}'_{lm}$, and the non diagonal ones, less singular. For all terms, the cornerstone is the Green function $G(x, y)$ defined by 1.2. $G(x, y)$ depends only on $|x - y|$, and its rapidly converging expansion into powers of ω^2 can be differentiated term by term at least twice. The first terms read

$$G_{jk}(x, y) = G_{jk}^{(0)}(x, y) + \omega g^1 + \omega^2 [g^2 \delta_{jk} + g'^2 Z_{jk}]r + \dots \quad (4.11)$$

where

$$G_{jk}^{(0)}(x, y) = \frac{\rho}{2\mu(\lambda + 2\mu)} [(\lambda + 3\mu)\delta_{jk} + (\lambda + \mu)Z_{jk}] \frac{1}{r} \quad (4.12)$$

while g^1, g^2, g'^2 , are numbers, $r = |x - y|$, $Z_{jk} = \frac{\partial r}{\partial x_j} \frac{\partial r}{\partial x_k}$, and the following terms of 4.11 are bounded, together with their first and second derivatives, for any finite x, y . It follows that the most singular behavior comes from $G^{(0)}$. Hence we shall first study the analytic properties of surface operators at $\omega = 0$, and (as regards $\mathcal{K}'$ and $\mathcal{T}$) in the diagonal case. Other cases will be sketched. In the first case, we use the abbreviated notations $\mathcal{S}_m^{(0)}, \mathcal{K}_m^{(0)}, \mathcal{K}'_m^{(0)}, \mathcal{T}_m^{(0)}$, (being understood that $m=n$), with the convenient upper index.

4.4 Properties of $\mathcal{S}_m^{(0)}$

We meet surface integrals as:

$$[\mathcal{S}_m^{(0)}(\phi)]_i = \int_{S_m} G_{ij}^{(0)}(x, y) \phi_j(y) ds(y) \quad (4.13)$$

where $G_{ij}^{(0)}(x, y)$ is a weakly singular kernel, absolutely bounded by $\frac{C}{r}$, and depending on $|x - y|$ only. As in the scalar case ([Colton, Kress, 1983], th. 2.6 and 2.9), it follows that $\mathcal{S}_m^{(0)}$ is a compact operator from $C^{0,\alpha}(S_m)$ to $C^{0,\alpha}(S_m)$. The gradient of $[\mathcal{S}_m^{(0)}(\phi)]_i$ involves kernels as $\frac{x_i - y_i}{r^3}$ and $\frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{r^5}$. They satisfy conditions of the lemmas 2.10 and 2.11 of [Colton, Kress, 1983]. Indeed the only nonobviously satisfied condition for these lemmas, which correspond respectively to a domain D_h bordered by S_m and a surface parallel to S_m (i.e. $\{x | \exists z \in S_m, x - z = h\nu(z), h \in [0, 1]\}$), or respectively to S_m itself, is that there exists an integer m such that the kernel K satisfies

$$|K(x, y) - K(z, y)| \leq M \sum_{j=1}^m |x - y|^{-2-j} |x - z|^j \quad (4.14)$$

for all $x, z \in D_h$ (resp S_m), $z \in S_m$, with $2|x - z| \leq |x - y|$. Let us prove it for a kernel as $\frac{X_i X_j X_k}{|X|^5}$, $X = x - y$, and $Z = z - y$. Clearly, the assumption implies that

$|z - y| \leq \frac{3}{2}|x - y|$, and it suffices to use this in the equality

$$\begin{aligned} & \frac{X_i X_j X_k}{|X|^5} - \frac{Z_i Z_j Z_k}{|Z|^5} = \\ &= \frac{X_i X_j (X_k - Z_k) |Z|^5 + X_i Z_k (X_j - Z_j) |Z|^5 + Z_i Z_k (X_i - Z_i) |Z|^5}{|X|^5 |Z|^5} \\ &+ \frac{Z_i Z_k Z_j (|Z|^5 - |X|^5)}{|X|^5 |Z|^5}. \end{aligned} \quad (4.15)$$

After the lemmas conditions are satisfied, the surface integral can be defined in the usual way as a principal value and if $\phi \in C^{0,\alpha}(S_m)$, it defines a function of $C^{0,\alpha}(D_h)$, respectively $C^{0,\alpha}(S_m)$ and the operator is bounded in the normed space $C^{0,\alpha}(S_m)$. It follows that $S_m^{(0)}$ is a bounded operator from $C^{0,\alpha}(S_m)$ to $C^{1,\alpha}(S_m)$, or to $C^{1,\alpha}(D_h)$, respectively.

4.5 Properties of $\mathcal{K}_m^N$ and $\mathcal{K}_m'^N$

We apply to (4.11) the operator N_ν , which was defined by 2.6, and is such that

$$[N_\nu(u)]_i = \frac{\mu}{\rho} \left(Bu + \frac{\eta+1}{\eta+3} Cu + \frac{(\eta+1)(\eta+2)}{\eta+3} Au \right)_i, \quad (4.16)$$

where $\eta + \frac{\lambda}{\mu}$, and the operators A, B, C are

$$\left. \begin{aligned} (Au)_i &= \nu_i \frac{\partial u_j}{\partial x_j}, \\ (Bu)_i &= \nu_j \frac{\partial u_i}{\partial x_j}, \\ (Cu)_i &= \nu_j \frac{\partial u_j}{\partial x_i}. \end{aligned} \right\} \quad (4.17)$$

The result is

$$[N_\nu(G^{(0)}(x, y))]_{ij} = \left[\frac{2}{\eta+3} \delta_{ij} + \frac{3(\eta+1)}{\eta+3} \frac{\partial r}{\partial x_i} \frac{\partial r}{\partial x_j} \right] \frac{\nu_y \cdot (x - y)}{|x - y|^3}. \quad (4.18)$$

It is well known that for a C^2 surface, $|\nu_y \cdot (x - y)|$ is $O(|x - y|^2)$ ([Colton, Kress, 1983]). Hence the kernel of $\mathcal{K}_m^N$ is weakly singular as $\frac{1}{r}$, and $\mathcal{K}_m^N$ is a compact operator from $C^{0,\alpha}(S_m)$ to $C^{0,\alpha}(S_m)$. So is $\mathcal{K}_m'^N$ for similar reasons. If we apply $\mathcal{K}_m^N$ to a function of $C^{1,\alpha}(S_m)$, Betti formula gives a single layer potential on $C^{1,\alpha}(S_m)$, whose derivatives are bounded in $C^{0,\alpha}(S_m)$. Hence $\mathcal{K}^N$ is bounded from $C^{1,\alpha}(S_m)$ to $C^{1,\alpha}(S_m)$.

4.6 Properties of $\mathcal{K}_m^T, \mathcal{K}_m^Q, \mathcal{K}_m'^T, \mathcal{K}_m'^Q$

The operator T_ν is such that

$$[T_\nu(u)]_i = \frac{\mu}{\rho} [(B + C + \eta A)u]_i. \quad (4.19)$$

Consistently with (4.16), (4.18), and (2.13), we can write down:

$$[Q_\nu(u)]_i = [(-A + C)u]_i \quad (4.20)$$

which does not depend on λ, μ, ρ . The effect of $(-A + C)$ on $G^{(0)}$ produces terms as:

$$\frac{x_i - y_i}{r^3} \text{ and } \frac{(x_i - y_i)(x_j - y_j)(x_k - y_k)}{r^5}$$

as those which we already saw in the gradient of $(S\phi)$. It follows that $\mathcal{K}_m^T, \mathcal{K}_m^Q, \mathcal{K}_m'^T, \mathcal{K}_m'^Q$, after they are defined properly by means of principal values, are bounded operators from $C^{0,\alpha}(S_m)$ to $C^{0,\alpha}(S_m)$.

4.7 Properties of $\mathcal{T}^{TN}$

It is proved in the appendix B that $\mathcal{T}_{mm}^{TN}$ is a bounded operator from $C^{1,\alpha}(S_m)$ to $C^{0,\alpha}(S_m)$.

4.8 Properties of the remainders and of the "nondiagonal" terms

It is easy to see that the properties of compactness of $S_m, \mathcal{K}_m^N, \mathcal{K}_m'^N$ hold if we take into account the remainders in (4.11). The boundedness of $\mathcal{K}^T, \mathcal{K}'^T, \mathcal{K}^Q$, and $\mathcal{K}'^Q$ from $C^{0,\alpha}(S_m)$ into $C^{0,\alpha}(S_m)$ also holds: in fact the remainder add compact operators. As for $\mathcal{T}_m^{TN}$, taking into account the remainder add a compact operator from $C^{0,\alpha}(S_m)$ (or $C^{1,\alpha}(S_m)$) to $C^{0,\alpha}(S_m)$. The non diagonal operators $\mathcal{K}^T, \mathcal{K}'^T, \mathcal{K}^Q$, and $\mathcal{K}'^Q$ are compact operators from $C^{0,\alpha}(S_m)$ into $C^{0,\alpha}(S_m)$, and so are the nondiagonal operators $\mathcal{T}_{mn}^{TN}, m \neq n$, from $C^{1,\alpha}(S_m)$ to $C^{0,\alpha}(S_n)$.

4.9 Algebraic properties of the surface operators.

There exist algebraic relations between scalar surface operators. The way they are proved (see e.g. [Sabatier, 1989]) readily extends to the vectorial operators, since it only involves the Green formula (in the scalar case) and here the Betti formula which readily generalizes the Green formula.

As an example, let us try to calculate, in the most difficult (diagonal) case, $S_{lm} \mathcal{T}_{lmm}^{TN}$. Let $\psi \in C^{1,\alpha}(S_m)$. We can write down

$$\begin{aligned} \mathcal{K}_m^N[\psi](y) &= 2 \int_{S_m} \psi(z) N_{\nu m}(z) [G(y, z)] ds(z) \\ &= u^+(y) + u^-(y) \end{aligned} \quad (4.21)$$

where we used (4.7). $u^+(y)$ and $u^-(y)$ are respectively the limit values of the N -double layer potential as y goes to S_m from the domain Ω enclosed by S_m and from $\mathbb{R}^3 \setminus \bar{\Omega}$. We dropped the index l to the Green's function and the operators.

Now let

$$\begin{aligned}\phi(y) &= \mathbf{T}_{\nu_m}(y) \mathcal{K}_m^N[\psi](y) \quad (y \in S_m) \\ &= \mathbf{T}_{\nu_m}(y) [\mathbf{u}^+(y) + \mathbf{u}^-(y)]\end{aligned}\quad (4.22)$$

We study

$$[S_m \phi](x) = 2 \int_{S_m} \mathbf{G}(x, y) \phi(y) ds(y) \quad (4.23)$$

For $y \in \Omega$, the N-double layer potential $\mathbf{u}^-(y)$ is a solution of the N.S. equation, and the representation theorem (2.18) yields

$$4\pi \mathbf{u}^-(x) = \int_{S_m} [\mathbf{G}(x, y) \mathbf{T}_{\nu_m} \mathbf{u}^-(y) ds(y) - \mathbf{u}^-(y) \mathbf{T}_{\nu_m}(y) \mathbf{G}(x, y)] ds(y) \quad (4.24)$$

Hence

$$2 \int_{S_m} [\mathbf{G}(x, y) \mathbf{T}_{\nu_m}(y) \mathbf{u}^-(y) ds(y) = \mathcal{K}_m^T[\mathbf{u}^-](x) + 4\pi \mathbf{u}^-(x) \quad (4.25)$$

For $y \in \mathbb{R}^3 \setminus \overline{\Omega}$, the N-double layer potential $\mathbf{u}^+(y)$ is an outgoing solution of the N.S. equation, and the representation theorem 2.18 yield in the same way

$$2 \int_{S_m} [\mathbf{G}(x, y) \mathbf{T}_{\nu_m}(y) \mathbf{u}^+(y) ds(y) = \mathcal{K}_m^T[\mathbf{u}^+](x) - 4\pi \mathbf{u}^+(x) \quad (4.26)$$

Inserting (4.24) and (4.25) into (4.22) and taking into account (4.7) yields the derived result.

$$S_m \mathcal{T}_{mm}^{TN} = \mathcal{K}_m^T \mathcal{K}_m^N \psi - (4\pi)^2 \psi \quad (4.27)$$

Quite similar formulas could be obtained for $\mathcal{T}^{TT}$ or $\mathcal{T}^{TN}$, with $\mathcal{K}_m^T$ and $\mathcal{K}_m^N$ in adequate positions. The generalisations of the other scalar case formulas are :

$$\mathcal{T}_{mm}^{TN} \mathcal{K}_m^N - \mathcal{K}_m^T \mathcal{T}_{mm}^{NN} = 0 \quad (4.28)$$

$$\mathcal{T}_{mm}^{TN} S_m - \mathcal{K}_m^T \mathcal{K}_m^N + (4\pi)^2 \mathcal{I} = 0 \quad (4.29)$$

$$\mathcal{K}_m^N S_m - S_m \mathcal{K}_m^N = 0 \quad (4.30)$$

As in the acoustical case, non diagonal term, or inclusion of $\mathcal{K}_m^N$ and $\mathcal{K}_m^N$ or others for values of x which are not in S_m , give formulas that are obtained from (4.27), (4.28), (4.29), by replacing 0 in the right hand side by a convenient operator which is compact from $C^{0,\alpha}(S_m)$ to $C^{0,\alpha}(S_m)$.

4.10 Dependence of the ($\omega = 0$) kernels on the Lamé parameters

Setting

$$\begin{aligned}a_{jk} &= r^{-1} \delta_{jk} & b_{jk} &= \frac{\mathbf{v}_y(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3} \delta_{jk} = Y(y) \delta_{jk} \\ c_{jk} &= r^{-1} Z_{jk} & d_{jk} &= \frac{\mathbf{v}_y(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3} Z_{jk} = Y(y) Z_{jk}\end{aligned}\quad (4.31)$$

We can write down (4.11),(4.16) and (4.17) as

$$\mathbf{G}_{ij}^{(0)} = \rho(2\mu)^{-1}(1 + \frac{1}{2}\eta)^{-1}[\mathbf{a} + \frac{1}{2}c + \frac{1}{2}\eta(a + c)] \quad (4.32)$$

$$[\mathbf{N}_y \mathbf{G}^{(0)}]_{ij} = \frac{1}{3}(1 + \frac{1}{3}\eta)^{-1}[2b + 3d + 3\eta d] \quad (4.33)$$

$$\rho[\mathbf{T}_x \mathbf{N}_y \mathbf{G}^{(0)}]_{ij} = \mu/3(1 + 1/3\eta)^{-1}[(B+C)(b+3/2d) + \eta[A(b+3/2d) + 3/2(B+C)d] \quad (4.34)$$

Since $\mathbf{G}$ is a symmetrical function of $\mathbf{x}$ and $\mathbf{y}$, and $\mathbf{Q}$ does not depend on the Lamé parameters, similar combinations hold with other operators. We remark that although $\mathbf{T}^{NN}$ in (4.17) involves ηA , (4.33), (4.32), are **only** first degree polynomials in η , because $\text{Ad}=0$. The equality $A(a+c)=0$ also explains **the** (slightly more complicated) reduction of (4.32). Had we calculated $\mathbf{Q} \mathbf{G}^{(0)}$, obviously **a similar decomposition** would have been obtained, and also for $\mathcal{K}^T$ because $A(a+c)=0$. It follows that each $\omega = 0$ operators can be written as a linear combination of two operators that do not depend on λ, μ, ρ as so (the surface index m is dropped):

$$S^{(0)} = \frac{\rho}{2\mu(\eta + 2)}(\tilde{S} + \eta \hat{S}) \quad (4.35)$$

where $\tilde{S}$ and $\hat{S}$ are compact operators from $C^{0,\alpha}(S)$ to $C^{0,\alpha}(S)$, and bounded operators to $C^{1,\alpha}(S)$.

$$\mathcal{K}^{(0)N} = \frac{1}{\eta + 3}(\tilde{\mathcal{K}}^N + \eta \hat{\mathcal{K}}^N) \quad (4.36)$$

where $\tilde{\mathcal{K}}^N$ and $\hat{\mathcal{K}}^N$ are compact operators from $C^{0,\alpha}(S)$ to $C^{0,\alpha}(S)$.

$$\mathcal{K}^{(0)T} = \frac{1}{2(\eta + 2)}(\tilde{\mathcal{K}}^T + \eta \hat{\mathcal{K}}^T) \quad (4.37)$$

where $\tilde{\mathcal{K}}^T$ is a bounded operator from $C^{0,\alpha}(S)$ to $C^{0,\alpha}(S)$. $\hat{\mathcal{K}}^T$ is in addition compact because its kernel reduces to $-6ZY$, which is weakly singular.

$$\mathcal{K}^{(0)Q} = \frac{\rho}{2\mu(\eta + 2)}(\tilde{\mathcal{K}}^Q + \eta \hat{\mathcal{K}}^Q) \quad (4.38)$$

where $\tilde{\mathcal{K}}^Q$ and $\hat{\mathcal{K}}^Q$ are bounded operators from $C^{0,\alpha}(S)$ to $C^{0,\alpha}(S)$. It is possible to find another formula for $\mathcal{K}^{(0)Q}$ where a compact operator appears say $\overline{\mathcal{K}}^Q$, with kernel

$$[(C - A)c]_{ij} = Y(\delta_{ij} - 3Z_{jk}) \quad (4.39)$$

This combination of two operators independent of λ, μ, ρ is

$$\mathcal{K}^{(0)Q} = \frac{\rho(\eta + 3)}{2(\lambda + 2\mu)}\tilde{\mathcal{K}}^Q + \frac{\rho(\eta + 1)}{2(\lambda + 2\mu)}\overline{\mathcal{K}}^Q \quad (4.40)$$

Same combinations respectively hold for $\mathcal{K}'^{(0)N}, \mathcal{K}'^{(0)T}, \mathcal{K}'^{(0)Q}$ and $\overline{\mathcal{K}'^Q}$ and $\hat{\mathcal{K}}^{iT}$ are compact from $C^{0,\alpha}(s)$ to $C^{0,\alpha}(s)$. Finally, $\mathcal{T}^{(0)T}$ shows the combination

$$\rho \mathcal{T}^{TN(0)} = \frac{\mu}{\nu + 3} (\tilde{\mathcal{T}}^{TN} + \eta \hat{\mathcal{T}}^{TN}) \quad (4.41)$$

where $\tilde{\mathcal{T}}$ and $\hat{\mathcal{T}}$ have the properties of $\mathcal{T}^{(0)}$.

The algebraic formulas (4.26) to (4.29) hold for $\omega = 0$, and we can insert into them the formulas (4.34)-(4.40). Noticing that for fixed μ and ρ (which disappear in the products), η can be chosen freely, we see that similar formulas, but with coefficients different from $(4\pi)^2$, are obtained for the hatted operators.

If the remainders of (4.10) after are introduced, the formulas (4.34) to (4.40) remain true modulo an operator which is compact from $C^{0,\alpha}(S)$ to $C^{0,\alpha}(S)$.

As an example of formulas which still contains η but no other Lamé parameter, we easily derive from the formulas above the following ones

$$(\tilde{\mathcal{S}} + \eta \hat{\mathcal{S}})(\tilde{\mathcal{T}}^{TN} + \eta \hat{\mathcal{T}}^{TN}) - (\tilde{\mathcal{K}}^T + \eta \hat{\mathcal{K}}^T)(\tilde{\mathcal{K}}^N + \eta \hat{\mathcal{K}}^N) + 2(\eta + 2)(\eta + 3)(4\pi)^2 \mathcal{I} = \mathbb{Q} \quad (4.42)$$

$$(\tilde{\mathcal{S}} + \eta \hat{\mathcal{S}})(\tilde{\mathcal{T}}^{TN} + \eta \hat{\mathcal{T}}^{TN}) - (\tilde{\mathcal{K}}^{iT} + \eta \hat{\mathcal{K}}^{iT})(\tilde{\mathcal{K}}^{iN} + \eta \hat{\mathcal{K}}^{iN}) + 2(\eta + 2)(\eta + 3)(4\pi)^2 \mathcal{I} = \mathbb{Q} \quad (4.43)$$

$$(\tilde{\mathcal{T}}^{TN} + \eta \hat{\mathcal{T}}^{TN})(\tilde{\mathcal{K}}^N + \eta \hat{\mathcal{K}}^N) - (\tilde{\mathcal{K}}^{iT} + \eta \hat{\mathcal{K}}^{iT})(\tilde{\mathcal{T}}^{NN} + \eta \hat{\mathcal{T}}^{NN}) = \mathbb{Q} \quad (4.44)$$

4.11 Regularising $\mathcal{T}$

Let $\mathbf{R}(x,y)$ be the kernel defined by $\mathbf{G}^{(2)}$ in (4.10) or one of the following terms (the constant kernel can usually be eliminated easily)- and let $\mathbf{R}$ be the corresponding surface operator. It is a matter of straightforward use of Betti's identity and evaluation of the volume integral to prove that $\mathcal{R}\mathcal{T}$ is equal to $(\mathbf{T}_x \mathcal{R})\mathcal{K}$ plus a compact operator (from $C^{0,\alpha}(S)$ to $C^{0,\alpha}(S)$) and that $(\mathbf{T}_x \mathcal{R})$ is itself compact. Same for $\mathcal{T}\mathcal{R}$. As it stands, we know from appendix A and B that $\mathcal{T}$ is bounded from $C^{1,\alpha}(S)$ to $C^{0,\alpha}(S)$. Suppose now we meet $\mathcal{T}\mathcal{K}^N S\phi$ when $\phi \in C^{0,\alpha}$, so that $S\phi \in C^{1,\alpha}$; suppose also that $\mathcal{K}, S$ and $\mathcal{T}$ do not necessarily correspond to the same Green function G_ρ , but to functions corresponding to different sets of Lamé parameters: we can nevertheless prove that $\mathcal{T}\mathcal{K}^N S$ is bounded from $C^{0,\alpha}(S)$ to $C^{0,\alpha}(S)$ by first rewriting

$$\mathcal{K}^N S\phi = \int_S \mathbf{N}_{\nu y} \mathbf{G}_1(x,y) ds(y) \int_S \mathbf{G}_2(y,z) \phi(z) dz \quad (4.45)$$

by using Betti's identity, as

$$\int_S \mathbf{G}_1(x,y) ds(y) \mathbf{N}_{\nu y} \int_S \mathbf{G}_2(y,z) \phi(z) dz \quad (4.46)$$

plus extra terms, which can separately be bounded for $\phi(z) \in C^{0,\alpha}(s)$ by taking advantage of the symmetry of $\mathbf{G}_2$ with respect to the exchange y,z . The result is that $\mathcal{K}^N S$ is bounded from $C^{0,\alpha}(S)$ to $C^{1,\alpha}(S)$ and therefore $\mathcal{T}\mathcal{K}^N S$ is bounded from $C^{0,\alpha}(S)$ to $C^{0,\alpha}(S)$.

5 Scattering theory

5.1 The transmission problem with constant Poisson Ratio

Saying that the Poisson ratio is constant simply means that $\eta = \lambda/\mu$ is an invariant. This is not badly wrong in very homogeneous geological structure but even in this case, it is at most a crude approximation. On the other hand, the problem is simple and will help the reader understanding of the general one. The problem to be solved is

$$\begin{cases} \mu_i [(\eta_i + 2)\text{grad}_x \text{div}_x \mathbf{u}(x) - \text{rot}_x \text{rot}_x \mathbf{u}(x)] + \rho_i \omega^2 \mathbf{u}(x) = 0 \\ \quad \quad \quad x \in \Omega_i \quad i = 0, 1, \dots, N \\ \mathbf{u}(x^+) - \mathbf{u}(x^-) = \sigma_i(x) \quad x \in S_i \quad i = 0, 1, \dots, N-1 \\ (\mathbf{t}_{\nu_i} \mathbf{u})^+ - (\mathbf{t}_{\nu_i} \mathbf{u})^- = \tau_i(x) \quad x \in S_i \quad i = 0, 1, \dots, N-1 \\ \mathbf{u}(x) - \mathbf{u}_\infty(x) \quad \text{outgoing} \end{cases} \quad (5.1)$$

where $\sigma_i(x)$ and $\tau_i(x)$ are prescribed jumps across S_i . Since this is an introduction problem, there is no point of taking $N > 1$. We note $\mathbf{u}(x)$ as $\mathbf{u}^-(x)$ for $x \in \Omega_0$, $\mathbf{u}^+(x)$ for $x \in \Omega_1$. We represent them by means of potentials:

$$\begin{cases} \mathbf{u}^-(x) = \alpha^- \int_{S_0} \phi(y) \cdot \mathbf{G}_0(x, y) ds(y) \\ \quad + \beta^- \int_{S_0} \psi(y) \cdot \mathbf{N}_{0\nu_y} \mathbf{G}_0(x, y) ds(y) \\ \mathbf{u}^+(x) - \mathbf{u}_\infty(x) = \alpha^+ \int_{S_0} \phi(y) \cdot \mathbf{G}_1(x, y) ds(y) \\ \quad + \beta^+ \int_{S_0} \psi(y) \cdot \mathbf{N}_{1\nu_y} \mathbf{G}_1(x, y) ds(y) \end{cases} \quad (5.2)$$

so that

$$\begin{cases} \mathbf{t}_{0x}(\mathbf{u}^-) = \alpha^- \rho_0 \mathbf{T}_{0\nu_x} \int_{S_0} \phi(y) \cdot \mathbf{G}_0(x, y) ds(y) \\ \quad + \beta^- \rho_0 \mathbf{T}_{0\nu_x} \int_{S_0} \psi(y) \cdot \mathbf{N}_{0\nu_y} \mathbf{G}_0(x, y) ds(y) \\ \mathbf{t}_{1x}(\mathbf{u}^+) - \mathbf{t}_{1x}(\mathbf{u}_\infty) = \alpha^+ \rho_1 \mathbf{T}_{1\nu_x} \int_{S_0} \phi(y) \cdot \mathbf{G}_1(x, y) ds(y) \\ \quad + \beta^+ \rho_1 \mathbf{T}_{1\nu_x} \int_{S_0} \psi(y) \cdot \mathbf{N}_{1\nu_y} \mathbf{G}_1(x, y) ds(y) \end{cases} \quad (5.3)$$

As $x \rightarrow S_0$, we obtain, because of (4.7), (4.8), (4.9),

$$\begin{cases} \mathbf{u}(x^-) = \alpha^- \mathcal{S}_0 \phi + \beta^- (\mathcal{K}_0^N \psi - 4\pi \psi) \\ \mathbf{u}(x^+) = \mathbf{u}_\infty + \alpha^+ \mathcal{S}_1 \phi + \beta^+ (\mathcal{K}_1^N \psi + 4\pi \psi) \end{cases} \quad (5.4)$$

$$\begin{cases} \mathbf{t}_0(\mathbf{u}^-) = \alpha^- \rho_0 (\mathcal{K}_{00}^T \phi + 4\pi \phi) + \beta^- \rho_0 \mathcal{T}_{00}^{TN} \psi - \frac{\beta^- \mu_0}{\eta_0 + 3} \theta(\psi) \\ \mathbf{t}_1(\mathbf{u}^+) - \mathbf{t}_\infty = \alpha^+ \rho_1 (\mathcal{K}_{10}^T \phi - 4\pi \phi) + \beta^+ \rho_1 \mathcal{T}_{10}^{TN} \psi - \frac{\beta^+ \mu_1}{\eta_1 + 3} \theta(\psi) \end{cases} \quad (5.5)$$

Setting now

$$a = 4\pi(\beta^+ \beta^-); \quad a\mathcal{K}^N = \beta^+ \mathcal{K}_1^N - \beta^- \mathcal{K}_0^N; \quad \mathcal{S} = \alpha^+ \mathcal{S}_1 - \alpha^- \mathcal{S}_0 \quad (5.6)$$

we see that the displacement continuity equation in (5.1) reads in condensed notations:

$$a(\mathcal{I} + \mathcal{K}^N)\psi + \mathbf{u}_\infty + \mathcal{S}\phi = \sigma \quad (5.7)$$

whereas the traction continuity reads

$$-b(\mathcal{I} - \mathcal{K}^T)\phi + \mathbf{t}_\infty + \mathcal{T}^{TN}\phi + \xi(\psi) = \tau \quad (5.8)$$

where

$$\begin{cases} b = 4\pi(\alpha^+ \rho_1 + \alpha^- \rho_0) \\ b\mathcal{K}'^T = \alpha^+ \rho_1 \mathcal{K}_{10}^{'T} - \alpha^- \rho_0 \mathcal{K}_{00}^{'T} \\ \mathcal{T}^{TN} = \beta^+ \rho_1 \mathcal{T}_{10}^{TN} - \beta^- \rho_0 \mathcal{T}_{00}^{TN} \end{cases} \quad (5.9)$$

and

$$\xi(\psi) = \left[\frac{\beta^+ \mu_1}{\eta_1 + 3} + \frac{\beta^- \mu_0}{\eta_0 + 3} \right] \theta(\psi) \quad (5.10)$$

Now, it follows from (4.37) (more exactly the similar equation holding for $\mathcal{K}$) that, up to a “remainder operator”,

$$\mathcal{K}_l^{'T} = \frac{1}{2(\eta_l + 2)} (\check{\mathcal{K}}_l^T + \eta_l \hat{\mathcal{K}}_l^T) \quad (5.11)$$

where $\hat{\mathcal{K}}_l^T$ is a compact operator from $C^\alpha(S)$ to $C^\alpha(S)$, and a bounded operator to $C^{1,\alpha}(S)$. Hence choosing

$$\frac{\alpha^+ \rho_1}{\eta_1 + 2} = \frac{\alpha^- \rho_0}{\eta_0 + 2} = \alpha \quad (5.12)$$

makes $\mathcal{K}'^T$ a compact operator, i.e.

$$b\mathcal{K}'^T = \frac{\alpha}{2}(\eta_1 - \eta_0)\hat{\mathcal{K}}'^T + r_{K'} \quad (5.13)$$

where $r_{K'}$ is a “remainder operator”.

Up to this point, we did not assume anything about η_1/η_0 . If it is equal to 1 (equal Poisson ratio), we can proceed in several ways. For instance the problem can be managed by setting $\beta^+ = \beta^-$, in a style which is somewhat similar to the general case treated below. However, if we want to take full advantage of the Poisson ratios equality, we have better to set

$$\alpha^+ \rho_1 = \alpha^- \rho_0 = 1; \quad \beta^+ = \left(\frac{\mu_0}{\mu_1} \right)^{1/2}; \quad \beta^- = \left(\frac{\mu_1}{\mu_0} \right)^{1/2} \quad (5.14)$$

so that

$$a = 4\pi \left[\left(\frac{\mu_1}{\mu_0} \right)^{1/2} + \left(\frac{\mu_0}{\mu_1} \right)^{1/2} \right] \quad (5.15)$$

$$a\mathcal{K}^N = \left[\left(\frac{\mu_1}{\mu_0} \right)^{1/2} - \left(\frac{\mu_0}{\mu_1} \right)^{1/2} \right] (\eta + 3)^{-1} (\check{\mathcal{K}}^N + \eta \hat{\mathcal{K}}^N) + r_K \quad (5.16)$$

$$\mathcal{S} = \frac{1}{2(\eta + 2)} \left(\frac{1}{\mu_1} - \frac{1}{\mu_0} \right) (\check{\mathcal{S}} + \eta \hat{\mathcal{S}}) + r_S \quad (5.17)$$

$$b = 8\pi; \quad \mathcal{K}'^T = r_{K'}; \quad \mathcal{T}^{TN} = r_T; \quad \xi(\psi) = \frac{(\mu_0 \mu_1)^{1/2}}{\eta + 3} \theta(\psi) \quad (5.18)$$

where the remainder operators $r_S, r_K, r_{K'}, r_T$ vanish at the limit $\mu_0 = \mu_1$.

We seek ψ in $C^{1,\alpha}(S_0)$, ϕ in $C^{0,\alpha}(S_0)$. It is obvious that $S\xi(\psi)$ is a bounded operator from $C^{1,\alpha}(S_0)$ to $C^{1,\alpha}(S_0)$. Now, the equation (5.1) simply reduces to

$$-b(1-r_{K'})\phi + (\xi + r_T)\psi = \tau - t_\infty \quad (5.19)$$

or

$$-b\phi + (\xi + r)\psi = (1+r)(\tau - t_\infty) \quad (5.20)$$

where r stands for any operator which has the properties of a remainder operator. Applying S to both sides of (5.10), and inserting the result in (5.7) yields the equation:

$$a \left[1 + \mathcal{K}^N + a^{-1}b^{-1}S\xi + r \right] \psi + u_\infty - \sigma + b^{-1}S[(1+r)(\tau - t_\infty)] = 0 \quad (5.21)$$

which is readily solved if $u_\infty - \sigma$ is in $C^{1,\alpha}$ and if the norm of $\mathcal{K}^N + a^{-1}b^{-1}S\xi + r$ as a bounded operator from $C^{1,\alpha}(S_0)$ to $C^{1,\alpha}(S_0)$ is small enough. It follows from (5.16) and (5.17) that this is always achieved if $\mu_0 - \mu_1$ is small enough. Hence there is a range of values of $\mu_0 - \mu_1$ where ψ can be constructed in a stable way in $C^1(S_0)$, and ϕ readily follows from (5.19).

The problem algebra suggests another choice of β^+ and β^- , which reduces $\xi(\psi)$ to zero in (5.10)

$$\beta^+ = \frac{\mu_1}{\eta + 1}; \quad \beta^- = -\frac{\mu_0}{\eta + 1} \quad (5.22)$$

choosing also

$$\alpha^+ \rho_1 = \alpha^- \rho_0 = \eta + 2 \quad (5.23)$$

we get $\mathcal{K}'_T = r_{K'}$

$$b = 8\pi(\eta + 2) \quad (5.24)$$

$$S = \frac{1}{2} \left(\frac{1}{\mu_1} - \frac{1}{\mu_0} \right) [(\tilde{S} + \eta \hat{S}) + r_S] \quad (5.25)$$

$$a = 4\pi(\eta + 3) \left(\frac{1}{\mu_1} - \frac{1}{\mu_0} \right) \quad (5.26)$$

$$\begin{cases} a\mathcal{K}^N = \left(\frac{1}{\mu_1} + \frac{1}{\mu_0} \right) [(\tilde{\mathcal{K}}^N + \eta \hat{\mathcal{K}}^N) + r_K] \\ \mathcal{T}^{TN} = 2 \left(\tilde{\mathcal{T}}^{TN} + \eta \hat{\mathcal{T}}^{TN} \right) + r_T \end{cases} \quad (5.27)$$

where $r_{K'}$ goes to zero as $\mu_0 \rightarrow \mu_1$. The equations (5.7),(5.1) reduce to

$$a(1 + \mathcal{K}^N)\psi + S\phi = \sigma - u_\infty \quad (5.28)$$

$$-b(1 - r_{K'})\phi + \mathcal{T}^{TN}\psi = \tau - t_\infty \quad (5.29)$$

Now, apply $S(1 - r_{K'})^{-1}$ to both sides of (5.29). Using (4.42) and the regularizing properties of remainder operator, we obtain

$$\begin{aligned} - \left[(\eta + 2)(\eta + 3) \left(\frac{1}{\mu_1} - \frac{1}{\mu_0} \right) (4\pi)^2 \right] (1 + r)\psi = \\ 8\pi(\eta + 2)S\phi + S(1 - r_{K'})^{-1}(\tau - t_\infty) \end{aligned} \quad (5.30)$$

which yields $\mathcal{S}\phi$ from ψ in a constructive stable way. It can then be inserted into (5.28), to give

$$\begin{aligned} \left(\frac{1}{\mu_1} + \frac{1}{\mu_0}\right) \left[(\tilde{\mathcal{K}}^N + \eta \hat{\mathcal{K}}^N) + r_K \right] \psi + \\ \left(\frac{1}{\mu_1} - \frac{1}{\mu_0}\right) [(p+r)\psi + \psi_0] = \sigma - u_\infty \end{aligned} \quad (5.31)$$

where p is a number easy to obtain, r a remainder operator, ψ_0 a function constructed from $\tau - t_\infty$. Since all the operators but $p\mathcal{I}$ are compact, (5.31) is a Fredholm equation for ψ and can be solved provided it is not precisely on an eigenvalue- which is not true in general. But as $\mu_0 \rightarrow \mu_1$, the relative value of the identity and the compact operator make the problem more and more ill-conditioned. The previous method should then be preferred.

5.2 The general scattering problem with one surface

We shall give up the jumps σ and τ to be a little bit simpler. It is clear that introducing them is trivial. Again, the problem is (5.1) and all equations up to (5.13) hold. But now η is no longer equal in Ω_0 and Ω_1 . We prefer not choosing β^+/β^- in order to cancel $\xi(\psi)$ because it yields an ill posed problem, nor in order to cancel $\mathcal{T}^{TN}(\psi)$ because it does not work. Since $\mathcal{T}^{TN}$ remains, it may be interesting, but not necessary, to choose β^+/β^- such that the ratio of the coefficients of $\tilde{\mathcal{T}}$ (respectively $\hat{\mathcal{T}}$) is equal to the coefficient of $\tilde{\mathcal{S}}$ (respectively $\hat{\mathcal{S}}$). This is done if

$$\beta^+ \mu_1 (\eta_0 + 3) = \mu_0, \quad \beta^- \mu_0 (\eta_1 + 3) = \mu_1. \quad (5.32)$$

with this choice (and $\alpha = 1$), we get

$$b\mathcal{K}^T = \frac{1}{2}(\eta_1 - \eta_0)\hat{\mathcal{K}}^T + r_K, \quad (5.33)$$

$$b = 4\pi(4 + \eta_1 + \eta_0) \quad (5.34)$$

$$\mathcal{S} = \frac{1}{2\mu_1\mu_0} [(\mu_0 - \mu_1)\tilde{\mathcal{S}} + (\mu_0\eta_1 - \mu_1\eta_0)\hat{\mathcal{S}} + r_S] \quad (5.35)$$

$$a = 4\pi \left(\frac{\mu_0}{\mu_1(\eta_0 + 3)} + \frac{\mu_1}{\mu_0(\eta_1 + 3)} \right) \quad (5.36)$$

$$a\mathcal{K}^N = \frac{1}{(\eta_1 + 3)(\eta_0 + 3)\mu_1\mu_0} [(\mu_0^2 - \mu_1^2)\tilde{\mathcal{K}}^N + (\eta_1\mu_0^2 - \eta_0\mu_1^2)\hat{\mathcal{K}}^N + r_K] \quad (5.37)$$

$$\mathcal{T}^{TN} = \frac{1}{(\eta_1 + 3)(\eta_0 + 3)} [(\mu_0 - \mu_1)\tilde{\mathcal{T}}^{TN} + (\eta_1\mu_0 - \eta_0\mu_1)\hat{\mathcal{T}}^{TN} + r_T] \quad (5.38)$$

where r_S, r_K, r_T , go to zero as $(\mu_0 - \mu_1)$ and $(\eta_0 - \eta_1)$ go to zero. Thanks to 4.37, we can write down

$$\mathcal{T}^{TN}\mathcal{S} = \zeta(\mathcal{I} + \mathcal{C}) + s \quad (5.39)$$

where s is a remainder operator which vanishes when $\mu_0 - \mu_1$ and $\eta_0 - \eta_1$ do, $\mathcal{C}$ is a compact operator, and

$$\zeta = -\frac{(4\pi)^2[\mu_0\eta_1 - \mu_1\eta_0 + 2(\mu_0 - \mu_1)][\mu_0\eta_1 - \mu_1\eta_0 + 3(\mu_0 - \mu_1)]}{(\eta_1 + 3)(\eta_0 + 3)\mu_1\mu_0} \quad (5.40)$$

Let us limit our study to the scattering problem ($\sigma = \tau = 0$). With $u_\infty \in C^{1,\alpha}(S)$ we can get ψ from (5.7) as

$$a\psi = -(\mathcal{I} + \mathcal{K}^N)^{-1}(u_\infty + \mathcal{S}\phi) \quad (5.41)$$

and insert the result into (5.1). For $(\mu_0 - \mu_1)$ small enough, $(\mathcal{I} + \mathcal{K}^N)^{-1}$ can be expanded in Neumann series, and we find operators as $\mathcal{T}(\mathcal{K}^N)^p \mathcal{S}$ which are bounded and whose norm is usually $O(\mu_0 - \mu_1)$ smaller than that of $\mathcal{T}^{TN}$ because of (5.37). In any case $\mathcal{T}^{TN}(\mathcal{I} + \mathcal{K}^N)^{-1} \mathcal{S}$ is a bounded operator from $C^{0,\alpha}(S)$ into $C^{0,\alpha}(S)$, whose norm goes to zero as $\mu_0 - \mu_1$ and $\eta_0 - \eta_1$ go to zero. Hence the equation (5.1) reads

$$\begin{aligned} -b(\mathcal{I} - \mathcal{K}'^T)\phi + t_\infty - \frac{1}{a}\mathcal{T}^{TN}(\mathcal{I} + \mathcal{K}^N)^{-1}(\mathcal{S}\phi + u_\infty) \\ - \frac{1}{a}\xi(\mathcal{I} + \mathcal{K}^N)^{-1}(\mathcal{S}\phi + u_\infty) = 0 \end{aligned} \quad (5.42)$$

where $b\mathcal{I}$ is given by (5.34) (and all the other operators norms go to zero as $\mu_0 - \mu_1$ and $\eta_0 - \eta_1$ goes to zero). The equation (5.42) can be solved at least in this neighbourhood. Then ψ is obtained from (5.41).

We shall not study the approach which is ill posed as $\mu_1 \rightarrow \mu_0, \eta_1\mu_1 \rightarrow \eta_0\mu_0$. The behaviour of the corresponding equations in this neighbourhood depends on the relative values of $\mu_0 - \mu_1$, $\lambda_0 - \lambda_1$, and $\rho_1 - \rho_0$.

5.3 The general scattering problem with several surfaces

The problem to be solved is (5.1) with $\sigma_i(x) = \tau_i(x) = 0$. In the external domain Ω_n , we represent u by means of two potentials on S_{n-1} :

$$\begin{aligned} u_n(x) - u_\infty(x) = \alpha_{n-1}^+ \int_{S_{n-1}} \phi_{n-1}(y) \cdot \mathbf{G}_n(x, y) ds(y) \\ + \beta_{n-1}^+ \int_{S_{n-1}} \psi_{n-1}(y) \cdot \mathbf{N}_{n\nu(y)} \mathbf{G}_n(x, y) ds(y), \end{aligned} \quad (5.43)$$

so that the traction is:

$$\begin{aligned} t_{\nu_x}(u_n) - t_{\nu_x}(u_\infty) = \alpha_{n-1}^+ \rho_n \mathbf{T}_{n\nu(x)} \int_{S_{n-1}} \phi_{n-1}(y) \cdot \mathbf{G}_n(x, y) ds(y) \\ + \beta_{n-1}^+ \rho_n \mathbf{T}_{n\nu(x)} \int_{S_{n-1}} \psi_{n-1}(y) \cdot \mathbf{N}_{n\nu(y)} \mathbf{G}_n(x, y) ds(y). \end{aligned} \quad (5.44)$$

In each of the internal domains Ω_i , we represent u by means of two potentials on its external boundary S_i , with density functions equal to those used in Ω_{i+1} , and two potentials

on its internal boundary S_{i-1} , with density functions equal to those used in Ω_{i-1} :

$$\begin{aligned} \mathbf{u}_i(x) = & \alpha_i^- \int_{S_i} \phi_i(y) \cdot \mathbf{G}_i(x, y) ds(y) \\ & + \beta_i^- \int_{S_i} \psi_i(y) \cdot \mathbf{N}_{i\nu(y)} \mathbf{G}_i(x, y) ds(y) \\ & + \alpha_{i-1}^+ \int_{S_{i-1}} \phi_{i-1}(y) \cdot \mathbf{G}_i(x, y) ds(y) \\ & + \beta_{i-1}^+ \int_{S_{i-1}} \psi_{i-1}(y) \cdot \mathbf{N}_{i\nu(y)} \mathbf{G}_i(x, y) ds(y), \end{aligned} \quad (5.45)$$

giving the tractions:

$$\begin{aligned} \mathbf{t}_i \mathbf{u}_i(x) = & \alpha_i^- \mathbf{T}_{i\nu(x)} \int_{S_i} \phi_i(y) \cdot \mathbf{G}_i(x, y) ds(y) \\ & + \beta_i^- \mathbf{T}_{i\nu(x)} \int_{S_i} \psi_i(y) \cdot \mathbf{N}_{i\nu(y)} \mathbf{G}_i(x, y) ds(y) \\ & + \alpha_{i-1}^+ \mathbf{T}_{i\nu(x)} \int_{S_{i-1}} \phi_{i-1}(y) \cdot \mathbf{G}_i(x, y) ds(y) \\ & + \beta_{i-1}^+ \mathbf{T}_{i\nu(x)} \int_{S_{i-1}} \psi_{i-1}(y) \cdot \mathbf{N}_{i\nu(y)} \mathbf{G}_i(x, y) ds(y), \end{aligned} \quad (5.46)$$

Preliminary remark:

Let $S = \bigcup_{i=0}^{i=n-1} S_i$. We can redefine $\phi_m(x)$ and $\psi_m(x)$ on S by setting their values to zero for $x \in S_p, p \neq m$. Hence $S_{lm}, \mathcal{K}_{lm}^N, \mathcal{K}_{lmn}^N, T_{lmn}^{TN}$ are surface operators on S , defined here for functions supported by S_m . We are interested by their values in $C^{0,\alpha}(S)$. However, we can consider separately their restrictions to $x \in S_m$ and $x \in S \setminus S_m$. The first restriction has the analytic properties studied in part 4, for instance S_{lm} is compact from $C^{0,\alpha}(S_m)$ into itself, and bounded from $C^{0,\alpha}(S_m)$ into $C^{1,\alpha}(S_m)$. The second restriction has always the best analytic properties. In fact, it is compact from $C^{0,\alpha}(S_m)$ into $C^{1,\alpha}(S \setminus S_m)$. This should be kept in mind in the analysis below, where we shall use the condensed notations $\mathcal{K}_{lm}^N, T_{lm}$ for $\mathcal{K}_{lm}^N, T_{lm}$.

Let us now write down the continuity equations

$$\mathbf{u}_i(x) = \mathbf{u}_{i+1}(x), \quad \mathbf{t}_i \mathbf{u}_i(x) = \mathbf{t}_{i+1} \mathbf{u}_{i+1}(x) \quad x \in S_i. \quad (5.47)$$

They read, for $x \in S_i$:

$$\begin{aligned} & \alpha_i^- S_{ii}(\phi_i) + \beta_i^- [\mathcal{K}_{ii}^N(\psi_i) - 4\pi\psi_i] \\ & + \alpha_{i-1}^+ S_{ii-1}(\phi_{i-1}) + \beta_{i-1}^+ \mathcal{K}_{ii-1}^N(\psi_{i-1}) \\ & = \alpha_i^+ S_{i+1i}(\phi_i) + \beta_i^+ [\mathcal{K}_{i+1i}^N(\psi_i) + 4\pi\psi_i] \\ & + \alpha_{i+1}^- S_{i+1i+1}(\phi_{i+1}) + \beta_{i+1}^- \mathcal{K}_{i+1i+1}^N(\psi_{i+1}), \end{aligned} \quad (5.48)$$

and

$$\alpha_i^- \rho_i [\mathcal{K}_{ii}^{TN}(\phi_i) + 4\pi\phi_i] + \beta_i^- \rho_i T_{ii}^{TN}(\psi_i) - \frac{\beta_i^- \mu_i}{\eta_i + 3} \Theta(\psi_i)$$

$$\begin{aligned}
& +\alpha_{i-1}^+ \rho_i \mathcal{K}_{ii-1}'^T(\phi_{i-1}) + \beta_{i-1}^+ \rho_i \mathcal{T}_{ii-1}^{TN}(\psi_i) \\
& = \alpha_i^+ \rho_{i+1} [\mathcal{K}_{i+1i}'^T(\phi_i) - 4\pi\phi_i] + \beta_i^+ \rho_{i+1} \mathcal{T}_{i+1i}^{TN}(\psi_i) - \frac{\beta_i^+ \mu_{i+1}}{\eta_{i+1} + 3} \Theta(\psi_i) \\
& + \alpha_{i+1}^- \rho_{i+1} \mathcal{K}_{i+1i+1}'^T(\phi_{i+1}) + \beta_{i+1}^- \rho_{i+1} \mathcal{T}_{i+1i+1}^{TN}(\psi_{i+1}), \tag{5.49}
\end{aligned}$$

whereas, for $i = n - 1$:

$$\begin{aligned}
& \alpha_{n-1}^- \mathcal{S}_{n-1n-1}(\phi_{n-1}) + \beta_{n-1}^- [\mathcal{K}_{n-1n-1}^N(\psi_{n-1}) - 4\pi\psi_{n-1}] \\
& + \alpha_{n-2}^+ \mathcal{S}_{n-1n-2}(\phi_{n-2}) + \beta_{n-2}^+ \mathcal{K}_{n-1n-2}^N(\psi_{n-2}) \\
& = \alpha_{n-1}^+ \mathcal{S}_{nn-1}(\phi_{n-1}) + \beta_{n-1}^+ [\mathcal{K}_{nn-1}^N(\psi_{n-1}) + 4\pi\psi_{n-1}] + u_\infty, \tag{5.50}
\end{aligned}$$

and

$$\begin{aligned}
& \alpha_{n-1}^- \rho_i [\mathcal{K}_{n-1n-1}'^T(\phi_{n-1}) + 4\pi\phi_{n-1}] + \beta_{n-1}^- \rho_{n-1} \mathcal{T}_{n-1n-1}^{TN}(\psi_{n-1}) - \frac{\beta_{n-1}^- \mu_{n-1}}{\eta_{n-1} + 3} \Theta(\psi_{n-1}) \\
& + \alpha_{n-2}^+ \rho_{n-1} \mathcal{K}_{n-1n-2}'^T(\phi_{n-2}) + \beta_{n-2}^+ \rho_{n-1} \mathcal{T}_{n-1n-2}^{TN}(\psi_{n-1}) \\
& = \alpha_{n-1}^+ \rho_n [\mathcal{K}_{nn-1}'^T(\phi_{n-1}) - 4\pi\phi_{n-1}] + \beta_{n-1}^+ \rho_n \mathcal{T}_{nn-1}^{TN}(\psi_{n-1}) + t_{infy} + \frac{\beta_{n-1}^+ \mu_n}{\eta_n + 3} \Theta(\psi_n) \tag{5.51}
\end{aligned}$$

Hence, if we introduce the vector valued density **functions**

$$\begin{pmatrix} \psi_0 \\ \vdots \\ \psi_{n-1} \end{pmatrix} \text{ and } \begin{pmatrix} \phi_0 \\ \vdots \\ \phi_{n-1} \end{pmatrix}, \tag{5.52}$$

that we shall try to derive respectively in $C^{0,\alpha}(S)$ and $C^{1,\alpha}(S)$, we can write down the continuity equations by means of the following operator matrices, where $i, k = 0, 1, \dots, n-1$

$$\mathcal{S}_i^k = -\delta_{i-1}^k \alpha_{i-1}^+ \mathcal{S}_{ii-1} + \delta_i^k (\alpha_i^+ \mathcal{S}_{i+1i} - \alpha_i^- \mathcal{S}_{ii}) + \delta_{i+1}^k \alpha_{i+1}^- \mathcal{S}_{i+1i+1} \tag{5.53}$$

$$\mathcal{K}_i^k = -\delta_{i-1}^k \beta_{i-1}^+ \mathcal{K}_{ii-1}^N + \delta_i^k (\beta_i^+ \mathcal{K}_{i+1i}^N - \beta_i^- \mathcal{K}_{ii}^N) + \delta_{i+1}^k \beta_{i+1}^- \mathcal{K}_{i+1i+1}^N \tag{5.54}$$

$$\alpha_i^k = 4\pi(\beta_i^+ - \beta_i^-) \delta_i^k \mathcal{I} \tag{5.55}$$

$$\mathcal{K}_i'^k = -\delta_{i-1}^k \alpha_{i-1}^+ \rho_i \mathcal{K}_{ii-1}'^T + \delta_i^k (\alpha_i^+ \rho_{i+1} \mathcal{K}_{i+1i}'^T - \alpha_i^- \rho_i \mathcal{K}_{ii}'^T) + \delta_{i+1}^k \alpha_{i+1}^- \rho_{i+1} \mathcal{K}_{i+1i+1}'^T \tag{5.56}$$

$$b_i^k = 4\pi(\alpha_i^+ \rho_{i+1} + \alpha_i^-) \delta_i^k \mathcal{I} \tag{5.57}$$

$$\mathcal{T}_i^k = -\delta_{i-1}^k \beta_{i-1}^+ \rho_i \mathcal{T}_{ii-1}^{TN} + \delta_i^k (\beta_i^+ \rho_{i+1} \mathcal{T}_{i+1i}^{TN} - \beta_i^- \rho_i \mathcal{T}_{ii}^{TN}) + \delta_{i+1}^k \beta_{i+1}^- \rho_{i+1} \mathcal{T}_{i+1i+1}^{TN} \tag{5.58}$$

$$\xi_i^k = \left(\frac{\beta_i^+ \mu_{i+1}}{\eta_{i+1} + 3} + \frac{\beta_i^- \mu_i}{\eta_i + 3} \right) \delta_i^k \mathcal{I} \tag{5.59}$$

$$u_\infty = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ u_\infty \end{pmatrix} \quad t_\infty = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ t_\infty \end{pmatrix} \tag{5.60}$$

The result is the two equations

$$\left. \begin{aligned} \alpha(\mathbf{I} + \mathcal{K})\psi + \mathbf{u}_\infty + S\phi &= 0 \\ -b(1 - \mathcal{K}')\phi + \mathbf{t}_\infty + T\psi + \xi(\psi) &= 0 \end{aligned} \right\} \quad (5.61)$$

It is obvious that the most general transmission could be readily written by including the σ_i and τ_i in the definition of $\mathbf{u}_\infty$ and $\mathbf{t}_\infty$ by (5.60). It follows from the remarks we made on the operators, and from the diagonal character of ξ , that all the methods displayed in the one surface case can be applied: the analytic properties of the operators which are involved are the same (from $C^{0,\alpha}(S)$ or $C^{1,\alpha}(S)$ into $C^{0,\alpha}(S)$) as they were in the simpler case. The small norms cases are the same, and hold provided we remain in a convenient domain with all μ_i 's, λ_i 's, ρ_i 's reasonably close to each other. Outside this domain, which is a common physical one, we can use the Fredholm approach (which would be ill-posed at the limit where all μ_i 's, all λ_i 's, all ρ_i 's are clustered).

5.4 The scattering amplitude

We start with the expression of $\mathbf{u}(\mathbf{x})$ outside S_{n-1} :

$$\begin{aligned} \mathbf{u}(x) = u_\infty(x) + \alpha_{n-1}^+ \int_{S_{n-1}} \phi_{n-1} \cdot \mathbf{G}_n ds \\ + \beta_{n-1}^+ \int_{S_{n-1}} \psi_{n-1} \cdot (\mathbf{N}_{n\nu} [\mathbf{G}]) ds. \end{aligned} \quad (5.62)$$

In this formula, ϕ_{n-1} and ψ_{n-1} are supposed to be known as solutions of 5.61. To obtain the asymptotic form, we simply use, as $|x| \rightarrow +\infty$:

$$(\mathbf{G}_n)_{ij}(x, y) = \frac{1}{4\pi} \frac{1}{(c_n^p)^2} \hat{x}_i \hat{x}_j e^{-ik^p \hat{x} \cdot y} \frac{e^{ik^p |x|}}{|x|} \quad (5.63)$$

$$+ \frac{1}{4\pi} \frac{1}{(c_n^s)^2} (\delta_{ij} - \hat{x}_i \hat{x}_j) e^{-ik^s \hat{x} \cdot y} \frac{e^{ik^s |x|}}{|x|} \quad (5.64)$$

$$+ o\left(\frac{1}{|x|}\right). \quad (5.65)$$

where $\hat{x} = \frac{x}{|x|}$. But if $(\hat{\tau}^1, \hat{\tau}^2, \hat{x})$ is an orthonormal system in $\mathbb{R}^3$, one can write

$$\delta_{ij} - \hat{x}_i \hat{x}_j = \hat{\tau}_i^1 \hat{\tau}_j^1 + \hat{\tau}_i^2 \hat{\tau}_j^2. \quad (5.66)$$

as can be seen from left and right contraction with elements of any element of $\mathbb{R}^3$.

So

$$\begin{aligned} \mathbf{G}_n(x, y) = \frac{1}{4\pi} \frac{1}{(c_n^p)^2} \hat{x} \otimes \hat{x} e^{-ik_n^p \hat{x} \cdot y} \frac{e^{ik_n^p |x|}}{|x|} \\ + \frac{1}{4\pi} \frac{1}{(c_n^s)^2} \hat{\tau}^1 \otimes \hat{\tau}^1 e^{-ik_n^s \hat{x} \cdot y} \frac{e^{ik_n^s |x|}}{|x|} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4\pi} \frac{1}{(c_n^s)^2} \hat{\tau}^2 \otimes \hat{\tau}^2 e^{-ik_n^s \hat{x} \cdot y} \frac{e^{ik_n^s |x|}}{|x|} \\
& + o\left(\frac{1}{|x|}\right),
\end{aligned} \tag{5.67}$$

where $\otimes$ stands for the tensorial product. In (5.62), this gives:

$$\begin{aligned}
u_i(x) &= u_{\infty i}(x) + \frac{1}{4\pi} \alpha_{n-1}^+ \frac{1}{(c_n^p)^2} \frac{e^{ik_n^p |x|}}{|x|} \hat{x}_i \int_{S_{n-1}} (\phi_{n-1})_j \hat{x}_j e^{-ik_n^p \hat{x} \cdot y} ds \\
& + \frac{1}{4\pi} \alpha_{n-1}^+ \frac{1}{(c_n^s)^2} \frac{e^{ik_n^s |x|}}{|x|} (\hat{\tau}^1)_i \int_{S_{n-1}} (\phi_{n-1})_j (\hat{\tau}^1)_j e^{-ik_n^s \hat{x} \cdot y} ds \\
& + \frac{1}{4\pi} \alpha_{n-1}^+ \frac{1}{(c_n^s)^2} \frac{e^{ik_n^s |x|}}{|x|} (\hat{\tau}^2)_i \int_{S_{n-1}} (\phi_{n-1})_j (\hat{\tau}^2)_j e^{-ik_n^s \hat{x} \cdot y} ds \\
& + \frac{1}{4\pi} \beta_{n-1}^+ \frac{1}{(c_n^p)^2} \frac{e^{ik_n^p |x|}}{|x|} \hat{x}_i \int_{S_{n-1}} (\psi_{n-1})_j \left(N_{n\nu} \left[\hat{x} e^{-ik_n^p \hat{x} \cdot y} \right] \right)_j ds \\
& + \frac{1}{4\pi} \beta_{n-1}^+ \frac{1}{(c_n^s)^2} \frac{e^{ik_n^s |x|}}{|x|} (\hat{\tau}^1)_i \int_{S_{n-1}} (\psi_{n-1})_j \left(N_{n\nu} \left[(\hat{\tau}^1) e^{-ik_n^s \hat{x} \cdot y} \right] \right)_j ds \\
& + \frac{1}{4\pi} \beta_{n-1}^+ \frac{1}{(c_n^s)^2} \frac{e^{ik_n^s |x|}}{|x|} (\hat{\tau}^2)_i \int_{S_{n-1}} (\psi_{n-1})_j \left(N_{n\nu} \left[(\hat{\tau}^2) e^{-ik_n^s \hat{x} \cdot y} \right] \right)_j ds \\
& = u_{\infty i}(x) + \frac{1}{4\pi} \frac{1}{(c_n^p)^2} \frac{e^{ik_n^p |x|}}{|x|} D^P_i(u_0, k_n^p) \\
& + \frac{1}{4\pi} \frac{1}{(c_n^s)^2} \frac{e^{ik_n^s |x|}}{|x|} D^{S1}_i(u_0, k_n^s) \\
& + o\left(\frac{1}{|x|}\right).
\end{aligned} \tag{5.68}$$

The $\hat{x}$, $\hat{\tau}^1$, $\hat{\tau}^2$ parts correspond respectively to a longitudinal (or compressive) part of the diffusion amplitude, and two transverse (or shear) components. In order to obtain the scattering channels, we must introduce an incoming P wave u_0^P , or S wave u_0^S , which, after resolution of (5.61), gives two densities ϕ_n^P, ψ_n^P or ϕ_n^S, ψ_n^S . With these conventions, we will obtain four channels of scattering: P→P, P→S, S→P, S→S. For example the P→P amplitude will be:

$$\begin{aligned}
D^P_i(u_0^p, k_n^p) &= \hat{x}_i \alpha_{n-1}^+ \int_{S_{n-1}} (\phi_{n-1}^P)_j \hat{x}_j e^{-ik_n^p \hat{x} \cdot y} ds \\
& + \hat{x}_i \beta_{n-1}^+ \int_{S_{n-1}} (\psi_{n-1}^P)_j \left(N_{n\nu} \left[\hat{x} e^{-ik_n^p \hat{x} \cdot y} \right] \right)_j ds
\end{aligned} \tag{5.69}$$

5.5 The response to a source

For the sake of simplicity, we only consider the case of a point source localised at y in Ω_n . The problem to be solved is:

$$\begin{cases} \mu_i[(\eta_i + 2)grad_x div_x u(x) - rot_x rot_x u(x) + \rho_i \omega^2 u(x) = 0, x \in \Omega_i, i = 1, 2, \dots, N - 1; \\ \mu_N[(\eta_N + 2)grad_x div_x u(x) - rot_x rot_x u(x) + \rho_N \omega^2 u(x) = -\delta(x - y)\mathcal{I}; \\ u(x^+) - u(x^-) = (t_{\nu_i} u)^+ - (t_{\nu_i} u)^- = 0, x \in S_i, i = 1, 2, \dots, N - 1; \\ u(x) \text{ outgoing.} \end{cases} \quad (5.70)$$

Hence it is nothing but the Green function problem restricted to Ω_N for the perturbed medium. If the source may be localised everywhere but on a discontinuity surface, the problem becomes the general Green function problem for the perturbed medium. Setting:

$$v(x) = \begin{cases} u(x), x \in \mathbb{R}^3 \setminus \overline{\Omega}_N \\ u(x) - G(x, y), x \in \Omega_N, \end{cases} \quad (5.71)$$

we see that $v(x)$ is a solution of the problem (5.1) with

$$\begin{cases} \sigma_i(x) = \tau_i(x) = 0, i = 1, 2, \dots, N - 2; \\ \sigma_{N-1}(x) = -G(x, y), x \in S_{N-1}; \\ \tau_{N-1}(x) = -t_{\nu_{N-1}x} G_{N-1}(x, y), x \in S_{N-1}; \\ v(x) + G(x, y) \text{ outgoing} \end{cases} \quad (5.72)$$

Which is nothing but (5.1) It is clear that a source localised inside another Ω_i could be managed in a similar way. Beyond its direct physical meaning, and applications in earthquakes analysis, these "perturbed Green functions" may be used to manage approximately special kinds of inhomogeneities.

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A Jump properties of $\mathcal{T}^{TT}$ and analytic properties

A.1 Jump theorem

We will write the jump of $\mathcal{T}^{TT}$ ($\omega \neq 0$) defined with the same set of parameters for $\mathbf{T}_{\nu(y)}$, $\mathbf{T}_{\nu(x)}$, and $\mathbf{G}$, on each side of a surface S .

Given the expressions (2.5) of $\mathbf{T}_{\nu}[\mathbf{u}]$ and $\mathbf{T}_{\nu}[\mathbf{G}]$ it is obvious that $\mathcal{T}^{TT}[\mathbf{B}]$, for a density $\mathbf{B} \in C^{1,\alpha}(S_m)$, will contain nine terms. Using the tensorial notation to write the exterior product, and remembering that $\epsilon_{rst}\epsilon_{lqt} = \delta_{qr}\delta_{st} - \delta_{rt}\delta_{sq}$, one can, after some algebraic manipulation, group the terms in three families

The $(\lambda)^2$ terms: There is only one of them

$$\mathbf{A}_i = (\lambda)^2 \nu_i(x) \partial_x \int_{\partial\Omega} \mathbf{B}_l(y) \nu_l(y) \partial_{y_p} \mathbf{G}_{pj}(x, y) ds(y) \quad (\text{A.1})$$

which, using

$$\left. \begin{aligned} (\text{div}_y \mathbf{G})_j &= \partial_{y_i} \mathbf{G}_{ij}(x, y) = \frac{(k^p)^2}{\omega^2} \partial_{y_j} \Phi^{k^p}(x, y) \\ \partial_x \partial_{y_j} \Phi^{k^p}(x, y) &= (k^p)^2 \Phi^{k^p}(x, y) \end{aligned} \right\} \quad (\text{A.2})$$

(one can differentiate inside the sum since x is outside Ω) reduces to:

$$\mathbf{A}_i = (\lambda)^2 \left(\frac{k^p}{c^p}\right)^2 \nu_i(x) \int_{\partial\Omega} (\mathbf{B} \cdot \nu)(y) \Phi^{k^p}(x, y) ds(y) \quad (\text{A.3})$$

and thus is continuous through $\mathbb{R}^3$ as a single layer potential.

The $(\mu)^2$ terms: they can be written as:

$$\begin{aligned} \mathbf{B}_i &= \int_{\partial\Omega} \nu_q(x) \nu_k(y) \mathbf{B}_l(y) \left[\partial_{x_i} \partial_{y_l} \mathbf{G}_{kq}(x, y) + \partial_{x_i} \partial_{y_k} \mathbf{G}_{lq}(x, y) \right. \\ &\quad \left. + \partial_{x_q} \partial_{y_l} \mathbf{G}_{ki}(x, y) + \partial_{x_q} \partial_{y_k} \mathbf{G}_{li}(x, y) \right] ds(y) \end{aligned} \quad (\text{A.4})$$

Inserting the $\mathbf{G}$ explicit expression, one gets:

$$\begin{aligned} \mathbf{B}_i &= \frac{4}{\omega^2} \int_{\partial\Omega} \mathbf{B}_l(y) \frac{\partial}{\partial \nu(x)} \frac{\partial}{\partial \nu(y)} \partial_{x_i} \partial_{y_l} (\Phi^{k^p}(x, y) - \Phi^{k^s}(x, y)) \\ &\quad + \frac{1}{(c^s)^2} \int_{\partial\Omega} \nu_q(x) \nu_k(y) \mathbf{B}_l(y) \partial_{x_i} \partial_{y_l} \Phi^{k^s}(x, y) \delta_{kq} \\ &\quad + \frac{1}{(c^s)^2} \int_{\partial\Omega} \nu_q(x) \nu_k(y) \mathbf{B}_l(y) \partial_{x_i} \partial_{y_k} \Phi^{k^s}(x, y) \delta_{lq} \\ &\quad + \frac{1}{(c^s)^2} \int_{\partial\Omega} \nu_q(x) \nu_k(y) \mathbf{B}_l(y) \partial_{x_q} \partial_{y_l} \Phi^{k^s}(x, y) \delta_{ik} \\ &\quad + \frac{1}{(c^s)^2} \int_{\partial\Omega} \nu_q(x) \nu_k(y) \mathbf{B}_l(y) \partial_{x_q} \partial_{y_k} \Phi^{k^s}(x, y) \delta_{il} \end{aligned} \quad (\text{A.5})$$

The third and fifth terms in this expression can be treated by the scalar jump theorems, being a gradient and a normal derivative of a double layer (for the gradient of a double layer see [Colton, Kress 1983, Th 2.23]. The fifth term is continuous through $\mathbb{R}^3$ as a normal derivative of a double layer, and thus can be forgotten in our following jump calculations.

We easily write down the singular part of the first term:

$$\begin{aligned} & \frac{1}{\omega^2} \frac{\partial}{\partial \nu(x)} \frac{\partial}{\partial \nu(y)} \partial_{x_i} \partial_{y_l} (\Phi^{k^p}(x, y) - \Phi^{k^s}(x, y)) \\ &= \frac{1}{2} \left(\frac{1}{(c^p)^2} - \frac{1}{(c^s)^2} \right) \frac{\partial}{\partial \nu(x)} \frac{\partial}{\partial n(y)} \left(\frac{\delta_{il}}{|x - y|} + \frac{(\mathbf{x}_i - \mathbf{y}_i)(\mathbf{y}_l - \mathbf{x}_l)}{|x - y|^3} \right). \end{aligned}$$

Putting aside, as a normal derivative of a scalar double layer the $\frac{\delta_{il}}{|x - y|}$ term, and using

$$\frac{(\mathbf{x}_i - \mathbf{y}_i)}{|x - y|} = \partial_{y_i} \frac{1}{|x - y|} \quad (\text{A.6})$$

we write $\mathbf{B}$ as:

$$\mathbf{B}_i = 2 \left(\frac{1}{(c^p)^2} - \frac{1}{(c^s)^2} \right) \int_{\partial\Omega} \mathbf{B}_l(y) \frac{\partial}{\partial \nu(x)} \frac{\partial}{\partial \nu(y)} (\mathbf{y}_l - \mathbf{x}_l) \partial_{y_i} \frac{1}{|x - y|} ds \quad (\text{A.7})$$

$$+ \frac{\nu_p(x)}{(c^s)^2} \partial_{x_i} \int_{\partial\Omega} \nu_p(y) \mathbf{B}_l(y) \partial_{y_l} \Phi^{k^s}(x, y) ds(y) \quad (\text{A.8})$$

$$+ \frac{\nu_p(x)}{(c^s)^2} \partial_{x_i} \int_{\partial\Omega} \mathbf{B}_p(y) \frac{\partial}{\partial \nu(y)} \Phi^{k^s}(x, y) ds(y) \quad (\text{A.9})$$

$$+ \frac{1}{(c^s)^2} \frac{\partial}{\partial \nu(x)} \int_{\partial\Omega} \nu_i(y) \mathbf{B}_l(y) \partial_{y_l} \Phi^{k^s}(x, y) ds(y). \quad (\text{A.10})$$

We must evaluate (A.7), (A.8), (A.10), for (A.9) is the gradient of an acoustic double layer. The first one can be written

$$\mathbf{B}_{1i} = 2 \left(\frac{1}{(c^p)^2} - \frac{1}{(c^s)^2} \right) \nu_p(x) \partial_{x_p} \partial_{x_q} \int_{\partial\Omega} [\mathbf{B} \cdot (\mathbf{y} - \mathbf{x}) \partial_{x_i} \frac{1}{|x - y|}] \nu_q(y) ds(y) \quad (\text{A.11})$$

where the antisymmetry of a $x - y$ function has been used twice. Next, from

$$\text{grad}_x \text{div}_x = \Delta_x + \text{rot}_x \text{rot}_x \quad (\text{A.12})$$

we obtain A.7 as:

$$\begin{aligned} \mathbf{B}_{1i} &= 2 \left(\frac{1}{(c^p)^2} - \frac{1}{(c^s)^2} \right) \nu_p(x) \left[\Delta_x \int_{\partial\Omega} [\mathbf{B} \cdot (\mathbf{y} - \mathbf{x}) \partial_{x_i} \frac{1}{|x - y|}] \nu(y) ds(y) \right]_p \\ &+ 2 \left(\frac{1}{(c^p)^2} - \frac{1}{(c^s)^2} \right) \nu_p(x) \left[\text{rot}_x \text{rot}_x \int_{\partial\Omega} [\mathbf{B} \cdot (\mathbf{y} - \mathbf{x}) \partial_{x_i} \frac{1}{|x - y|}] \nu(y) ds(y) \right]_p \end{aligned} \quad (\text{A.13})$$

$$\begin{aligned}
&= 4\left(\frac{1}{(c^p)^2} - \frac{1}{(c^s)^2}\right)\nu_p(x) \int_{\partial\Omega} \mathbf{B}_l(y) \partial_{y_l} \partial_{x_i} \frac{1}{|x-y|} \nu_p(y) ds(y) \\
&- 2\left(\frac{1}{(c^p)^2} - \frac{1}{(c^s)^2}\right)\nu_p(x) \\
&\quad \left[\text{rot}_x \int_{\partial\Omega} \left[\mathbf{B}_l(y) \text{Grad}_y \left((y_l - x_l) \partial_{x_i} \frac{1}{|x-y|} \right) \wedge \nu(y) \right] ds(y) \right]_p
\end{aligned}$$

Now for a *closed* surface integral, with continuous density ϕ , one has:

$$\int_{\partial\Omega} [\text{Grad}_y \phi(y) \wedge \nu(y)] = 0. \quad (\text{A.14})$$

(See [Colton, Kress, 1983] p.34) Thus $\mathbf{B}_{li}$ can be expressed as:

$$\begin{aligned}
\mathbf{B}_{li} &= 4\left(\frac{1}{(c^p)^2} - \frac{1}{(c^s)^2}\right)\nu_p(x) \int_{\partial\Omega} \mathbf{B}_l(y) \partial_{y_l} \partial_{x_i} \frac{1}{|x-y|} \nu_p(y) ds(y) \\
&+ 2\left(\frac{1}{(c^p)^2} - \frac{1}{(c^s)^2}\right)\nu_p(x) \\
&\quad \left[\text{rot}_x \int_{\partial\Omega} [\text{Grad}_y [\mathbf{B}_l(y)] (y_l - x_l) \partial_{x_i} \frac{1}{|x-y|}, \nu(y)] ds(y) \right]_p \\
&= 4\left(\frac{1}{(c^p)^2} - \frac{1}{(c^s)^2}\right)\nu_p(x) \partial_{x_i} \int_{\partial\Omega} \mathbf{B}_l(y) \nu_p(y) \partial_{y_l} \frac{1}{|x-y|} ds(y) \\
&- 2\left(\frac{1}{(c^p)^2} - \frac{1}{(c^s)^2}\right)\nu_p(x) \epsilon_{pqr} \epsilon_{rst} \partial_{x_q} \int_{\partial\Omega} [\text{Grad}_y \mathbf{B}_l(y)]_s (y_l - x_l) \nu_t(y) \partial_{x_i} \frac{1}{|x-y|} ds(y)
\end{aligned}$$

The two terms in the right hand side are of the form:

$$\partial_{x_i} \int_{\partial\Omega} \phi(y) \partial_{y_j} \frac{1}{|x-y|} ds(y). \quad (\text{A.15})$$

We want to obtain an expression with only double and single layers, and thus we must reduce the degree of differentiation. We use for it the properties of the surface gradient :

$$\partial_{y_j} = \nu_j(y) \frac{\partial}{\partial \nu(y)} + [\text{Grad}_y]_j \quad (\text{A.16})$$

The right hand side of (A.15) thus becomes:

$$\begin{aligned}
&\partial_{x_i} \int_{\partial\Omega} \phi(y) \nu_j(y) \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} ds(y) + \partial_{x_i} \int_{\partial\Omega} [\text{Grad}_y \phi(y) \frac{1}{|x-y|}]_j ds(y) \\
&- \partial_{x_i} \int_{\partial\Omega} [\text{Grad}_y \phi(y)]_j \frac{1}{|x-y|} ds(y)
\end{aligned} \quad (\text{A.17})$$

Using then the integral property for a closed surface

$$\int_S \text{Grad} \phi ds = -2 \int_S \phi H n ds \quad (\text{A.18})$$

(See [Colton, Kress, 1983] p.34) we reformulate A.17 as a combination of tractable terms:

$$\begin{aligned} & \partial_{x_i} \int_{\partial\Omega} \phi(y) \nu_j(y) \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} ds(y) \\ & - 2 \partial_{x_i} \int_{\partial\Omega} \nu_j(y) \phi(y) H(y) \frac{1}{|x-y|} ds(y) \\ & - \partial_{x_i} \int_{\partial\Omega} [\text{Grad}_y \phi(y)]_j \frac{1}{|x-y|} ds(y) \end{aligned}$$

Thus B_{1i} finally becomes:

$$\begin{aligned} B_1 = & 4 \left(\frac{1}{(c^p)^2} - \frac{1}{(c^s)^2} \right) \nu_p(x) \partial_{x_i} \int_{\partial\Omega} (B \cdot \nu)(y) \nu_p(y) \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} ds(y) \\ & - 8 \left(\frac{1}{(c^p)^2} - \frac{1}{(c^s)^2} \right) \nu_p(x) \partial_{x_i} \int_{\partial\Omega} (B \cdot \nu)(y) \nu_p(y) H(y) \frac{1}{|x-y|} ds(y) \\ & - 4 \left(\frac{1}{(c^p)^2} - \frac{1}{(c^s)^2} \right) \nu_p(x) \partial_{x_i} \int_{\partial\Omega} [\text{Grad}_y B_l(y) \nu_p(y)]_l \frac{1}{|x-y|} ds(y) \\ & + 2 \left(\frac{1}{(c^p)^2} - \frac{1}{(c^s)^2} \right) \nu_p(x) \epsilon_{pqr} \epsilon_{rst} \\ & \quad \partial_{x_q} \int_{\partial\Omega} (y_l - x_l) [\text{Grad}_y B_l(y)]_s \nu_t(y) \nu_i(y) \frac{\partial}{\partial \nu(y)} \frac{1}{|x-y|} ds(y) \\ & - 4 \left(\frac{1}{(c^p)^2} - \frac{1}{(c^s)^2} \right) \nu_p(x) \epsilon_{pqr} \epsilon_{rst} \\ & \quad \partial_{x_q} \int_{\partial\Omega} (y_l - x_l) [\text{Grad}_y B_l(y)]_s \nu_t(y) \nu_i(y) H(y) \frac{1}{|x-y|} ds(y) \\ & - 2 \left(\frac{1}{(c^p)^2} - \frac{1}{(c^s)^2} \right) \nu_p(x) \epsilon_{pqr} \epsilon_{rst} \\ & \quad \partial_{x_q} \int_{\partial\Omega} [\text{Grad}_y ((y_l - x_l) [\text{Grad}_y (B_l(y)) \nu_t(y)])]_i \frac{1}{|x-y|} ds(y) \end{aligned} \quad (A.19)$$

We proceed now to the examination of the term (A.8). It is also of the form A.15 and is easily obtained as

$$\begin{aligned} B_{2i} = & \frac{\nu_p(x)}{(c^s)^2} \partial_{x_i} \int_{\partial\Omega} \nu_p(y) (B \cdot \nu)(y) \frac{\partial}{\partial \nu(y)} \Phi^{k^s}(x, y) ds(y) \\ & - 2 \frac{\nu_p(x)}{(c^s)^2} \partial_{x_i} \int_{\partial\Omega} \nu_p(y) (B \cdot \nu)(y) H(y) \Phi^{k^s}(x, y) ds(y) \\ & - \frac{\nu_p(x)}{(c^s)^2} \partial_{x_i} \int_{\partial\Omega} [\text{Grad}_y \nu_p(y) B_l(y)]_l \Phi^{k^s}(x, y) ds(y) \end{aligned} \quad (A.20)$$

Last but not least A.10 (which is not a gradient of a double layer...) shows the same problems, and is obtained as:

$$B_4 = \frac{1}{(c^s)^2} \frac{\partial}{\partial \nu(x)} \int_{\partial\Omega} \nu_i(y) (B \cdot \nu)(y) \frac{\partial}{\partial \nu(y)} \Phi^{k^s}(x, y) ds(y)$$

$$\begin{aligned}
& - \frac{2}{(c^s)^2} \frac{\partial}{\partial \nu(x)} \int_{\partial\Omega} \nu_i(y) (\mathbf{B} \cdot \nu)(y) H(y) \Phi^{k^s}(x, y) ds(y) \\
& - \frac{1}{(c^s)^2} \frac{\partial}{\partial \nu(x)} \int_{\partial\Omega} [\nu_i(y) \mathbf{B}_l(y)]_l \Phi^{k^s}(x, y) ds(y)
\end{aligned} \tag{A.21}$$

To conclude, we apply the scalar jump theorems to the expressions of $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \mathbf{B}_4$, and after some manipulations, one obtains:

$$\begin{aligned}
[\mathbf{B}] &= [\mathbf{B}]^{\partial\Omega} \\
&\pm 4 \left(\frac{1}{(c^p)^2} - \frac{1}{2} \frac{1}{(c^s)^2} \right) H(x) (\mathbf{B} \cdot \nu)(x) \nu_i(x) \\
&\pm 2 \left(\frac{1}{(c^p)^2} - \frac{1}{2} \frac{1}{(c^s)^2} \right) [\mathbf{Grad}_x \mathbf{B}_l]_l(x) \nu_i(x) \\
&\pm 2 \left(\frac{1}{(c^p)^2} - \frac{1}{2} \frac{1}{(c^s)^2} \right) [\mathbf{Grad}_x (\mathbf{B} \cdot \nu)]_i(x)
\end{aligned} \tag{A.22}$$

The $\mu\lambda$ terms: After some algebra, they write:

$$\begin{aligned}
C &= \int_{\partial\Omega} \nu_i(x) \mathbf{B}_l(y) \nu_k(y) (\partial_{y_k} \partial_{x_j} \mathbf{G}_{lj} + \partial_{y_l} \partial_{x_j} \mathbf{G}_{kj}) ds(y) \\
&+ \int_{\partial\Omega} \nu_q(x) (\mathbf{B} \cdot \nu)(y) (\partial_{y_p} \partial_{x_i} \mathbf{G}_{qp} + \partial_{y_p} \partial_{x_q} \mathbf{G}_{pi}) ds(y)
\end{aligned}$$

Using the definition of $\mathbf{G}$ and the formula:

$$(\operatorname{div}_y \mathbf{G})_j = \partial_{y_i} \mathbf{G}_{ij}(x, y) = \frac{(k^p)^2}{\omega^2} \partial_{y_j} \Phi^{k^p}(x, y) \tag{A.23}$$

one can express it as:

$$\begin{aligned}
C &= \frac{2}{(c^p)^2} \int_{\partial\Omega} \nu_i(x) \mathbf{B}_l(y) \partial_{x_l} \frac{\partial}{\partial \nu(y)} \Phi^{k^p}(x, y) ds(y) \\
&+ \frac{2}{(c^p)^2} \int_{\partial\Omega} \nu_q(x) (\mathbf{B} \cdot \nu)(y) \partial_{y_i} \partial_{x_q} \Phi^{k^p}(x, y) ds(y)
\end{aligned} \tag{A.24}$$

and, again by the (A.15) equation method, we can expand the result into classical scalar potentials:

$$\begin{aligned}
C &= \frac{2}{(c^p)^2} \int_{\partial\Omega} \nu_i(x) \mathbf{B}_l(y) \partial_{x_l} \frac{\partial}{\partial \nu(y)} \Phi^{k^p}(x, y) ds(y) \\
&+ \frac{2}{(c^p)^2} \frac{\partial}{\partial \nu(x)} \int_{\partial\Omega} (\mathbf{B} \cdot \nu)(y) \nu_i(y) \frac{\partial}{\partial \nu(y)} \Phi^{k^p}(x, y) ds(y) \\
&- \frac{4}{(c^p)^2} \frac{\partial}{\partial \nu(x)} \int_{\partial\Omega} (\mathbf{B} \cdot \nu)(y) \nu_i(y) H(y) \Phi^{k^p}(x, y) ds(y) \\
&- \frac{2}{(c^p)^2} \frac{\partial}{\partial \nu(x)} \int_{\partial\Omega} [\mathbf{Grad}_y (\mathbf{B} \cdot \nu)(y)]_i \Phi^{k^p}(x, y) ds(y).
\end{aligned} \tag{A.25}$$

With the jumps theorems of section 4-1, we conclude that:

$$\begin{aligned}
[C] &= [C]^{\partial\Omega} \\
&\pm \frac{2}{(c^p)^2} H(x) (\mathbf{B} \cdot \nu)(x) \nu_i(x) \\
&\pm \frac{1}{(c^p)^2} [\text{Grad}_x \mathbf{B}_l]_l(x) \nu_i(x) \\
&\pm \frac{1}{(c^p)^2} [\text{Grad}_x (\mathbf{B} \cdot \nu)]_i(x)
\end{aligned} \tag{A.26}$$

Adding now the expressions (A.22) and (A.26), multiplied respectively by $(\mu)^2$ and $\lambda\mu_2$, one obtains the following theorem

Theorem 17 *The generalized derivative of the elastic double layer vectorial potential, with density $\mathbf{B}$ in $C^{1,\alpha}(\partial\Omega)$, is Hölder continuous in $\mathbb{R}^3$.*

A.2 analytic properties

We have succeeded in writing the jump of the $\mathcal{T}^{TT}$ operator by means of an explicit decomposition of this potential in terms of acoustic single and double layer and their derivatives, for $\mathbf{x} \in \mathbb{R}^3 \setminus S$. The analytic properties of these operators can be found in [Colton, Kress, 1983], or [colton, Kress, 1992]. The most singular operator in the decomposition is the double layer gradient:

$$\partial_{x_i} \int_S \phi(y) \frac{\partial}{\partial \nu(y)} \Phi(x, y) ds(y) \tag{A.27}$$

But according to theorem of [Colton, Kress, 1983], it is bounded from $C^{1,\alpha}(S)$ into $C^{0,\alpha}(S)$. It follows that the same property holds for our elastic vectorial double layer potential $\mathcal{T}^{TT}$ from $(C^{1,\alpha}(S))^3$ into $(C^{0,\alpha}(S))^3$.

B Jumps and analytic properties of $\mathcal{T}^{TN}, \mathcal{T}^{NN}$

B.1 Jump theorem for $\mathcal{T}^{TN}$

from the definitions (2.5) and (2.6) of $\mathbf{t}_{\nu(\mathbf{x})}$ and $\mathbf{N}_{\nu(\mathbf{y})}$, we can write down

$$\mathbf{N}_{\nu(\mathbf{y})} = \frac{\lambda + 2\mu}{\rho(\lambda + 3\mu)} \left[2\mu \frac{\partial}{\partial \nu(\mathbf{y})} + \mu \nu(\mathbf{y}) \wedge \text{rot}_{\mathbf{y}} + \lambda \nu(\mathbf{y}) \text{div}_{\mathbf{y}} \right] \quad (\text{B.1})$$

$$+ \frac{\mu}{\rho(\lambda + 3\mu)} \left[(\lambda + 2\mu) \nu(\mathbf{y}) \text{div}_{\mathbf{y}} - \mu \nu(\mathbf{y}) \wedge \text{rot}_{\mathbf{y}} \right], \quad (\text{B.2})$$

so that

$$\mathbf{t}_{\nu(\mathbf{x})} \mathbf{N}_{\nu(\mathbf{y})} = \frac{\lambda + 2\mu}{\rho(\lambda + 3\mu)} \mathbf{t}_{\nu(\mathbf{x})} \mathbf{t}_{\nu(\mathbf{y})} + \frac{\mu}{\rho(\lambda + 3\mu)} \mathbf{t}_{\nu(\mathbf{x})} \left[(\lambda + 2\mu) \nu(\mathbf{y}) \text{div}_{\mathbf{y}} - \mu \nu(\mathbf{y}) \wedge \text{rot}_{\mathbf{y}} \right]. \quad (\text{B.3})$$

Because of the theorem stated at the end of Appendix A, we need not to study the jump related to $\mathbf{t}_{\nu(\mathbf{x})} \mathbf{t}_{\nu(\mathbf{y})}$. We look at the jump of:

$$\mathcal{T}_1^{TN} = \mathbf{t}_{\nu(\mathbf{x})} \int_S \mathbf{B}(\mathbf{y}) \cdot \left[(\lambda + 2\mu) \nu(\mathbf{y}) \text{div}_{\mathbf{y}} - \mu \nu(\mathbf{y}) \wedge \text{rot}_{\mathbf{y}} \right] \left(\mathbf{G}(\mathbf{x}, \mathbf{y}) \right) ds(\mathbf{y}) \quad (\text{B.4})$$

From

$$\left. \begin{aligned} \left(\text{div}_{\mathbf{y}} \mathbf{G}(\mathbf{x}, \mathbf{y}) \right)_i &= \left(\frac{k^p}{\omega} \right)^2 \partial_{y_i} \Phi^{kp}(\mathbf{x}, \mathbf{y}) \\ \left(\text{rot}_{\mathbf{y}} \mathbf{G}(\mathbf{x}, \mathbf{y}) \right)_{ij} &= \left(\frac{k^s}{\omega} \right)^2 \epsilon_{ilj} \partial_{y_l} \Phi^{ks}(\mathbf{x}, \mathbf{y}) = \left(\frac{k^s}{\omega} \right)^2 \left(\text{rot} \Phi^{kp}(\mathbf{x}, \mathbf{y}) \right)_{ij} \end{aligned} \right\} \quad (\text{B.5})$$

we get

$$\mathcal{T}_1^{TN} = \left(\frac{k^p}{\omega} \right)^2 \mathbf{t}_{\nu(\mathbf{x})} \int_S \mathbf{B}(\mathbf{y}) \cdot \left[(\lambda + 2\mu) \nu(\mathbf{y}) \otimes \text{grad}_{\mathbf{y}} \Phi^{kp}(\mathbf{x}, \mathbf{y}) \right] ds(\mathbf{y}) \quad (\text{B.6})$$

$$- \left(\frac{k^s}{\omega} \right)^2 \mathbf{t}_{\nu(\mathbf{x})} \int_S \mathbf{B}(\mathbf{y}) \cdot \left[\mu \nu(\mathbf{y}) \wedge \text{rot} \Phi^{ks}(\mathbf{x}, \mathbf{y})(\mathbf{x}, \mathbf{y}) \right] ds(\mathbf{y}). \quad (\text{B.7})$$

Six terms are to be studied. First

$$\left(\frac{k^p}{\omega} \right)^2 (\lambda + 2\mu) \lambda \nu(\mathbf{x}) \text{div}_{\mathbf{x}} \int_S \left(\mathbf{B} \cdot \nu \right)(\mathbf{y}) \text{grad}_{\mathbf{y}} \Phi^{kp}(\mathbf{x}, \mathbf{y}) ds(\mathbf{y}). \quad (\text{B.8})$$

We can differentiate under the integral as long as $\mathbf{x}$ does not belongs to S , and from

$$\text{div}_{\mathbf{x}} \text{grad}_{\mathbf{y}} \Phi^{kp}(\mathbf{x}, \mathbf{y}) = -\Delta_{\mathbf{x}} \Phi^{kp}(\mathbf{x}, \mathbf{y}) = (k^p)^2 \Phi^{kp}(\mathbf{x}, \mathbf{y}) \quad (\text{B.9})$$

we obtain this term as a non jumping acoustic single layer.

Second term:

$$\mu(\lambda + 2\mu) \left(\frac{k^p}{\omega} \right)^2 \nu(\mathbf{x}) \text{rot}_{\mathbf{x}} \int_S \mathbf{B} \cdot \left[\nu(\mathbf{y}) \otimes \text{grad}_{\mathbf{y}} \Phi^{kp}(\mathbf{x}, \mathbf{y}) \right] ds(\mathbf{y}) \quad (\text{B.10})$$

Since

$$\text{grad}_{\mathbf{y}} \Phi^{kp}(\mathbf{x}, \mathbf{y}) = -\text{grad}_{\mathbf{x}} \Phi^{kp}(\mathbf{x}, \mathbf{y}) \quad (\text{B.11})$$

the rotationnal of a gradient appears under the integral, so the term is equal to zero.
The third term writes in coordinates:

$$-\mu\lambda\left(\frac{k^s}{\omega}\right)^2\nu_i(x)\partial_{x_i}\int_S\mathbf{B}_l\epsilon_{lpq}\nu_p(y)\epsilon_{qmj}\partial_{y_m}\Phi^{k^s}(x,y)ds(y). \quad (\text{B.12})$$

This term also vanish from

$$\epsilon_{qmj}\partial_{x_j}\partial_{y_m}\Phi^{k^s}(x,y)=-\epsilon_{qmj}\partial_{y_j}\partial_{y_m}\Phi^{k^s}(x,y)=0. \quad (\text{B.13})$$

The fourth term writes:

$$-\mu^2\left(\frac{k^s}{\omega}\right)^2\nu(x)\wedge\text{rot}_x\int_S\mathbf{B}\cdot[\nu(y)\wedge\text{rot}\mathbf{G}(x,y)]ds(y). \quad (\text{B.14})$$

From

$$\left(-\mu\nu(y)\wedge\text{rot}_y\mathbf{G}\right)_{ij}=-\rho\left(\nu_j(y)\partial_{y_i}\Phi^{k^s}(x,y)-\delta_{ij}\frac{\partial}{\partial\nu(y)}\Phi^{k^s}(x,y)\right) \quad (\text{B.15})$$

we obtain

$$-\rho\mu\left(\frac{k^s}{\omega}\right)^2\nu_t(x)\partial_{x_i}\int_S\nu_t(y)\mathbf{B}_l(y)\partial_{y_l}\Phi^{k^s}(x,y)ds(y) \quad (\text{B.16})$$

$$+\rho\mu\left(\frac{k^s}{\omega}\right)^2\nu_p(x)\partial_{x_p}\int_S\nu_i(y)\mathbf{B}_l(y)\partial_{y_l}\Phi^{k^s}(x,y)ds(y) \quad (\text{B.17})$$

$$+\rho\mu\left(\frac{k^s}{\omega}\right)^2\nu_t(x)\partial_{x_i}\int_S\mathbf{B}_l(y)\frac{\partial}{\partial\nu(y)}\Phi^{k^s}(x,y)ds(y) \quad (\text{B.18})$$

$$-\rho\mu\left(\frac{k^s}{\omega}\right)^2\frac{\partial}{\partial\nu(y)}\int_S\mathbf{B}_i(y)\frac{\partial}{\partial\nu(y)}\Phi^{k^s}(x,y)ds(y) \quad (\text{B.19})$$

In this decomposition, (B.19) is continuous and (B.18) is a derivative of double layer, whose jumps are well known. We transform (B.16) and (B.17) with our usual "surface integration by parts" tool, to give respectively:

$$-\rho\mu\left(\frac{k^s}{\omega}\right)^2\nu_t(x)\partial_{x_i}\int_S\nu_t(y)(\mathbf{B}\cdot\nu)(y)\frac{\partial}{\partial\nu(y)}\Phi^{k^s}(x,y)ds(y) \quad (\text{B.20})$$

$$+\rho\mu\left(\frac{k^s}{\omega}\right)^2\nu_t(x)\partial_{x_i}\int_S[\text{Grad}\nu_t(y)\mathbf{B}_l(y)]_l\Phi^{k^s}(x,y)ds(y) \quad (\text{B.21})$$

$$+2\rho\mu\left(\frac{k^s}{\omega}\right)^2\nu_t(x)\partial_{x_i}\int_S\nu_t(y)(\mathbf{B}\cdot\nu)(y)H(y)\Phi^{k^s}(x,y)ds(y) \quad (\text{B.22})$$

and

$$-\rho\mu\left(\frac{k^s}{\omega}\right)^2\frac{\partial}{\partial\nu(x)}\int_S[\text{Grad}\nu_i(y)\mathbf{B}_l(y)]_l\Phi^{k^s}(x,y)ds(y) \quad (\text{B.23})$$

$$-2\rho\mu\left(\frac{k^s}{\omega}\right)^2\frac{\partial}{\partial\nu(x)}\int_S\nu_i(y)(\mathbf{B}\cdot\nu)(y)H(y)\Phi^{k^s}(x,y)ds(y) \quad (\text{B.24})$$

$$+\rho\mu\left(\frac{k^s}{\omega}\right)^2\frac{\partial}{\partial\nu(x)}\int_S\nu_i(y)(\mathbf{B}\cdot\nu)(y)\frac{\partial}{\partial\nu(y)}\Phi^{k^s}(x,y)ds(y) \quad (\text{B.25})$$

From the scalar jump theorems, we obtain the limit values of (B.16), (B.17), (B.18), and using the relation

$$[Grad\nu_i]_j = [Grad\nu_j]_i, \quad (B.26)$$

(a consequence of the definition of the surface gradient, see [Colton, Kress, 1983] p32-33) we see that the jump is zero, giving the continuity in $\mathbb{R}^3$ of the fourth term.

The fifth term is

$$2\mu(\lambda + 2\mu)\left(\frac{k^p}{\omega}\right)^2 \frac{\partial}{\partial\nu(x)} \int_S (\mathbf{B}.\nu)(y) grad\Phi^{k^p}(x, y) ds(y). \quad (B.27)$$

Always by the same trick, it reads:

$$2\mu(\lambda + 2\mu)\left(\frac{k^p}{\omega}\right)^2 \frac{\partial}{\partial\nu(x)} \int_S (\mathbf{B}.\nu)(y) \nu(y) \frac{\partial}{\partial\nu(y)} \Phi^{k^p}(x, y) ds(y) \quad (B.28)$$

$$-2\mu(\lambda + 2\mu)\left(\frac{k^p}{\omega}\right)^2 \frac{\partial}{\partial\nu(x)} \int_S [Grad(\mathbf{B}.\nu)(y)] \Phi^{k^p}(x, y) ds(y) \quad (B.29)$$

$$-4\mu(\lambda + 2\mu)\left(\frac{k^p}{\omega}\right)^2 \frac{\partial}{\partial\nu(x)} \int_S (\mathbf{B}.\nu)(y) \nu(y) H(y) \Phi^{k^p}(x, y) ds(y). \quad (B.30)$$

The scalar theorems gives us its jump:

$$\pm \rho\mu [Grad\mathbf{B}.\nu](x) \pm 2\rho\mu (\mathbf{B}.\nu H)(x) \nu(x). \quad (B.31)$$

The last term has also a jump. Its cartesian expression is:

$$-2\mu^2 \frac{\partial}{\partial\nu(x)} \int_S \mathbf{B}_j (\nu \wedge rot\mathbf{G})_{ij} ds(y) \quad (B.32)$$

wich gives

$$-2\rho\mu \frac{\partial}{\partial\nu(x)} \int_S (\mathbf{B}.\nu)(y) \nu_i(y) \frac{\partial}{\partial\nu(y)} \Phi^{k^s}(x, y) ds(y) \quad (B.33)$$

$$+2\rho\mu \frac{\partial}{\partial\nu(x)} \int_S [Grad\mathbf{B}_j(y) \nu_i(y)]_j \Phi^{k^s}(x, y) ds(y) \quad (B.34)$$

$$+4\rho\mu \frac{\partial}{\partial\nu(x)} \int_S (\mathbf{B}.\nu)(y) \nu_i(y) H(y) \Phi^{k^s}(x, y) ds(y) \quad (B.35)$$

$$+2\mu \frac{\partial}{\partial\nu(x)} \int_S \mathbf{B}_i(y) \frac{\partial}{\partial\nu(y)} \Phi^{k^s}(x, y) ds(y). \quad (B.36)$$

Its jump:

$$\mp \rho\mu [Grad\mathbf{B}_j \nu_i]_j(x) \mp 2\rho\mu (\mathbf{B}.\nu H)(x) \nu_i(x). \quad (B.37)$$

Adding (B.31) and (B.37), using again (B.26) , and multiplying by $\mu/\rho(\lambda + 3\mu)$, gives , for $\mathbf{x} \in S$ a decomposition of the jump in tangential and normal parts:

$$[\mathcal{T}^{TN}(\mathbf{B})]_i^\pm(x) = [\mathcal{T}^{TN}(\mathbf{B})]_i(x) \quad (B.38)$$

$$\pm \frac{\mu^2}{\lambda + 3\mu} \{ \nu_j [Grad\mathbf{B}_j]_i - \nu_i [Grad\mathbf{B}_j]_j \}. \quad (B.39)$$

B.2 Analytic properties

By the same argument as in the first appendix, its clear that $\mathcal{T}^{TN}$ is bounded from $C^{1,\alpha}$ into $C^{0,\alpha}$.

B.3 Remark on $\mathcal{T}^{NN}$

The same manipulations can lead one to show that $\mathcal{T}^{NN}$ is continuous throught $\mathbb{R}^3$. Of course its also bounded from $C^{1,\alpha}$ into $C^{0,\alpha}$.

C miscellaneous proofs

Proof of theorem 12

Simply write that the expansion of $\mathbf{u}^s$ is divergence-free , and that of $\mathbf{u}^p$ is rotationnal free, which reads explicitley, for the first one:

$$\begin{cases} \frac{1}{r \sin \theta} \left[\partial_\theta \left(\sin \theta \sum_0^{+\infty} \frac{e^{ik^p r}}{r} \frac{\mathbf{u}_{n,p}^\phi}{r^n} \right) - \partial_\phi \left(\sum_0^{+\infty} \frac{e^{ik^p r}}{r} \frac{\mathbf{u}_{n,p}^\theta}{r^n} \right) \right] = 0 \\ \sum_0^{+\infty} \frac{e^{ik^p r}}{r} \frac{1}{r^n} \frac{1}{\sin \theta} \partial_\phi \mathbf{u}_{n,p}^r - \partial_r \left(e^{ik^p r} \sum_0^{+\infty} \frac{\mathbf{u}_{n,p}^\phi}{r^n} \right) = 0 \\ \partial_r \left(e^{ik^p r} \sum_0^{+\infty} \frac{\mathbf{u}_{n,p}^\theta}{r^n} \right) - \sum_0^{+\infty} \frac{e^{ik^p r}}{r} \frac{1}{r^n} \frac{1}{\sin \theta} \partial_\theta \mathbf{u}_{n,p}^r = 0 \end{cases}$$

From the lowest order of the last two ones we get here:

$$\begin{aligned} \mathbf{u}_{0,p}^\phi &= 0, \\ \mathbf{u}_{0,p}^\theta &= 0. \end{aligned}$$

In the same way the second expansion:

$$\operatorname{div}_x \mathbf{u}^s = 0$$

reads

$$\begin{aligned} \frac{1}{r^2} \partial_r \left(r^2 \frac{e^{ik^s r}}{r} \sum_{n=0}^{\infty} \frac{\mathbf{u}_{n,s}^r}{r^n} \right) + \frac{1}{r \sin \theta} \frac{e^{ik^s r}}{r} \sum_{n=0}^{\infty} \frac{\partial_\theta (\sin \theta \mathbf{u}_{n,s}^\theta)}{r^n} \\ + \frac{1}{r \sin \theta} \frac{e^{ik^s r}}{r} \sum_{n=0}^{\infty} \frac{\partial_\phi (\mathbf{u}_{n,s}^\phi)}{r^n} = 0 \end{aligned}$$

so that at the order r^{-1} , one obtains

$$\mathbf{u}_{0,s}^r = 0 \quad \square]$$

Proof of theorem 13

From the expression of $\mathbf{t}_\nu[\mathbf{u}](x)$, we must show that:

$$\frac{\partial \mathbf{u}}{\partial n} - n \operatorname{div} \mathbf{u} - [\operatorname{rot} \mathbf{u}, n] = O\left(\frac{1}{r^2}\right). \quad (\text{C.1})$$

On the sphere $|x| = r$, one has

$$\frac{\partial}{\partial n(x)} = \frac{\partial}{\partial r}$$

We insert the expansion of $\mathbf{u}(x)$ obtained from the Theorem 8 and Lemma 10:

$$\mathbf{u}(x) = \frac{e^{ik_N^p r}}{r} \mathbf{u}_{o,p}^r(\theta, \phi) \frac{\vec{r}}{r} \quad (\text{C.2})$$

$$+ \left(\mathbf{u}_{o,s}^\theta(\theta, \phi) \vec{\theta} + \mathbf{u}_{o,s}^\phi(\theta, \phi) \vec{\phi} \right) + O\left(\frac{1}{r^2}\right) \quad (\text{C.3})$$

into the expressions of $\frac{\partial}{\partial n(x)}$, div_x , rot_x , obtaining:

$$\begin{aligned}\frac{\partial}{\partial n(x)} &= ik_N^p \frac{e^{ik_N^p r}}{r} \mathbf{u}_{o,p}^r(\theta, \phi) \frac{\vec{r}}{r} \\ &\quad + ik_N^s (\mathbf{u}_{o,s}^\theta(\theta, \phi) \vec{\theta} + \mathbf{u}_{o,s}^\phi(\theta, \phi) \vec{\phi}) + O\left(\frac{1}{r^2}\right) + O\left(\frac{1}{r^2}\right), \\ div_x u &= \frac{e^{ik_N^p r}}{r} \mathbf{u}_{o,p}^r(\theta, \phi) \frac{\vec{r}}{r} + O\left(\frac{1}{r^2}\right), \\ rot_x u &= ik_N^s (\mathbf{u}_{o,s}^\theta(\theta, \phi) \vec{\theta} + \mathbf{u}_{o,s}^\phi(\theta, \phi) \vec{\phi}) + O\left(\frac{1}{r^2}\right).\end{aligned}$$

From these expressions C.1 follows and the lemma is straightforward. $\square$]

Proof of lemma 15

From the equivalence of our asymptotic condition for $\mathbf{u}$ with the Sommerfeld one for its components $\mathbf{u}_i$, it is obvious (See Colton, Kress, 1983], chap 3) that

$$\mathbf{u}(x) = O\left(\frac{1}{|x|}\right),$$

which simply means that we consider a physically outgoing wave, so that the second order term in the expression 3.14 will be

$$\begin{aligned}\overline{(k^p \mathbf{u}^p + k^s \mathbf{u}^s)} \cdot t_\nu[\mathbf{u}](x) &= \overline{(k^p \mathbf{u}^p + k^s \mathbf{u}^s)} \cdot (\mathbf{n}(x) div \mathbf{u}^p + [rot \mathbf{u}^s, \mathbf{n}(x)]) \\ &= k^p \overline{\mathbf{u}^p} \cdot n div \mathbf{u}^p + k^p \overline{\mathbf{u}^p} \cdot [rot \mathbf{u}^s, \mathbf{n}] + k^s \overline{\mathbf{u}^s} \cdot n div \mathbf{u}^p \\ &\quad + k^s \overline{\mathbf{u}^s} \cdot [rot \mathbf{u}^s, \mathbf{n}].\end{aligned}$$

Applying the rotational and divergence operators (written in spherical coordinates) to the expansions of $\mathbf{u}^p$ and $\mathbf{u}^s$, taking theorem 11 into account, we get:

$$[rot_x \mathbf{u}^p(x), \mathbf{n}(x)] = O\left(\frac{1}{r}\right) \vec{\theta} + O\left(\frac{1}{r}\right) \vec{\phi} + O\left(\frac{1}{r^2}\right),$$

so that

$$\overline{\mathbf{u}^p} \cdot [rot \mathbf{u}^s, \mathbf{n}] = O\left(\frac{1}{r^3}\right).$$

And in the same way:

$$\overline{\mathbf{u}^s} \cdot (n div \mathbf{u}^p) = O\left(\frac{1}{r^3}\right).$$

Hence the lowest order terms will be the $\frac{1}{r^2}$ ones:

$$\begin{aligned}\overline{\mathbf{u}^p} \cdot n div \mathbf{u}^p &= ik^p |\mathbf{u}_{o,p}^r|^2 \frac{1}{r^2} + O\left(\frac{1}{r^3}\right), \\ \overline{\mathbf{u}^s} \cdot [rot \mathbf{u}^s, \mathbf{n}] &= ik^s (|\mathbf{u}_{o,s}^\theta|^2 + |\mathbf{u}_{o,s}^\phi|^2) \frac{1}{r^2} + O\left(\frac{1}{r^3}\right).\end{aligned}$$

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