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Hyperbolic complex structures in physics

Joseph Hucks*

Institute for Theoretical Physics
Santa Barbara, California 93106 USA(To appear in *Journal of Mathematical Physics*, December 1993)

ABSTRACT: A brief introduction to the parabolic and hyperbolic complex number systems is given. Parabolic and hyperbolic phase transformations are shown to be equivalent to two-dimensional Galilean and Lorentzian relativity transformations, respectively. Basic properties and definitions for the hyperbolic complex numbers are given, and they are then applied to special relativistic physics and the Dirac equation in $1+1$ dimensions. Turning to string theory, it is briefly shown that for Minkowski signature the string world sheet possesses an integrable almost hyperbolic complex structure, with a metric that is generalized Hermitian. Next, the hyperbolic complex numbers are applied to the formalism of Dirac spinors in $3+1$ dimensions. It is shown that a four component Dirac spinor is equivalent to a two component hyperbolic complex spinor, that the Lorentz group is equivalent to a generalized $SU(2)$, and that the Dirac adjoint corresponds to a generalized Hermitian adjoint. A complete formalism is presented for the hyperbolic complex two-component spinors. Lastly, it is shown that the operations of C , P , and T on Dirac spinors are closely related to the three types of complex conjugation which exist when both hyperbolic and ordinary imaginary units are present.

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*Bitnet address: JCHucks@sbitp.bitnet

I. INTRODUCTION

In this paper I will discuss the complex number systems of the parabolic and hyperbolic types and some of their applications to physics, with the spotlight on the use of hyperbolic complex numbers in relativistic physics in two and four space-time dimensions.

The mathematical segment of this paper begins in Section II with a proof that there are only three types of abelian complex number systems with a single imaginary unit. They are: the ordinary (or elliptic) complex numbers, the *dual* or parabolic complex numbers, and the *double* or hyperbolic complex numbers. Of the three, only the ordinary complex numbers form a field; the other two complex number systems form rings since not all elements possess multiplicative inverses.

It is shown in Section III that Galilean relativity transformations in one space and one time dimension are equivalent to parabolic $U(1)$ transformations. The emphasis then shifts to the hyperbolic complex numbers in Section IV, beginning with an analogous demonstration that hyperbolic $U(1)$ transformations are equivalent to Lorentz transformations in two space-time dimensions. It is interesting that the phase transformations associated with the three types of number systems correspond to Euclidean rotations, Galilean relativity transformations, and Lorentz transformations. There then follows a complete discussion of the basic properties of hyperbolic complex numbers, with the introduction of some new notations, and a brief discussion of hyperbolic analyticity. The Dirac algebra in two space-time dimensions is shown to be equivalent to an antilinear algebra with the hyperbolic imaginary unit and hyperbolic complex conjugation as generators. The massless Dirac equation in one space and one time dimension is shown to be equivalent to a hyperbolic holomorphicity condition on the Dirac spinor.

In Section V, the hyperbolic complex numbers are very briefly applied to string theory, where it is shown that the string world sheet possesses an integrable almost *hyperbolic* complex structure (for Minkowski signature), and is hence a hyperbolic complex manifold. There is then a legitimate (hyperbolic) complex structure on the world-sheet, and there is no need to Euclideanize in order to use complex methods.

In Section VI, the applicability of the hyperbolic complex numbers to the Dirac algebra and equation in $3+1$ dimensions is discussed. It is shown how a four component Dirac spinor may be represented as a two component hyperbolic complex spinor, with γ_5 shown to be a representation of the hyperbolic imaginary unit. A complete set of conventions for working with two component hyperbolic complex spinors is given. The Lorentz group is shown to be equivalent to a generalized special unitary group, and the Dirac adjoint to a generalized Hermitian adjoint. Lastly, some interesting light is shed on the operations of C , P , and T on Dirac spinors: each is shown to be closely related to the three types of conjugation which exist when one has both ordinary and hyperbolic imaginary units. At the end of Section VI, the reader should be left with the impression that physicists have been working around the use of hyperbolic complex numbers in their dealings with Dirac spinors.

II. CLASSIFICATION OF ABELIAN COMPLEX NUMBER SYSTEMS: THE ELLIPTIC, PARABOLIC, AND HYPERBOLIC COMPLEX NUMBER SYSTEMS

To begin with, we briefly repeat an argument given by Yaglom¹ and also by Kantor and Solodovnikov². Let us assume that we have a general abelian complex number system where each element is of the form $w = a + tb$, where a and b are ordinary real numbers, and t is some non-real imaginary unit. Assuming that the complex number system is closed under multiplication, we then have that $t^2 = c + 2td$, with c and d real. We can complete the square and then obtain $(t-d)^2 = c + d^2$. There are then three possible cases: $c + d^2 < 0$, $c + d^2 = 0$, or $c + d^2 > 0$.

In the first case, we define a new imaginary unit $i = (t-d)/(-c-d^2)^{1/2}$ with the property that $i^2 = -1$. Then we have a system of complex numbers equivalent to the ordinary complex numbers, which we will also call the *elliptic* complex numbers.

In the second case, we define a new imaginary unit $\zeta = t-d$, which has the property that $\zeta^2 = 0$. The quantity ζ acts like a Grassmann number in that it is nilpotent. This number system is called the parabolic complex or the *dual* numbers.

The third case will be the one this paper will be mostly concerned with. In this last case, we may define a new imaginary unit $\varepsilon = (t-d)/(c+d^2)^{1/2}$, which has the property that $\varepsilon^2 = +1$. This is the hyperbolic complex number system, also called the *double* numbers by Clifford³, who was the first to discover them.

We have shown that there are only three types of abelian complex number systems in which the square of the imaginary unit is affinely related to itself (i.e., $t^2 = c + 2td$). The reader is already familiar with the ordinary complex numbers, so that I will only discuss the parabolic and hyperbolic cases at any length, concentrating on the hyperbolic complex numbers and their applications to relativistic physics for most of this paper.

The elliptic, parabolic, and hyperbolic imaginary units may be represented by the following real 2×2 matrices

$$i = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \zeta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2.1)$$

in a suitable basis, or by any matrices related to the above by similarity transformations.

In the case where one wishes to use a mixture of hyperbolic and complex numbers and have the imaginary units commute, we could use the following real representations for the imaginary units

$$i = -\sqrt{-1}(\sigma_2 \otimes I) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \quad \varepsilon = I \otimes \sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \quad (2.2)$$

where I denotes the 2×2 identity matrix. [Recall that the rule for multiplication of matrices which are outer (Kronecker) products of smaller matrices is⁴ $(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)$, which is most easily seen by writing out the components in tensor notation]. This will be the case which will be of greatest interest. If instead one had i and ε anti-commute, then one would be led to the use of quaternion-like objects. In this paper i and ε will commute for all cases of interest, although this is certainly not necessary from a mathematical standpoint. When one has only a hyperbolic imaginary unit, the number system will be called the *pure* hyperbolic complex numbers, and when

one has both hyperbolic and ordinary imaginary units, the number system will be called *mixed* or just hyperbolic complex for convenience.

There are two points of view which one can take about the complex number systems described above. The first is more abstract and looks at the complex number systems as mathematical entities which exist independently of any representation. The second point of view is more pragmatic, and is that all complex number systems may be thought of as shorthands for more cumbersome expressions involving real matrices. From either viewpoint, the parabolic and hyperbolic complex numbers possess as valid an existence as do ordinary complex numbers. As is well known, the regular complex numbers make many results more elegant as well as add a unifying element to the presentation of many formulae. One of the purposes of this paper is to show that the same is true for the hyperbolic complex numbers.

III. THE GALILEAN RELATIVITY GROUP IN TWO DIMENSIONS AND THE PARABOLIC COMPLEX NUMBERS

We know that if we multiply an ordinary complex coordinate, $z = x + iy$, by a phase $e^{-i\theta}$, then we obtain a new coordinate,

$$z' = x' + iy' = e^{-i\theta}(x + iy) = (x \cos \theta + y \sin \theta) + i(-x \sin \theta + y \cos \theta) \quad (3.1)$$

which yields the same answer as a rotation by θ in the Euclidean plane:

$$x' = x \cos \theta + y \sin \theta, \quad y' = -x \sin \theta + y \cos \theta \quad (3.2)$$

Now let us explore an interesting relationship between the parabolic complex numbers and the Galilean relativity group in $1+1$ dimensions. Let us combine the time and space coordinates into a single parabolic complex coordinate, $w = t + \zeta x$ (for simplicity, let ζ have units of inverse speed). Using the power series expansion for the exponential function, and the property $\zeta^2 = 0$, we can show that the analogue of de Moivre's theorem is

$$e^{\pm \zeta \theta} = 1 \pm \zeta \theta \quad (3.3)$$

since higher powers of the imaginary unit ζ vanish. Setting $\theta = -v$, with v the relative velocity between two Galilean reference frames, we see that a parabolic 'phase' acting on the parabolic complex coordinate w yields a Galilean relativity transformation:

$$w' = t' + \zeta x' = e^{-\zeta v} w = (1 - \zeta v)(t + \zeta x) = t + \zeta(x - vt) \quad \text{or} \quad t' = t, \quad x' = x - vt \quad (3.4)$$

Thus we can think of the Galilean relativity group as a parabolic 'U(1)', and we could even designate it 'U(1, p)'. In a generalized way, it is 'unitary' in that the 'phase', denoted below by V , satisfies

$$V \& V = (e^{-\zeta v}) \& e^{-\zeta v} = (1 - \zeta v) \& (1 - \zeta v) = (1 + \zeta v)(1 - \zeta v) = 1 \quad (3.5)$$

where in the above equation parabolic complex conjugation is denoted by an ampersand (&) and takes ζ to $-\zeta$.

Note that

$$V(v_1) V(v_2) = e^{-\zeta v_1} e^{-\zeta v_2} = (1 - \zeta v_1)(1 - \zeta v_2)$$

$$= I - \zeta(v_1 + v_2) = e^{-\zeta(v_1 + v_2)} = V(v_1 + v_2) \quad (3.6)$$

so we see easily that the Galilean relativity group in $I+I$ dimensions is isomorphic to the group of real numbers with the group operation of ordinary addition.

Further results for the parabolic complex numbers and their application to Galilean physics will be presented in future work, as the main focus of this paper is on hyperbolic complex numbers and their application to relativistic physics. For more details on the parabolic complex or dual numbers the interested reader should refer to the texts by Yaglom^{5, 6}.

IV. RELATIVISTIC PHYSICS IN TWO DIMENSIONS AND BASIC PROPERTIES OF THE HYPERBOLIC COMPLEX NUMBERS

In this section, the basic properties of the hyperbolic complex numbers are derived and applied to special relativistic physics in $I+I$ dimensions. For instance, we see that the condition that a spinor field satisfy the massless Dirac equation is simply that it be a holomorphic function of a hyperbolic complex coordinate. For more information on hyperbolic complex numbers, refer to the texts by Yaglom^{7, 8}.

A. Special relativity in two space-time dimensions and the hyperbolic complex numbers

Let us now consider the hyperbolic complex numbers. First we will see that they have some relevance to the Lorentz group in $I+I$ dimensions, which is $SO(I, I, r) \cong GL(I, r)/Z_2$. Let us define a hyperbolic complex coordinate $w = t + \varepsilon x$, and consider the action of a hyperbolic 'phase' acting on it. To do this, we need the analogy to de Moivre's theorem in the hyperbolic case. If we use the power series expansion of the exponential function and the property $\varepsilon^2 = +I$, we easily obtain the result

$$e^{\pm \varepsilon \theta} = \cosh \theta \pm \varepsilon \sinh \theta \quad (4.1)$$

Setting

$$\cosh \theta = \gamma \equiv \frac{1}{\sqrt{1-v^2}}, \quad \sinh \theta = \gamma v = \frac{v}{\sqrt{1-v^2}} \quad (4.2)$$

where $v = \tanh \theta$ is the speed in units where the speed of light is one, we then have

$$\begin{aligned} w' = t' + \varepsilon x' &= e^{-\varepsilon \theta} w = (\cosh \theta - \varepsilon \sinh \theta)(t + \varepsilon x) = (t \cosh \theta - x \sinh \theta) + \varepsilon (x \cosh \theta - t \sinh \theta) \\ &= \gamma(t - vx) + \varepsilon \gamma(x - vt) \quad \text{or} \quad t' = \gamma(t - vx), \quad x' = \gamma(x - vt) \end{aligned} \quad (4.3)$$

So we see that a hyperbolic 'phase' transformation corresponds to a Lorentz transformation in $I+I$ dimensions. It is remarkable that the phase transformations on the different types of complex coordinates discussed correspond to Euclidean rotations, Galilean relativity transformations, and Lorentz transformations. This leads one to believe that parabolic and hyperbolic complex numbers may be useful in Galilean and Lorentzian physics, respectively.

In a generalized sense, the group of Lorentz transformations is 'unitary' in that if $\Lambda = \exp(-\varepsilon \theta)$ represents a Lorentz transformation in one space and one time dimension, then $\Lambda^\# \Lambda = I$:

$$\begin{aligned} \Lambda^\# \Lambda &= (e^{-\varepsilon \theta})^\# e^{-\varepsilon \theta} = (\cosh \theta - \varepsilon \sinh \theta)^\# (\cosh \theta - \varepsilon \sinh \theta) \\ &= (\cosh \theta + \varepsilon \sinh \theta)(\cosh \theta - \varepsilon \sinh \theta) = \cosh^2 \theta - \sinh^2 \theta = I \end{aligned} \quad (4.4)$$

In the above, hyperbolic complex conjugation is denoted by a pound symbol ($\#$) and replaces ε with $-\varepsilon$. In general, a square matrix V with pure hyperbolic complex entries will be considered hyperbolic 'unitary' if $(V^T)^\# V = V (V^T)^\# = I$.

The 'squared norm' of a hyperbolic complex number is given by

$$w^\# w = (t - \varepsilon x)(t + \varepsilon x) = t^2 - x^2 \quad (4.5)$$

The hyperbolic complex numbers possess a natural Minkowski metric:

$$d\tau^2 \equiv dw^\# dw = (dt - \varepsilon dx)(dt + \varepsilon dx) = dt^2 - dx^2 \quad (4.6)$$

Note that the hyperbolic squared norm can be positive, zero, or negative, in distinction to the case for ordinary complex numbers, in which all non-zero complex numbers have positive squared norm. Hyperbolic complex numbers with positive squared norm will be termed *time-like*, those with zero squared norm *light-like* or *null*, and those with negative squared norm *space-like*, in accordance with the standard terminology of special relativity. Hyperbolic complex numbers which are space-like or time-like possess inverses, while null ones do not:

$$(t + \varepsilon x)^{-1} = \frac{1}{t^2 - x^2} (t - \varepsilon x) \quad (4.7)$$

Since every non-zero element does not possess a multiplicative inverse, the hyperbolic complex numbers form an *abelian ring with unity* and *not* a field, unlike the ordinary complex numbers, which form a field. Recall that the main difference between a ring and a field is that non-zero elements of a ring need not have multiplicative inverses.

B. Null basis elements and basic properties of the hyperbolic complex numbers

Every null hyperbolic complex number is either proportional to $I + \varepsilon$ ($t = x$) or to $I - \varepsilon$ ($t = -x$), so we can define new null basis elements^{10, 11} proportional to these:

$$\alpha \equiv \frac{1}{2}(I + \varepsilon), \quad \alpha^\# \equiv \bar{\alpha} \equiv \frac{1}{2}(I - \varepsilon) \quad (4.8)$$

In this section, an 'overbar' is used to denote the hyperbolic complex conjugate of a quantity when that quantity has no ordinary imaginary part.)

The new basis elements have the special properties that

$$\alpha^2 = \alpha, \quad (\bar{\alpha})^2 = \bar{\alpha}, \quad \alpha \bar{\alpha} = 0, \quad \alpha + \bar{\alpha} = I \quad (4.9)$$

and thus form a complete set of projection operators for the hyperbolic complex numbers. Recall that N operators form a complete set of projection operators if

$$P_i^2 = P_i, \quad P_i P_j = 0 \quad (i \neq j), \quad \sum_{i=1}^N P_i = I \quad (4.10)$$

Each hyperbolic complex number hence decomposes into two pieces, each proportional to one of the null basis elements:

$$w = t + \varepsilon x = (t + x)\frac{1}{2}(I + \varepsilon) + (t - x)\frac{1}{2}(I - \varepsilon) = (t + x)\alpha + (t - x)\bar{\alpha} \equiv x^+ \alpha + x^- \bar{\alpha} \quad (4.11)$$

In terms of the new basis elements, the multiplication of hyperbolic complex numbers is particularly simple:

$$(a\alpha + b\bar{\alpha})(c\alpha + d\bar{\alpha}) = (ac)\alpha^2 + (ad)\alpha\bar{\alpha} + (bc)\bar{\alpha}\alpha + (bd)\bar{\alpha}^2 = (ac)\alpha + (bd)\bar{\alpha} \quad (4.12)$$

Because of the properties (4.9), there are no cross terms in the product. It is then simple to see that the formula for the inverse of a hyperbolic complex number in terms of the null basis elements is

$$(a\alpha + b\bar{\alpha})^{-1} = a^{-1}\alpha + b^{-1}\bar{\alpha} \quad (4.13)$$

so that a hyperbolic complex number possesses an inverse if it is not in one of the null directions.

Let us consider a hyperbolic complex function of a real variable which possesses the Taylor series expansion

$$f(t) = g(t)\alpha + h(t)\bar{\alpha} = \sum_{n=0}^{\infty} [g^{(n)}(0)\alpha + h^{(n)}(0)\bar{\alpha}] \frac{t^n}{n!} \quad (4.14)$$

We extend the domain of f to the hyperbolic complex numbers through its power series:

$$f(a\alpha + b\bar{\alpha}) \equiv \sum_{n=0}^{\infty} [g^{(n)}(0)\alpha + h^{(n)}(0)\bar{\alpha}] \frac{1}{n!} (a\alpha + b\bar{\alpha})^n = \sum_{n=0}^{\infty} \frac{1}{n!} [g^{(n)}(0)a^n\alpha + h^{(n)}(0)b^n\bar{\alpha}]$$

$$= g(a)\alpha + h(b)\bar{\alpha} = f(a)\alpha + f(b)\bar{\alpha} \quad (4.15)$$

This property will be used as the definition by which the argument of an arbitrary function is extended to the hyperbolic complex numbers. It will be called the property of 'functional linearity', and is given explicitly by

$$f(a\alpha + b\bar{\alpha}) \equiv f(a)\alpha + f(b)\bar{\alpha} \quad (4.16)$$

with a and b real or ordinary complex numbers.

A real diagonal 2×2 matrix representation for the imaginary unit ε of the hyperbolic complex numbers is given by

$$\varepsilon \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.17)$$

so that an arbitrary hyperbolic complex number has the matrix representation

$$w = t + \varepsilon x = \begin{pmatrix} t+x & 0 \\ 0 & t-x \end{pmatrix} = \begin{pmatrix} x^+ & 0 \\ 0 & x^- \end{pmatrix} \quad (4.18)$$

The null basis elements α and $\bar{\alpha}$ have the matrix representations

$$\alpha \equiv \frac{1}{2}(1 + \varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{\alpha} \equiv \frac{1}{2}(1 - \varepsilon) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad (4.19)$$

Hyperbolic complex conjugation is accomplished in the 2×2 matrix representation through the following outer automorphism

$$w^\# \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t+x & 0 \\ 0 & t-x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} t-x & 0 \\ 0 & t+x \end{pmatrix} = \begin{pmatrix} x^- & 0 \\ 0 & x^+ \end{pmatrix} \quad (4.20)$$

which interchanges the diagonal entries. We thus see that the pure hyperbolic complex numbers are isomorphic to the ring of real diagonal 2×2 matrices, and that the 'mixed' hyperbolic complex numbers, in which we allow the coefficients of 1 and ε to be ordinary complex numbers, are isomorphic to the ring of complex diagonal 2×2 matrices. In both cases, hyperbolic complex conjugation for the matrix rings is defined by the outer automorphism in (4.20).

The null basis elements α and $\bar{\alpha}$ are easier to work with than the basis 1 and ε and it is worthwhile to introduce a shorthand notation for the hyperbolic complex numbers based upon the diagonal matrix representation given above. We write

$$w = a\alpha + b\bar{\alpha} = \text{diag}(a, b) \equiv (a, b) \quad (4.21)$$

thus introducing an ordered pair notation for the hyperbolic complex numbers.

Multiplication is denoted by juxtaposition, and has the simple form

$$(a, b)(c, d) = (ac, bd) \quad (4.22)$$

'Scalar' multiplication by a real or ordinary complex number is given by

$$c(a, b) = (ca, cb) \quad (4.23)$$

Hyperbolic complex conjugation has the action of switching the left and right entries:

$$(a, b)^\# = (b, a) \quad (4.24)$$

while ordinary complex conjugation merely acts on each component:

$$(a, b)^* = (a^*, b^*) \quad (4.25)$$

The double conjugate in either order is given by

$$(a, b)^{\#*} = (a^*, b^*)^\# = (b^*, a^*) = (b, a)^* = (a, b)^{\#*} \quad (4.26)$$

The extension of the domain of a function to the hyperbolic complex numbers is given by the definition (see Eq. (4.16))

$$f((a, b)) = (f(a), f(b)) \quad (4.27)$$

One example of the above result is that for the inverse of a hyperbolic complex number:

$$(a, b)^{-1} = (a^{-1}, b^{-1}) \quad (4.28)$$

In terms of the new notation, the basis elements are given by

$$\alpha = (1, 0), \quad \bar{\alpha} = (0, 1); \quad 1 = (1, 1), \quad \varepsilon = (1, -1) \quad (4.29)$$

Note that a number with no hyperbolic imaginary part is of the form

$$(a, a) = a(1, 1) = a1 = a \quad (4.30)$$

Other results follow easily from the above, or are readily obtained through the isomorphism between the hyperbolic complex numbers and the 2×2 diagonal matrices.

C. Hyperbolic analyticity and harmonicity

Armed with the above results, we see that if we have a single hyperbolic complex coordinate $w = t + \varepsilon x$, the following hold:

$$\begin{aligned} w = t + \varepsilon x &= (t+x, t-x) \equiv (x^+, x^-), \quad \bar{w} \equiv w^\# = t - \varepsilon x = (t-x, t+x) \equiv (x^-, x^+) \\ \partial &\equiv 2 \frac{\partial}{\partial w} = \partial_t + \varepsilon \partial_x = (\partial_t + \partial_x, \partial_t - \partial_x) = 2(\partial_+, \partial_-), \\ \bar{\partial} &\equiv 2 \frac{\partial}{\partial \bar{w}} = \partial_t - \varepsilon \partial_x = (\partial_t - \partial_x, \partial_t + \partial_x) = 2(\partial_-, \partial_+) \end{aligned} \quad (4.31)$$

If we generalize the concept of holomorphicity (analyticity) to the pure hyperbolic complex numbers, we can define a holomorphic function to be a function which only depends on the hyperbolic coordinate w , and is sufficiently well behaved:

$$\psi \text{ holomorphic} \Rightarrow \psi = \psi(w) \equiv u(t, x) + \varepsilon v(t, x) \quad (4.32)$$

where u and v are real or ordinary complex functions, which are constrained by (4.32). The holomorphicity condition is also given by

$$\bar{\partial} \psi = (\partial_t - \varepsilon \partial_x)(u + \varepsilon v) = [(\partial_t u - \partial_x v) + \varepsilon(\partial_t v - \partial_x u)] = 0 \quad (4.33)$$

Note that the condition that a function can be written as a function of w only is a necessary but not sufficient condition for holomorphicity; for instance note that $\bar{\partial}(w^{-1}) = 2\pi i \delta(t) \delta(x)$ (see Green, Schwarz, and Witten¹²).

Equation (4.33) yields the hyperbolic analogy to the Cauchy-Riemann equations

$$\partial_t u = \partial_x v, \quad \partial_x u = \partial_t v \quad (4.34)$$

As a consequence, both u and v satisfy the following second order differential equation,

$$(\partial_t^2 - \partial_x^2) u = (\partial_t^2 - \partial_x^2) v = 0 \quad (4.35)$$

which is the hyperbolic analogy to harmonicity, which is just that the real and imaginary parts (relative to the hyperbolic imaginary unit) satisfy the one-dimensional wave equation. So we see that a consequence of hyperbolic analyticity is that a function's real and imaginary parts satisfy the wave equation, which is indeed an interesting result. This means that a generalization of the conformal mapping techniques of the ordinary complex numbers could be used to solve the wave equation in $1+1$ dimensions.

D. The Dirac equation and algebra in two space-time dimensions and hyperbolic complex numbers

Now that we have the basic properties of the hyperbolic complex numbers behind us, we turn to another interesting relationship between the hyperbolic complex numbers and special relativistic physics in two space-time dimensions. Consider the $1+1$ dimensional massless Dirac equation

$$i\gamma^\mu \partial_\mu \psi = 0 \quad (4.36)$$

($\mu, \nu, \dots = 0, 1$) in a chiral basis where

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$$\gamma_5 = \gamma^5 \equiv \frac{1}{2} \epsilon_{\mu\nu} \gamma^\mu \gamma^\nu = -\gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.37)$$

(in the above $\epsilon^{01} = \epsilon_{10} = +1$) and the anti-commutation relations and metric convention are given by

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} = 2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (4.38)$$

With the above conventions, the massless Dirac equation becomes explicitly

$$i \begin{pmatrix} 0 & \partial_t + \partial_x \\ \partial_t - \partial_x & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = i \begin{pmatrix} (\partial_t + \partial_x) \psi_R \\ (\partial_t - \partial_x) \psi_L \end{pmatrix} = 0 \quad (4.39)$$

which may be rewritten as

$$i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial_t - \partial_x & 0 \\ 0 & \partial_t + \partial_x \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = i\gamma^0(\partial_t - \gamma_5 \partial_x) \psi = 0 \quad (4.40)$$

Let us define the following hyperbolic complex quantity

$$\Psi \equiv (\psi_L, \psi_R) = \alpha \psi_L + \bar{\alpha} \psi_R = \frac{1}{2}(\psi_L + \psi_R) + \frac{1}{2}\epsilon(\psi_L - \psi_R) \quad (4.41)$$

and make the following identification

$$\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \equiv (\psi_L, \psi_R) \quad (4.42)$$

between the Dirac spinor and the hyperbolic complex spinor. Then we see that the action of γ^0 on the Dirac spinor is equivalent to the action of hyperbolic complex conjugation on the hyperbolic complex spinor:

$$\gamma^0 \psi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} \equiv (\psi_R, \psi_L) = (\psi_L, \psi_R)^\# = \Psi^\# \quad (4.43)$$

and the action of γ_5 is equivalent to multiplication by the hyperbolic imaginary unit ϵ :

$$\gamma_5 \psi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_L \\ -\psi_R \end{pmatrix} \equiv (\psi_L, -\psi_R) = (1, -1)(\psi_L, \psi_R) = \epsilon \Psi \quad (4.44)$$

A quick look at Eq. (4.40) reveals that $\gamma^1 = -\gamma^0 \gamma_5$, so that the identifications made between the two formalisms are given by

$$\gamma^0 \equiv \#, \quad \gamma^1 = -\gamma^0 \gamma_5 \equiv -\# \epsilon = \epsilon \# \equiv \gamma_5 \gamma^0, \quad \gamma_5 \equiv \epsilon \quad (4.45)$$

Note that the Dirac matrices are real in the chiral representation given in $1+1$ dimensions, and that the relationship between them and the chirality operator (Eq. (4.37)) only involves real numbers, so that the almost hyperbolic structure on Dirac spinor space is genuine (as defined in the next section).

It is clear that the hyperbolic complex conjugation operator obeys the following properties

$$\#^2 = 1, \quad \# \epsilon = -\epsilon \#, \quad \# A = A^\# \#, \quad \# A \pm A \# = (A^\# \pm A) \# \quad (4.46)$$

with A being an arbitrary operator. There is an ambiguity in the way that the $\#$ operator acts on quantities, in the same way that $\partial_t f = (\partial_t f)$ or $(\partial_t f) + f \partial_t$ depending on whether the derivative operator is acting on another 'operator' or on a function. In the same sense, the identities in (4.46) hold when the final quantities are considered as operators, but when the final quantity is considered to be a set of hyperbolic complex numbers, the $\#$ operator is defined to act as $\# \Psi = \Psi^\#$ and *not* as $\# \Psi = \Psi^\# \#$.

The properties given in (4.46) imply that the identifications made in (4.45) preserve the Dirac anti-commutation relations:

$$\{\gamma^0, \gamma^0\} \equiv \{\#, \#\} = 2, \quad \{\gamma^1, \gamma^1\} \equiv \{\epsilon \#, \epsilon \#\} = -2, \quad \{\gamma^0, \gamma^1\} \equiv \{\#, \epsilon \#\} = 0, \\ \{\gamma^0, \gamma^5\} \equiv \{\#, \epsilon\} = 0, \quad \{\gamma^1, \gamma^5\} \equiv \{\epsilon \#, \epsilon\} = 0, \quad (\gamma_5)^2 \equiv \epsilon^2 = +1 \quad (4.47)$$

In other words, the algebra of the operators $\{1, \#, \epsilon \#, \epsilon\}$ is isomorphic to the $1+1$ -dimensional Dirac algebra: $\{1, \#, \epsilon \#, \epsilon\} \equiv \{1, \gamma^0, \gamma^1, \gamma^5\}$.

The massless Dirac equation then becomes in terms of hyperbolic complex notation

$$0 = i\gamma^\mu \partial_\mu \psi = i\gamma^0(\partial_t - \gamma_5 \partial_x) \psi \\ \equiv i\#(\partial_t - \epsilon \partial_x) \Psi = i\# \bar{\partial} \Psi = i\partial \Psi^\# = 0 \quad (4.48)$$

or, hyperbolic complex conjugating both sides, and dropping the factor of i , we have finally

$$\bar{\partial} \Psi = 2 \frac{\partial}{\partial w} \Psi = 0 \quad (4.49)$$

This is merely the statement that the field Ψ is holomorphic, that is that it is only a function of the pure hyperbolic complex coordinate w and not its conjugate (and is assumed to be well-behaved). So we see that the massless Dirac equation in one space and one time dimension has this aesthetic interpretation in terms of the hyperbolic complex numbers, which would not be possible without them. Note that in the massive case the Dirac equation in hyperbolic form is

$$i\bar{\partial} \Psi = m \Psi^\# \quad (4.50)$$

One last remark concerns the bosonization relation between a Dirac spinor and a scalar field

in $I+1$ dimensions:

$$j^\mu = \bar{\psi} \gamma^\mu \psi = \frac{1}{\sqrt{\pi}} \varepsilon^{\mu\nu} \partial_\nu \phi \quad (4.51)$$

It is not difficult to show that this takes the very simple form

$$\Psi^* \Psi = \frac{1}{\sqrt{4\pi}} \varepsilon \partial \phi \quad (4.52)$$

in terms of hyperbolic complex quantities. For more details on bosonization in $I+1$ dimensions, refer to Fradkin¹³.

V. TWO-DIMENSIONAL SPACE-TIME MANIFOLDS, THEIR INHERENT HYPERBOLIC COMPLEX STRUCTURE, AND STRING THEORY

Now let us consider an orientable two-dimensional manifold possessing a non-degenerate metric. We will show that the manifold has a natural almost complex structure that is integrable if the metric has Euclidean signature, and possesses a natural almost hyperbolic complex structure which is integrable if the metric has Minkowski signature. This means in the first case that the manifold is a complex manifold, and that the introduction of a complex coordinate ('z') is justified. In the second case, this means that the manifold is a pure hyperbolic complex manifold, and the introduction of a pure hyperbolic complex coordinate ('w') is justified.

Before going further, we should define exactly what is meant by an 'almost hyperbolic complex structure'. An almost hyperbolic complex structure is defined to be a special case of another type of structure which has already been studied by mathematicians, called an almost product structure (refer to Yano¹⁴ for a more complete discussion and other references, and also see Flaherty¹⁵ for a brief outline). An almost product structure K^λ_μ is defined to be a *real* rank (I, I) tensor field on a real N -dimensional manifold or vector space $(\lambda, \mu, \dots = 1, \dots, N)$ which satisfies $K^\lambda_\rho K^\rho_\mu = +\delta^\lambda_\mu$, and K has p positive eigenvalues and $q = N - p$ negative eigenvalues. Note that N need *not* be even for an almost product structure. K is said to be integrable (and then is called a *locally* product structure) if there exists a coordinate system such that

$$K^\lambda_\mu = \text{diag}(\underbrace{+1, \dots, +1}_p, \underbrace{-1, \dots, -1}_{q=N-p}) \quad (5.1)$$

This is possible iff its Nijenhuis tensor,

$$N^\lambda_{\mu\nu} \equiv K^\lambda_\rho (K^\rho_{\mu,\nu} - K^\rho_{\nu,\mu}) + K^\rho_\mu K^\lambda_{\nu,\rho} - K^\rho_\nu K^\lambda_{\mu,\rho} \quad (5.2)$$

vanishes (this is proved in Yano¹⁶; see Flaherty¹⁷ also).

An almost *hyperbolic* complex manifold (space) is defined to be an even dimensional $(N = 2n)$ almost product manifold (space) with $p = q = N/2 = n$. The almost product structure is then called an almost *hyperbolic* complex structure^{18, 19}. By the results for almost product manifolds, an almost hyperbolic complex structure is integrable, i.e. takes the form

$$K^\lambda_\mu = \text{diag}(\underbrace{+1, \dots, +1}_n, \underbrace{-1, \dots, -1}_n) = \sigma_3 \otimes I_{n \times n} \quad (5.3)$$

in certain coordinate frames, iff its Nijenhuis tensor vanishes.

With the above discussion complete, we now return to our discussion of orientable two-dimensional manifolds with non-degenerate metric. Orientability of the two-manifold and non-degeneracy of the metric allows the following traceless rank (I, I) tensor field to be defined *everywhere* on the manifold:

$$K^\lambda_\rho \equiv \sqrt{|g|} g^{\lambda\sigma} \varepsilon_{\sigma\rho}, \quad \varepsilon_{\sigma\rho} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (5.4)$$

where small Greek letters run from 1 to 2 or from 0 to 1, depending on the signature of the metric, and $g \equiv \det(g_{\mu\nu})$. A simple calculation shows that

$$K^\lambda_\sigma K^\sigma_\rho = -\frac{g}{|g|} \delta^\lambda_\rho = -\text{sgn}(g) \delta^\lambda_\rho \quad (5.5)$$

If the metric has Euclidean signature then $\text{sgn}(g) = +1$ and K defines an almost complex structure on the manifold. If the metric has Minkowski signature, then $\text{sgn}(g) = -1$ and K defines an almost hyperbolic complex structure on the manifold. Because the Nijenhuis tensor

$$N^\lambda_{\mu\nu} \equiv K^\lambda_\rho (K^\rho_{\mu,\nu} - K^\rho_{\nu,\mu}) + K^\rho_\mu K^\lambda_{\nu,\rho} - K^\rho_\nu K^\lambda_{\mu,\rho} \quad (5.6)$$

vanishes identically in two dimensions for an almost complex or almost hyperbolic complex structure (all that is required for the vanishing of the Nijenhuis tensor in two dimensions is that $\text{tr} K = 0$ and $K^2 = \text{const} \times I$), every almost complex or almost hyperbolic complex structure in two dimensions is integrable²⁰. The integrability of K allows us to introduce a complex or pure hyperbolic complex coordinate on the manifold. This means that an orientable two-dimensional manifold with non-degenerate metric is either a complex manifold or a pure hyperbolic complex manifold, depending on the signature of the metric.

It also turns out that the metric is 'Hermitian' with respect to the structure K . The condition of 'Hermiticity' is $g(KX, KY) = \text{sgn}(g) g(X, Y)$ (I am not using the ordered pair notation for hyperbolic complex numbers here!). Notice in the hyperbolic complex case that there is a minus sign; this is basically due to the fact that the hyperbolic imaginary element has the property $\varepsilon^\# \varepsilon = -1$, whereas the ordinary imaginary unit satisfies $i^* i = +1$. In components, the Hermiticity condition is just

$$g_{\lambda\rho} K^\lambda_\mu K^\rho_\nu = \text{sgn}(g) g_{\mu\nu} \quad (5.7)$$

which can readily be shown to hold for the K defined in Eq. (5.5). If a metric is 'Hermitian' with respect to a given integrable almost complex (or hyperbolic complex) structure, then the diagonal components of the metric vanish in the complex coordinate system which is introduced through integration of the structure K .

In string theory, the world sheet is treated as a one-dimensional complex manifold after Euclidean continuation. However, this is not completely satisfactory since the metric on the worldsheet has Minkowski signature. The above discussion shows that the string worldsheet is a pure hyperbolic complex manifold, assuming that it is orientable with non-degenerate metric. It would thus seem that hyperbolic complex numbers would be useful in the description of world sheet dynamics. Indeed, the string wave equation for a flat background metric is just $\partial \bar{\partial} X = 0$, and the constraints are equivalent to the requirement that the induced metric be 'Hermitian' with respect to the hyperbolic complex structure. There is not space enough here to rewrite all of string theory using hyperbolic complex numbers, but to do so would be straightforward.

In this section it has been shown that an orientable manifold of one space and one time dimension with a non-degenerate metric of Minkowski signature possesses a natural almost

hyperbolic complex structure which is *integrable*. The application to string theory is immediate; since the string world sheet in some theories satisfies these criteria, we can then state positively that in those cases it is a *hyperbolic* complex manifold, and the use of a hyperbolic complex coordinate is justified.

VI. DIRAC SPINORS AND CPT IN FOUR SPACE-TIME DIMENSIONS

In this section, hyperbolic complex numbers are used to simplify the representation of Dirac spinors. It is shown that a four-component complex Dirac spinor may be represented by a two-component hyperbolic complex spinor, that the Lorentz group is homomorphic to a generalized $SU(2)$ and that the Dirac adjoint corresponds to a generalized Hermitian adjoint. A complete formalism for working with two component hyperbolic complex spinors is given. The operations of charge conjugation, parity, and time reversal are shown to be closely related to the three types of complex conjugation which exist for hyperbolic complex numbers (in the mixed case). Throughout this section, our two-component spinor conventions are the same as those given in S. P. Misra's text on supersymmetry and supergravity²¹.

A. Hyperbolic complex numbers, the Dirac algebra and the Dirac equation in 3+1 dimensions

In this section, we will work in a chiral basis, where the components of a Dirac spinor are given by

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} L\alpha \\ \bar{R}\dot{\alpha} \end{pmatrix} \quad (6.1)$$

in terms of its left- and right-handed components. In this basis, we will define the Dirac gamma matrices to be

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot{\beta}} \\ \bar{\sigma}^\mu^{\dot{\alpha}\beta} & 0 \end{pmatrix} \quad \text{or} \quad \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & +\sigma^i \\ -\sigma^i & 0 \end{pmatrix} \quad (6.2)$$

where σ^i denote the usual Pauli matrices, I the 2×2 unit matrix, and we also define $\sigma^0 = \bar{\sigma}^0 = I$, $\bar{\sigma}^i = -\sigma^i$. The anti-commutation relations and metric convention are given by

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} = 2 \text{diag}(+1, -1, -1, -1) \quad (6.3)$$

As usual, we define

$$\gamma_5 = \gamma^5 \equiv \frac{i}{4!} \epsilon_{\lambda\rho\mu\nu} \gamma^\lambda \gamma^\rho \gamma^\mu \gamma^\nu = -i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} +\delta_\alpha^\beta & 0 \\ 0 & -\delta_{\dot{\alpha}}^{\dot{\beta}} \end{pmatrix} = \begin{pmatrix} +I & 0 \\ 0 & -I \end{pmatrix} \quad (6.4)$$

where $\epsilon^{0123} = -\epsilon_{0123} \equiv +1$. In analogy to the $1+1$ -dimensional case, we make the following identifications:

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \equiv \# = I\# = \sigma^0\#, \quad \gamma_5 = \begin{pmatrix} +I & 0 \\ 0 & -I \end{pmatrix} \equiv \epsilon$$

$$\gamma^i = \begin{pmatrix} 0 & +\sigma^i \\ -\sigma^i & 0 \end{pmatrix} = -\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} +I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \equiv -\# \epsilon \sigma^i = \epsilon \sigma^i \# \quad (6.5)$$

Before proceeding further, it is important to settle a technical point relating to the genuineness of the almost hyperbolic complex structure provided by γ_5 . Let us define $J = i\gamma_5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$. In a Majorana basis, the Dirac matrices are all real, and J is a real matrix satisfying $J^2 = -1$. A Majorana spinor is also real in this basis, and J is seen to be a genuine almost complex structure on Majorana spinor space. Dirac spinor space is the complexification of Majorana spinor space, and in general a Dirac spinor is complex in a Majorana basis. It is then necessary to have two commuting imaginary units, i from the complexification of the Majorana spinor space, and j arising from the almost complex structure J , with $i^2 = j^2 = -1$. For Majorana spinors, we may set $i = j$, but for Dirac spinors, we may not (Majorana spinor space is precisely that subspace of Dirac spinor space for which we can set $i = j$). We will then have numbers of the form $a + ib + jc + ijd$, with a, b, c , and d real. We may then define a hyperbolic imaginary unit $\epsilon = -ij$, so that we have an equivalent number system with numbers of the form $a + ib + i\epsilon c - \epsilon d = a + ib + jc + ijd$. This is what was done in (6.4).

Technicalities dispensed with, let us now define the following 2×2 hyperbolic complex matrices

$$\beta^0 = I, \quad \beta^i = \epsilon \sigma^i; \quad (\beta^\mu)^\dagger = \beta^\mu \quad (6.6)$$

with the further definitions

$$\bar{\beta}^\mu \equiv (\beta^\mu)^\dagger \# = (\beta^\mu)^\#; \quad \bar{\beta}^0 = I, \quad \bar{\beta}^i = -\epsilon \sigma^i \quad (6.7)$$

we have the following identities

$$\beta^\mu \bar{\beta}^\nu + \beta^\nu \bar{\beta}^\mu = 2\eta^{\mu\nu}, \quad \bar{\beta}^\mu \beta^\nu + \bar{\beta}^\nu \beta^\mu = 2\eta^{\mu\nu} \quad (6.8)$$

Comparing to (6.5) above, we see that we may define the following 2×2 matrix operator

$$\gamma^\mu = \beta^\mu \# \quad (6.9)$$

which is 'antilinear'²² in a sense because it involves hyperbolic complex conjugation:

$$\gamma^\mu (a\psi) = \beta^\mu \# (a\psi) = a^\# (\beta^\mu \# \psi) = a^\# (\gamma^\mu \psi)$$

Recall from Section IV that the hyperbolic complex conjugate operator has the properties

$$\#^2 = I, \quad \# \epsilon = -\epsilon \#, \quad \# A = A^\# \#, \quad \# A \pm A \# = (A^\# \pm A) \# \quad (6.10)$$

with A being an arbitrary operator. Given the above, we see that

$$\begin{aligned} \gamma^5 &\equiv -i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \equiv -i (\beta^0 \#) (\beta^1 \#) (\beta^2 \#) (\beta^3 \#) = -i (\beta^0 \bar{\beta}^1 \#^2) (\beta^2 \bar{\beta}^3 \#^2) \\ &= -i (\beta^0 \bar{\beta}^1 \beta^2 \bar{\beta}^3) = -i (I) (-\epsilon \sigma^1) (\epsilon \sigma^2) (-\epsilon \sigma^3) = -i \epsilon^3 (\sigma^1 \sigma^2 \sigma^3) \\ &= -i \epsilon (iI) = \epsilon I = \epsilon \end{aligned} \quad (6.11)$$

which is an elegant and simple result, and validates the identifications in (6.5).

While it is true that the anti-commutation relations of the Dirac algebra cannot be obeyed by 2×2 matrices, they *are* obeyed by the 2×2 matrix operators given in Eq. (6.9):

$$\begin{aligned} \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu &= (\beta^\mu \#) (\beta^\nu \#) + (\beta^\nu \#) (\beta^\mu \#) = \beta^\mu \bar{\beta}^\nu \#^2 + \beta^\nu \bar{\beta}^\mu \#^2 \\ &= \beta^\mu \bar{\beta}^\nu + \beta^\nu \bar{\beta}^\mu = 2\eta^{\mu\nu} \end{aligned} \quad (6.12)$$

and we also have

$$\{\gamma^\mu, \gamma^5\} = \gamma^\mu \gamma^5 + \gamma^5 \gamma^\mu = (\beta^\mu \#) \epsilon + \epsilon (\beta^\mu \#) = -\epsilon \beta^\mu \# + \epsilon \beta^\mu \# = 0 \quad (6.13)$$

If we define the following differential operator,

$$\partial \equiv \gamma^\mu \partial_\mu = (\beta^\mu \#) \partial_\mu = \beta^\mu \partial_\mu \# \quad (6.14)$$

and the following two component hyperbolic complex spinor,

$$\Psi \equiv (\psi_L, \psi_R) = \alpha \psi_L + \bar{\alpha} \psi_R = \frac{1}{2}(\psi_L + \psi_R) + \frac{1}{2}\varepsilon(\psi_L - \psi_R) \quad (6.15)$$

(note that the real and imaginary parts of Ψ relative to ε are proportional to the components of the Dirac spinor in the standard representation²³) then the Dirac equation may simply be written as

$$i \partial \Psi = i \gamma^\mu \partial_\mu \Psi = i(\beta^\mu \#) \partial_\mu \Psi = i \beta^\mu \partial_\mu \Psi^\# = m \Psi \quad (6.16)$$

Taking the hyperbolic complex conjugate of both sides of the above, we have the equivalent result

$$i \bar{\partial} \bar{\Psi} = m \bar{\Psi}^\# \quad (6.17)$$

If we make the following notational definitions,

$$\partial \equiv \beta^\mu \partial_\mu = \partial_t + \varepsilon \sigma^i \partial_i = (\partial_t + \sigma^i \partial_i, \partial_t - \sigma^i \partial_i) \quad (6.18a)$$

$$\bar{\partial} \equiv \bar{\beta}^\mu \partial_\mu = \partial_t - \varepsilon \sigma^i \partial_i = (\partial_t - \sigma^i \partial_i, \partial_t + \sigma^i \partial_i) \quad (6.18b)$$

we have the identity

$$\bar{\partial} \partial = \bar{\beta}^\mu \beta^\nu \partial_\mu \partial_\nu = \frac{1}{2}(\bar{\beta}^\mu \beta^\nu + \bar{\beta}^\nu \beta^\mu) \partial_\mu \partial_\nu = \eta^{\mu\nu} \partial_\mu \partial_\nu \quad (6.19)$$

which follows from (6.8). The Dirac equation may then be written as

$$i \bar{\partial} \Psi = m \Psi^\# \quad (6.20)$$

In the massless case, we have $\bar{\partial} \Psi = 0$, which resembles the holomorphicity condition in the $1+1$ -dimensional case (see Section IV of this paper).

B. The group $SL(2, c)$, its generalized unitary equivalent, and their spinor representations

In this section, it will be shown that the group $SL(2, c)$ is isomorphic to a generalized $SU(2)$ group, and that the spinor representations of the group $SL(2, c)$ may be written as generalized unitary representations of the generalized $SU(2)$ group.

If we allow hyperbolic complex entries (here I mean the mixed case, i.e., numbers of the form $a+ib+\varepsilon c+i\varepsilon d$ with a, b, c, d real), then it turns out that the group of 2×2 matrices which satisfies the following two conditions

$$U^\dagger T^{**} U = U U^\dagger T^{**} = I, \quad \det U = I \quad (6.21)$$

(T denotes the usual matrix transpose) has as its most general element

$$U = \exp\left(\frac{i}{2}\bar{\sigma} \cdot (\bar{\omega} - i\varepsilon \bar{\nu})\right) \quad (\bar{\omega} \text{ and } \bar{\nu} \text{ real}) \quad (6.22)$$

and will be defined to be the group $SU(2, hc)$, the group of generalized special unitary 2×2 matrices with mixed complex hyperbolic complex entries.

The combined operations of complex conjugation, hyperbolic complex conjugation, and transposition of a matrix are combined into a new symbol, the 'double dagger',

$$\ddagger \equiv \dagger \# = T^{**} \quad (6.23)$$

The 'dagger' symbol ($\dagger$) is reserved for the usual Hermitian adjoint, $\dagger = T^* = *T$. When necessary, we will use an 'H' superscript to denote the hyperbolic adjoint, $H \equiv T\# = \#T$. Since the three operations ($*$, $\#$, and T) all commute, there is no ambiguity in the order in which they are performed, so that the $\ddagger$ operation is well defined. $\ddagger$ will be referred to as the generalized Hermitian adjoint, or the 'Dirac' adjoint, for reasons which will be given shortly. Eq. (6.21) may then be rewritten as

$$U^\ddagger U = U U^\ddagger = I, \quad \det U = I \quad (6.24)$$

Using the ordered pair notation introduced in Section IV of this paper, and the property of functional linearity (see Eqns. (4.16) and (4.27) above), we see that (6.22) may be written as

$$U = \exp\left(\frac{i}{2}\bar{\sigma} \cdot (\bar{\omega} - i\varepsilon \bar{\nu})\right) = \exp\left(\frac{i}{2}\bar{\sigma} \cdot (\bar{\omega} - i\bar{\nu})\right) \exp\left(\frac{i}{2}\bar{\sigma} \cdot (\bar{\omega} + i\bar{\nu})\right) \\ = (\exp\left(\frac{i}{2}\bar{\sigma} \cdot (\bar{\omega} - i\bar{\nu})\right), \exp\left(\frac{i}{2}\bar{\sigma} \cdot (\bar{\omega} + i\bar{\nu})\right)) \equiv (\Lambda_L, \Lambda_R) \quad (6.25)$$

where Λ_L and Λ_R are the usual transformation matrices for left- and right-handed two-component Weyl spinors, respectively:

$$\Lambda_L, \Lambda_R \in SL(2, c); \quad \Lambda_L^{-1} = \Lambda_R^\dagger, \quad \Lambda_R^{-1} = \Lambda_L^\dagger \quad (6.26)$$

(For more details, see Ramond²⁴ or Ryder²⁵). Using the properties of hyperbolic complex numbers developed in Section IV and Eq. (6.26), we see that the generalized Hermitian adjoint of U in Eq. (6.22) is given by

$$U^\ddagger = (\Lambda_L, \Lambda_R)^\ddagger = (\Lambda_L^\dagger, \Lambda_R^\dagger)^\# = (\Lambda_R^\dagger, \Lambda_L^\dagger) = (\Lambda_L^{-1}, \Lambda_R^{-1}) = U^{-1} \quad (6.27)$$

so that U is indeed unitary in the generalized sense, and satisfies both conditions in (6.24) due to the following property of the determinant of a hyperbolic complex matrix

$$\det U = \det (\Lambda_L, \Lambda_R) = (\det \Lambda_L, \det \Lambda_R) = (I, I) = I \quad (6.28)$$

Indeed if a general square matrix $M = (A, B)$ with hyperbolic complex entries (A and B are ordinary complex matrices) is to be unitary (in the generalized sense), it must satisfy the following condition

$$I = (I, I) = M^\ddagger M = (A, B)^\ddagger (A, B) = (B^\dagger, A^\dagger)(A, B) = (B^\dagger A, A^\dagger B) \quad (6.29)$$

so that generalized unitarity requires that

$$A^{-1} = B^\dagger, \quad B^{-1} = A^\dagger \quad (6.30)$$

Any Lie group is equivalent to a generalized unitary group through the isomorphism $A \equiv U = (A, (A^\dagger)^{-1})$ where A is any element of the Lie group. Any hyperbolic complex matrix group is a subgroup of $GL(N, c) \times GL(N, c)$ for some N , since an arbitrary square invertible hyperbolic complex matrix can be written as $M = (A, B)$, with A and B complex invertible $N \times N$ matrices (i.e., elements of $GL(N, c)$).

The matrix in Eq. (6.25) is the transformation matrix for a two-component spinor representation of the group $SU(2, hc)$:

$$\Psi \equiv (\psi_L, \psi_R), \quad \Psi' \equiv (\psi'_L, \psi'_R) = U \Psi = (\Lambda_L, \Lambda_R)(\psi_L, \psi_R) = (\Lambda_L \psi_L, \Lambda_R \psi_R) \quad (6.31)$$

So we thus see that a two-component hyperbolic complex spinor contains both left- and right-handed Weyl spinors in the same representation. However, the spinor representation in (6.31), which we will refer to as a ' II ', is reducible under $SU(2, hc)$, since the left- and right-handed parts do not mix under $SU(2, hc)$ transformations, as can be seen from (6.31). Recall that a Dirac spinor is also reducible into left- and right-handed parts unless the Lorentz group is extended by parity.

We can have pure left-handed spinors $\Psi = (\psi_L, 0)$ which satisfy the following equivalent set of conditions

$$\alpha \Psi = (I, 0)(\psi_L, 0) = (\psi_L, 0) = \Psi \\ \varepsilon \Psi = (I, -I)(\psi_L, 0) = (\psi_L, 0) = \Psi \\ \bar{\alpha} \Psi = (0, I)(\psi_L, 0) = 0 \quad (6.32)$$

as well as pure right-handed spinors $\Psi = (0, \psi_R)$ which satisfy

$$\bar{\alpha} \Psi = (0, I)(0, \psi_R) = (0, \psi_R) = \Psi \\ \varepsilon \Psi = (I, -I)(0, \psi_R) = (0, -\psi_R) = -\Psi$$

$$\alpha \Psi = (1, 0) (0, \Psi_R) = 0 \quad (6.33)$$

These conditions, (6.32) and (6.33), are preserved by $SU(2, hc)$ transformations.

The above results are clarified if we recall the identification between the spinor representation of $SU(2, hc)$ and the Dirac bi-spinor in the Weyl representation

$$(\Psi_L, \Psi_R) \equiv \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} \quad (6.34)$$

and note that we have the following identifications

$$\alpha \equiv \frac{1}{2}(1 + \gamma_5), \quad \bar{\alpha} \equiv \frac{1}{2}(1 - \gamma_5), \quad \varepsilon \equiv \gamma_5 \quad (6.35)$$

The null basis elements $(\alpha, \bar{\alpha})$ thus play the role of chirality projection operators. This is another example of the power of hyperbolic complex numbers to simplify the algebra of Dirac spinors.

There is another independent two-component spinor rep of $SU(2, hc)$, unlike the case for $SU(2)$ in which the 2 and $\bar{2}$ reps are equivalent because of the ability to raise and lower indices with the $SU(2)$ -invariant Levi-Civita symbol. For $SU(2, hc)$, it turns out that U^* is independent of U and also generalized unitary, since $(U^*)^\dagger (U^*) = (U^\dagger U)^* = 1$, so that there is an independent two-component spinor rep of $SU(2, hc)$, which we will refer to as the II^* :

$$\Psi^* \equiv (\Psi_L^*, \Psi_R^*)$$

$$\Psi^{**} \equiv (\Psi_L^{**}, \Psi_R^{**}) = U^* \Psi^* = (\Lambda_L^*, \Lambda_R^*) (\Psi_L^*, \Psi_R^*) = (\Lambda_L^* \Psi_L^*, \Lambda_R^* \Psi_R^*) \quad (6.36)$$

However, the two dimensional Levi-Civita symbol is also preserved for the group $SU(2, hc)$, and makes the following pairs of reps equivalent

$$\begin{aligned} II: \Psi' &= U \Psi \quad \equiv \quad II^{**}: \Psi'^* = \Psi^{**T} U^* = \Psi^{**T} U^{*T} = \Psi^{\dagger} U^{\dagger} \\ II^*: \Psi'^* &= U^* \Psi^* \quad \equiv \quad II^*: \Psi'^H = \Psi^{*T} U^{\dagger} = \Psi^{*T} U^{\dagger T} = \Psi^H U^H \end{aligned} \quad (6.37)$$

The above equivalences will be made more apparent after what follows, but first we must make some conventions.

In this section, regular Weyl two-component $SL(2, c)$ spinor indices will be denoted by the Greek letters $\alpha, \beta, \chi, \delta, \varepsilon, \phi, \gamma, \eta, \iota, \varphi, \theta, \nu$ and run from 1 to 2; space-time indices run from 0 to 3 and will be denoted by the Greek letters $\kappa, \lambda, \mu, \nu, \pi, \rho, \sigma, \tau, \omega, \xi, \psi, \zeta$; and $SU(2, hc)$ spinor indices will be denoted by the Latin indices $i, j, k, l, m, n, o, p, q, r, s, t$ and run from 1 to 2. In what follows, the spinor conventions are the same as found in S. P. Misra's text²¹.

With the above conventions the four different types of $SU(2, hc)$ spinors are given below, in equivalent pairs of representations (see (6.37) above), with the $SL(2, c)$ content indicated:

$$\begin{aligned} II: \Psi &= (L_\alpha, \bar{R}^\alpha) \quad \equiv \quad II^{**}: \Psi^i = (R^\alpha, \bar{L}_\alpha) \\ II^*: \Psi^* &= (\bar{L}_\alpha, R^\alpha) \quad \equiv \quad II^*: \Psi^{i*} = (\bar{R}^\alpha, L_\alpha) \end{aligned} \quad (6.38)$$

where we will be using starred and un-starred upper and lower Latin indices for the representations of $SU(2, hc)$.

Raising and lowering of indices will be accomplished by the second index of the following $SU(2, hc)$ -invariant symbols:

$$\varepsilon^{ik} = (\varepsilon^{\alpha\beta}, \varepsilon_{\dot{\alpha}\dot{\beta}}) = (i\sigma_2, -i\sigma_2) = i\sigma_2(1, -1) = i\varepsilon\sigma_2 = \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix}$$

$$\begin{aligned} \varepsilon^{i*k*} &= (\varepsilon^{ik})^* = (\varepsilon^{\dot{\alpha}\dot{\beta}}, \varepsilon_{\alpha\beta}) = (i\sigma_2, -i\sigma_2) = i\varepsilon\sigma_2 = \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix} \\ \varepsilon_{ik} &= (\varepsilon^{ik})^{**} = (\varepsilon_{\alpha\beta}, \varepsilon^{\dot{\alpha}\dot{\beta}}) = (-i\sigma_2, i\sigma_2) = -i\varepsilon\sigma_2 = \begin{pmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{pmatrix} \\ \varepsilon_{i*k*} &= (\varepsilon^{ik})^\# = (\varepsilon_{\alpha\dot{\beta}}, \varepsilon^{\alpha\dot{\beta}}) = (-i\sigma_2, i\sigma_2) = -i\varepsilon\sigma_2 = \begin{pmatrix} 0 & -\varepsilon \\ \varepsilon & 0 \end{pmatrix} \\ \varepsilon^{I2} &= \varepsilon^{I*2*} = \varepsilon_{2I} = \varepsilon_{2*I*} = \varepsilon \end{aligned} \quad (6.39)$$

Observe that no special ansatz is needed for the ε 's of $SU(2, hc)$; all cases follow from appropriate conjugation of ε^{ik} . This is not the case for $SL(2, c)$, where different conventions are sometimes used for the dotted and un-dotted Levi-Civita symbols, resulting in confusing raising and lowering rules. Note that ordinary complex conjugation turns an index from starred to un-starred, and vice-versa. Hyperbolic complex conjugation turns an upper (lower) starred index into a lower (upper) un-starred index, and vice-versa. Complex conjugation together with hyperbolic complex conjugation, or 'double' conjugation, turns an upper index into a lower index of the same type, and vice-versa, and we will call this 'Dirac' conjugation for reasons given below.

Note that the entire development of spinor reps of $SU(2, hc)$ mirrors what happens when $SL(2, c)$ spinor reps are discussed, with the existence of upper and lower starred and un-starred indices paralleling that of the upper and lower dotted and un-dotted indices of $SL(2, c)$. This is because the groups $SL(2, c)$ and $SU(2, hc)$ are isomorphic. What we have done, in essence, is to find a generalization of the concept of unitarity so that the covering group for the Lorentz group, $SL(2, c)$, is equivalent to a generalized unitary group, $SU(2, hc)$, which possesses finite dimensional generalized unitary representations.

C. C, P, T and the Dirac adjoint

In this section the operations of charge conjugation, parity, and time reversal on $SU(2, hc)$ spinors will be discussed. It will be shown that C, P , and T correspond to simple operations on the $SU(2, hc)$ spinors discussed above.

First, let us make a new convention, for convenience and to make contact with the usual treatment of Dirac spinors. We will use an overbar to denote either: the Dirac adjoint, if we are working in matrix notation; the double conjugate, if we are working in components; the Hermitian adjoint, if we are working in matrix notation and the quantity has vanishing hyperbolic imaginary part; and finally, the complex conjugate if we are working in components and the quantity has vanishing hyperbolic imaginary part.

Consider an $SU(2, hc)$ spinor in the II representation

$$\Psi_i = (L_\alpha, \bar{R}^\alpha) \quad (6.40)$$

The operation of charge conjugation produces a spinor in the same representation by taking the generalized Hermitian adjoint (the 'Dirac' adjoint), which turns a lower into an upper index, and then lowering the index:

$$(C\Psi)_i \equiv \bar{\Psi}_i = \varepsilon_{ik} \bar{\Psi}^k = \varepsilon_{ik} (\Psi_k)^{**} = (\varepsilon_{\alpha\beta}, \varepsilon^{\dot{\alpha}\dot{\beta}}) (L_\beta, \bar{R}^{\dot{\beta}})^{**}$$

$$= (\varepsilon_{\alpha\beta}, \varepsilon^{\alpha\beta})(R^\beta, \bar{L}_\beta) = (\varepsilon_{\alpha\beta} R^\beta, \varepsilon^{\alpha\beta} \bar{L}_\beta) = (R_\alpha, \bar{L}^\alpha) \quad (6.41)$$

To charge conjugate other representations of $SU(2, hc)$, one simply takes the Dirac (double) conjugate, and then raises lower indices and lowers upper indices.

The Majorana condition is simply

$$\Psi_i \equiv \bar{\Psi}_i \quad \text{or} \quad (L_\alpha, \bar{R}^\alpha) = (R_\alpha, \bar{L}^\alpha) \Rightarrow L_\alpha = (\bar{R}^\alpha)^* = R_\alpha \quad (6.42)$$

Notice from the above that the ‘Dirac’ adjoint does indeed correspond to the usual Dirac adjoint:

$$\begin{aligned} \Psi^\dagger &\equiv (L_\alpha, \bar{R}^\alpha) \equiv \Psi = \begin{pmatrix} L_\alpha \\ \bar{R}^\alpha \end{pmatrix} \\ \bar{\Psi}^\dagger &\equiv (L_\alpha, \bar{R}^\alpha)^* = (R^\alpha, \bar{L}_\alpha) \equiv \bar{\Psi} = \Psi^\dagger \gamma^0 = (R^\alpha, \bar{L}_\alpha) \end{aligned} \quad (6.43)$$

The need for ‘barring’ a Dirac spinor corresponds to the necessity of taking the generalized Hermitian (Dirac) adjoint in the $SU(2, hc)$ picture. In most texts which discuss the Dirac equation, it is usually noted with some sense of surprise that ‘†’ is not the appropriate adjoint to use to form invariants, and it seems somewhat unaesthetic that the matrix γ^0 must appear in the Dirac adjoint. In the present formulation, it is obvious that the usual Hermitian adjoint is not the proper operation, and that the Dirac adjoint is. For example, we see that under an $SU(2, hc)$ (Lorentz) transformation

$$\begin{aligned} (\bar{\Psi} \Psi)' &= \bar{\Psi}' \Psi' = (\Psi')^\dagger \Psi' = (U \Psi)^\dagger U \Psi \\ &= \Psi^\dagger (U^\dagger U) \Psi = \Psi^\dagger I \Psi = \Psi^\dagger \Psi = \bar{\Psi} \Psi \end{aligned} \quad (6.44)$$

whereas

$$(\Psi^\dagger \Psi)' = \Psi'^\dagger \Psi' = (U \Psi)^\dagger U \Psi = \Psi^\dagger (U^\dagger U) \Psi \neq \Psi^\dagger \Psi \quad (\text{in general}) \quad (6.45)$$

since $U^\dagger U \neq I$ for a general element of $SU(2, hc)$. The use of hyperbolic complex numbers has made the need for barring in the usual treatment of Dirac spinors more transparent, and turned the unaesthetic into the aesthetic.

Let us digress momentarily to demonstrate that the chiral $U(1)$ symmetry is not a symmetry that is generalized unitary. We may represent a chiral $U(1)$ transformation as

$$\Psi' = (L', \bar{R}') = e^{i\varepsilon\theta} \Psi = (e^{i\theta}, e^{-i\theta})(L, \bar{R}) = (e^{i\theta} L, e^{-i\theta} \bar{R}) \quad (6.46)$$

from which we see that

$$\begin{aligned} (\Psi')^\dagger \Psi' &= (e^{i\varepsilon\theta} \Psi)^\dagger (e^{i\varepsilon\theta} \Psi) = \Psi^\dagger ((e^{i\varepsilon\theta})^\dagger e^{i\varepsilon\theta}) \Psi = \Psi^\dagger ((e^{i\varepsilon\theta})^* e^{i\varepsilon\theta}) \Psi \\ &= \Psi^\dagger ((e^{-i\varepsilon\theta})^* e^{i\varepsilon\theta}) \Psi = \Psi^\dagger (e^{i\varepsilon\theta} e^{i\varepsilon\theta}) \Psi = e^{2i\varepsilon\theta} \Psi^\dagger \Psi \neq \Psi^\dagger \Psi \quad (\text{in general}) \end{aligned} \quad (6.47)$$

Also note that

$$\begin{aligned} \bar{\Psi} \Psi &= (R^\alpha, \bar{L}_\alpha)(L_\alpha, \bar{R}^\alpha) = (R^\alpha L_\alpha, \bar{L}_\alpha \bar{R}^\alpha) = \frac{1}{2}(R^\alpha L_\alpha + \bar{L}_\alpha \bar{R}^\alpha) + \frac{1}{2}\varepsilon(R^\alpha L_\alpha - \bar{L}_\alpha \bar{R}^\alpha) \\ &= \frac{1}{2}(\bar{\Psi} \Psi + \varepsilon \bar{\Psi} \gamma_5 \Psi) \end{aligned} \quad (6.48)$$

so that the inner product $\bar{\Psi} \Psi$ contains both the scalar $\bar{\Psi} \Psi$ and pseudo-scalar $\bar{\Psi} \gamma_5 \Psi$ terms. A mass term, $m \bar{\Psi} \Psi + n (\bar{\Psi} \Psi)^\dagger$, is seen to break the chiral symmetry. The chiral symmetry is unitary, but not generalized unitary, and only elements of $U(2, hc) \equiv GL(2, c)$ leave the mass term invariant.

The operation of parity on a Dirac spinor switches left- and right-handed fields, so that parity simply corresponds to hyperbolic complex conjugation for all representations in the $SU(2, hc)$

picture, e.g.:

$$(P\Psi)^i \equiv (\Psi_i)^\# = (L_\alpha, \bar{R}^\alpha)^\# = (\bar{R}^\alpha, L_\alpha) \equiv P\Psi \equiv \gamma^0 \Psi = \begin{pmatrix} \bar{R}^\alpha \\ L_\alpha \end{pmatrix} \quad (6.49)$$

Finally, the operation of time reversal corresponds to taking the complex conjugate, and then raising lower indices and lowering upper indices, e.g.:

$$\begin{aligned} (T\Psi)^i &\equiv \varepsilon^{i*k} \Psi_k^* \equiv \varepsilon^{i*k} (\Psi_k)^* = (\varepsilon^{\alpha\beta}, \varepsilon_{\alpha\beta})(L_\beta, \bar{R}^\beta)^* \\ &= (\varepsilon^{\alpha\beta}, \varepsilon_{\alpha\beta})(\bar{L}_\beta, R^\beta) = (\varepsilon^{\alpha\beta} \bar{L}_\beta, \varepsilon_{\alpha\beta} R^\beta) = (\bar{L}^\alpha, R_\alpha) \end{aligned} \quad (6.50)$$

The combined operations of C, P , and T in any order on any representation of $SU(2, hc)$ leave that quantity invariant.

To summarize, the operations of C, P , and T on any representation are given below:

Charge conjugation: Take the Dirac (double conjugate) and then raise lower indices and lower upper indices. C leaves the quantity in the same representation, e.g., takes a spinor in the II representation to its charge conjugate spinor which is also in the II representation.

Parity: Take the hyperbolic complex conjugate. This takes a spinor in the II representation to a spinor in the II^* representation, and in general changes the representation.

Time reversal: Take the ordinary complex conjugate and then raise lower indices and lower upper indices. This takes a spinor in the II representation to a spinor in the II^* representation, and in general gives you a different representation.

Note that none of the operations of C, P , or T are generalized unitary in the $SU(2, hc)$ picture, and are not elements of the group $SU(2, hc)$ as they each involve some kind of conjugation. Requiring invariance under any combination of C, P , and T other than all three together requires a separate assumption from that of just simple $SU(2, hc)$ symmetry. As it is well known that only the combination CPT is an invariance of nature, and that chiral symmetries are broken by mass terms, one would be led to believe that the only exact spinor symmetry of nature is $SU(2, hc)$.

In this section, we have seen that we may describe the operations of C, P , and T in a simpler fashion with the $SU(2, hc)$ picture. The Majorana condition for a Dirac spinor is simplified, and the Dirac adjoint is shown to be a more natural operation in the hyperbolic complex picture. Dealing with Dirac spinors without the use of hyperbolic complex numbers is very analogous to doing Schroedinger wave mechanics without complex numbers. Just as the use of complex numbers simplifies and clarifies wave mechanics, hyperbolic complex numbers can do so for Dirac spinors. We have seen that the use of hyperbolic complex numbers simplifies the mathematics of Dirac spinors in a very conceptual way. We have only been able here to touch upon the basics of the usage of hyperbolic complex numbers in the Dirac algebra.

VII. SUMMARY AND DISCUSSION

In this paper, we have seen how there are only three types of abelian complex numbers systems where the square of the imaginary unit is affinely related to itself. Each type was shown to be related to a different kind of geometry: the elliptic, or ordinary complex numbers to

Euclidean geometry, parabolic complex numbers to Galilean geometry, and most importantly, the hyperbolic complex numbers to the relativistic geometry of Einstein and Minkowski.

Basic results and new conventions were given for the hyperbolic complex numbers, and it was shown how the Dirac algebra in two space-time dimensions is greatly simplified by the use of hyperbolic complex numbers. It was shown that the massless Dirac equation was equivalent to a hyperbolic holomorphicity condition. It was then shown that a string world sheet of Minkowski signature possesses a natural hyperbolic complex structure, and that hyperbolic complex numbers could be used in string theory when one did not wish to Euclideanize.

Next, it was demonstrated how a four complex-component Dirac spinor in four space-time dimensions could be faithfully represented by a two-component hyperbolic complex spinor, and the Dirac algebra was cast in hyperbolic complex form, with the noteworthy result that γ_5 is merely a representation of the hyperbolic imaginary unit ε . Dealing with Dirac spinors without the use of hyperbolic complex numbers is seen to be akin to doing wave mechanics without the use of complex numbers. Having to use the matrices γ_5 for a chirality operator and γ^0 to take the Dirac adjoint seem to amount to working around the use of hyperbolic complex numbers.

It was then shown that the Lorentz group is homomorphic to a generalized $SU(2)$ group, and the representations of this group were discussed and compared to the $SL(2, \mathbb{C})$ representations. The exposition of charge conjugation, parity and time reversal is shown to be greatly simplified in the hyperbolic complex picture, since each is directly related to one of the three types of conjugation which exist when one has both hyperbolic complex and ordinary complex numbers. A very aesthetic result obtained was that the Dirac adjoint was shown to be equivalent to a generalized Hermitian adjoint in the hyperbolic framework. This clarifies the need for 'barring' in the usual picture.

I hope that I have made a strong case for the use of hyperbolic complex numbers in relativistic physics. Some of the simplifications achieved are certainly remarkable. After exposure to hyperbolic complex numbers, it does seem that physicists have been working around them in their treatment of Dirac spinors. Hyperbolic complex numbers could also be used in string theory to streamline many formulae, instead of using the cumbersome $+$ and $-$ notation when one is working with strings of Minkowski worldsheet signature.

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