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Differential Renormalization and Dimensional Regularization

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Abstract

Original recipes of differential renormalization are supplied with simple prescriptions which enable one to avoid infrared problems in writing down renormalized Feynman amplitudes and to calculate involved coefficients at least at the level of calculability of counterterms within dimensional renormalization. To evaluate these coefficients it is worthwhile to exploit calculational experience based on dimensional regularization.

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1 Introduction

The recently invented differential renormalization [1, 2] which was studied and applied in a series of papers [3] is in fact based on ideas implicitly contained in the earliest works by Bogoliubov and Parasiuk. It rests on the representation of a product of massless propagators in coordinate space²

$$\Pi_\Gamma(x_1, \dots, x_n) = \prod_{i < j} \frac{1}{(x_i - x_j)^{2L_{ij}}} \quad (1)$$

for a graph Γ (here L_{ij} is the number of lines that connect vertices i and j) through derivatives of some order of locally integrable functions. Thus, by this procedure, one can explicitly obtain an extension of the functional $\Pi_\Gamma(x_1, \dots, x_n)$ from the subspace of test functions which vanish in a vicinity of points where the coordinates x_i coincide to the whole space $\mathcal{D}(R^{4n})$.

An essential step involved in the initial version of differential renormalization [1, 2] is the reduction of the problem of writing down the renormalized quantities to the case of diagrams with two vertices. Then one uses the following prescriptions:

$$\frac{1}{x^4} \rightarrow R \frac{1}{x^4} = -\frac{1}{4} \square \frac{\ln \mu^2 x^2}{x^2}, \quad (2)$$

$$\frac{\ln \mu^2 x^2}{x^4} \rightarrow R \frac{\ln \mu^2 x^2}{x^4} = -\frac{1}{8} \square \frac{\ln^2 \mu^2 x^2 + 2 \ln \mu'^2 x^2}{x^2}, \quad (3)$$

$$\frac{1}{x^6} \rightarrow R \frac{1}{x^6} = -\frac{1}{32} \square^2 \frac{\ln \mu^2 x^2}{x^2}, \quad (4)$$

$$\frac{\ln \mu^2 x^2}{x^6} \rightarrow R \frac{\ln \mu^2 x^2}{x^6} = -\frac{1}{64} \square^2 \frac{\ln^2 \mu^2 x^2 + 2 \ln \mu'^2 x^2 + 3 \ln \mu^2 x^2}{x^2}, \quad (5)$$

$$\frac{x_\alpha}{x^6} \rightarrow R \frac{x_\alpha}{x^6} = \frac{1}{16} \partial_\alpha \square \frac{\ln \mu^2 x^2}{x^2}, \quad (6)$$

$$\frac{x_\alpha \ln \mu^2 x^2}{x^6} \rightarrow R \frac{x_\alpha \ln \mu^2 x^2}{x^6} = \frac{1}{32} \partial_\alpha \square \frac{\ln^2 \mu^2 x^2 + 3 \ln \mu'^2 x^2}{x^2} \quad (7)$$

etc., where $x^4 = (x^2)^2$, $x^6 = (x^2)^3$, and μ, μ' are massive parameters which play the role of subtraction points. Here the symbol R denotes the extension of the corresponding functional from the subspace of test functions which vanish near $x = 0$ to the whole space. For the case of diagrams with two vertices, this is just the differential renormalization. Thus, for $n > 2$, the renormalization is defined, by use of (2-7), through renormalization of two-vertex diagrams (with additional logarithms) and thereby it is explicitly realized as an operation of the extension of the initial functional. In accordance with prescriptions of differential renormalization, derivatives involved are understood in the distributional sense, i.e. a derivative $D^\alpha f$ of a

²Euclidean and Minkowski spaces can be treated on the same footing. For simplicity in what follows Feynman amplitudes will be considered in four-dimensional Euclidean space-time. However the Laplacian will be denoted by $\square$. We shall also omit the factor $1/4\pi^2$ in the propagators

distribution f acts on a test function ϕ as $(D^\alpha f, \phi) = (-1)^{|\alpha|}(f, D^\alpha \phi)$, $|\alpha|$ being the order of the derivative.

However the way one gets the case $n = 2$ from the case of a given n was not completely characterized in [2]. When defining this procedure by integration over all coordinates except two chosen ones it is possible to run into infrared problems since this ‘naive’ integration generally induces infrared divergences. In fact this reduction to the case $n = 2$ resembles very much a technical (and practically rather important) standard trick [5] in renormalization group calculations in the framework of dimensional renormalization: the main idea of this trick is to nullify all the massive parameters (masses and external momenta) except one parameter. However, when applying such a trick one cannot avoid infrared problems (at least starting from the five-loop level in ϕ_4^4 -theory). The general device that enables us to ‘fight’, in this situation, against infrared divergences is the so-called R^* -operation³ [6].

Although there is a similarity of the reduction to the case $n = 2$ in RG calculations, on one hand, and in writing down renormalized quantities in the original version of differential renormalization, on the other hand, there are certain distinctions. It happens that there are no infrared problems in differential renormalization for writing down renormalized Feynman amplitudes corresponding to those five-loop diagrams [7] that generated infrared problems in RG calculations within dimensional renormalization. We shall show however that, for differential renormalization, one should overcome infrared problems in another type of diagrams.

More precisely, to write down the differentially renormalized Feynman amplitude for a given graph one first writes down an ‘incompletely’ renormalized quantity: it includes all the counterterms for subgraphs except the overall counterterm. Then one tries to perform the above mentioned integration over coordinates in order to arrive at expressions of the type of the left-hand side of (2-7), with some coefficients. The following step is to apply (2-7) to obtain the overall counterterm. Thus the problem reduces to the calculation of the above mentioned coefficients. They are generally hardly calculable if one does not consider separately different terms contributing to the incompletely renormalized Feynman amplitude. The corresponding calculable integrals happen to be divergent, in the infrared sense. However we shall see that there are simple tricks that enable us to fight against such infrared divergences.

The main goal of the present paper is to supply the recipes of the original version of differential renormalization with simple prescriptions for writing down renormalized Feynman amplitudes and calculating involved coefficients at least at the level of calculability of counterterms within dimensional renormalization. This will be done in Sections 3 and 4, using three-, four- and five-loop examples from ϕ_4^4 -theory. Before

³In [4] it was claimed that a generalization of the R^* -operation for the case of differential renormalization is hardly possible because when choosing different order of integration (with some renormalization prescription implied) the authors obtained different results. However this situation does not look inconsistent: these different results correspond in fact to different definitions of the infrared renormalization which involves, as its ultraviolet analog, a natural finite arbitrariness.

that, in Section 2, the reduction of the problem to the case $n = 2$ will be described in detail. Another important point is to argue that it is worthwhile to exploit calculational experience based on dimensional regularization when evaluating the coefficients in the renormalized quantities.

2 Reducing to $n = 2$

Feynman amplitudes are obtained from the products Π_Γ by integrating over coordinates associated with internal vertices. This integration does not influence the problem of renormalization so that we shall treat all the vertices as external. However, in calculating coefficients which enter differentially renormalized quantities, some of the vertices will be certainly considered as internal ones.

The differential renormalization is defined recursively [2]. Suppose that it is known how to renormalize all proper subgraphs of a given graph Γ . This means that we know the counterterms $\Delta(\gamma)\Pi_\gamma$ for all $\gamma \subset \Gamma$. The R -operation (renormalization at diagrammatic level) acts on the Feynman amplitude Π_Γ for the graph Γ as

$$\begin{aligned} R\Pi_\Gamma &= \sum_{\mathcal{V}=\mathcal{V}_1 \cup \dots \cup \mathcal{V}_j} \Delta(\gamma_1) \dots \Delta(\gamma_j) \Pi_\Gamma \\ &\equiv R'\Pi_\Gamma + \Delta(\Gamma)\Pi_\Gamma. \end{aligned} \tag{8}$$

The sum is over all decompositions of the set of vertices $\mathcal{V}$ of the graph Γ into non-empty non-intersecting subsets $\mathcal{V}_1, \dots, \mathcal{V}_j$. Moreover, γ_i is the subgraph composed of vertices $\mathcal{V}_i$ and all lines that are internal to these vertices. Remember that $\Delta(\gamma_i) = 1$ if $\mathcal{V}_i$ is an isolated vertex, and $\Delta(\gamma_i) = 0$, if γ_i is not an 1PI divergent subgraph.

The operation R' is called incomplete R -operation. This operation removes all subdivergences of the diagram but does not include the overall counterterm $\Delta(\Gamma)$. This implies that the function $R'\Pi_\Gamma$ is locally integrable in the space of coordinates except the point where all the coordinates coincide. Thus, the problem reduces to the extension of this function considered as a distribution to the whole space. Suppose that the graph Γ is logarithmically divergent. Then $R'\Pi_\Gamma$ becomes locally integrable (and therefore naturally interpreted as a distribution) after multiplication by any coordinate difference.

According to the assumption, the R' -operation is known for the given graph Γ . Suppose also that the number of vertices n is greater than two. Following the recipes of [2] it is necessary to choose a pair of vertices, with $x_i - x_j = x$, and perform integration over all other coordinate differences. Let us denote them by $\underline{y}$. Before writing down the renormalized Feynman amplitude $R\Pi_\Gamma(x, \underline{y})$ one writes down a renormalized value $Rf(x)$ for

$$f(x) = \int d\underline{y} R'\Pi_\Gamma(x, \underline{y}), \tag{9}$$

using prescriptions (2-7) etc. The corresponding ‘two-vertex counterterm’ is equal to $(R - 1)f$. Then the renormalization of Π_Γ is given by [2]

$$R\Pi_\Gamma(x, \underline{y}) = R'(x, \underline{y}) + \delta(\underline{y})[Rf(x) - f(x)], \quad (10)$$

where $\delta(\underline{y}) = \prod_{i' \neq i, j} \delta(x_{i'} - x_i)$. Thus, the counterterm for Γ

$$\Delta\Pi_\Gamma(x, \underline{y}) = \delta(\underline{y})[Rf(x) - f(x)] \quad (11)$$

happens to be local since the square brackets are zero unless $x = \underline{y} = 0$ (because $Rf(x) = f(x)$ for $x \neq 0$).

A generalization of (10) for quadratically divergent diagrams looks as follows:

$$R\Pi_\Gamma(x, \underline{y}) = R'(x, \underline{y}) + \delta(\underline{y})[Rf(x) - f(x)] - \sum_i \partial_{y_i}^\alpha \delta(\underline{y})[Rf_\alpha^i(x) - f_\alpha^i(x)] + \frac{1}{2} \sum_{ij} \partial_{y_i}^\alpha \partial_{y_j}^\beta \delta(\underline{y})[Rf_{\alpha\beta}^{ij}(x) - f_{\alpha\beta}^{ij}(x)], \quad (12)$$

with

$$f_\alpha^i(x) = \int d\underline{y} y_{i\alpha} R'\Pi_\Gamma(x, \underline{y}), \quad (13)$$

$$f_{\alpha\beta}^{ij}(x) = \int d\underline{y} y_{i\alpha} y_{j\beta} R'\Pi_\Gamma(x, \underline{y}). \quad (14)$$

Formulae (10) and (12), with (9), (13) and (14), describe differentially renormalized Feynman amplitudes $R\Pi_\Gamma$ as far as the functions $f, \dots$ are renormalized according to (2-7), etc. The renormalized amplitudes are well-defined as distributions. For instance, the action of functional (10) on a test function ϕ is given by

$$(R\Pi_\Gamma(x, \underline{y}), \phi(x, \underline{y})) = \int dx d\underline{y} R'\Pi_\Gamma(x, \underline{y})(\phi(x, \underline{y}) - \phi(x, \underline{0})) + \int dx Rf(x)\phi(x, \underline{0}). \quad (15)$$

The second term on the right-hand side of (15) is well-defined. The first term is locally convergent because of the logarithmical character of divergence. However, it is integration at large $\underline{y}$ that can generate new (infrared) divergences. Consider, for example, the graph of Fig. 1 which can arise in the product of three composite operators $\psi_\alpha(x_1)\psi_\beta(x_2)\phi(x_3)$, with $\psi_\alpha = \partial_\alpha\phi$. Here the bottom line involves two derivatives of the propagator. If we try to reduce the problem of writing down the differentially renormalized value for such diagram by integrating over x_3 we shall obtain an infrared divergent integral.

Of course, it is possible to integrate over x_1 or x_2 without running into infrared problems. On the other hand, there is a possibility of running into infrared problems in other situations. This will be the subject of the following two sections.

3 Three- and four-loop examples

Let us consider the three-loop ‘open envelope’ graph of Fig. 2a as an example. We have

$$\Pi_\Gamma = (x_1 - x_2)^{-2}(x_1 - x_3)^{-2}(x_1 - x_4)^{-2}(x_2 - x_3)^{-2}(x_2 - x_4)^{-2}(x_3 - x_4)^{-2}. \quad (16)$$

It is logarithmically divergent and does not contain subdivergences. Thus we apply (10) where $R'\Pi_\Gamma = \Pi_\Gamma$. The problem then reduces to evaluation of the massless master two-loop diagram. It can be calculated by various methods (integration by parts within dimensional regularization, Gegenbauer polynomial technique, star-triangle identity etc.). As a result, for the function f we obtain, with $x = x_1 - x_2$,

$$f(x) = \int dx_3 dx_4 \Pi_\Gamma(\underline{x}) = 6\zeta(3)\pi^4 \frac{1}{x^4}, \quad (17)$$

where $\zeta(z)$ is the Riemann ζ -function. Using (2) one comes to the following renormalized value [2]

$$R\Pi_\Gamma(\underline{x}) = \Pi_\Gamma(\underline{x}) - 6\zeta(3)\pi^4 \left\{ \frac{1}{(x_1 - x_2)^4} + \frac{1}{4} \square \frac{\ln \mu^2(x_1 - x_2)^2}{(x_1 - x_2)^2} \right\}. \quad (18)$$

The corresponding counterterm is

$$\Delta\Pi_\Gamma(\underline{x}) = -6\zeta(3)\pi^4 \left\{ \frac{1}{(x_1 - x_2)^4} + \frac{1}{4} \square \frac{\ln \mu^2(x_1 - x_2)^2}{(x_1 - x_2)^2} \right\}. \quad (19)$$

Note that we have not yet run into infrared problems because the integral (17) was convergent. Consider now, as Γ , the graph of Fig. 2b. It has two (logarithmically) divergent subgraphs: the subgraph $\gamma = (1, 2, 3, 4)$ (i.e. Fig. 2a) and the graph Γ itself. In accordance with general formula (8)

$$R = 1 + \Delta(\gamma) + \Delta(\Gamma) = R' + \Delta(\Gamma)$$

so that

$$R'\Pi_\Gamma(\underline{x}) = \Pi_\Gamma(\underline{x}) - 6\zeta(3)\pi^4 \delta(x_1 - x_3)\delta(x_1 - x_4) \frac{1}{(x_1 - x_5)^4} \left\{ \frac{1}{(x_1 - x_2)^4} + \frac{1}{4} \square \frac{\ln \mu^2(x_1 - x_2)^2}{(x_1 - x_2)^2} \right\}. \quad (20)$$

Now, according to prescriptions of [2], we should choose a pair of vertices, with $x = x_i - x_j$. We are not allowed to choose $i, j \leq 4$ — one of the obstacles is that an infrared divergence in the integral at large x_1 will appear. So, let us choose, say, 1 and 5. Then one observes that the resulting integral (9) is hardly calculable unless we separate terms in the integrand. However, we then arrive at IR-divergent

integrals that correspond to the two terms in braces of (20). In this situation one can apply the following trick. Anticipating that we would run into IR-problems, we could insert, as a factor of braces in (18), some function ϕ which is equal to unity at the point $x_1 = \dots = x_4$. We are allowed to do this because the braces are equal to zero everywhere except at this point. Let us choose

$$\phi(\underline{x}) = \frac{(x_1 - x_5)^2}{(x_2 - x_5)^2}. \quad (21)$$

As a result we may consider two integrals separately:

$$\begin{aligned} & \int dx_2 dx_3 dx_4 R' \Pi_\Gamma(\underline{x}) \\ &= \int dx_2 dx_3 dx_4 \Pi_\gamma \left\{ \frac{1}{(x_3 - x_5)^2 (x_4 - x_5)^2} - \frac{1}{(x_1 - x_5)^2 (x_2 - x_5)^2} \right\} \\ & \quad - \frac{3}{2} \zeta(3) \pi^4 \frac{1}{(x_1 - x_5)^2} \int dx_2 \frac{1}{(x_2 - x_5)^2} \square \frac{\ln \mu^2 (x_1 - x_2)^2}{(x_1 - x_2)^2}, \end{aligned} \quad (22)$$

with Π_γ given by the right-hand side of (16). Applying the Laplacian in the distributional sense and using the identity $\square \frac{1}{x^2} = -4\pi^2 \delta(x)$ we come to the following value for the corresponding term:

$$f(x) = 6\zeta(3) \pi^6 \frac{\ln \mu^2 (x_1 - x_5)^2}{(x_1 - x_5)^4}. \quad (23)$$

Let us now calculate, at $x \neq 0$, the first two terms on the right-hand side of (22) — they are graphically shown in Fig. 2c. This can be easily done with the help of dimensional regularization. Two graphs Γ and Γ' are topologically identical and differ only by a choice of external vertices. Let us consider the corresponding Feynman amplitudes $F_\Gamma(q)$ and $F_{\Gamma'}(q)$ in momentum space. Within dimensional regularization they have the form

$$F_\Gamma(q) = g(\epsilon)(q^2)^{-4\epsilon}, \quad F_{\Gamma'}(q) = g'(\epsilon)(q^2)^{-4\epsilon}, \quad (24)$$

with meromorphic functions g and g' . Since the Fourier transform of $(q^2)^\epsilon$ evaluated at $x \neq 0$ is proportional to ϵ it is sufficient to calculate the singular part in ϵ of the functions g and g' . To do this let us note that the MS-renormalized Feynman integral for Γ is written as $R^{\text{MS}} F_\Gamma = F_\Gamma + Z_\gamma F_{\Gamma/\gamma} + Z_\Gamma$, where Z_γ and Z_Γ are the corresponding counterterms. Thus, for the singular part given by the operator $\hat{K}_\epsilon$ that picks up the pole part of the Laurent series in ϵ , we have

$$\hat{K} F_\Gamma = -Z_\gamma F_{\Gamma/\gamma} - Z_\Gamma. \quad (25)$$

Observe now that in a similar equation for Γ' the counterterms are the same as for Γ . Moreover, the Feynman integral for the reduced graph Γ/γ is also the same. Therefore we come to the zero value of the first term on the right-hand side of (22).

Using (3), (10) and (23) we obtain the following differentially renormalized value for the graph of Fig. 2b:

$$\begin{aligned}
R\Pi_\Gamma(\underline{x}) &= \Pi_\Gamma(\underline{x}) \\
&- 6\zeta(3)\pi^4\delta(x_1-x_3)\delta(x_1-x_4)\frac{1}{(x_1-x_5)^4}\left\{\frac{1}{(x_1-x_2)^4}+\frac{1}{4}\square\frac{\ln\mu^2(x_1-x_2)^2}{(x_1-x_2)^2}\right\} \\
&- 6\zeta(3)\pi^6\delta(x_1-x_2)\delta(x_1-x_3)\delta(x_1-x_4)\left\{\frac{\ln^2\mu^2(x_1-x_5)^2}{(x_1-x_5)^4}\right. \\
&\quad \left.+\frac{1}{8}\square\frac{\ln^2\mu^2(x_1-x_5)^2+2\ln\mu'^2(x_1-x_5)^2}{(x_1-x_5)^2}\right\}. \quad (26)
\end{aligned}$$

4 Five-loop example

Now, let us consider as Γ the five-loop graph of Fig. 3a. It has an overall quadratic divergence as well as two logarithmically divergent overlapping subgraphs $\gamma_1 = (1, 2, 3, 4)$ and $\gamma_2 = (2, 3, 4, 5)$. In accordance with (8), using the value (19) for the counterterm of Fig. 2a, we have

$$\begin{aligned}
R'\Pi_\Gamma(\underline{x}) &= (1 + \Delta(\gamma_1) + \Delta(\gamma_2))\Pi_\Gamma(\underline{x}) \\
&= \Pi_\Gamma(\underline{x}) - 6\zeta(3)\pi^4\delta(x_1-x_3)\delta(x_1-x_4)\frac{1}{(x_1-x_5)^6}\left\{\frac{1}{(x_1-x_2)^4}\right. \\
&\quad \left.+\frac{1}{4}\square\frac{\ln\mu^2(x_1-x_2)^2}{(x_1-x_2)^2}\right\} + \{1 \leftrightarrow 5\}. \quad (27)
\end{aligned}$$

Then it is necessary to choose a pair of vertices. It is not difficult to see that this reduction strategy works only if we choose 1 and 5 — these are the vertices that do not simultaneously belong to one of subgraphs of the given graph. Let us now apply (12) and evaluate the corresponding integrals (9), (13) and (14). Let us begin with (9). As in the case of Fig. 2b, we shall run into infrared problems if we consider the terms in braces separately. However we need this to perform calculations. Thus we apply the same trick as before: we insert the function (21) as a factor of the braces and similar function, with $1 \leftrightarrow 5$ into the last term.

The evaluation of two (identical) terms with Laplacian gives

$$f^{(1)}(x) = 12\zeta(3)\pi^6\frac{\ln\mu^2(x_1-x_5)^2}{(x_1-x_5)^6}. \quad (28)$$

The contribution of the other terms, given by the integral over x_2, x_3 and x_4 , is graphically shown in Fig. 3b where external lines are distinguished by (external) lines attached. Let us once again introduce dimensional regularization which will enable us to consider these two terms separately. As before it is sufficient to calculate the

pole part of the difference of the corresponding Feynman integrals $F_\Gamma(q) - 2F_{\Gamma'}(q)$ in momentum space:

$$F_\Gamma(q) = g(\epsilon)(q^2)^{1-5\epsilon}, \quad F_{\Gamma'}(q) = g'(\epsilon)(q^2)^{1-5\epsilon}. \quad (29)$$

Using results of [8] we have

$$g(\epsilon) = -(4\pi)^{-5d/2} \left(\frac{2}{5}\zeta(3)\frac{1}{\epsilon^2} + \frac{34}{5}\zeta(3)\frac{1}{\epsilon} + \frac{3}{5}\zeta(4)\frac{1}{\epsilon} \right) + O(\epsilon^0). \quad (30)$$

The calculation of the Feynman integral for Γ' can be, for example, performed with the information about the ϵ -expansion of the master two-loop diagram [10]. The result is

$$g'(\epsilon) = -(4\pi)^{-5d/2} \left(\frac{1}{5}\zeta(3)\frac{1}{\epsilon^2} + 4\zeta(3)\frac{1}{\epsilon} + \frac{3}{10}\zeta(4)\frac{1}{\epsilon} \right) + O(\epsilon^0). \quad (31)$$

Calculating the difference $g - 2g'$ and turning to the Fourier transform at $x \neq 0$ we obtain the following contribution:

$$f^{(2)}(x) = -12\zeta(3)\pi^6 \frac{1}{(x_1 - x_5)^6}. \quad (32)$$

To calculate the integral (13) for the given graph let us observe that the polynomial factor nullifies one of the counterterms. Moreover, the dimensionally regularized Feynman integral (without counterterms) with this factor is nothing but half the value of the corresponding scalar integral multiplied by $(x_5 - x_1)_\alpha$. Then the problem reduces to evaluation of the previous integral of the type (9). The final result is equal to one half of the sum of (28) and (32) multiplied by x_α :

$$f_\alpha^i(x) = 6\zeta(3)\pi^6(x_5 - x_1)_\alpha \frac{\ln \mu^2(x_1 - x_5)^2 - 1}{(x_1 - x_5)^6}. \quad (33)$$

Let us now calculate integrals (14), with $i, j = 2, 3, 4$. From Lorentz covariance it follows that the result should have the form

$$a_1 g_{\alpha\beta} \frac{1}{x^4} + a_2 \frac{x_\alpha x_\beta - \frac{1}{4} g_{\alpha\beta} x^2}{x^4}. \quad (34)$$

However the second term is well-defined as a distribution (without renormalization) because the transversal part is proportional to $\partial_\alpha \partial_\alpha \frac{1}{x^2}$ (with no logarithms!). So, it suffices to calculate $\frac{1}{4} g_{\alpha\beta} f_{\nu\nu}^{ij}(x)$ instead of $f_{\alpha\beta}^{ij}$.

For $j = i$ one of the counterterms in (27) is nullified. The factor $(x_i - x_1)^2$ is taken into account by deleting the corresponding line. Then the problem reduces to calculations performed for the graph of Fig. 2b. After using the same ‘infrared tricks’

we see that the term with the Laplacian gives the value (23) multiplied by $g_{\alpha\beta}/4$ while the other terms are zero at $x \neq 0$. Thus

$$f_{\alpha\beta}^{ii}(x) = \frac{3}{2}\zeta(3)\pi^6 g_{\alpha\beta} \frac{\ln \mu^2(x_1 - x_5)^2}{(x_1 - x_5)^4}. \quad (35)$$

For $i \neq j$ we use the identity

$$(x_i - x_1)(x_j - x_1) = \frac{1}{2}((x_i - x_1)^2 + (x_j - x_1)^2) - \frac{1}{2}(x_i - x_j)^2. \quad (36)$$

The first term on the right-hand side gives the same value as in the case $i = j$. When calculating the contribution from the second term we see that both counterterms in (27) are nullified. Thus it suffices to compute the pole part in ϵ of the integral for the graph Γ with one of the middle lines deleted (e.g. $\{23\}$). This is just the closed envelope diagram [9] shown in Fig. 3c. Taking into account the factors $-\frac{1}{2}$ and $\frac{1}{4}$ we obtain the following contribution

$$f_{\alpha\beta}^{ij}(x) = \frac{1}{2}\pi^6 g_{\alpha\beta} \left\{ 3\zeta(3) \frac{\ln \mu^2(x_1 - x_5)^2}{(x_1 - x_5)^4} - 5\zeta(5) \frac{1}{(x_1 - x_5)^4} \right\}; \quad i \neq j. \quad (37)$$

Substituting the values (28), (32), (33), (35) and (37) into (12) we come to the following differentially renormalized value for the graph of Fig. 3a:

$$\begin{aligned} R\Pi_\Gamma(\underline{x}) = & \Pi_\Gamma(\underline{x}) - 6\zeta(3)\pi^4 \delta(x_1 - x_3)\delta(x_1 - x_4) \frac{1}{x^6} \left\{ \frac{1}{(x_1 - x_2)^4} \right. \\ & \left. + \frac{1}{4}\square \frac{\ln \mu^2(x_1 - x_2)^2}{(x_1 - x_2)^2} \right\} + \{1 \leftrightarrow 5\} \\ & - 12\zeta(3)\pi^6 \prod_{i=2,3,4} \delta(x_i - x_1) \left\{ \frac{\ln \mu^2 x^2}{x^6} + \frac{1}{64}\square^2 \frac{\ln^2 \mu^2 x^2 + 2 \ln \mu'^2 x^2 + 3 \ln \mu^2 x^2}{x^2} \right. \\ & \left. - \frac{1}{x^6} - \frac{1}{32}\square^2 \frac{\ln \mu'^2 x^2}{x^2} \right\} \\ & - 6\zeta(3)\pi^6 \sum_{i=2,3,4} \partial_{x_i}^\alpha \prod_{j=2,3,4} \delta(x_j - x_1) \left\{ x_\alpha \frac{\ln \mu^2 x^2}{x^6} - \frac{1}{32}\partial_\alpha \square \frac{\ln^2 \mu^2 x^2 + 3 \ln \mu'^2 x^2}{x^2} \right. \\ & \left. - \frac{x_\alpha}{x^6} + \frac{1}{16}\partial_\alpha \square \frac{\ln \mu'^2 x^2}{x^2} \right\} \\ & - \frac{3}{4}\zeta(3)\pi^6 \sum_{i=2,3,4} \square_{x_i} \prod_{j=2,3,4} \delta(x_j - x_1) \left\{ \frac{\ln \mu^2 x^2}{x^4} + \frac{1}{8}\square \frac{\ln^2 \mu^2 x^2 + 2 \ln \mu'^2 x^2}{x^2} \right\} \\ & - \frac{1}{8}\pi^6 \sum_{i < j; i,j=2,3,4} \partial_{x_i} \partial_{x_j} \prod_{j=2,3,4} \delta(x_j - x_1) \times \\ & \times \left\{ 3\zeta(3) \left(\frac{\ln \mu^2 x^2}{x^4} + \frac{1}{8}\square \frac{\ln^2 \mu^2 x^2 + 2 \ln \mu'^2 x^2}{x^2} \right) - 5\zeta(5) \left(\frac{1}{x^4} + \frac{1}{4}\square \frac{\ln \mu'^2 x^2}{x^2} \right) \right\}, \quad (38) \end{aligned}$$

with $x = x_1 - x_5$.

5 Conclusions

We have seen that when reducing the problem to $n = 2$ we are generally not allowed to choose a pair of vertices in an arbitrary way. It seems rather plausible that such choice always exists at least for S -matrix diagrams in renormalizable theories. It is not difficult to check this statement up to the five-loop level: in fact, Fig. 3a turns out to be one of the most difficult diagrams up to this order. Using experience of Sections 3 and 4 one can also observe that, at these orders, it is possible to calculate coefficients, which enter differentially renormalized Feynman amplitudes, by use of simple tricks that help us to avoid generating spurious infrared divergences.

It happens that the problem of writing down differentially renormalized quantities demands calculation of these coefficients. It has been shown that calculational methods based on dimensional regularization are rather helpful for this purpose. Of course, this does not mean coming back to dimensional renormalization. Let us clearly distinguish the problem of *renormalization* and a calculational problem that naturally arises. To resolve the second one it is worthwhile to apply powerful methods of calculation developed by many authors and based, in particular, on dimensional *regularization*.

Note that the authors of [1, 2, 3] dealt mostly with two- and three-loop diagrams. In fact the reduction to the case $n = 2$ was trivial, at these levels, because in the corresponding diagrams (except Fig. 2a) one distinguished simple loops as subgraphs. A standard device applied was to contract one of two incident lines by use of the Laplacian involved in the differential renormalization of the simple loop (see (2)).

It should be stressed that two new versions of differential renormalization [11] based on another differential operator, instead of the Laplacian, contain general recipes which are applicable to *arbitrary* Feynman amplitudes just from the very beginning. Within this framework there is no need to reduce the problem of the renormalization to the case of two-point diagrams. For the same reason there is no asymmetry in treating different vertices of diagrams.

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Figures

Figure 1: Integrating over x_3 generates an infrared divergence.

Figure 2: (a) The open envelope diagram. (b) A four-loop diagram. (c) The first term on the right-hand side of (22).

Figure 3: (a) Five-loop propagator-type diagram from ϕ^4 -theory. (b) The difference of the Feynman integrals $F_{\Gamma} - 2F_{\Gamma'}$ that appears in calculations for Fig. 3a. (c) The closed envelope diagram.