QUANTUM AND THERMAL FLUCTUATIONS
IN FIELD THEORY *

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ABSTRACT

Blocking transformation is performed in quantum field theory at finite temperature. It is found that the manner temperature deforms the renormalized trajectories can be used to understand better the role played by the quantum fluctuations. In particular, it is conjectured that domain formation and mass parameter generation can be observed in theories without spontaneous symmetry breaking.

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I. INTRODUCTION

The understanding of the genuine quantum effects depends crucially on our ability to describe field fluctuations, the presence of which attributes to the fundamental difference between the classical and the quantum systems. A straightforward method to study fluctuations for a given system is by means of introducing some relevant physical parameters which have influence on the fields. For example, one may consider a finite region of space-time \( R \), and monitor the field fluctuations within it. If \( \phi(x) \) denotes the original field variable which parameterizes the system, then the exploration of the physical phenomena in the region \( R \) can be facilitated by introducing the coarse-grained average field:

\[
\phi_R = \frac{1}{\Omega_R} \int_R d^d x \phi(x),
\]

where \( \Omega_R \) is the volume of \( R \). The manner in which the quantum fluctuations affect other degrees of freedom coupled to \( \phi(x) \) can then be probed by studying the \( R \) dependence of the corresponding distribution

\[
\varrho_R(\Phi) = \langle \delta(\Phi - \phi_R) \rangle = \int D[\phi] e^{-S[\phi]} \delta(\Phi - \phi_R).
\]

Since it is difficult to compute directly (1.2) in the conventional perturbation expansion, we adopt an alternative scheme analogous to the Wilson-Kadanoff blocking procedure in statistical mechanics [1]. In this formulation, the averaged blocked field variable in (1.1) is written as

\[
\phi_R(x) = \int dy \rho_R(x - y) \phi(y),
\]

where \( \rho_R(x - y) \) is a smearing function which vanishes rapidly for \( x - y \notin R \). This leads to a blocked lagrangian \( L_R(\phi_R) \) satisfying

\[
\langle \hat{O}[\phi_R] \rangle = \int D[\phi] e^{-S[\phi]} \hat{O}[\phi_R] = \int D[\phi_R] e^{-S_R[\phi_R]} \hat{O}[\phi_R],
\]

where

\[
S_R = \int d^d x L_R.
\]

In general, \( L_R \) can be written in the spirit of Ginsburg-Landau approach as

\[
L_R[\phi] = \sum_{n=0}^{\infty} U_R^{(n)}(\phi(x)),
\]

where \( U_R^{(n)}(\phi(x)) \) is a homogeneous polynomial of order \( n \) in space-time derivatives \( \frac{\partial}{\partial x^\mu} \). However, as long as the inhomogeneities of the configuration \( \phi_R(x) \) can be neglected, we then have \( U^{(0)} \gg U^{(n)} \) for \( n \neq 0 \), and (1.2) becomes

\[
\varrho_R(\Phi) \approx e^{-\Omega_R U_R^{(0)}(\Phi)}.
\]
One notable feature of (1.2) is that it develops non-trivial structure when $\Omega_R$ becomes sufficiently small. This can be realized by examining the ‘blocked potential’ $V_R$ in

$$\varrho_R(\Phi) = e^{-\Omega_R V_R(\Phi)}. \quad (1.8)$$

On the one hand, $V_R$ approaches the usual effective potential $V_{\text{eff}}$ in the thermodynamical limit, $\Omega_R \to \infty$, where all the inhomogeneities of $\phi_R$ tend to zero. On the other hand, one recovers for $\Omega_R \to 0$, the bare potential $V_{\text{bare}}$ which exhibits stronger divergence than the kinetic energy term. A negative mass squared term for sufficiently large values of the UV regulator would imply the existence of degenerate minima in the potential $V_R(\Phi)$ for small enough $R$. Such scenario can indeed take place for a symmetric theory in which the manifest realization of a global symmetry in the vacuum requires the effective potential to have a non-degenerate minimum at $\Phi = 0$. In this case the distribution is characterized by a length-scale $\zeta$ such that $V_R(\Phi)$ exhibits a unique minimum at $\Phi = 0$ only if the typical size of $\mathcal{R}$ is greater than $\zeta$ [2]. For smaller domains the potential $V_R(\Phi)$ has minima at $\Phi = \Phi_j \neq 0$ which can be related to each other by symmetry transformations.

A straightforward explanation of such behavior of $\varrho_R(\Phi)$ in (1.8) can be given by noting that the theory contains “domains” of typical size $\zeta$ which dominate the path integral. One naturally finds non-vanishing field average inside each domain, and that different domains have different symmetry transformations relating to other field average values. The smaller the volume $\Omega_R$ we probe, the stronger the dominance of the non-vanishing domain values $\Phi$ in the distribution. We emphasize here that the appearance of degenerate minima in $V_R(\Phi)$ is not the signal of spontaneous breakdown of symmetry. Contrary to the former case which occurs only for small length scale, the latter phenomenon refers to the low-energy, large-distance behavior of the theory in the limit $\Omega_R \to \infty$.

In inquiring more details concerning the domain structures in the symmetric phase, it is also crucial to examine the role of thermal fluctuations in the system. We shall choose the region $\mathcal{R}$ to be translationally invariant along the time direction in Minkowski space-time. The symmetry implies that (1.3) is an averaged blocked field in the three space having a projection of the zero frequency component of the field variable. The same interpretation applies when going over to the Euclidean space-time except now $\phi_R$ contains only the zero Matsubara frequency modes. Adopting the latter reference frame, we write

$$\phi_k(x) = T \int_0^\beta dy \int d^3 y \rho_k(x - y) \phi(y), \quad (1.9)$$

where the smearing function is taken to be

$$\rho_k(x) = \int_{|p| < k} \frac{d^3 p}{(2\pi)^3} e^{ip \cdot x}, \quad (1.10)$$

with $k^{-1}$ being the characteristic linear dimension of the 3-space volume $\Omega_3$. Note that (1.10) which eliminates fast-fluctuating spatial modes and leaves unconstrained the Matsubara frequency modes is only $O(3)$ symmetric. This can be compared with that of the $O(4)$ invariant smearing function considered in [2]. The full $O(4)$ symmetry can be recovered only
in the limit $k \to 0$ and the effect of temperature is switched off. Blocking transformation has been used to study various facets of finite-temperature theory in [3].

One interesting issue to be addressed here is the competition of the length scales $k^{-1}$ and $\beta = 1/T$. The fact that excitations are strongly screened at high $T$ leads one to expect qualitatively the same features for small $R$ and high $T$ limits, that is, (1.8) has similar dependence on the two length scales. Hence, the value of the most probable field average increases with $T$. However, this seems to contradict the common conviction that fluctuations are more "symmetrical" at high temperature, thereby implying diminishing $\Phi_j$ as $T$ increases. Indeed, $\Phi_j$ increases as the spatial length scale, $k^{-1}$ approaches zero. We shall explicitly show in this paper that the one-loop computation lends support to the latter scenario.

As the temperature is raised, the contribution of the non-static modes in Euclidean space-time becomes more suppressed by the large values of the Matsubara frequencies and a three dimensional effective theory emerges for the remaining static modes [4]. Hence, at high $T$, the distribution (1.8) reflects the features of a three dimensional theory. In the low temperature limit the system is four dimensional since the $T$ dependence of $\varrho_R(\Phi)$ is suppressed by a factor

$$O(e^{-m_L/T}),$$

where $m_L$ is the mass gap. The exponent in (1.11) can be interpreted as the ratio of the "time extent" and the "correlation length" of the system.

Another interesting application one may consider is how the presence of local domain structure in an overall symmetric background Higgs field can influence the mass parameter of a particle propagating through it. Suppose that a test particle is coupled to the Higgs field $\phi(x)$ and is moving along $\mathcal{R}$ which has the shape of a "tube" extending along the time direction with a finite width. If the potential $V_\mathcal{R}(\Phi)$ corresponding to this tube has degenerate minima at the values $\Phi = \Phi_j$, $j = 1, \cdots, n$ which are related by the symmetry transformations, then an observable $\hat{O}$ associated with the test particle can be obtained in the mean field approximation for the Higgs field as

$$\langle\hat{O}\rangle = \frac{1}{n} \sum_j <\hat{O}>_{\phi(x) = \Phi_j},$$

where $<\hat{O}>_{\phi(x)}$ is the expectation value of $\hat{O}$ on the background Higgs field $\phi(x)$. In another word, within the region $\mathcal{R}$, only the most probable average values of the Higgs field influences the behavior of the test particle within the framework of mean field approximation. The approximation (1.12) can be justified by choosing a small $\mathcal{R}$ such that $\Phi \approx \Phi_j \neq 0$.

Consider now the special case where a scalar test particle is coupled to the Higgs field in such a way that its mass squared parameter in the lagrangian, $M^2 = \mathcal{G} \Phi^2$ is the same for all domains, i.e., $\Phi_j^2 = \Phi^2$. The constant $\mathcal{G}$ is the measure of the coupling strength between the Higgs field and the test particle. In this situation one readily finds the mass parameter of the test particle $M^2 \neq 0$ even when the Higgs field has symmetrical vacuum expectation value $<\phi> = 0$. In fact, whatever "important" Higgs field configuration being taken in (1.12), each leads to the same non-vanishing squared mass parameter. As for the Higgs field, its "blocked" average decreases as the volume is enlarged to account for the more independent fluctuations at large separations. This scenario amounts to the generation of a mass parameter for the test particle without spontaneous symmetry breaking in the vacuum.
While this simple argument works well for scalars, the situation is slightly more complicated when the test particle is a fermion. The reason is that the coupling between the Higgs field and fermion is linear, contrary to quadratic coupling for scalars. With

$$\sum_j \Phi_j = 0,$$  \hspace{1cm} (1.13)

according to the orthogonality relations for the symmetry group, the average mass parameter computed by (1.12) is vanishing. For example the usual Higgs field possesses discrete symmetry $\phi \rightarrow -\phi$. Therefore, while the chirality-odd term in the fermion propagator remain massless, the chirality-even sector in (1.12) will be that of a massive fermion. We again caution the reader that the usual problem of mass generation is twofold: First, one must give an account on the appearance of some non-zero mass parameter in the lagrangian which possesses symmetries characteristic of massless particles. The second step which can be proceeded with more conventional methods involves establishing a connection between the mass parameter $m$ and the actual physical mass $m_{\text{phy}}$, i.e., the location of the pole in the Minkowski real space-time propagator. We address only the first problem in the present work, keeping in mind that $m_{\text{phy}}$ differs from $m$ by additional radiative corrections.

Alas, this type of mass parameter generation mechanism should not be expected because the field average is arbitrary small for sufficiently long observation time. In another words, within a long annular world-tube $\mathcal{R}$, the practically uncorrelated fluctuations of the Higgs field should yield vanishing most probable average. Nevertheless the mass parameter might be generated at finite temperature. In fact, at finite temperature the scalar fields are periodic in the Euclidean time directions with period length $1/T$. Therefore, the local domains align themselves and there can be no uncorrelated fluctuations in the Euclidean time direction at sufficiently high temperature. Naturally all one can see in this case is the finite-temperature Euclidean mass parameter $m(\Phi_{\beta,k})$ since the real physical mass $m_{\text{phy}}(\Phi_\beta)$ including the thermal contribution can only be determined by locating the pole of the propagator after an analytic continuation back to real Minkowski space-time at finite $T$, i.e.,

$$\frac{1}{p^2 + m^2(\Phi_{\beta,k})} \rightarrow \frac{i}{p^2 - m^2_{\text{phy}}(\Phi_\beta) + i\epsilon}.$$  \hspace{1cm} (1.14)

Although our analyses yield directly a non-vanishing $m(\Phi_{\beta,k})$, we believe that it will also contribute to $m_{\text{phy}}(\Phi_\beta)$ in the analytically continued real-time propagator as well. Notice that in computing $m_{\text{phy}}$, the scale $k$ will be absorbed via a set of self-consistent equations.

The organization of the paper is the following. In Section II we illustrate the formalism of finite-temperature blocking transformation using the scalar $\lambda\phi^4$ theory and generate a renormalization group flow equation for the finite-temperature blocked potential $U_{\beta,k}$. The existence of domain structure in the symmetric phase is examined in the context of Ising model. The possibility of mass parameter generation without symmetry breaking is discussed for the Yukawa model and the scalar QED. Theory with fermionic matter field coupled to the scalar Higgs potential is presented in Section III where we compute the critical scale $k_{cr}$ above which domain structure is detected. In Section IV where scalar QED is studied, we introduce the blocking scale $k$ in a gauge invariant manner by inserting a smearing function into the proper-time integration variable. Our choice of smearing function which has similar
origin as the Pauli-Villars regularization method, avoids the complication of UV divergences. We discuss in Section V the issue of dynamical mass generation without symmetry breaking by an optimization method in the particle propagator. It is shown that in the mean field limit, with presence of domain structure in the background Higgs field, the test particles can acquire a non-vanishing mass parameter albeit symmetry is preserved. Section VI is reserved for summary and discussion. In Appendix A, we give an estimate of the width which defines the effective region traversed by the free test particle. For completeness, various scenarios for finite-temperature massless scalar QED are included in Appendix B.

II. \( \lambda \phi^4 \) Theory

We begin our investigation with the \( \lambda \phi^4 \) theory described by:

\[
\mathcal{L}(\phi) = \frac{1}{2} (\partial_\mu \phi)^2 + V(\phi),
\]

(2.1)

where

\[
V(\phi) = \frac{\mu^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4.
\]

(2.2)

Before computing the finite temperature blocked action \( \tilde{S}_{\beta,k} \), which is the effective action at the scale \( k \) and temperature \( T \), we give a brief review of blocking transformation at zero temperature in \( d = 4 \) Euclidean space-time.

The first step is to separate the slowly-varying background fields \( \sigma \) from the fast-fluctuating modes \( \xi \), i.e.,

\[
\phi(p) = \begin{cases} 
\sigma(p), & 0 \leq p \leq k \\
\xi(p), & k < p < \Lambda.
\end{cases}
\]

(2.3)

This can be achieved most easily by applying an \( O(4) \) symmetric sharp momentum smearing function \( \rho_k(p) = \Theta(k-p) \). Given a set of blocked variables \( \phi_k(x) \), the blocked action \( \tilde{S}_k \) can be derived from

\[
e^{-\tilde{S}_k[\Phi]} = \int \mathcal{D}[\phi] \prod_x \delta(\phi_k(x) - \Phi(x)) e^{-S[\phi]},
\]

(2.4)

where the field average \( \Phi \) of a given block is chosen to coincide with the slowly varying background since \( \phi_k(p) = \rho_k(p) \phi(p) = \sigma(p) \). One then integrates out the fast-fluctuating modes \( \xi(x) \) using the loop expansion to obtain \( \tilde{S}_k[\Phi] \):

\[
\tilde{S}_k[\Phi] = -\ln \int \mathcal{D}[\sigma] \mathcal{D}[\xi] \prod_x \delta(\phi_k(x) - \Phi(x)) \exp \left\{ -S[\sigma + \xi] \right\}
\]

\[
= -\ln \int \mathcal{D}[\sigma] \prod_p \delta(\sigma(p) - \Phi(p)) \int \mathcal{D}[\xi] \exp \left\{ -S[\sigma] - \frac{1}{2} \int_p \xi(p) K(\sigma) \xi(-p) + \cdots \right\}
\]

\[
= -\ln \int \mathcal{D}[\sigma] \prod_p \delta(\sigma(p) - \Phi(p)) \exp \left\{ -S[\sigma] - \frac{1}{2} \text{Tr} \ln K(\sigma) + \cdots \right\}
\]

\[
= S[\Phi] + \frac{1}{2} \text{Tr} \ln K(\Phi),
\]

(2.5)
where

\[ K(\Phi) = \frac{\partial^2 \mathcal{L}}{\partial \phi^2(x)} \bigg|_{\Phi} = -\partial^2 + V''(\Phi), \]  

\[ \int_p = \int_k \frac{d^4 p}{(2\pi)^4}, \]  

and \( \text{Tr} \) implies that the trace in momentum space is to be carried out for \( k \leq p \leq \Lambda \), i.e., the modes which are to be eliminated by blocking transformation. \( \tilde{S}_k[\Phi] \) can readily be seen to interpolate smoothly between the bare action \( S[\Phi] \) and the renormalized one-loop effective action as \( k \) evolves from \( \Lambda \) to 0.

Going to the finite-temperature formalism, the field periodic in the time with period \( \beta \) can be written as

\[ \phi(x) = \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3 p}{(2\pi)^3} e^{-i(\omega_n \tau - p \cdot x)} \phi(\omega_n, p) \]  

\[ \rightarrow \frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3 p}{(2\pi)^3} e^{i p x} \phi(p), \]  

where \( p^\mu = (i \omega_n, p) \) and \( \omega_n = \frac{2\pi n}{\beta} \), the Matsubara frequencies for bosons. The analogous coarse-grained blocked field is defined as:

\[ \phi_k(x) = T \int_0^\beta dy \int d^3 y \rho_k(x - y) \phi(y), \]  

where

\[ \rho_k(x) = \int_{|p| < k} \frac{d^3 p}{(2\pi)^3} e^{i p x}. \]

A comparison between the manner in which momentum modes are eliminated using the \( O(4) \) and \( O(3) \) symmetric smearing functions is depicted in Fig. 1. The finite-temperature blocked action \( \tilde{S}_{\beta, k}[\Phi] \), can be computed in the same manner as the zero temperature case, with the exception of evaluating the kernel \( K(\Phi) \). Summing over the discrete Matsubara frequency modes, we have

\[ \frac{1}{2\Omega} \text{Tr} \ln K(\Phi) = \frac{1}{2\beta} \sum_n \int_k^\infty \frac{d^3 p}{(2\pi)^3} \ln \left[ \omega_n^2 + p^2 + V''(\Phi) \right] \]  

\[ = \frac{1}{2\beta} \int_k^\infty \frac{d^3 p}{(2\pi)^3} \left\{ \beta \sqrt{p^2 + V''(\Phi)} + 2 \ln \left[ 1 - e^{-\beta \sqrt{p^2 + V''(\Phi)}} \right] \right\}. \]  

By separating the zero temperature contribution in (2.11) via the relation

\[ \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \ln [p_0^2 + E^2] = E, \]

(2.12)
which holds up to an infinite $E$-independent constant, one obtains

$$U_{\beta,k}(\Phi) = V(\Phi) + \frac{1}{2} \int_{k}^{\infty} \frac{d^3 p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \ln \left[ p^2 + V''(\Phi) \right]$$

$$+ T \int_{k}^{\infty} \frac{d^3 p}{(2\pi)^3} \ln \left[ 1 - e^{-\frac{\beta}{2} p^2 + V''(\Phi)} \right].$$

(2.13)

This shows explicitly how the blocked potential for the static Euclidean modes is given by the analytic continuation of the blocked potential for the static modes in Minkowski space-time. In fact, the latter contains the integration over each frequency and is just the second term in the right hand side of (2.13).

Since the manipulation above applies to arbitrary potential $V(\Phi)$, the renormalization group equation for the finite-temperature local potential $U_{\beta,k}(\Phi)$ can be obtained by differentiating (2.13) with respect to $k$:

$$k \frac{\partial U_{\beta,k}}{\partial k} = - \frac{k^3}{4\pi^2} \sqrt{k^2 + U''_{\beta,k}} - T \frac{k^3}{2\pi^2} \ln \left[ 1 - e^{-\beta \sqrt{k^2 + U''_{\beta,k}}} \right].$$

(2.14)

The higher loop contributions are vanishing in the renormalization group equation (2.14) since they are suppressed by $\frac{\partial k}{k} \to 0$. To obtain a flow equation for the full theory, however, one needs to take into accounts the wave function renormalization constant as well as the higher order derivative terms in the blocked lagrangian. The simplifications, we believe, represent only small errors in the infrared limit of the four dimensional theories.

While at low temperature, $T << \sqrt{k^2 + U''_{\beta,k}}$, (2.14) reduces to

$$k \frac{\partial U_{\beta,k}}{\partial k} = - \frac{k^3}{4\pi^2} \sqrt{k^2 + U''_{\beta,k}},$$

(2.15)

in the high $T$ limit where $T >> \sqrt{k^2 + U''_{\beta,k}}$, we find

$$k \frac{\partial U_{\beta,k}(\Phi)}{\partial k} = -T \frac{k^3}{4\pi^2} \ln \left[ \frac{k^2 + U''_{\beta,k}(\Phi)}{k^2 + U''_{\beta,k}(0)} \right].$$

(2.16)

These expressions could have been guessed easily. At high temperature the large Matsubara frequency suppresses the contribution of the non-static modes. This can be seen in the first equation of (2.11) where the terms with $n \neq 0$ have diminishing dependence on $\Phi$.

When the contribution of the non-static modes is altogether neglected then the static modes are described by a three dimensional theory with the potential $\tilde{U}_k = \beta U_{\beta,k}$. The natural field variable is $\tilde{\Phi} = \sqrt{\beta} \Phi$ in three dimensions and the renormalization group equation of the three-dimensional theory becomes (c.f. (2.16)):

$$k \frac{\partial \tilde{U}_k(\tilde{\Phi})}{\partial k} = - \frac{k^3}{4\pi^2} \ln \left[ \frac{k^2 + \tilde{U}_k(\tilde{\Phi})}{k^2 + \tilde{U}_k(0)} \right].$$

(2.17)
where the dot denotes differentiation with respect to \( \hat{\Phi} \). By writing the renormalization group equation (2.14) as

\[
k \frac{\partial U_{\beta,k}}{\partial k} = - \frac{k^3}{4\pi^2} \sqrt{k^2 + U_{\beta,k}''} \\
- \frac{k^3}{4\pi^2} \left\{ 2 \ln \left[ 1 - e^{-\beta \sqrt{k^2 + U_{\beta,k}''}} \right] - \ln \left[ \frac{k^2 + U_{\beta,k}''}{k^2 + U_{\beta,k}''(0)} \right] \right\}
\]

we separate the \( T = 0 \), the finite temperature and the \( T = \infty \) contributions. While first two lines on the right-hand side correspond to the non-static modes, the last term contains the contribution of the static modes to the renormalization of the potential. It is rather surprising to find among these terms a non-logarithmic temperature-independent contribution. The high temperature system is close to the three dimensional one and modifications in the infrared region becomes more enhanced for lower dimensions. Such qualitative difference is clearly visible between the first and the last line since the power-like function increases more rapidly than that of logarithm.

Instead of integrating out the renormalization group equation (2.14), we can compute the blocked potential easily in the independent-mode approximation which yields:

\[
U_{\beta,k}(\Phi) = \frac{\mu_R^2}{2} \Phi^2 \left( 1 - \frac{\lambda_R}{64\pi^2} \right) + \frac{\lambda_R}{4!} \Phi^4 \left( 1 - \frac{9\lambda_R}{64\pi^2} \right) \\
+ \frac{1}{32\pi^2} \left\{ -k \left( 2k^2 + \mu_R^2 + \frac{1}{2} \lambda_R \Phi^2 \right) \left( k^2 + \mu_R^2 + \frac{1}{2} \lambda_R \Phi^2 \right)^{1/2} \right. \\
+ \left( \mu_R^2 + \frac{1}{2} \lambda_R \Phi^2 \right)^2 \ln \left[ \frac{k + \sqrt{k^2 + \mu_R^2 + \lambda_R \Phi^2}}{\mu_R} \right] \\
+ \frac{1}{2\pi^2} \int_0^\infty dp p^2 \ln \left[ 1 - e^{-\beta \sqrt{p^2 + \mu_R^2 + \lambda_R \Phi^2}} \right],
\]

where \( \mu_R^2 \) and \( \lambda_R \) are the renormalized parameters satisfying

\[
\left\{ \begin{array}{l}
\mu_R^2 = \frac{\partial^2 U_{\beta,k}}{\partial \Phi^2} \mid_{\Phi = 1/\beta = k = 0} \\
\lambda_R = \frac{\partial^4 U_{\beta,k}}{\partial \Phi^4} \mid_{\Phi = 1/\beta = k = 0}.
\end{array} \right.
\]

One can however write down the general effective scale- and temperature-dependent mass squared \( \mu_R^2(\beta, k) \) and coupling constant \( \lambda_R(\beta, k) \):

\[
\mu_R^2(\beta, k) = \mu_R^2 - \frac{\lambda_R}{64\pi^2} \sqrt{k^2 + \mu_R^2} \left\{ 4k^3 + \mu_R^2 \left( 3k - \sqrt{k^2 + \mu_R^2} \right) - \frac{\mu_R^4}{k + \sqrt{k^2 + \mu_R^2}} \right\} \\
- 4 \mu_R^2 \sqrt{k^2 + \mu_R^2} \ln \left( \frac{k + \sqrt{k^2 + \mu_R^2}}{\mu_R} \right) + \frac{\lambda_R}{4\pi^2} \int_0^\infty dx \frac{x^2 - \beta^2 \mu_R^2}{e^x - 1},
\]

\( 9 \)
and

$$
\lambda_R(\beta, k) = \lambda_R - \frac{3\lambda_R^2}{16\pi^2} \left\{ k \left[ 2k + (2k^2 + \mu_R^2)(k^2 + \mu_R^2)^{-1/2} \right] \left( k + \sqrt{k^2 + \mu_R^2} \right)^2 - \ln \left( k + \sqrt{k^2 + \mu_R^2} \frac{\mu_R}{\mu_R^2} \right) \right\} - \frac{3\lambda_R^3}{8\pi^2} \int_{\beta \sqrt{k^2 + \mu_R^2}}^{\infty} dx \frac{\sqrt{x^2 - \beta^2 \mu_R^2}}{x^2} \ln \left( \frac{e^x - 1 + xe^x}{e^x - 1} \right),
$$

where \( x = \beta \sqrt{\mu_R^2 + \mu_R^2} \). In the limit \( 1/\beta = k = 0 \), we readily recover the renormalized quantities and the Coleman-Weinberg potential [6]:

$$
U_{\text{eff}}(\Phi) = \frac{\mu_R^2}{2} \Phi^2 \left( 1 - \frac{\lambda_R}{64\pi^2} \right) + \frac{\lambda_R \Phi^4}{4!} \left( 1 - \frac{9\lambda_R}{64\pi^2} \right) + \frac{1}{64\pi^2} \left( \mu_R^2 + \frac{1}{2} \lambda_R \Phi^2 \right)^2 \ln \left( \frac{\mu_R^2 + \lambda_R \Phi^2}{\mu_R^2} \right).
$$

If one is only interested in the small \( \beta \) and large \( k \) physics, it is possible to approximate the above integral expressions. Setting \( k^2 \gg \mu_R^2 + \frac{1}{2} \lambda_R \Phi^2 \), we find, keeping only the leading order contributions:

$$
U_{\beta, k}(\Phi) \approx \frac{1}{2} \mu_R^2 (\beta, k) \Phi^2 + \frac{\lambda_R}{4!} \Phi^4,
$$

$$
\mu_R^2 (\beta, k) = \frac{\lambda_R}{16\pi^2} k^2 + \frac{\lambda_R}{4\pi^2 \beta^2} \sum_{n=1}^{\infty} e^{-n^2 k} \left( \frac{1}{n^2} + \frac{\beta k}{n} \right) + \cdots,
$$

where the following integrations have been used [7]:

$$
\int_0^r \frac{du \ u^\ell}{e^u - 1} = r^\ell \left[ \frac{1}{\ell} - \frac{r}{2(\ell + 1)} + \sum_{n=1}^{\infty} \frac{B_{2n+1}}{(2n+1)(2n)!} \right] \quad (\ell \geq 1),
$$

$$
\int_r^\infty \frac{du \ u^\ell}{e^u - 1} = \sum_{n=1}^{\infty} e^{-n r} \left[ \frac{r^\ell}{n + \frac{\ell (r - 1)}{r^2} \ell^2 + \cdots + \frac{\ell!}{n^\ell+1}} \right] \quad (\ell \geq 1).
$$

Note that the diverging structures in \( \mu_R^2 (\beta^{-1} = 0, k) \) and \( \lambda_R (\beta^{-1} = 0, k) \) resemble that of the bare quantities. However, discrepancy exists between the coefficients since for renormalizing the theory, it suffices to use \( T \)-independent counterterms which are usually derived with an \( O(4) \) symmetric smearing function \( \rho_k(x) \). However, \( \rho_k(x) \) is only \( O(3) \) symmetric for the present theory.

The most probable value of the average field according to (1.7) satisfies

$$
0 = \frac{\partial U_{\beta, k}}{\partial \Phi} = \Phi \left[ \mu_R^2 (\beta, k) + \frac{\lambda_R}{6} \Phi^2 \right],
$$

which shows that in addition to \( \Phi = 0 \), there exists other non-trivial solutions if \( \mu_R^2 (\beta, k) < 0 \). Consider first the case \( \mu_R^2 > 0 \) such that the theory is in the symmetric phase. Then \( U_{\beta, k}(\Phi) \) has only a unique minimum at \( \Phi = 0 \) for positive \( \mu_R^2 (\beta, k) \). However, for a fixed \( \beta \), there
always exists a critical value $k_{cr} \sim O(\mu_R/\sqrt{\lambda_R})$ beyond which $\mu_R^2(\beta, k)$ becomes negative. When this happens, $U_{\beta, k}(\Phi)$ has non-trivial minima at

$$\pm \Phi_{\beta, k} \approx \pm \left( -\frac{6\mu_R^2(\beta, k)}{\lambda_R} \right)^\frac{1}{2}.$$

(2.29)

This can be interpreted as having non-vanishing average coarse-grained background fields that are distributed primarily around $+\Phi_{\beta, k}$ or $-\Phi_{\beta, k}$ with equal probability, giving an overall zero average field $\Phi = 0$. In the same time the system as a whole retains its full symmetry.

Do we really have such a peculiar distribution of the collective variable $\Phi_{\beta, k}$ in the path integral for the Higgs field? To understand this issue better, let us turn to the four dimensional Ising model in the high temperature phase. The distribution of the average magnetization $s_v$ for the lattice of volume $v$ is given by

$$g_v(\Phi) = \langle \delta(0; v(\Phi - s_v)) \rangle = \sum_{\{s_x\}} \delta(0; v(\Phi - s_v)) e^{-\beta H},$$

(2.30)

where $H$ is the hamiltonian and $\delta(i; j)$ is the Kronecker delta and

$$s_v = \frac{1}{v} \sum_{a \in v} s_a.$$

(2.31)

In the ultraviolet limit with $v = 1$, it is easy to find individual spins according to (2.30):

$$g_1(\Phi) = \frac{1}{2} \left( \delta(1; \Phi) + \delta(-1; \Phi) \right).$$

(2.32)

On the other hand, in the infrared limit where $v \to \infty$, (2.30) gives the distribution of the global magnetization which, in the symmetrical phase, has a maximum at $\Phi = 0$ and a curvature $m_L^2 v$, where $m_L$ is the inverse correlation length, i.e. mass gap in lattice spacing units. It is not difficult to understand such “transmutation” of distribution from the double-peak to the single-peak shape as we move towards the infrared domain. In fact, if we divide the space-time into hypercubes with characteristic linear width $k^{-1}$, then for $k^{-1} > m_L^{-1}$ the magnetization $s_{k^{-1}}$ can be obtained by averaging over the magnetization $s_{m_L^{-1}}$ of hypercubes with size $m_L^{-1}$ which are inside of the hypercube of size $k^{-1}$. The correlation of the magnetization of these smaller hypercubes is suppressed by $O(e^{-m_L/\ell})$ and the central limit theorem can be invoked to give an approximate description of the distribution of their average, $s_{k^{-1}}$. Thus, $g_k(\Phi)$ will be a Gaussian distribution centered at $\langle s \rangle = 0$. Cross-over between the double-peak feature in the ultraviolet and the single-peak shape in the infrared limit can take place at

$$k_{cr} = O(m_L).$$

(2.33)

The long-range instability present in the symmetry broken phase of the theory would, however, prevents the distribution function $g(\Phi)$ from approaching the Gaussian limit in the infrared regime. We give a plot of $U_{\beta, k}(\Phi)$ in Fig. 2 for various values of $k$ at a fixed $T$. 
In the above treatment, $k_{cr}$ is qualitatively the scale in which the “effective mass” becomes zero. We associate it with the “correlation length” $\xi$ which defines the domain size via $\xi \sim k_{cr}^{-1}$. Note that this definition of correlation length differs from the usual one. Clearly, $\xi$ is $T$-dependent. This implies that $k_{cr}$ also depends on $T$. The manner in which $\xi$ varies with $T$ can be seen from:

$$
T \frac{\partial \xi}{\partial T} = -\frac{2}{\coth\left(\frac{\theta}{2}\right)} \left[ \frac{1}{e^\theta - 1} + \frac{2}{\nu} \sum_{n=1}^{\infty} \frac{e^{-n\nu}}{n} \left(1 + \frac{1}{n}\right) \right],
$$

(2.34)

where $\nu = \beta \xi^{-1}$. The negative sign in (2.34) shows that domain size decreases with increasing $T$. In Fig. 3, we demonstrate that for a large $k$ where the blocked potential is double-welled, it is always possible to adjust $T$ such that for $T > T_k$, $U_{\beta,k}(\Phi)$ again has unique minimum at $\Phi = 0$. Physically, this implies that an increase of temperature leads to a greater extent of randomness and results in a smaller domain size. That the two scale parameters, $k$ and $T$ generate opposite kind of changes in the distribution function can easily be seen from the way $\Phi_{\beta,k}$ varies. Namely, using (2.23), one has:

$$
k \frac{\partial \Phi_{\beta,k}^2}{\partial k} = -\left(\frac{6}{\lambda_R}\right) k \frac{\partial \mu_{\beta,k}^2}{\partial k} = \frac{3k^2}{4\pi^2} \left[ 1 + 2 \sum_{n=1}^{\infty} e^{-n\theta} \right] = \frac{3k^2}{4\pi^2} \coth\left(\frac{\theta}{2}\right) > 0,
$$

(2.35)

and

$$
T \frac{\partial \Phi_{\beta,k}^2}{\partial T} = -\left(\frac{6}{\lambda_R}\right) T \frac{\partial \mu_{\beta,k}^2}{\partial T} = -\frac{3k^2}{2\pi^2} \left[ \frac{1}{e^\theta - 1} + \frac{2}{\nu} \sum_{n=1}^{\infty} \frac{e^{-n\theta}}{n} \left(1 + \frac{1}{n}\right) \right] < 0,
$$

(2.36)

where $\theta = \beta k$. Adding these two terms together, one finds:

$$
R = k \frac{\partial \Phi_{\beta,k}^2}{\partial k} + T \frac{\partial \Phi_{\beta,k}^2}{\partial T} = \frac{3k^2}{4\pi^2} \left[ 1 - \frac{4}{\nu} \sum_{n=1}^{\infty} \frac{e^{-n\theta}}{n} \left(1 + \frac{1}{n}\right) \right],
$$

(2.37)

which has a maximum value of $R = \frac{3k^2}{4\pi^2}$ at $\theta = \infty$ ($T = 0$), and decreases as $T$ is raised. Numerically, the critical value at which $R = 0$ is found to be $\theta_{cr} = 1.55$. Below $\theta_{cr}$, $R$ becomes negative. This shows that with the given initial values for $T$ and $k$ which give a non-vanishing $\Phi_{\beta,k}$, an increase in $T$ that shifts $\Phi_{\beta,k}$ toward the origin can be overcome with a corresponding increase in $k$ at a ratio larger than or equal to $\theta_{cr}$.

While (2.21) includes the leading order self-energy for the particle in the heat bath, one must be careful in interpreting the temperature effect on the coupling constant using (2.22). Naive differentiation of (2.22) would give:

$$
T \frac{\partial \lambda_R(\beta, k)}{\partial T} = -\frac{3\lambda_R^2}{8\pi^2} \int_{\beta \sqrt{k^2 + \mu_R^2}}^{\infty} dx \sqrt{x^2 - \beta^2 \mu_R^2} \frac{e^x(e^x + 1)}{(e^x - 1)^3},
$$

(2.38)

which shows that within the framework of one-loop approximation, $\lambda_R(\beta, k)$ diminishes with increasing $T$. This feature can be substantiated by the physical observation of screening effect. However, it is not clear how $\lambda_R(\beta, k)$ behaves at very high $T$. As pointed out in [8] and [9],
the validity of perturbation theory becomes questionable at very high temperature, and that higher loop effects may also become important [10].

In passing, we show how the above arguments are connected to the case where discrete reflection symmetry \( \Phi \rightarrow -\Phi \) is spontaneously broken by requiring \( \mu_R^2 < 0 \). In this symmetry broken regime, the potential \( U_{\text{eff}}(\Phi) \) in (2.23) develops imaginary part for small \( \Phi \) due to the negative argument \( \mu_R^2 + \lambda_R \Phi^2 / 2 \) in the logarithm. This instability indicates that the dominant field configurations are not homogeneous anymore. The vacuum state is then dominated by configurations where the sign of the field is different. However, if we consider a blocked system instead, then the imaginary part of the local potential can be "smared out" by choosing a scale \( k^2 > \mu_R^2 + \lambda_R \Phi^2 / 2 \). For such \( k \) the domain size is smaller than the inhomogeneous fluctuations of the vacuum and the instability does not show up in the blocked potential. In this symmetry-broken regime where \( \mu_R^2(\beta, k) < 0 \) at low temperature, \( U_{\beta, k}(\Phi) \) has non-trivial local minima at \( \pm \Phi_{\beta, k} \). Now by universality we anticipate a second order phase transition at high temperature which restores the symmetry. The critical temperature above which symmetry is restored is given by

\[
\mu_R^2(\beta_c, k = 0) = 0,
\]

which yields [11]

\[
\frac{1}{\beta_c^2} \approx -\frac{24\mu_R^2}{\lambda_R}.
\]

Indeed \( \beta_c \) is a physical quantity independent of \( k \). Above the critical temperature, the potential again becomes reflection-symmetric with a minimum at \( \Phi = 0 \). Nevertheless, if we continue to increase \( k \), field fluctuations around some non-trivial minima will eventually occur.

### III. YUKAWA MODEL

We shall now consider the Yukawa model with massless fermions coupled to the \( \lambda \phi^4 \) scalar sector. The lagrangian for the theory is

\[
\mathcal{L}(\bar{\psi}, \psi, \phi) = \bar{\psi} \mathcal{M}(\phi) \psi + V(\phi),
\]

where

\[
\mathcal{M}(\phi) = \partial + g \phi,
\]

\[
V(\phi) = \frac{1}{2}(\partial \phi)^2 + \frac{\mu^2}{2} \phi^2 + \frac{\lambda}{4!} \phi^4,
\]

and \( g \) is the Yukawa coupling constant. Following the presentation of Section II, we decompose \( \phi \) into the fast modes \( \xi \) and the slow background \( \sigma \), and expand the fermionic fields about their classical configurations \( \tilde{\eta} \) and \( \eta \). The lagrangian can then be rewritten as:

\[
\mathcal{L}(\bar{\tilde{\eta}} + \tilde{\zeta}, \eta + \zeta, \sigma + \xi) = \mathcal{L}(\bar{\tilde{\eta}}, \eta, \sigma) + \tilde{\zeta} \mathcal{M}(\sigma) \xi + g \xi \left( \bar{\tilde{\eta}} \zeta + \tilde{\zeta} \eta \right) + \frac{1}{2} \xi K(\sigma) \xi
\]

\[
= \mathcal{L}(\bar{\tilde{\eta}}, \eta, \sigma) + \tilde{\zeta}' \mathcal{M}(\sigma) \xi' + \frac{1}{2} \xi K'(\tilde{\eta}, \eta, \sigma) \xi,'
\]
where
\[
\mathcal{K}'(\bar{\eta}, \eta, \sigma) = -\partial^2 + V''(\sigma) + 2g^2 \bar{\eta}(\bar{\varphi} + g\sigma)^{-1} \eta = \mathcal{K}(\sigma) + 2g^2 \bar{\eta}(\bar{\varphi} + g\sigma)^{-1} \eta, \tag{3.5}
\]
and
\[
\begin{cases}
\zeta' = \zeta + g \xi \bar{\eta} \mathcal{M}(\sigma)^{-1} \\
\zeta' = \zeta + g \xi \mathcal{M}(\sigma)^{-1} \eta.
\end{cases}
\tag{3.6}
\]

The equations above show that the effect of coupling between the scalar and the fermion sectors is to add to the original scalar field determinant \(\mathcal{K}\) an extra term proportional to the background fermion fields.

With the Jacobian of such field transformation being unity:
\[
D[\zeta'] D[\bar{\zeta}'] = D[\zeta] D[\bar{\zeta}],
\tag{3.7}
\]
one can easily perform the Gaussian integrations and obtain:
\[
\hat{S}_k[\bar{\eta}, \eta, \Phi] = -\ln \left\{ \int D[\sigma] D[\xi] D[\zeta'] D[\bar{\zeta}'] \prod_p \delta(\sigma - \Phi) \times \exp \left( -S[\bar{\eta}, \eta, \sigma] - \int_p \zeta' \mathcal{M}(\sigma) \zeta' - \frac{1}{2} \int_p \xi \mathcal{K}'(\bar{\eta}, \eta, \sigma) \xi + \cdots \right) \right\}
\tag{3.8}
\]
\[
= S[\bar{\eta}, \eta, \Phi] - \text{Tr} \ln \mathcal{M}(\Phi) + \frac{1}{2} \text{Tr} \ln \mathcal{K}'(\bar{\eta}, \eta, \Phi).
\]

Note that the one-loop contributions from the fermion and the scalar fields appear opposite in sign.

Concentrating only on the scalar sector, we set the background fermion fields to be zero and obtain:
\[
U_{\beta,k}(\Phi) = \frac{1}{2} \left( \mu_R^2 + \delta \mu^2 \right) \Phi^2 + \frac{1}{4!} \left( \lambda_R + \delta \lambda \right) \Phi^4
+ \frac{1}{2\beta} \sum_n \int_k^\infty \frac{dp p^2}{2\pi^2} \ln \left[ \omega_n^2 + p^2 + \mu_R^2 + \frac{\lambda_R}{2} \Phi^2 \right]
- \frac{2}{\beta} \sum_n \int_0^\infty \frac{dp p^2}{2\pi^2} \ln \left[ \omega_n^2 + p^2 + g^2 \Phi^2 \right].
\tag{3.9}
\]

Imposing the usual off-shell renormalization condition
\[
\lambda_R = \left. \frac{\partial^4 U_k}{\partial \Phi^4} \right|_{\Phi = \mu_R},
\tag{3.10}
\]
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then leads to

\[
U_k(\Phi) = \frac{\mu_R^2}{2} \Phi^2 \left(1 - \frac{\lambda_R}{64 \pi^2}\right) + \frac{\lambda_R}{4!} \Phi^4 \left(1 - \frac{9 \lambda_R}{64 \pi^2}\right) - \frac{g^4 \Phi^4}{16 \pi^2} \ln \left(\frac{\Phi^2}{\mu_R^2}\right) - \frac{25}{6}
\]

\[
+ \frac{1}{32 \pi^2} \left\{-k \left(2 k^2 + \mu_R^2 + \frac{\lambda_R}{2} \Phi^2\right) \left(k^2 + \mu_R^2 + \frac{\lambda_R}{2} \Phi^2\right)^{1/2}
\right.
\]

\[
+ \left(\mu_R^2 + \frac{\lambda_R}{2} \Phi^2\right)^2 \ln \left[\frac{k + \sqrt{k^2 + \mu_R^2 + \lambda_R \Phi^2/2}}{\mu_R}\right]
\]

\[
- \frac{1}{2 \pi^2 \beta} \int_k^\infty dp p^2 \ln \left[1 - e^{-\beta \sqrt{p^2 + \mu_R^2 + \lambda_R \Phi^2/2}}\right]
\]

\[
- \frac{2}{\pi^2 \beta} \int_0^\infty dp p^2 \ln \left[1 - e^{-\beta \sqrt{p^2 + g^2 \Phi^2}}\right].
\]

(3.11)

Keeping only the leading terms in the limit where \(k^2, T^2 \gg \mu_R^2 + \lambda_R \Phi^4/2\) and \(g^2 \Phi^2\), we arrive at

\[
U_{\beta,k}(\Phi) \approx \frac{1}{2} \mu_R^2(\beta, k) \Phi^2 + \frac{\lambda_R}{4!} \Phi^4 - \frac{g^4 \Phi^4}{16 \pi^2} \ln \left(\frac{\Phi^2}{\mu_R^2}\right) - \frac{25}{6}.
\]

(3.12)

where, by the help of (2.27),

\[
\mu_R^2(\beta, k) = \mu_R^2 - \frac{\lambda_R}{16 \pi^2} k^2 + \frac{1}{12 \beta^2} \left[\frac{3 \lambda_R}{\pi^2} \sum_{n=1}^\infty e^{-n \beta k} \left(\frac{1}{n^2} + \frac{\beta k}{n}\right) - 4 \beta^2\right].
\]

(3.13)

Here, one must be careful in choosing \(\lambda_R\) and \(g\), to avoid \(U_{\beta,k}(\Phi)\) from becoming unbound from below [12]. However, the general feature of field fluctuations around non-trivial minima at some observational length scale \(k^{-1}\) remains unaffected, since the field where the local maximum of \(U_{\beta,k}(\Phi)\) is attained, lies considerably far away from the origin. The condition for existence of non-trivial minimum for the potential implies

\[
0 = \mu_R^2(\beta, k) + \Phi^2 \left(\frac{\lambda_R}{6} + \frac{11 g^4}{12 \pi^2}\right) - \frac{g^4 \Phi^2}{4 \pi^2} \ln \left(\frac{\Phi^2}{\mu_R^2}\right),
\]

(3.14)

which becomes, for \(k = 0\),

\[
0 = \mu_R^2 + \frac{\lambda_R - 8 g^2}{24 \beta^2} + \Phi^2 \left(\frac{\lambda_R}{6} + \frac{11 g^4}{12 \pi^2}\right) - \frac{g^4 \Phi^2}{4 \pi^2} \ln \left(\frac{\Phi^2}{\mu_R^2}\right).
\]

(3.15)

We shall choose \(\lambda_R > 8 g^2\) such that an increase in temperature will tend to restore the symmetry. We illustrate the dependence of field fluctuations on \(k\) at a fixed temperature in Fig. 4.
IV. MASSIVE SCALAR ELECTRODYNAMICS

The results derived in Section II for the $\lambda\phi^4$ theory can be readily applied to scalar QED which is defined by the following lagrangian density:

$$L_{\text{SQED}} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - \frac{1}{2\alpha} (\partial_{\mu} A_{\mu})^2$$

$$+ \frac{1}{2} m^2 \phi(x)^2 + \frac{\lambda}{6} (\phi(x))^4,$$

where $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$. The complex field, $\phi(x)$, will be parameterized as $(\phi_1(x) + i\phi_2(x))/\sqrt{2}$, where $\phi_1$ and $\phi_2$ are real. In computing the scalar QED blocked potential at finite temperature, it is desirable that the blocking scale $k$ be introduced into the theory in a gauge invariant manner for preserving gauge symmetry. A sharp cut-off smearing function $\rho_k(p)$ which explicitly violates such symmetry should not be used. The scheme that contains the scale $k$ and yet is gauge invariant does exist, and it is similar to the Pauli-Villars regularization method \[13\]. The way how $\Lambda$ and $k$ enter gauge invariantly as the UV and IR regulator, respectively, can easily be seen by invoking the Schwinger proper-time formalism, in which the functional determinant of a general second-order operator $\mathcal{H}$ is represented by:

$$\ln \left( \frac{\mathcal{H}}{\mathcal{H}_0} \right) = -\int_0^\infty ds \left( e^{-\mathcal{H}s} - e^{-\mathcal{H}_0s} \right),$$

where the expression is normalized with respect to the corresponding operator $\mathcal{H}_0$ evaluated at vanishing field. If we introduce into the proper-time integration a smearing function

$$\rho(z) = 1 - (1 + z)e^{-z},$$

(4.3) then becomes, setting $z = \Lambda^2 s$,

$$-\int_0^\infty ds \rho(\Lambda^2 s) \left( e^{-\mathcal{H}s} - e^{-\mathcal{H}_0s} \right) = \ln \left[ \frac{\mathcal{H}}{\mathcal{H}_0} \right] - \frac{\Lambda^2 (\mathcal{H} - \mathcal{H}_0)}{(\Lambda^2 + \mathcal{H})(\Lambda^2 + \mathcal{H}_0)},$$

which is the Pauli-Villars regularized version containing a UV cut-off. The manner in which the propagator is modified can also be seen by the following relation:

$$\frac{1}{\mathcal{H}} = \int_0^\infty ds e^{-\mathcal{H}s} \rightarrow \int_0^\infty ds \rho(\Lambda^2 s) e^{-\mathcal{H}s} = \frac{1}{\mathcal{H}} \left( 1 - \frac{\mathcal{H}}{\Lambda^2 + \mathcal{H}} \right)^2.$$

(4.5) If we now simply choose

$$\rho_k(s, \Lambda) = \rho(\Lambda^2 s) - \rho(k^2 s),$$

(4.6) then $k$ enters into the theory without breaking gauge symmetry. Therefore, when evaluating one loop contribution of the blocked potential, one simply makes the substitution:

$$\int_k^{\Lambda} d^4 p \frac{d^4 p}{(2\pi)^4} \ln \left( \frac{p^2 + V''}{p^2 + V_0''} \right) \rightarrow -\int_0^\infty d^4 p \frac{d^4 p}{(2\pi)^4} \int_0^\infty \frac{ds}{s} \rho_k(s, \Lambda) e^{-p^2 s} \left( e^{-V''s} - e^{-V_0''s} \right)$$

$$= \int \frac{d^4 p}{(2\pi)^4} \left\{ \frac{k^2 V'' - V_0''}{k^2 + p^2 + V''} \right\} - \frac{\Lambda^2 (V'' - V_0'')}{(\Lambda^2 + p^2 + V'')(\Lambda^2 + p^2 + V_0'')}$$

$$+ \ln \left[ \frac{k^2 + p^2 + V'}{k^2 + p^2 + V_0'} \right] \right\} \right\}$$

$$= \frac{1}{32 \pi^2} \left\{ (\Lambda^2 - k^2)V'' - (V'')^2 \ln \left( \frac{\Lambda^2 + V''}{k^2 + V''} \right) + \Lambda^4 \ln \left( \frac{\Lambda^2 + V''}{\Lambda^2 + V_0''} \right) - k^4 \ln \left( \frac{k^2 + V''}{k^2 + V_0''} \right) \right\}. $$

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Incidentally, both gauge-invariant (proper-time) and gauge non-invariant (sharp momentum regularized) expressions give the same result, up to some $\phi$-independent constants. The two blocking procedures differ in the sense that one classifies the modes according to their length scale, $k^{-1}$ and the other according to their weight in the "partition function", $s$. In the ultraviolet regime where the eigenvalue spectra of the fluctuation operator is dominated by the kinetic energy term, blocking transformation introduced in the preceding Sections and the present construction yield the same result. However, as one approaches the infrared regime the contribution from the potential term becomes more important in determining the eigenvalues of the fluctuation operator. In this limit, the length scale $k^{-1}$ would only be characteristic of a smooth cut-off procedure in space-time. As long as the underlying low-energy excitations of the system are extended, the two cut-off methods can be comparable. However, for a system having localized low-energy states, the functional similarity exhibited in the two different blocking approaches is slightly misleading.

In computing the finite-temperature blocked potential it is convenient to work with Landau gauge $a = 0$. With $\Phi^a = \Phi^a e^{i \frac{1}{2} \mu^2}$ and by performing blocking for the scalar field only one has the following result:

$$U_{\beta, k}(\Phi) = \frac{\mu_R^2}{2} \Phi^2 \left(1 - \frac{\lambda_R}{48 \pi^2}\right) + \frac{\lambda_R}{4!} \Phi^4 \left(1 - \frac{5 \lambda_R}{32 \pi^2}\right)$$

$$+ \frac{1}{32 \pi^2} \left\{ -k \left(2 k^2 + \mu_R^2 + \frac{\lambda_R}{6} \Phi^2 \right) \left(2 k^2 + \mu_R^2 + \frac{\lambda_R}{6} \Phi^2 \right)^{1/2} \right\}$$

$$- k \left(2 k^2 + \mu_R^2 + \frac{\lambda_R}{6} \Phi^2 \right) \left(2 k^2 + \mu_R^2 + \frac{\lambda_R}{6} \Phi^2 \right)^{1/2}$$

$$+ \left(\frac{\mu_R^2 + \lambda_R}{2} \Phi^2 \right)^2 \ln \left[\frac{k + \sqrt{k^2 + \mu_R^2 + \lambda_R \Phi^2}}{\mu_R^2} \right]$$

$$+ \left(\frac{\mu_R^2 + \lambda_R}{6} \Phi^2 \right)^2 \ln \left[\frac{k + \sqrt{k^2 + \mu_R^2 + \lambda_R \Phi^2}}{\mu_R^2} \right]$$

$$+ \frac{1}{2 \pi^2} \beta \left\{ \int_k^\infty dpp^2 \ln \left[1 - e^{-\beta \sqrt{p^2 + \mu_R^2 + \lambda_R \Phi^2}}\right] \right\}$$

$$+ \int_k^\infty dpp^2 \ln \left[1 - e^{-\beta \sqrt{p^2 + \mu_R^2 + \lambda_R \Phi^2}}\right]$$

$$+ 3 \int_0^\infty dpp^2 \ln \left[1 - e^{-\beta \sqrt{p^2 + \epsilon^2 \Phi^2}}\right]$$

where the renormalized coupling constant $\lambda_R$ is again defined by an off-shell description (3.10). In the regime where $k^2, T^2 \gg \mu_R^2 + \lambda_R \Phi^4/2$ and $\epsilon^2 \Phi^2$,

$$U_{\beta, k}(\Phi) \approx \frac{1}{2} \mu_R^2(\beta, k) \Phi^2 + \frac{\lambda_R}{4!} \Phi^4 + \frac{3 \epsilon^4}{64 \pi^2} \Phi^4 \left[\ln \left(\frac{\Phi^2}{\mu_R^2}\right) - \frac{25}{6}\right],$$

where

$$\mu_R^2(\beta, k) = \mu_R^2 - \frac{\lambda_R}{12 \pi^2} k^2 + \frac{1}{6 \beta^2} \left[\frac{2 \lambda_R}{\pi^2} \sum_{n=1}^\infty \epsilon^{-n \beta k} \left(\frac{1}{n^2} + \frac{\beta k}{n}\right) + \frac{3}{2} \epsilon^2\right].$$
The minimum of \( U_{\beta,k}(\Phi) \) is again located by

\[
0 = \mu_R^2(\beta, k) + \Phi^2 \left( \frac{\lambda_R}{6} - \frac{11\epsilon^4}{16\pi^2} \right) + \frac{3\epsilon^4\Phi^2}{16\pi^2} \ln \left( \frac{\Phi^2}{\mu_R^2} \right),
\]

(4.11)

We first note that for \( U_{1/\beta=0}(\Phi) \) to have non-trivial minimum, the condition

\[
\lambda_R < \frac{9\epsilon^4}{8\pi^2} \left\{ \frac{8}{3} - \ln \left( \frac{16\pi^2}{3\epsilon^4} \right) \right\}
\]

(4.12)

must be satisfied. For small \( \epsilon \), the expression above leads to a negative quartic coupling \( \lambda_R \) for the theory, which is in contradiction with our original assumption. Therefore, as long as \( \lambda_R > 0 \), \( U_{1/\beta=0}(\Phi) \) always has unique minimum at \( \Phi = 0 \) for the theory in the symmetric phase, with \( \mu_R^2 > 0 \). Again, we plot the \( k \) dependence of \( U_{\beta,k}(\Phi) \) for a given set of parameters in Fig. 5.

**V. DYNAMICAL MASS GENERATION**

We explore in this Section the mechanism of mass parameter generation according to the results obtained before. As stated in the Introduction, the novelty of this construction is that it is in effect even when the expectation value of the scalar field is vanishing. The foundation of this mechanism is based on the following three arguments: (1) within a sufficiently small region \( \mathcal{R} \) such that the linear dimension of \( \mathcal{R} \) is smaller than \( \zeta \sim k_{\text{c}}^{-1} \), the correlation length, the scalar field fluctuates around different locally non-vanishing values even in the symmetrical phase with an overall zero field expectation value; (2) what is relevant from the point of view of the mass generation is the behavior of the scalar field within the region where the particle propagates, and not the whole space-time; and (3) at sufficiently high temperature, \( \mathcal{R} \) is so short in the time direction that the local field values are correlated inside. As we shall see below, if this effective propagation region is smaller than the domain size, then within a certain range of coupling constants it is possible to obtain the non-trivial solution for mass parameter in a self-consistent manner.

Consider now a fermionic test particle described by the field \( \psi(x) \) and the lagrangian

\[
\mathcal{L} = \bar{\psi}(x)(\partial_x \psi(x) + U(x) + G\phi(x))\psi(x).
\]

(5.1)

We choose a fermion field here but our qualitative remarks hold for scalar as well as vector fields. Suppose the static external potential, \( U \) is strong enough to create a bound state of size \( L \) consisting of a very heavy anti-fermion and the massless test particle. The bound state wave function will then be localized in a space-time “tube” \( \mathcal{R} \) which is infinitely long in the time direction and has a width \( L \) in the spatial direction. The fermion, due to the binding effect, is constrained to stay in the vicinity of the heavy one. Thus, the region of space-time traced by the fermion is practically a tube consisting of points with \( x^2 < 1/k^2 \), where \( 1/k \) is the tube radius characterizing the size of the bound state.

What determines the mass parameter for the fermion in this bound state? It is perhaps much easier to answer this question by inquiring what is unimportant from the fermion’s point of view. Obviously, how the scalar field behaves outside the tube is irrelevant since
the fermion is never there. The mass parameter should then be determined solely by the dynamics of the scalar field, \( \phi(x) \), within the tube.

The binding mechanism alone is sufficient to generate a non-vanishing mass parameter which otherwise would have been prohibited from the symmetry argument of the lagrangian. A renowned example to substantiate the effect of binding on mass generation is the chiral symmetry breaking and the generation of valence quark mass in the bag model. In this model, it is just the boundary condition on the quark wave functions which leads to the mixing of small and large spinor components and generation of mass parameter by "binding" the wave functions to fit inside the bag.

Since it is the fluctuations of the scalar field \( \phi(x) \) within the tube that determine the mass parameter of the bound test particle through the Yukawa coupling in (5.1), one simply concentrates on the interior of the tube and isolates the "most important" mean field values for \( \phi(x) \) from determining the maxima of the distribution function:

\[
\varrho_M(\Phi) = \langle \delta(\Phi - \phi_R) \rangle = \int D[\phi] e^{iS_M[\phi]} \delta(\Phi - \phi_R),
\]

where \( S_M[\phi] \) is the Minkowski space-time action and the blocked variable \( \phi_R \) is constructed by (1.1) in Minkowski space-time as well. The relation between the distribution function (5.2) and the local potential in the blocked action, \( \mathcal{U}_R^{(0)}(\Phi) \) is given by (1.7). The surface to volume ratio of the tube \( R \) stays finite as the time extent of the tube tends to infinity. Thus the coupling between "neighboring" tubes is strong and the approximation given in (1.7) becomes invalid.

However, we have good reason to trust (1.7) at high temperature. This is because then the fluctuations are strongly damped and suppress the contributions of the derivative terms in the blocked action. A more formal way to see this is to note that the time extent of the Euclidean system is small at high temperature and thus (1.7) is justified. Since the distribution of the static modes agree in the Minkowski and the Euclidean space-time, (1.7) holds for both cases as well. The results presented above indicates that there is a non-zero mass parameter generated in this manner for small enough localized states either in the Yukawa-model or in scalar QED.

Fundamentally, there are differences in the way bosons and fermions acquire mass; namely, the former couples quadratically to the scalar field, while the latter couples only linearly. For the bosons, although the two most probable Higgs field averages \( \pm \Phi_{\beta,k} \), differ in sign, the mass squared generated from either contribution nevertheless agree; however, for fermions, the two background field values lead to opposite signs for the generated mass. We argue that the fermionic Green’s functions must retain the chirality-even sector only. In fact, since chiral transformation changes the sign of the mass term due to the presence of \( \gamma_5 \), only the contributions with even chirality will survive the averaging over \( \pm \Phi_{\beta,k} \). In that case, we expect the single-particle fermionic propagator to be different from the usual one. The propagator for the bound state of even number of fermions will remain the same, however.

The above consideration for a bound state system is not particularly illuminating. The truly challenging and important issue that remains, is how to implement such mechanism to point-like particles, in which the presence of a non-zero mass parameter is excluded at the
tree level by the gauge symmetry of the lagrangian. Let us consider now a point-like, free test particle described by the lagrangian:

\[ \mathcal{L} = \bar{\psi}(x)(\partial + G\phi(x))\psi(x) = \bar{\psi}(x)(\partial + m)\psi(x) + (G\phi(x) - m)\bar{\psi}(x)\psi(x) \]  

(5.3)

where the "variational" mass parameter, \( m \), is to be chosen in a \( \phi(x) \)-dependent manner such that the perturbation expansion in \( \delta K \) for the propagator,

\[ < G > = \frac{1}{K_0 + \delta K} = \int D[\phi]e^{-S[\phi]} \frac{1}{K_0[\phi] + \delta K[\phi]}, \]

(5.4)

is optimized.

In order to determine the form of \( m[\phi] \), we must first find approximatively the space-time region where the test particle propagates. In the zeroth order of \( \delta K \), we follow the proper-time parametrization due to Schwinger and write

\[ G(x, y) = \left< \frac{m - \partial}{m^2 - \partial^2} \right| y \right> = (m - \partial) \int_0^\infty ds < x | e^{-s(m^2 - \partial^2)} | y >. \]  

(5.5)

The matrix element in the second equation can be obtained in the path integral representation. In fact, it is the Euclidean time evolution operator matrix element corresponding to the "hamiltonian"

\[ H = m^2 - \partial^2, \]  

(5.6)

which acts on the "wave functions" defined over the four dimensional space-time. The repetition of the standard steps lead to the path integral formulae for our free 4+1 dimensional problem,

\[ G(x, y) = (m - \partial) \int_0^\infty ds \int z(s) = y D[z''(\tau)]e^{-s m^2 - \frac{1}{4} \int_0^s d\tau z''(\tau)}. \]  

(5.7)

Here the functional integration is performed over trajectories in space-time which connect the points \( x \) and \( y \) as the proper time, \( \tau \), moves from 0 to \( s \). The functional integral is Gaussian and can be carried out to give:

\[ G(x, y) = \frac{1}{16\pi^2}(m - \partial) \int_0^\infty ds e^{-s m^2 - \frac{1}{4s} \frac{|x - y|^2}{s}}. \]  

(5.8)

This is the immediate consequence of the relation

\[ < x | e^{\frac{\partial^2}{s^2}} | y > = \frac{1}{\sqrt{4\pi s}} e^{-\frac{|x - y|^2}{4s}}, \]  

(5.9)

for the one dimensional propagator in quantum mechanics.

The trajectories in the functional integration are restricted to be within a finite-width world tube, the shape of which would then determine the distribution of the field \( \phi(x) \) experienced by the test particle. Having obtained the world tube that corresponds to the test
particle propagator, it is sufficient to consider only the behavior of the scalar field within this tube. The non-Poincaré invariant aspect of this reasoning poses no difficulty since the location and shape of the tube introduced here are also dependent on the locality of creation and annihilation of the test particle.

In evaluating $G(x,y)$, the propagator of a free, unbound particle at large space-time separation, $x - y \to \infty$, there are in principle, two relevant length scales which appear in the distribution of the particle location in the hyperplane perpendicular to $x - y$. The first one is the width of the world tube, $\ell$, which gives account of the region of space-time traversed by the particle. The other length scale, $S$, is the characteristic size of the particle itself. These two length scales are independent. In fact, $S$ depends on the dynamics of the constituents of the particle. This is obvious for a composite particle. For a point particle we have the unavoidable particle-anti particle cloud which generates the finite extent $S$. It will be sufficient to determine $\ell$ for our purposes. Since $m$ is the only scale parameter of the problem we must have

$$\ell \approx \frac{c}{m}. \quad (5.10)$$

We are now in the position to determine the optimal choice of $m$. To minimize the effect of $\delta K$ we obviously have to choose

$$m = \pm G\Phi_{\beta,k}, \quad (5.11)$$

where $\Phi_{\beta,k}$ is the most probable average field for the world tube of the unbound test particle. The sign of $m$ follows that of the actual average field in (5.4). This relation leaves the width of the world tube, $\ell = k^{-1}$, undetermined. The self-consistent choice of $k$ is such that in the corresponding world tube, the most probable scalar field average, $\Phi_{\beta,k}$, reproduces the same mass parameter, i.e.

$$G\Phi_{\beta,k}=m/c = m. \quad (5.12)$$

Note that this relation holds in the symmetry broken phase as well.

We can now turn to the models discussed in the previous Sections. For the Yukawa theory considered in Section III, the self-consistent equations for the fermion mass parameter are

$$\begin{cases} 
0 = \mu_R^2(\beta,k) + \Phi_{\beta,k}^2 \left( \frac{\lambda_R}{6} + \frac{11g^4}{12\pi^2} \right) - \frac{g^4\Phi_{\beta,k}^2}{4\pi^2} \ln \left( \frac{\Phi_{\beta,k}^2}{\mu_R^2} \right) \\
M_\psi = g\Phi_{\beta,k} = c_1 k, 
\end{cases} \quad (5.13)$$

which is reduced to

$$0 = \frac{\mu_R^2}{48\pi^2g^2c_1^2} \left\{ 2g^2 \left[ 1 + \frac{4g^2 c_1^2}{\lambda_R} \ln \left( \frac{M_\psi^2}{\mu_R^2} \right) \right] - 8c_1^2 \pi^2 \left( 1 + \frac{11g^4}{2\lambda_R \pi^2} \right) \right\}$$

$$- \frac{g^2 T^2}{3} \left[ 1 - \frac{3\lambda_R}{4g^2 \pi^2} \sum_{n=1}^{\infty} e^{-\frac{\mu_R}{T c_1}} \left( \frac{1}{n^2} + \frac{M_\psi^2}{n T c_1} \right) \right], \quad (5.14)$$

upon use of (3.14). The mass parameter $M_\psi(T)$ is plotted in Fig. 6 for $\lambda_R = 0.4$, $c = 0.01$ and $g = 0.2$ and 0.15. Note that the validity of $M_\psi$ for low $T$ may become questionable since the formalism of mass parameter is expected to break down. Qualitatively, our result is in agreement with [14] in which the author obtained a larger Euclidean mass parameter with increasing Matsubara frequencies, although our conclusion is based on a different, mean-field.
argument. The decrease of the mass parameter at lower temperature may not be significant since (1.7) is no longer appropriate.

For the massive scalar QED, we shall consider its impact on the mass parameter of another scalar particle $\varphi$. The lagrangian of the entire system is written as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2}G\dot{\varphi}^2 + \mathcal{L}_{\text{SQED}},$$

where $\mathcal{L}_{\text{SQED}}$ is given by (4.1). Even though the background Higgs field may have vanishing vacuum expectation value, $<\varphi(x)> = 0$, $<\varphi^2(x)>$ needs not necessarily be zero, as we have shown before. The leading mean-field contribution to the mass parameter of $\varphi$ would then be $G\dot{\varphi}^2$, where $G$ is the interaction strength. In this case, the self consistent equations for the scalar particle in scalar QED would be

$$\begin{cases}
0 = \mu_R^2(\beta, k) + \Phi_{\beta, k}^2 \left( \frac{\lambda_R}{6} - \frac{11e^4}{16\pi^2} \right) + \frac{3e^4\Phi_{\beta, k}^2}{16\pi^2} \ln \left( \frac{\Phi_{\beta, k}^2}{\mu_R^2} \right) \\
M_\varphi = G\Phi_{\beta, k} = c_2 k,
\end{cases}$$

or

$$0 = \mu_R^2 - \frac{\lambda_R M_\varphi^2}{48\pi^2 G^2 e^2} \left( 4G^2 \left[ 1 - \frac{9e^4}{4\lambda_R} \ln \left( \frac{M_\varphi^2}{G^2 \mu_R^2} \right) \right] - 8e^2 \pi^2 \left( 1 - \frac{33e^4}{8\lambda_R \pi^2} \right) \right)$$

$$+ \frac{e^2 T^2}{4} \sum_{n=1}^{\infty} e^{-\frac{M_\varphi}{n^2 + M_\varphi}} \left( \frac{1}{n^2 + M_\varphi} + \frac{M_\varphi}{n T c_2} \right).$$

Once more, this non-linear equation can only be solved for certain values of coupling constants, and in general they may not be unique. In Fig. 7, we depict the $T$ dependence of the mass parameter for $c_2 = 0.05$ and $c_2 = 0.1$. There, we observe a very slow increase in $M_\varphi$ for low $T$, followed by a rapid growth. The interpretation is qualitatively the same as that of Yukawa model.

In summary, we find that at high temperature where the static modes dominate the Euclidean path integral, a test particle which propagates and interacts with the background Higgs field can acquire a non-vanishing mass parameter in the mean-field approximation if its path of propagation is a narrow tube within which the scalar Higgs field has non-vanishing most probable values. If it scans through a large region of space-time, then the average Higgs field would certainly vanish. However, noting that how the particle propagates is directly influenced by this mass parameter, we solve the coupled self-consistent equations and conclude that the mass parameter for the test particle may be non-vanishing for a chosen set of coupling constants.
V. SUMMARY

We have shown that in a certain temperature range, the finite-temperature blocked potential \( U_{\beta,k}(\Phi) \) develops minima at non-vanishing field values for short enough length scale, \( k^{-1} < k_{cr}^{-1} \), even in the symmetrical phase, and that the appearance of such degenerate minima differs from the phenomenon of spontaneous symmetry breaking. Although the field average computed in the finite region fluctuates around two symmetrical non-zero values, this has nothing to do with the behavior of the extreme infrared mode, \( \phi_{k=0} = \frac{1}{\Omega} \int d^4 x \phi(x) \), \( \Omega = \int d^4 x \). All we see is a distinct scale dependence of the fluctuations which suggests that the typical field configurations of the symmetrical phase contain domains, \( \phi(x) \sim \pm \Phi_{\beta,k,cr} \), of the size \( \zeta \sim k_{cr}^{-1} \). This is because the blocked variables fluctuate around zero or \( \pm \Phi_{\beta,k} \) for \( \zeta > k_{cr}^{-1} \) or \( \zeta < k_{cr}^{-1} \), respectively, in the presence of the domains.

Various interesting directions may be pursued starting with this result. One is that the Landau-Ginsburg theory may exhibit domain structure even in the symmetrical phase. That is, the fields may fluctuate around non-zero values even in the absence of long-range order. For example, although an Ising system becomes paramagnetic with average magnetization \( M = 0 \) at \( T > T_c \), where \( T_c \) is the Curie temperature, there may exist local, ferromagnetic domains with \( M = \pm 1 \). The size of a typical domain should be \( O(k_{cr}^{-1}) \), a decreasing function of the temperature.

The possibility of mass parameter generation without symmetry breaking was explored in this paper by considering a Higgs field coupled to massless fermions via Yukawa coupling, or to another scalar particle in the scalar QED background. Within a flux tube of characteristic width \( k^{-1} \) along the world line, the fermions feel the fluctuations of the Higgs field within a "radius" \( \sim k^{-1} \). If this is smaller than \( k_{cr}^{-1} \), then the Higgs field may generate mass for fermion even if the theory is in the symmetrical phase. It was found that the temperature generates a thermal mass and constrains the blocked variable to be within distance \( O(1/T) \) from zero. This result is in agreement with the expectation that the domain size should decrease as the temperature is increased. Therefore, when the effects of the heat bath are included, we need a larger scale energy in order to observe mass generation. Consequently, the production of small mass particles is suppressed.

It seems that this alternative mechanism is no more realistic than the one based on spontaneous symmetry breaking, for it too, involves a scalar "Higgs" field, albeit in a manner which breaks no symmetry. Despite the fact that our presentation here offers only a rough, mean-field solution to mass parameter generation, we believe that this novel approach for the old problem is worth communicating even at its rudimentary level, in order to stimulate further thoughts and explorations along this direction. The fact that the self-consistent conditions given in Section V for deducing mass parameters for unbound particles could only be satisfied by choosing small \( c \) may also be undesirable for investigating the feed-back to the scalar field from the test particle. Nevertheless, it can be taken into account with strong coupling expansion and we plan to return to this problem in a separate publication.

We also comment that phenomena similar to our discussion of mass generation have also been considered before for two-dimensional theories in which dynamical breakdown of continuous symmetries is not possible due to the severe infrared divergences \([15]\). Mass generation in the \( SU(N) \) Thirring \([16]\) and Yukawa \([17]\) models in \( d = 2 \) has been attained, with long-range correlations in the absence of any symmetry breaking effect.

We conclude that the temperature dependence is a useful tool to trace back the role the field fluctuations play in the theory. It allows us to vary domain size and mass parameters in a distinctive manner. It would be interesting to observe such effects experimentally in some easily accessible systems.
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FIGURE CAPTIONS

Fig. 1. Momentum space representation of $O(3)$ and $O(4)$ symmetric smearing function.

Fig. 2. Dependence of $U_{\beta,k}(\Phi)$ on $k$ for $\lambda_R = 0.1$ and $T = 20$, in units of $\mu_R$.

Fig. 3. Dependence of $U_{\beta,k}(\Phi)$ on $T$ for $\lambda_R = 0.1$ and $k = 50$ in unit of $\mu_R$. Fluctuations around non-trivial minima will disappear for $T \geq T_k$.

Fig. 4. Dependence of $U_{\beta,k}(\Phi)$ on $k$ for Yukawa model with $\lambda_R = 0.4$, $g = 0.2$ and $T = 15$, in units of $\mu_R$.

Fig. 5. Dependence of $U_{\beta,k}(\Phi)$ on $k$ for Massive scalar QED with $\lambda_R = 0.4$, $\epsilon = 0.02$ and $T = 20$, in units of $\mu_R$.

Fig. 6. Temperature dependence of the self-consistent mass parameter $M_\phi$ in Yukawa model.

Fig. 7. Temperature dependence of the self-consistent mass parameter $M_\varphi$ in massive scalar QED.

REFERENCES


[5] We thank M. Tsypin for pointing this out for us.


APPENDIX A

In this Appendix, we give an estimate of the width, $\ell$, of the tube where the test particle with mass $M$ propagates by computing $\langle\langle z_{tr}^2 \rangle\rangle_{tr}$. Here $z_{tr}$ stands for the transverse part of $z - x$:

$$z_{tr} = z - x - \frac{(y - x)(y - x)}{(y - x)^2}.$$  \hspace{1cm} (A.1)

The double bracket, $\langle\langle \cdots \rangle\rangle_{tr}$, denotes the normalized averaging over the trajectories in the functional integration (5.7) of the average along the trajectories. The latter, the averaging along the trajectories can be obtained by splitting the proper time, $s$, into two subsequent parts, $s = t + u$. The point, $z$, on the trajectory is then taken at the proper time $t$ and the values of $t$ and $u$ are integrated over,

$$\langle\langle z_{tr}^2 \rangle\rangle_{tr} = \mathcal{N}^{-1} \int_0^\infty \frac{ds}{s} \int_0^\infty dt \int_0^\infty du \delta(s - t - u) \int D[z] z^2(t)_{tr} e^{-m^2(t + u) - \frac{1}{4} \int_0^t d\tau \tau^2(\tau)}$$

$$= \mathcal{N}^{-1} \int_0^\infty dt \int_0^\infty du \frac{1}{t + u} \int D[z] z^2(t)_{tr} e^{-m^2(t + u) - \frac{1}{4} \int_0^{t+u} d\tau \tau^2(\tau)},$$  \hspace{1cm} (A.2)

where $\mathcal{N}^{-1}$ is the normalization constant,

$$\mathcal{N} = \int_0^\infty ds \int D[z] e^{-m^2 s - \frac{1}{4} \int_0^s d\tau \tau^2(\tau)}.$$  \hspace{1cm} (A.3)

The functional integration is now written as the integral over trajectories corresponding to the proper time intervals $0 < \tau < t$ and $t < \tau < t + u$ and connecting the space-time points $x, z$ and $z, y$, respectively when $z$ is integrated over, as well. The functional integration can be carried out yielding

$$\langle\langle z_{tr}^2 \rangle\rangle_{tr} = \frac{1}{(4\pi)^4} \int \frac{dt}{(t + u)^2 u^2} \int d^4 z z_{tr}^2 e^{-m^2(t + u) - \frac{(x - z)^2}{4u} - \frac{(y - z)^2}{4u}}.$$  \hspace{1cm} (A.4)

The $t$ and $u$ integrals are evaluated in the saddle point approximation. By retaining the exponent in the selection of the saddle point we have, $t_0 = |z - x|/2m$ and $u_0 = |y - z|/2m$, which give

$$\langle\langle z_{tr}^2 \rangle\rangle_{tr} = \frac{m^2 \mathcal{N}^{-1}}{16 \pi^3} \int d^4 z z_{tr}^2 \frac{\sqrt{|z - x||y - z|}}{|z - x| + |y - z|} e^{-m(|z - x| + |y - z|)}.$$  \hspace{1cm} (A.5)

Upon rescaling the variables, $y \rightarrow my$, $z \rightarrow mz$, we obtain, for $x = 0$,

$$\langle\langle z_{tr}^2 \rangle\rangle_{tr} = \frac{1}{m^2} \mathcal{N}^{-1} \int d^4 z (z^2 - (n \cdot z)^2) \frac{\sqrt{|z - x||y - z|}}{|z - x| + |y - z|} e^{-|z - y|},$$  \hspace{1cm} (A.6)

where $n = \frac{y}{|y|}$. The normalization can be obtained in a similar manner,

$$\mathcal{N} = \frac{1}{16 \pi^3} \int d^4 z \frac{\sqrt{|z - x||y - z|}}{|z - x| + |y - z|} e^{-|z - y|},$$  \hspace{1cm} (A.7)
yielding

\[
\langle \langle z_{tr}^2 \rangle \rangle_{tr} = \frac{1}{m^2} \lim_{y \to -\infty} \frac{\int d^4z \left( z^2 - (n \cdot z)^2 \right) \frac{\sqrt{|x-y||x-z|}}{|x-y|+|y-z|}(y-z)\tau e^{-\frac{1}{2}(|x-y|+|y-z|)}}{\int d^4z \frac{\sqrt{|x-y||x-z|}}{|x-y|+|y-z|}(y-z)\tau e^{-\frac{1}{2}(|x-y|+|y-z|)}}. \quad (A.8)
\]

Note that the ratio on the right hand side is convergent. Thus we have

\[
\langle \langle |z_{tr}| \rangle \rangle_{tr} \approx \frac{c}{m}. \quad (A.9)
\]

**APPENDIX B**

For completeness, we turn to massless scalar QED with \( \mu_R^2 = 0 \), and consider the following cases.

(1) \( k^2 = 1/\beta^2 = 0 \):

The potential here reads as

\[
U_{1/\beta=k=0}(\Phi) = \frac{\lambda_R}{4!} \Phi^4 + \frac{3\epsilon^4 \Phi^4}{64\pi^2} \ln\left( \frac{\Phi^2}{M^2} \right) - \frac{25}{6},
\]

where we have assumed a small value of \( \lambda_R \sim O(\epsilon^4) \). As shown in [6], the symmetry of the system is spontaneously broken due to the radiative corrections, with minimum of the potential being \( M \). Dimensional transmutation also takes place with the scalar field coupling related to the gauge field coupling strength by

\[
\lambda_R = \frac{33\epsilon^4}{8\pi^2}, \quad (B.2)
\]

which leads to the familiar result:

\[
U_{1/\beta=k=0}(\Phi) = \frac{3\epsilon^4 \Phi^4}{64\pi^2} \left[ \ln\left( \frac{\Phi^2}{M^2} \right) - \frac{1}{2} \right]. \quad (B.3)
\]

The parameter space spanned by the potential is now \((\epsilon, M)\), instead of the original \((\epsilon, \lambda_R)\).

(2) \( k^2 = 0, 1/\beta^2 \neq 0 \):

If we consider only the temperature effects, then for \( 1/\beta^2 \gg \mu_R^2 + \lambda_R \Phi^4/2 \) and \( \epsilon^2 \Phi^2 \),

\[
U_{\beta,k=0}(\Phi) = \frac{\epsilon^2}{8\beta^2} \Phi^2 + \frac{\lambda_R}{4!} \Phi^4 + \frac{3\epsilon^4 \Phi^4}{64\pi^2} \left[ \ln\left( \frac{\Phi^2}{M^2} \right) - \frac{25}{6} \right]. \quad (B.4)
\]

Using (B.2), the expression above is reduced to

\[
U_{\beta,k=0}(\Phi) = \frac{\epsilon^2}{8\beta^2} \Phi^2 + \frac{3\epsilon^4 \Phi^4}{64\pi^2} \left[ \ln\left( \frac{\Phi^2}{M^2} \right) - \frac{1}{2} \right]. \quad (B.5)
\]
The symmetry for this theory is broken initially due to radiative correction. However, we anticipate its restoration at a temperature above $1/\beta_c$, which may be found by solving the equation

$$\frac{1}{\beta_c^2} = \min \left\{ \frac{3e^2\Phi^2}{4\pi^2 \ln \left( \frac{M^2}{\Phi^2} \right)} \right\}. \quad (B.6)$$

With $\ln \left( M^2/\Phi^2 \right) = 1$, we obtain

$$\ln \left( \frac{3e^2M^2\beta_c^2}{4\pi^2} \right) = 1$$

for $\beta_c$. The pattern of symmetry restoration is first order. Above the critical temperature, $U_{\beta,k=0}(\Phi)$ again has unique minimum at $\Phi = 0$.

(3) $k^2 \neq 0, 1/\beta^2 = 0$:

As shown in case (1), the symmetry of the system is spontaneously broken at $k = 1/\beta = 0$, and shall remain so for non-zero $k$, since, being an internal parameters characterizing the scale of observation, it cannot change the phase of the system. Taking into account the dominant contribution of $k$ in the limit $k^2 \gg \lambda_R \Phi^2/2$ and $e^2\Phi^2$ gives

$$U_{1/\beta=0,k}(\Phi) = -\frac{\lambda_R k^2}{24\pi^2} \Phi^2 + \frac{\lambda_R}{4!} \Phi^4 + \frac{3e^4\Phi^4}{64\pi^2} \left[ \ln \left( \frac{\Phi^2}{M^2} \right) - \frac{25}{6} \right], \quad (B.8)$$

from which we obtain

$$0 = \frac{\partial}{\partial \Phi} U_{1/\beta=0,k}(\Phi) = \Phi \left\{ -\frac{\lambda_R k^2}{12\pi^2} + \Phi^2 \left[ \frac{\lambda_R}{6} - \frac{11e^4}{16\pi^2} \right] + \frac{3e^4 \Phi^2}{16\pi^2} \ln \left( \frac{\Phi^2}{M^2} \right) \right\}. \quad (B.9)$$

The presence of the $k$-dependent term is to shift the minimum of $U_{1/\beta=0,k}(\Phi)$ from $M$ to $\Phi_k > M$. which may be found by solving

$$\Phi_k^2 \ln \left( \frac{\Phi_k^2}{M^2} \right) = \frac{11k^2}{6\pi^2} \quad (B.10)$$

The result suggests that in the case of spontaneous symmetry breaking, depending on the scale, the “effective” vacuum expectation value $\Phi_k$ may actually be greater than the usual $M = 246$ GeV in the Standard Model. The smaller the $k^{-1}$, the larger the $\Phi_k$.

(4) $k^2, 1/\beta^2 < 1/\beta_c^2 \neq 0$:

When both the effects due to temperature and scale are considered, we arrive at

$$U_{\beta,k}(\Phi) = \frac{1}{2} \tilde{\mu}_R^2(\beta, k) \Phi^2 + \frac{\lambda_R}{4!} \Phi^4 + \frac{3e^4\Phi^4}{64\pi^2} \left[ \ln \left( \frac{\Phi^2}{M^2} \right) - \frac{25}{6} \right], \quad (B.11)$$

where

$$\tilde{\mu}_R^2(\beta, k) = -\frac{\lambda_R}{12\pi^2} k^2 + \frac{1}{6\beta^2} \left[ \frac{2\lambda_R}{\pi^2} \sum_{n=1}^{\infty} e^{-n\beta k} \left( \frac{1}{n^2} + \frac{3e^2}{2} \right) \right]. \quad (B.12)$$

As in the massive case, the field fluctuations due to the scale-dependent term is much suppressed by the high-temperature contribution. To observe field fluctuations around non-trivial minima, we again must go to the large $k$ limit.