LUMINOSITY MEASUREMENTS AND CALCULATIONS

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Abstract
The luminosity of a single-ring electron-positron collider is defined and an expression in terms of the beam intensity and geometry obtained. The equivalent expression for the case of coasting beams with a finite crossing angle is also derived. The use of the root mean square as a measure of particle density distributions is discussed. Methods of measuring the luminosity are described and indications of the important parameters for the accuracy of the measurements given.

1. INTRODUCTION

After the beam energy the most important parameter at a colliding beam facility, as far as the high energy physics user is concerned, is the counting rate. This is usually expressed by the term luminosity, which I shall define in a moment, but which as one can guess indicates the brilliance of the source. I shall first of all discuss the expressions which allow the collider builder to know what the luminosity will be in terms of his machine parameters. It is of course beyond the scope of this note to discuss the variation of these parameters with a view to increasing the luminosity.

Once the machine is built and operating the high energy physicist needs to know the machine luminosity in order to be able to normalise his measurements and he usually requires a more accurate value than can be obtained directly from machine parameters. I shall therefore go on to discuss methods of measuring luminosity from the experimenters' point of view.

2. DEFINITION OF LUMINOSITY

At a colliding beam facility the total interaction rate depends on the geometry of the beams, their density and energy, but the last is usually fixed by other requirements.

Consider first the interaction of a beam with a target of length \( l \) and particle number density \( n_2 \), as sketched in Fig. 1. Then the number of interactions \( (R) \) per beam particle is proportional to \( n_2 \times l \) the total number of particles it can collide with and the constant of proportionality is defined as the cross section \( q \) for the type of interaction concerned.

\[
R = q n_2 l
\]

Fig. 1  Schematic of a particle beam of \( n_1 \) particles per second incident
on a stationary target with \( n_2 \) particles per unit volume

where \( q \) has the dimension cm\(^2\). The transverse dimensions of the beam and target do not enter
as the target is assumed to be wider than the beam. If the beam consists of \( n_1 \) particles per
second the rate of interactions

\[
\frac{dR}{dt} = q n_1 n_2 l .
\] (2)

All the characteristics of the incident beam and target can be combined into a single term defined
as the luminosity \( L \) by writing

\[
\frac{dR}{dt} = q \ L
\] (3)

where \( L = n_1 n_2 l \) and has the dimensions cm\(^2\) s\(^{-1}\). Hence luminosity is simply the interaction
rate per unit crosssection

3. LUMINOSITY OF A SINGLE-RING COLLIDER

In a colliding beam machine the expression for \( L \) is more complicated because the target is
moving and we cannot always assume that the target is wider than the beam.

In a single-ring collider \((e^+ e^-)\) the two beams circulate in opposite directions. Suppose we
have \( N \) particles per beam and the beams have equal r.m.s. radii of \( \sigma_x \) (horizontal) and \( \sigma_z \)
(vertical). The cross-sectional area of the beam is then \( 2\pi \sigma_x \sigma_z \) and the number of positrons
which one electron encounters in one turn of the machine, assuming the beams follow identical
paths, is

\[
\frac{q}{2\pi \sigma_x \sigma_z}
\]

where \( q \) is an effective cross section of the electron.

The total rate \( \frac{dR_T}{dt} \) if the revolution frequency is \( f \) is

\[
\frac{dR_T}{dt} = \frac{NfqN}{2\pi \sigma_x \sigma_z}
\] (4)

We have not said anything yet about the azimuthal distribution of the particles around the
machine and for continuous or DC beams these interactions would be spread around the whole
machine circumference. Electrons must be bunched for other reasons and if we have \( K \) bunches in each beam there are \( 2K \) crossing points around the machine hence the interaction
rate per crossing \( \frac{dR}{dt} \) is given by

\[
\frac{dR}{dt} = \frac{N^2fq}{4\pi k \sigma_x \sigma_z}
\] (5)

From our previous definition of luminosity in (3)

\[
L = \frac{N^2f}{4\pi k \sigma_x \sigma_z}
\] (6)
or since it is more usual to measure beam currents $I$ where $I = N \cdot e \cdot f$

$$L = \frac{I^2}{4\pi kf\sigma_x\sigma_z}$$  \hspace{1cm} (7)

$\sigma_x$ and $\sigma_z$ are the r.m.s. dimensions of the two beams at the crossing points and we have assumed that they are constant over the effective crossing region which is of course a function of the bunch length. We have also assumed identical positron and electron beams which is normally the case. If one beam has a much larger cross section than the other, has been blown up by the beam-beam interaction for example, then the $\sigma_x\sigma_z$ term must be replaced by an effective area $A_{\text{eff}}$ which will be approximately the area of the larger beam. In the case of a proton-antiproton collider there is no reason to expect equal beams and the same remarks apply.

As an example of a more complicated formulation of these geometric factors, that given [1] for the electron proton ring HERA is perhaps instructive

$$L = \frac{f \cdot N_p \cdot N_e}{k \cdot 2\pi(\sigma_{xp,\text{eff}}^2 + \sigma_{xe}^2)^{1/2} \cdot (\sigma_{zp}^2 + \sigma_{ze}^2)^{1/2}}$$  \hspace{1cm} (8)

where $\sigma_{xp,\text{eff}} = (\sigma_{xp}^2 + (\sigma_{sp} \cdot \phi)^2)^{1/2}$, $\sigma_{sp}$ is the proton bunch length with $\phi$ the half crossing angle ($\pm 5$ mrad). $\sigma_{xp}$ and $\sigma_{xe}$ are the r.m.s. widths of the proton and electron beam respectively and $\sigma_{zp}$ and $\sigma_{ze}$ are the r.m.s. heights.

4. THE RMS AS A MEASURE OF BEAM HEIGHT

In all cases it has been assumed that the r.m.s. value of the beam density distribution is a good measure of beam size. It is interesting to consider to what extent this is true for different distributions or are we implying always a Gaussian distribution? To examine this further consider the simple case of two coasting beams interacting at an angle in one plane and with identical distributions $\rho(y)$ in the other plane. In the plane of the crossing there is no vertical dispersion and all particles of one beam intersect all particles of the other beam, hence the luminosity depends only on the distribution $\rho(y)$. For convenience we can consider $\rho(y)$ as normalised to unity so that:

$$\int_{-\infty}^{\infty} \rho(y) \, dy = 1.$$  \hspace{1cm} (9)

The mean square width of this distribution is then

$$\langle y^2 \rangle = \int_{-\infty}^{\infty} y^2 \rho(y) \, dy$$  \hspace{1cm} (10)

For given circulating currents (fixed numbers of particles) the interaction rate in this simple case is proportional to:

$$L = \int_{-\infty}^{\infty} \rho^2(y) \, dy$$  \hspace{1cm} (11)

which we notice means that $L = \langle \rho \rangle$ since the general definition of an average is
\[ \langle f \rangle = \int_{-\infty}^{\infty} f(y) \rho(y) \, dy \quad (12) \]

In other words \( L \) is the density of the target beam, as seen on average by the bombarding beam.

For a given distribution, we have seen (Eq. (6)) that \( L \) is inversely proportional to the width of the distribution parametrised by the r.m.s. width \( \langle y^2 \rangle^{1/2} \).

Therefore \( L \langle y^2 \rangle^{1/2} \) must be a constant. Suppose \( \rho \) is a Gaussian distribution

\[
\rho = \frac{1}{\sigma \sqrt{2\pi}} \exp \left(-\frac{y^2}{2\sigma^2}\right) \quad (13)
\]

then using (10)

\[
\langle y^2 \rangle = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 \exp \left(-\frac{y^2}{2\sigma^2}\right) \, dy = \sigma^2 \quad (14)
\]

and

\[
L = \frac{1}{2\pi \sigma^2} \int_{-\infty}^{\infty} \exp \left(-\frac{y^2}{\sigma^2}\right) \, dy = \frac{1}{2\sigma \sqrt{\pi}} \quad (15)
\]

and

\[
L \langle y^2 \rangle^{1/2} = \frac{1}{2 \sqrt{\pi}} = 0.28209
\]

The interesting observation made by H. Hereward [2] is that the same constant is approximately valid for many different distributions. For example consider a rectangular distribution defined by

\[
\rho = \frac{1}{2y_o} \quad \text{for} \quad -y_o \leq y \leq y_o \quad (16)
\]

Hence

\[
\langle y^2 \rangle = \frac{1}{2y_o} \int_{-y_o}^{+y_o} y^2 \, dy = \frac{1}{3} y_o^2 \quad (17)
\]

and

\[
L = \frac{1}{4y_o^2} \int_{-y_o}^{+y_o} y \, dy = \frac{1}{2y_o} \quad . \quad (18)
\]

Therefore

\[
L \langle y^2 \rangle^{1/2} = \frac{1}{2} \sqrt{\frac{\sqrt{3}}{3}} = \frac{\sqrt{3}}{6} = 0.28868
\]

Similarly for the distributions of Table 1.
The range of the results given in the last column of Table 1 is only \( \pm 3.7\% \). In fact Hereward [2] was able to show that \( L <y^2>^{1/2} \) is never less than 0.2683 (a parabolic distribution) for any distribution \( \rho \geq 0 \). Unfortunately he also showed that there is no equivalent limit to the maximum value and the distribution illustrated in Fig. 2 can have an arbitrarily large \( L <y^2>^{1/2} \).

### Table 1

\( L <y^2>^{1/2} \) for some typical distributions

<table>
<thead>
<tr>
<th>Distribution within ( -y_0 \leq y \leq y_0 )</th>
<th>( L &lt;y^2&gt;^{1/2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parabola ( \rho = \frac{3}{4y_0} \left(1 - \frac{y^2}{y_0^2}\right) )</td>
<td>( \frac{3\sqrt{5}}{25} = 0.2683 )</td>
</tr>
<tr>
<td>Cosine ( \rho = \frac{\pi}{4y_0} \cos \frac{\pi y}{2y_0} )</td>
<td>( \frac{\pi^2}{16} \sqrt{\left(1 - \frac{8}{\pi^2}\right)} = 0.2685 )</td>
</tr>
<tr>
<td>Triangle ( \rho = \frac{1}{y_0} \left(1 - \frac{y}{y_0}\right) )</td>
<td>( \frac{2}{3} \frac{1}{\sqrt{6}} = 0.2722 )</td>
</tr>
<tr>
<td>Truncated Gaussian ( \rho = \frac{1}{\sigma \sqrt{2\pi}} \alpha \left(\frac{y_0}{\sigma}\right) \exp \frac{-y^2}{2\sigma^2} )</td>
<td>Limit ( \frac{y_0}{\sqrt{\sigma}} \rightarrow 0 ), ( \frac{\sqrt{3}}{6} = 0.2887 )</td>
</tr>
<tr>
<td>Where ( \alpha(x) ) is the &quot;normal probability integral&quot;</td>
<td>Limit ( \frac{y_0}{\sqrt{\sigma}} \rightarrow \infty ), ( \frac{1}{2\sqrt{\pi}} = 0.2809 )</td>
</tr>
<tr>
<td>( [\alpha(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} \exp \frac{-t^2}{2} , dt] )</td>
<td>Minimum ( \frac{y_0}{\sqrt{\sigma}} (1.63) ), ( 0.2694 )</td>
</tr>
</tbody>
</table>

It can be argued that particle beams are much more likely to be something like a truncated Gaussian or one of the other distributions of Table 1 than that of Fig. 2. In particular since gas scattering or other random kick processes increase the width of a narrow beam making it more and more Gaussian.

### 5. Luminosity with Coasting Beams and Finite Crossing Angle

Colliding beam machines with two separate rings and a crossing angle can operate with coasting, that is to say unbunched beams. The interaction region length is then defined by the beam dimension in the plane of the crossing angle as illustrated in Fig. 3 for the ISR. We can again show that the luminosity is inversely proportional to the beam height.
In order to derive the luminosity formula assume two ribbon beams of height $h$ and width $w$ crossing in the horizontal plane with angle $\alpha$ as in Fig. 3. We first note that since the number of particles encountered must be the same for all observers we can choose any convenient system such as the rest frame of beam 2. From Eq. (1) we know that we have to determine the effective particle density and the length of traversal of the test particle $Q$ of beam 1.

![Fig. 2](image1.png) A particle distribution that can have $L <y^2>^{1/2}$ arbitrarily large. A fraction of the particles are spread out over a constant width and contribute a constant amount of $<y^2>$, while the remaining fraction is in a peak that contributes increasingly to $L$ as it is made narrower,

$$L = \int_{-\infty}^{\infty} \rho^2(y) \, dy.$$  

![Fig. 3](image2.png) Schematic of coasting beams of width $W$ and height $h$ colliding with a crossing angle $\alpha$ in the horizontal plane

The Lorentz transformation needed to bring beam 2 to rest when applied to particle $Q$ of beam 1 changes the angle of traversal $\alpha$ to $\alpha'$ given by
\[
\tan \alpha' = \frac{\sin \alpha}{\gamma \left( \cos \alpha + \frac{\beta_2}{\beta_1} \right)}
\]

(19)

where \(\beta_1\) and \(\beta_2\) are the usual ratios of the velocities of beam 1 and beam 2 respectively to that of light \((\beta = v/c)\) and \(\gamma = (1 - \beta^2)^{-1/2}\).

The length of traversal \(l'\) is then \(W_2/\sin \alpha'\).

Since for equal energy beams \(\beta_1 = \beta_2 = \beta\) and \(\gamma_1 = \gamma_2 = \gamma\), Eq. (19) becomes

\[
\tan \alpha' = \frac{\sin \alpha}{\gamma (\cos \alpha + 1)} = \frac{\tan \alpha/2}{\gamma}
\]

and by simple trigonometry

\[
\sin \alpha' = \frac{\tan \alpha/2}{\gamma} \sqrt{1 + \frac{\tan^2 \alpha/2}{\gamma^2}}
\]

For high energy beams \(\beta \sim 1\) and \(\gamma\) is large so that the second term can be neglected, giving

\[
\sin \alpha' = \frac{\tan \alpha/2}{\gamma}
\]

and

\[
l' = \frac{W_2 \gamma}{\tan \alpha/2}
\]

(20)

The unit volume of beam 2 is also reduced by the Lorentz contraction so that the particle density \((n_2')\) seen by \(Q\) is given by

\[
n_2' = \frac{n_2}{\gamma}
\]

(21)

Using (20) and (21) in Eq. (1) gives directly the number of interactions per traversal as

\[
q \cdot \frac{n_2 w_2}{\tan \alpha/2}
\]

(22)

The number of particles per second in beam 1 is

\[
n_1 v_1 w_1 h
\]

and the total interaction rate is therefore

\[
\frac{dN}{dt} = q \cdot \frac{n_1 n_2 v_1 w_1 w_2 h}{\tan \alpha/2}
\]

(23)

in terms of current
\[ n = \frac{I}{\text{ewhc}} \quad \text{again assuming} \ v = c \]
\[
\frac{dN}{dt} = \frac{\sigma}{c e^2 h \tan \alpha/2} I_1 I_2
\]  
(24)

Since we have two rings the beams can have different heights and profiles and are not necessarily well aligned, so that \(h\) must be replaced by \(h_{\text{eff}}\) where

\[
h_{\text{eff}} = \frac{\int \rho_1 \, dz \int \rho_2 \, dz}{\int \rho_1 \rho_2 \, dz}
\]  
(25)

Using Eqs. (3) and (24) the luminosity \(L\) becomes

\[
L = \frac{I_1 I_2}{c e^2 h_{\text{eff}} \tan \alpha/2}
\]  
(26)

and once again the problem of determining the luminosity is essentially that of determining the beam geometry term \(h_{\text{eff}}\).

6. MEASUREMENT OF THE LUMINOSITY

In principle Eq. (7) implies that at least in the case of a single-ring collider the luminosity is known provided the transverse beam dimensions can be measured. However, it is clear that if the beam profiles are measured elsewhere in the machine they must be transformed to the interaction point using a knowledge of \(\beta_y\) at the point of measurement and at the beam crossing point.

In practice, while machine designers make use of such expressions, at an operating collider the experimenters require a more precise knowledge of the luminosity for normalization purposes and must find other means. At an electron machine the standard technique is to set up a monitor consisting of two small-angle electron telescopes to observe elastic (Bhabha) scattering as in Fig. 4. This process has a well-known, exactly calculable, cross section, \(\sigma_B\) so that from a measurement of the counting rate the luminosity can be determined using Eq. (3). The \(\sigma\) must of course be calculated for the acceptance of the detector

\[
\sigma = \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} \frac{d\sigma_B}{d\Omega} \, d\Omega
\]  
(27)

with

\[
\frac{d\sigma_B}{d\Omega} = \frac{\alpha^2}{8E^2} \frac{(2 - \sin^2 \theta) \, (4 - \sin^2 \theta)}{\sin^4 \theta}
\]  
(28)

from quantum electrodynamics, where \(\theta_{\text{min}}\) and \(\theta_{\text{max}}\) are the minimum and maximum scattering angles accepted by the monitor, \(E\) is the beam energy and \(\alpha\) is the fine structure constant (\(e^2/\hbar c\)). If care is taken to choose a counter configuration which minimises the effects
of interaction region displacements and other geometrical effects the luminosity can be
determined to a few per cent [3]. But, since counting rates are low, less sophisticated monitors
with larger counters are often used to give rapid values of the luminosity to \( \sim 10\% \) accuracy.
These monitors are then very useful diagnostic devices of the machine, giving information on
beam sizes via Eq. (7) for example.

\[
\text{BHABHA SCATTERING } e^+ e^- \rightarrow e^+ e^- \\
\text{BHABHA SCATTERING } e^- e^- \rightarrow e^+ e^-
\]

7. THE VAN DER MEER METHOD OF LUMINOSITY MEASUREMENT

For two-ring colliders Eq. (25) allows a calculation of \( h_{\text{eff}} \) from a knowledge of the
vertical beam profiles but for the evaluation of the numerator the vertical distance between the
centres of the beam distributions at the crossing must be known. In the ISR it was out of the
question to use the standard bunched beam pick-up system to give this, mainly because of the
dependence of the closed orbit on intensity as a result of space charge forces. The only
possibility was to steer the beams vertically while observing a suitable monitor counting rate in
order to maximise the luminosity and obtain the \( h = 0 \) point.

This idea of vertical beam steering enabled S. van der Meer [4] to point out a much
cleverer way of measuring the luminosity.

If one measures the counting rate in a luminosity monitor, of a similar layout to that of
Fig. 4, as a function of relative vertical separation \( h \) of the two beams, a curve similar to that
of Fig. 5 will result with a maximum at zero separation. Van der Meer showed that the area
under this curve divided by the value at \( h = 0 \) is \( h_{\text{eff}} \).

At separation \( h \) the counting rate is

\[
A \cdot \int_0^\infty \rho_1 (z) \rho_2 (z - h) \, dz
\]

where \( A \) is an unknown constant which includes the interaction cross section and the
acceptance of the monitor but which can be assumed constant at fixed energy.

Then the area under the counting rate curve is
\[
\int [A \int \rho_1(z) \rho_2(z-h) \, dz \, dh] \, dz = A \int [\rho_1(z) \int \rho_2(z-h) \, dh \, dz] \, dz \tag{30}
\]

and the rate at \( h = 0 \) is

\[
A \cdot \int \rho_1(z) \rho_2(z) \, dz \tag{31}
\]

If the integrals are taken over the entire non-zero region then

\[
\int \rho_2(z-h) \, dh = \int \rho_2(z) \, dz
\]

and therefore Eq. (30) divided by (31) is

\[
\frac{\int [\rho_1(z) \int \rho_2(z) \, dz] \, dz}{\int \rho_1(z) \rho_2(z) \, dz} = \frac{\int \rho_1(z) \rho_2(z) \, dz}{\int \rho_1(z) \rho_2(z) \, dz} = h_{\text{eff}} \tag{32}
\]

With \( h_{\text{eff}} \) determined the luminosity can be calculated using Eq. (26). The currents \( I_1 \) and \( I_2 \) can be very accurately measured using a DC current transformer [5] and the crossing angle of the beams is known to high precision.

Since from the definition of luminosity the counting rate \( R_M \) in a monitor is given by

\[
R_M = \sigma_M \cdot L
\]
the monitor cross section $\sigma_M$ has in effect been measured by the above procedure and to the extent that the monitor rate is unaffected by backgrounds, geometry of the crossing region and its efficiency does not change, this monitor rate can be used as a direct measure of luminosity.

At the ISR this technique worked extremely well and with occasional calibrations of their monitors the experimenters always knew the luminosity to within a few per cent. For particular experiments such as the measurement of the total p-p and p-$\bar{p}$ cross section special care was taken and an error of less than 1% was achieved. In particular this required a calibration of the beam displacement ($h$) used in the luminosity measurement.

8. VERTICAL BEAM DISPLACEMENTS AT THE ISR

The most convenient way of creating a local closed orbit bump is to place two dipole magnets a quarter of a wavelength before and after the crossing point. In practice, however, the phase advances cannot be exact because of lack of space or simply because the tune of the machine is not fixed. At the ISR this problem was solved by using the horizontal field steering magnets of the adjacent intersecting regions to make the necessary corrections. The orbit distortion was then as illustrated by Fig. 6. The problem was to ensure that the one millimetre nominal displacement at the intersection was correct to a few parts per thousand and was reproducible to a similar level. This is a much better precision than is obtainable with the usual programs such as AGS [6] or MAD [7] and in addition great care with power supply setting and magnet hysteresis was required.

The solution was to calibrate the beam displacements using a scraper driven by a precision screw [8]. At the ISR special scrapers [9] were used for obtaining vertical beam profiles but as in this case only the precise centre of the distribution was required the technique was simplified so that it amounted to giving the beam a sharp edge with say the upper scraper and then finding that edge with the lower scraper. By repeating this procedure many times a very precise determination of the beam centre was possible. The location of the beam centre to $\pm 3\, \mu m$ was more than adequate to establish the linearity of the displacements, the success of a hysteresis correction routine, and to provide an absolute calibration to within $\pm 4\, \%$.

Dependence of the bump amplitude on all sorts of parameters such as machine tune, beam intensity, betatron coupling and even the horizontal closed orbit were all studied in order to make sure that the displacement scale measured with the scrapers using single pulses was the same as for the beams of a few amperes which were used to measure the total cross sections.

9. CONCLUDING REMARKS

The use of the van de Meer method to measure $h_{\text{eff}}$ and calibrate luminosity monitors was a great success at the ISR; where, as explained above, the luminosity depended only on the beam dimensions in one plane. In the more usual case of bunched beams, where both horizontal and vertical beam dimensions influence the luminosity it is in principle still possible to use the van der Meer method by scanning in both planes. However, the problems alluded to above become even more important and for finite crossing angles monitor acceptance as a function of longitudinal position has to be considered very carefully as the collision region is displaced as the beams are scanned in the plane of the crossing. In practice a combination of several methods including comparing small angle scattering with the total cross section, through the optical theorem, are used to establish the luminosity at colliders. For the next generation of hadron colliders such as the LHC and SSC there will be additional problems arising from the very small angles of elastic scattering in the 10 TeV region. Indeed most secondaries also stay in the beam pipe, which will not only make the use of the optical theorem very difficult but will also make the choice of a good luminosity monitor less than obvious. At this time it seems doubtful if the true luminosity of these future colliders will ever be known with an absolute accuracy better than a few percent.
Fig. 6 An AGS tracking of a 1-mm closed orbit bump as
used for luminosity measurements at an ISR intersection.

REFERENCES


