BEAM TRANSFER LINES

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Abstract

After making some distinctions between transfer lines and circular machines, certain problems, typical of the type met by a transfer line designer, are discussed. The topics chosen include: steering, measurement of emittance and mismatch, setting tolerances for magnet alignment and excitation, emittance dilution due to mismatches and scattering in thin windows and lastly emittance-exchange insertions.

1. DISTINCTIONS BETWEEN TRANSFER LINES AND PERIODIC CIRCULAR MACHINES

Transmission of the position-velocity vector of a particle through a section of a transfer line, or circular machine, can be simply represented by a $2 \times 2$ matrix (Fig. 1).

Fig. 1  Transmission through a section of lattice
(y represents either transverse coordinate)

$$
\begin{pmatrix}
  y_2 \\
  y'_2
\end{pmatrix}
= \begin{pmatrix}
  C & S \\
  C' & S'
\end{pmatrix}
\begin{pmatrix}
  y_1 \\
  y'_1
\end{pmatrix}
= M_{1\rightarrow 2}
\begin{pmatrix}
  y_1 \\
  y'_1
\end{pmatrix}.
$$

The transfer matrix $M_{1\rightarrow 2}$ can be found by multiplying together the transfer matrices for the individual elements in the appropriate order. The individual matrices have the form,

$$
M_y = \begin{pmatrix}
  \cos (\text{or} \cosh) \phi & \frac{s}{\phi} \sin (\text{or} \sinh) \phi \\
  -\frac{s}{\phi} \sin (\text{or} -\sinh) \phi & \cos (\text{or} \cosh) \phi
\end{pmatrix}
$$

where

$$
\phi = s \sqrt{|K|} \quad \text{and} \quad |K| = \left| \frac{1}{B\rho} \frac{dB}{dx} + \frac{1}{\rho^2} \right|
$$

in accordance with the earlier lectures by J. Rossbach and P. Schmüser in these proceedings.
However, we often use a parameterized form for the matrix for a section of line, which was also given by J. Rossbach and P. Schmüser.

\[
M_{1 \rightarrow 2} = \begin{pmatrix}
\sqrt{\beta_2} \frac{\cos \Delta \phi + \alpha_1 \sin \Delta \phi}{\sqrt{\beta_1}} & \sqrt{\beta_1 \beta_2} \sin \Delta \phi \\
(1 + \alpha_1 \alpha_2)\sin \Delta \phi + (\alpha_2 - \alpha_1)\cos \Delta \phi & \sqrt{\beta_1 \beta_2} \frac{\cos \Delta \phi - \alpha_2 \sin \Delta \phi}{\sqrt{\beta_2}}
\end{pmatrix}
\]

In the first case of Eq. (1), the matrix is unambiguously determined, but in the second case of Eq. (3), there are in fact an infinite number of sets of parameters \((\beta_1, \beta_2, \alpha_1, \alpha_2, \Delta \phi)\), which satisfy the numerical values of the matrix elements. This is the root of an important difference between circular machines and transfer lines, which sometimes leads to confusions.

### 1.1 Circular machines

A circular structure has an imposed periodicity, which imposes the same periodicity on the parameters \(\alpha\) and \(\beta\) and in fact determines them uniquely. If one samples the co-ordinates of an ion after each successive turn in a circular machine, the points will fill out an ellipse in phase space \((y, y')\). Only one set of \(\alpha\) and \(\beta\) values fit that ellipse. It is the periodicity of the structure which makes it possible for that specific ellipse to be returned unchanged turn after turn and for this reason it is called the matched ellipse [Fig. 2(a)]. Now suppose one injects a beam of particles, whose spatial distribution defines a different ellipse characterized by some other parameters, say \(\alpha^*\) and \(\beta^*\). The circular machine will not faithfully return this ellipse after each turn. Instead, the ellipse will tumble over and over filling out a much larger ellipse of the matched ellipse form [Fig. 2(b)]

In a truly linear system, the original ellipse will tumble round indefinitely inside the matched ellipse conserving its elliptical form and area, but in a practical system small non-linearities will cause an amplitude-frequency dependence, which will distort the ellipse. This is also shown in Fig. 2. Liouville’s theorem requires the phase-space density to be conserved and in a strict mathematical sense this is true, since as the figure becomes more wound-up the spiral arms become narrower and the area is indeed constant. However it does not take long before the beam is apparently uniformly distributed over the matched ellipse and for all practical purposes the beam emittance has been increased. This is called dilution of phase space by filamentation, which is present to a greater or lesser extent at the injection into all circular machines.

Since filamentation will quickly transpose any beam ellipse into the matched ellipse in a circular machine, there is no point in using any \(\alpha\) and \(\beta\) values other than the matched ones.

Since \(\alpha\) and \(\beta\) depend on the whole structure any change at any point in the structure will in general (matched insertions excepted) change all the \(\alpha\) and \(\beta\) values everywhere.

### 1.2 Transfer lines

In a transfer line, there is no such restriction. The beam passes once and the shape of the ellipse at the entry to the line determines its shape at the exit. Exactly the same transfer line injected first with one emittance ellipse and then a different ellipse has to be accredited with different \(\alpha\) and \(\beta\) functions to describe the two cases. Thus \(\alpha\) and \(\beta\) depend on the input beam and their propagation depends on the structure. Any change in the structure will only change the \(\alpha\) and \(\beta\) values downstream of that point. There is an infinite number of sets of \(\alpha\) and \(\beta\) values, which can be used to describe the motion of a single ion in a transfer line (see Fig. 3)
and the choice of a particular set depends on the input ellipse shape. The input ellipse must be chosen by the designer and should describe the configuration of all the particles in the beam.

Fig. 2 Matched ellipse, unmatched and filamenting beam ellipses

Fig. 3 Two ellipses from the infinite set that include the test ion

2. ORBIT CORRECTION IN TRANSFER LINES

Orbit correction, or steering, is basically straightforward in transfer lines, whereas in circular machines we could fill an entire course on the subject. The usual philosophy is illustrated in Fig. 4.
At the entry to the line, it is useful to have a very clear diagnosis of beam position and angle and qualitative information on the shape, since this is usually the ejection from an accelerator and often a boundary of responsibility between groups. A pair of pickups and knowledge of the transfer matrix between them is in principle all that is needed to find the entry angle and position, but in practice, the precision and reliability of this measurement and its credibility as a diagnostic tool are greatly improved by having only a drift space between the pickups. The qualitative knowledge of the beam shape is most easily obtained with a luminescent screen and is of obvious diagnostic use.

In the central section of the line, each steering magnet is paired with a pickup approximately a quarter of a betatron wavelength downstream, so that the trajectory can be corrected stepwise along the line. The direct application of this philosophy would lead to four pickups per betatron wavelength, but in practice, it is usual to find fewer pickups than this, especially if there are long straight sections. The measurement of beam emittance is usually made in the central part of the line in a dispersion-free section. The theory for the measurement of emittance and mismatches is treated in Section 4.

At the exit to the line, the last two dipole correctors are used as a doublet to steer the beam to the angle and position, dictated by the closed orbit of the following accelerator or by a target. For maximum sensitivity, the dipoles should be approximately a quarter betatron wavelength apart.

The horizontal and vertical planes should be independent for correction elements. For example, tilted dipoles are sometimes used in the lattice of a transfer line, but correction coils for steering should be avoided on such magnets. Skew quadrupoles are occasionally used to interchange emittances between the horizontal and vertical planes. Such insertions also exchange the planes for steering. While being novel, this is quite acceptable, as long as no corrector is placed inside the skew quadrupole insertion, which would cause a coupling of its effect to both planes rather than a simple exchange.

Some care is needed in the positioning elements for the best sensitivity. The monitor controlling a steering magnet should be on the adjacent peak of the downstream beam oscillation (see Fig. 5), i.e. for the section of line from the steering dipole to the pickup, the matrix element $S$ in Eq. (1) must be relatively large or in other terms $\Delta \phi = \pi/2$ in Eq. (3).

The monitors and magnets should be sited near maxima in the $\beta$-function, since these are the most sensitive points for controlling and observing. This depends on the choice of input beam ellipse.
Monitors can also be profitably placed in bends at points where off-momentum particles would have their maximum deviations. Using three well-placed pickups a bend can be used for momentum analysis. The simple linear matrices make the analysis of such systems very easy.

In a long line, a global correction may well be possible, followed by an exact beam steering at the end using two dipoles.

It should be possible to set the magnets in a transfer line and to be sure that the beam will be transmitted with 100% efficiency on the first try and that only a fine steering will be needed at the output to the line. If this is not the case, check that:

- the line is always cycled in the same way when it is powered and that the cycle saturates the magnets to set the hysteresis conditions.
- the current does not overshoot the requested value, especially when approaching the minimum value in the cycle and the final value. This is achieved by reducing the ramp rate when approaching set values.
- when a steering correction is made, that it is made using the standard excitation cycle. In this way, the value stored in the current file will reproduce the field exactly the next time the line is powered.
- check the position, angle and cross-section of the incoming beam.

Figure 6 shows computer output of the beam trajectory in the TT6-TT1 antiproton transfer line that was built in CERN. The first two pickups measure the incoming angle and position in both planes. These pickups are separated by exactly 10.75 m of free space. The next two pickups are in a long bend and act as a momentum analyser, in conjunction with an angle and position measurement made using the first two pickups. The remaining pickups have associated steering magnets. At the end of the line two dipoles match the beam to the ISR’s closed orbit.

For the example shown in Fig. 6, it was found that a single corrector could virtually correct the whole trajectory with the result shown in Fig. 6. This type of correction is only practical with non-destructive pickups, which reliably record the complete trajectory in one shot, and an online computer for logging, display, analysis and application. The correction was stable and was applied throughout the life of the transfer line. The cycling of the magnets ensured that the beam reached the ISR on the first shot and only a fine-tuning of the injection was required for each new run. The TT6 line achieved 0.1mm accuracy with as little as 10⁹ particles. All readings were logged and stored for later analysis and the detection of trends. The steering magnets were also equipped with Hall probes (temperature stabilized to +0.1 °C for outside ambient temperatures 15 °C to 34 °C). These probes made relative field changes extremely accurate, eliminating any hysteresis errors. This rather careful approach was justified by the scarcity of antiprotons and since setting-up could not be done with the reverse injection of protons.
Fig. 6  Trajectories in the TT6-TT1 antiproton transfer line at CERN

Before correction

PU  X    Y    X
602/602 -2.93  7.83  -2.86
603/603  0.91  7.97   0.24
606/606 10.04 -0.63  11.13
611/611  0.19  1.60   0.15
616/615  0.52 -2.05   6.56
619/619  ------  ------  ------
621/625 -1.34  3.43 -16.75
435/435 -4.61 -10.87  11.26
442/442 -0.32  9.63  -9.68
448/448  0.25  4.00  10.46
449/449 -1.42  2.59   7.91
450/450 -4.64  1.51   1.73
451/451 -6.75  1.11  -5.32

X  X'  Z  Z
ENTRY TT6
-5.37 -0.04 6.88 0.38
IN BH2010-PO
4.70 -8.09 -12.79 0.38

HOR. DELTA P / P (PER MIL)
PU606  PU611
-1.558  -0.466

CT 602:  131 18XX B
CT 449:  146 11XX B

SIGN CONVENTION:
LEFT-HANDED & UPWARD ARE positive

(lines are aids to visualisation not exact trajectories)
3. MATCHING TRANSFER LINES

Ideally long transfer lines consist of a regular cell structure over the majority of their length with matching sections at either end to coordinate them with their injector and user machines. The regular part of the structure is then regarded as periodic and the simple FODO cell theory, given in earlier lectures by J. Rossbach and P. Schmüser, applies. Usually thin-lens formulae are quite sufficient. The matching sections are complicated and a complete course could be given on this. Basically one needs to match $\beta$, $\alpha$, $D$, and $D'$ in both planes. In theory eight variables, that is eight quadrupole strengths and sometimes positions, need to be adapted. Some analytic solutions exist, but usually one uses a mixture of theory, intuition and computer optimization programmes.

4. EMITTANCE AND MISMATCH MEASUREMENT IN A DISPERSION-FREE REGION

With semi-destructive monitors, such as secondary emission grids or digitized luminescent screens, a density profile can be obtained of a beam. This profile is a projection of the population of the phase-space ellipse of the beam onto a transverse co-ordinate axis. In general, the profile is a near-Gaussian, but this is not really important for the following. From the profile, the standard deviation of the distribution, $\sigma$ can be found and this can be used to define a beam width, $W$. $W$ is then used to define the emittance $\varepsilon$, but unfortunately several definitions are current.

$$\varepsilon = \frac{W^2}{\beta} \pi = \begin{cases} \frac{(2\sigma)^2}{\beta} \pi & \text{Mostly used in proton machines, with or without } \pi \\ \frac{\sigma^2}{\beta} \pi & \text{Mostly used in electron machines, usually without } \pi \end{cases}$$

(4)

Somewhat arbitrarily, $\varepsilon = \sigma^2/\beta$ will be used in this paper.

If $\beta$ is known unambiguously as in a circular machine, then a single profile measurement determines $\varepsilon$ by Eq. (4), but as can be understood from Section 1.2, it is not easy to be sure in a transfer line which $\beta$ to use, or rather, whether the beam that has been measured is matched to the $\beta$-values used for the line. Indeed, the measurement of any mismatch is as important as the emittance itself. This problem can be resolved by using three monitors (see Fig. 7), i.e. the three width measurements determine the three unknowns $\alpha$, $\beta$ and $\varepsilon$ of the incoming beam.
Fig. 7 Layout for emittance measurement

By definition, Eq. (4),

$$\varepsilon = \pi \frac{\sigma_0^2}{\beta_0} = \pi \frac{\sigma_1^2}{\beta_1} = \pi \frac{\sigma_2^2}{\beta_2}$$

(5)

where $\beta_0$, $\beta_1$ and $\beta_2$ are the $\beta$-values corresponding to the beam and are therefore uncertain. Although we may not know $b$ and $a$, we do know the transfer matrices and how $b$ and $a$ propagate through the structure (see lectures by K. Steffen in these proceedings).

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix} = \begin{pmatrix} C^2 & -2CS & S^2 \\ -CC' & CS' + SC' & -SS' \\ C'^2 & -2C'S' & S'^2 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0$$

(6)

where $\gamma = (1 + \alpha^2)/\beta$. Thus, from Eq. (6)

$$\beta_1 = C_1^2 \beta_0 - 2C_1S_1\alpha_0 + \frac{S_1^2}{\beta_0}(1 + \alpha_0^2)$$

(7)

$$\beta_2 = C_2^2 \beta_0 - 2C_2S_2\alpha_0 + \frac{S_2^2}{\beta_0}(1 + \alpha_0^2)$$

(8)

and from Eq. (5),

$$\beta_0 = \frac{\pi \sigma_0^2}{\varepsilon}$$

(9)

$$\beta_1 = \left(\frac{\sigma_1}{\sigma_0}\right)^2 \beta_0$$

(10)

$$\beta_2 = \left(\frac{\sigma_2}{\sigma_0}\right)^2 \beta_0$$

(11)

From Eqs. (7) and (8), we can find $\alpha_0$ and using Eqs. (10) and (11), we can express $\alpha_0$ as,

$$\alpha_0 = \frac{1}{2} \beta_0 \Gamma$$

(12)

where

$$\Gamma = \frac{\left(\sigma_2 / \sigma_0\right)^2 / S_2^2 - \left(\sigma_1 / \sigma_0\right)^2 / S_1^2 - \left(C_2 / S_2\right)^2 + \left(C_1 / S_1\right)^2}{\left(C_1 / S_1\right) - \left(C_2 / S_2\right)}.$$ 

(13)

Since $\Gamma$ is fully determined, direct substitution back into Eq. (7) or Eq. (8), using Eq. (10) or Eq. (11) to re-express $\beta_1$ or $\beta_2$. yields $\beta_0$ which via Eq. (9) gives the emittance,

$$\beta_0 = 1 / \sqrt{\left(\sigma_1 / \sigma_0\right)^2 / S_1^2 - \left(C_1 / S_1\right)^2 + \left(C_1 / S_1\right) \Gamma - \Gamma^2 / 4}$$

(14A)

$$\varepsilon = (\pi \sigma_0^2) \sqrt{\left(\sigma_1 / \sigma_0\right)^2 / S_1^2 - \left(C_1 / S_1\right)^2 + \left(C_1 / S_1\right) \Gamma - \Gamma^2 / 4}.$$ 

(14B)

The mismatch parameters $\Delta \beta$ and $\Delta \alpha$, the differences between what is expected and what exists, can now be found directly from Eqs. (14B) and (12).

5. SMALL MISALIGNMENTS AND FIELD RIPPLE ERRORS IN DIPOLES AND QUADRUPOLES
One problem, which always faces a transfer line designer, is to fix the tolerances for magnet alignment and excitation currents. Although the following is rather idealistic and does not include such real-world problems as magnets having correlated ripple because they are on the same transformer, it does give a basis for fixing and comparing tolerances \[1\].

5.1 Dipole field and alignment errors in transfer lines

The motion of a particle in a transfer line can be written as

\[ y = A \sqrt{\beta} \sin (\phi + B) \]  \hspace{1cm} (15)

This motion is an ellipse in phase space with

\[ y' = \frac{A}{\sqrt{\beta}} \cos (\phi + B) - \frac{A\alpha}{\sqrt{\beta}} \sin (\phi + B) . \]  \hspace{1cm} (16)

Rearranging we have

\[ Y = y / \sqrt{\beta} = A \sin (\phi + B) \]
\[ Y' = y\alpha / \sqrt{\beta} + y'\sqrt{\beta} = A \cos (\phi + B) , \]  \hspace{1cm} (17)

where \((Y, Y')\) are known as normalized phase-space coordinates since with these variables particles follow circular paths. Note that \(y'\) denotes \(dy/ds\), while \(Y'\) denotes \(dY/d\phi\) and that \(\alpha = -1/2 \, d\beta/ds\). The transformation to \((Y, Y')\) is conveniently written in matrix form as

\[
\begin{pmatrix}
Y \\
Y'
\end{pmatrix}
= \begin{pmatrix}
1 / \sqrt{\beta} & 0 \\
\alpha / \sqrt{\beta} & \sqrt{\beta}
\end{pmatrix}
\begin{pmatrix}
y \\
y'
\end{pmatrix}.
\]  \hspace{1cm} (18)

Consider now a beam for which the equi-density curves are circles in normalized phase space. If this beam receives an unwanted deflection, \(D\), it will appear at the time of the deflection as shown in Fig. 8(a). However, this asymmetric beam distribution will not persist. As the beam continues along the transfer line, the particles will re-distribute themselves randomly in phase, while maintaining their distance from the origin, so as to restore rotational symmetry. This effect is known as filamentation (see also Section l.1). Thus after a sufficient time has elapsed the particles, which without the deflection \(D\) would have been at point \(P\) in Fig. 8(b), will be uniformly distributed at a radius \(D\) about the point \(P\).

For one of these particles the projection onto the \(Y\)-axis will be

\[ Y_2 = Y_1 + D \cos \theta \]

where the subscripts 1 and 2 denote the unperturbed and perturbed positions respectively. Taking the square of this amplitude

\[ Y_2^2 = Y_1^2 + 2Y_1D\cos \theta + D^2 \cos^2 \theta \]
and then averaging over the particles around the point P after filamentation has randomized the kick gives

\[ \langle Y_2^2 \rangle_p = \langle Y_1^2 \rangle_p + 2 \langle Y_1 D \cos \theta \rangle_p + \langle D^2 \cos^2 \theta \rangle_p . \]

Since \( Y_1 \) and \( D \) are uncorrelated (i.e. \( D \) does not depend on \( Y_1 \)), the second term can be written as

\[ 2 \langle Y_1 D \cos \theta \rangle_p = 2 \langle Y_1 \rangle_p \langle D \cos \theta \rangle_p . \]

The second factor is zero, since \( D \) is a constant [Fig. 8(a)], which gives,

\[ \langle Y_2^2 \rangle_p = \langle Y_1^2 \rangle_p + \frac{1}{2} \langle D^2 \rangle_p = \langle Y_1^2 \rangle_p + \frac{1}{2} D^2 . \]

However, this result is true for any \( P \) at any radius \( A \) and hence it is true for the whole beam and

\[ \langle Y_2^2 \rangle = \langle Y_1^2 \rangle + \frac{1}{2} D^2 . \]  (19)

Thus the emittance blow-up will be

\[ \varepsilon_2 = \varepsilon_1 + \frac{\pi}{2} D^2 , \]  (20)

where, by definition (4), \( \varepsilon = \pi \langle Y^2 \rangle \), since \( Y = y/\beta \) and \( \sigma^2 = \langle y^2 \rangle \). The subscripts 1 and 2 refer to the unperturbed and perturbed emittances respectively. The expansion of the deflection, \( D \), gives

\[ D^2 = (\Delta Y)^2 + (\Delta Y')^2 = (\Delta y)^2 \left( \frac{1 + \alpha^2}{\beta} \right) + (\Delta y')^2 \beta \]  (21)

so that (20) becomes

\[ \varepsilon_2 = \varepsilon_1 + \frac{\pi}{2} \left[ (\Delta y)^2 \left( \frac{1 + \alpha^2}{\beta} \right) + (\Delta y')^2 \beta \right] , \]  (22)

* \(<...>\) brackets indicate averaging over a distribution.
where $\Delta y$ is a magnet alignment error and $\Delta y' = l\Delta B/B\rho$ an angle error from a field error $\Delta B$ of length $l$.

### 5.2 Gradient errors in transfer lines

Consider once again a beam for which the equi-density curves are circles in normalized phase space. If this beam sees a gradient error, $k$, the equi-density curves directly after the perturbation will be ellipses as shown in Fig. 9(a). Since the object of this analysis is to evaluate the effects of small errors, it is sufficient to regard this gradient error as a thin lens with the transfer matrix

$$
\begin{pmatrix}
y_2 \\
y'_2
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
k & 1
\end{pmatrix}
\begin{pmatrix}
y_1 \\
y'_1
\end{pmatrix}
$$

(23)

where $k = -l\Delta G/B\rho$ an amplitude-dependent kick due to a gradient error $\Delta G$ of length $l$.

![Diagram of gradient error effect](image)

Fig. 9 Effect of a gradient error

Denoting the matrix in Eq. (18) as $T$, it is easy to show that

$$
\begin{pmatrix}
Y_2 \\
Y'_2
\end{pmatrix} = T
\begin{pmatrix}
1 & 0 \\
k & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
Y_1 \\
Y'_1
\end{pmatrix} =
\begin{pmatrix}
1 & 0 \\
k\beta & 1
\end{pmatrix}
\begin{pmatrix}
Y_1 \\
Y'_1
\end{pmatrix}.
$$

(24)

It is now convenient to find a new co-ordinate system $(YY, YY')$, which is at an angle $\theta$ to the $(Y, Y')$ system, and in which the perturbed ellipse is a right ellipse [see Fig. 9(b)].

$$
\begin{pmatrix}
YY_2 \\
YY'_2
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
k\beta & 1
\end{pmatrix}
\begin{pmatrix}
Y_1 \\
Y'_1
\end{pmatrix}.
$$

(25)

If the initial distribution $Y_1 = A \sin (\phi + B)$. $Y'_1 = A \cos (\phi + B)$, is introduced into the above expression, the new distribution will be

$$
\begin{align*}
YY_2 &= A \sqrt{1 + k^2 \beta^2} \sin^2 \theta - 2k\beta \sin \theta \cos \theta \sin (\phi + B + \Psi) \\
YY'_2 &= A \sqrt{1 + k^2 \beta^2} \cos^2 \theta + 2k\beta \sin \theta \cos \theta \sin (\phi + B + \Psi')
\end{align*}
$$

where
\[ \Psi = \tan^{-1}\left(\frac{-\sin \theta}{\cos \theta - k\beta \sin \theta}\right) \quad \text{and} \quad \Psi' = \tan^{-1}\left(\frac{\cos \theta}{\sin \theta + k\beta \cos \theta}\right). \]

The \((YY_2, YY'_2)\) ellipse will be a right ellipse when \((\Psi - \Psi') = \pi/2\), which gives the condition

\[ \tan (2\theta) = 2 / k\beta. \]  

Equations (25) can be simplified using (26) and the relationship \((\Psi - \Psi') = \pi/2\). Equation (24) can then be rewritten as

\[ \begin{pmatrix} YY_2 \\ YY'_2 \end{pmatrix} = \begin{pmatrix} \tan \theta & 0 \\ 0 & 1 / \tan \theta \end{pmatrix} \begin{pmatrix} YY_i \\ YY'_i \end{pmatrix} \]  

where

\[ YY_i = A \sin (\phi + B') \quad \text{i.e. } Y_i \quad \text{and } Y'_i \quad \text{with a phase shift} \]

\[ YY'_i = A \cos (\phi + B') \]

\[ B' = B + \Psi = B + \tan^{-1}(1 / \tan \theta). \]

Thus it has been possible to diagonalize Eq. (24) by introducing a phase shift \(\Psi\) into the initial distribution. Equation (27) is therefore not a true point-to-point transformation, as is Eq. (24) but since the initial distribution is rotationally symmetric the introduction of this phase shift has no effect.

The distance from the origin of a perturbed particle is given by Eq. (27) as

\[ YY^2 + YY'^2 = A^2 \sin^2(\phi + B') \tan^2 \theta + A^2 \cos^2(\mu + B') \frac{1}{\tan^2 \theta}. \]

Averaging over \(2\pi\) in \(\phi\) gives

\[ \langle YY^2 + YY'^2 \rangle = \frac{1}{2} \left( \tan^2 \theta + \frac{1}{\tan^2 \theta} \right) \langle A^2 \rangle, \]

but

\[ \langle A^2 \rangle = \langle YY^2 + YY'^2 \rangle = \langle Y_i^2 + Y'_i^2 \rangle \]

and from (26)

\[ \tan^2 \theta + \frac{1}{\tan^2 \theta} = k^2\beta^2 + 2. \]

Thus,

\[ \langle YY^2 + YY'^2 \rangle = \frac{1}{2} (k^2\beta^2 + 2) \langle YY_i^2 \rangle. \]  

As in the previous case for dipole errors, the asymmetric beam distribution will not persist. The beam will regain its rotational symmetry by filamentation. Each particle, however, will maintain its distance from the origin constant. Once filamentation has occurred, the distribution will not distinguish between the \(YY\) and \(YY'\) axes and Eq. (28) can be rewritten as

\[ \langle YY^2 \rangle = \frac{1}{2} (k^2\beta^2 + 2) \langle YY_i^2 \rangle \]  

and hence the emittance blow-up will be

\[ \varepsilon_2 = \frac{1}{2} (k^2\beta^2 + 2) \varepsilon_1. \]

5.3 Combining errors
If there is a circular machine at the end of the transfer line, filamentation will take place there and the above expressions will give the emittance blow-up due to a single error in the preceding transfer line. A series of errors can be treated by taking them in beam order and assuming complete phase randomization between each error, although this is unlikely to be true in practical cases. By themselves, transfer lines are usually too short for the effects of filamentation to show and certainly there is never complete randomization between elements in a line. In the real world adjacent magnets are often on the same transformer, which also gives correlated errors. Having pointed out these deficiencies, the above method still gives a basis upon which to compare errors and fix tolerances. The assumption that full randomization takes place between elements will give a pessimistic result for the usual case of many independent elements, which errs on the correct side for fixing tolerances. For small numbers of elements with correlated errors however, the analysis may underestimate the effect.

6. EMITTANCE BLOW-UP DUE TO THIN WINDOWS IN TRANSFER LINES

Transfer lines are often built with a thin metal window separating their relatively poor vacuum from that of the accelerator or storage ring that they serve. The beam must pass through this window with as little degradation as possible. Luminescent screens are also frequently put into beams with the same hope that they will have a negligibly small effect on the beam emittance. It is therefore interesting to know how to calculate the blow-up for such cases.

The root mean square projected angle $q_s^2$ due to multiple Coulomb scattering in a window is given by [2,3]

$$\sqrt{\langle q_s^2 \rangle} = \frac{0.0141}{\beta_c p [\text{MeV} / c]} Z_{inc} \sqrt{\frac{L}{L_{rad}}} \left( 1 + \frac{1}{9} \log_{10} \frac{L}{L_{rad}} \right) \text{[radian]},$$

(31)

where $Z_{inc}$ is particle charge in units of electron charge, $p$ is the particle momentum in MeV/c, $\beta_c = v/c$, $L$ is thickness of scatterer and $L_{rad}$ is radiation length of material of the scatterer.

Consider a particle with a projected angular deviation of $y_1'$ at the window due to the initial beam emittance. This particle receives a net projected kick in the window of $q_s$ and emerges with an angle $y_2'$ given by

$$y_2' = y_1' + q_s.$$

By squaring and averaging over the whole beam this becomes

$$\langle y_2'^2 \rangle = \langle y_1'^2 \rangle + \langle q_s^2 \rangle + 2 \langle y_1' q_s \rangle$$

but, since $y_1'$ is not correlated to $q_s$, $2 \langle y_1' q_s \rangle = 2 \langle y_1' \rangle \langle q_s \rangle = 0$ and the above simplifies to

$$\langle y_2'^2 \rangle = \langle y_1'^2 \rangle + \langle q_s^2 \rangle.$$

(32)

This describes the situation immediately after the scattering (see Fig. 10) when the beam is no longer matched. The position of the particles is unchanged since the scatterer is assumed to be thin.
Using the same arguments as in Section 5.1, we see that this initial distribution filaments and the average angular divergence becomes

\[ \langle y'^2 \rangle = \langle y^2 \rangle + \frac{1}{2} \langle \theta_s^2 \rangle. \]  

(33)

For conversion to emittance, the following relationship can be used,

\[ \varepsilon_2 = \pi \left[ \frac{\alpha}{\beta} \langle y'^2 \rangle + 2\alpha \langle y'_2 y_2' \rangle + \beta \langle y'^2 \rangle \right] \]  

(34)

which is found by re-writing (4) as \( \varepsilon = \pi \sigma^2 / \beta = \pi \langle y^2 \rangle = \pi \langle y'^2 \rangle \) and applying (17). The first term in (34) is unchanged by the scattering since the scatterer is assumed to be thin, so that \( \langle y'^2 \rangle = \langle y^2 \rangle \). The second term directly after the scattering yields,

\[ \langle y'_2 y'_2 \rangle = \langle y_2 (y'_2 + \theta_s) \rangle = \langle y'_2 y'_2 \rangle + \langle y_2 \theta_s \rangle \]

but since \( y_2 \) and \( \theta_s \) are uncorrelated the second term can be written as \( \langle y_2 \rangle \langle \theta_s \rangle \) and is zero. Finally the third term can be evaluated by (33) after filamentation, so that,

\[ \varepsilon_2 = \varepsilon_1 + \frac{\pi}{2} \beta \langle \theta_s^2 \rangle. \]  

(35)

7. EMITTANCE DILUTION FROM BETATRON MISMATCH

This is basically the gradient error problem of Section 5.2 seen from a slightly different viewpoint. It often happens that the constraints on the linear optics are such that an analytically perfect match cannot be found between the end of a transfer line and the accelerator its serves. It may also be that measurements of the beam ellipse reveal a mismatch of unknown origin. These situations pose the problem of what error in \( \beta \) and \( \alpha \) can be tolerated? The designer must therefore be able to convert the mismatch into an emittance increase that can be judged against criteria such as the acceptable loss of luminosity in a collider.
The transformation of the phase-space motion to normalised coordinates \((Y, Y')\) was given in (18). If this transformation is prepared for an ellipse characterised by \(a_1\) and \(b_1\), but is then applied to a mismatched ellipse characterised by \(a_2\) and \(b_2\) [using (15) and (16)] as indicated below,

\[
\begin{pmatrix} Y \\ Y' \end{pmatrix} = \begin{pmatrix} 1/\sqrt{\beta_1} & 0 \\ \alpha_1/\sqrt{\beta_1} & \sqrt{\beta_1} \end{pmatrix} \begin{pmatrix} Y' \\ \sqrt{\beta_1} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{\beta_1} & 0 \\ \alpha_1/\sqrt{\beta_1} & \sqrt{\beta_1} \end{pmatrix} \begin{pmatrix} A\sqrt{\beta_2} \sin(\phi + B) \\ A\sqrt{\beta_2} \cos(\phi + B) - A\alpha_2/\sqrt{\beta_2} \sin(\phi + B) \end{pmatrix}
\]

(36)

then an ellipse is obtained in the normalised phase space (see Fig. 11) with the equation†,

\[
Y^2 \left[ \frac{\beta_1}{\beta_2} + \left( \alpha_1 - \alpha_2 \frac{\beta_1}{\beta_2} \right)^2 \right] + Y'^2 \frac{\beta_2}{\beta_1} - 2YY' \left( \alpha_1 - \alpha_2 \frac{\beta_1}{\beta_2} \right) \frac{\beta_2}{\beta_1} = A^2
\]

(37)

that can be compared to the circle given by the matched beam

\[
Y^2 + Y'^2 = A^2
\]

(38)

† Multiply out (36) and then eliminate the sine and cosine terms by squaring and adding the two equations for \(Y\) and \(Y'\).

Equation (37) is exactly similar in form to a general phase space ellipse in normal space and if we apply the equivalents,

\[
\gamma \equiv \frac{\beta_1}{\beta_2} + \left( \alpha_1 - \alpha_2 \frac{\beta_1}{\beta_2} \right)^2 \frac{\beta_2}{\beta_1}, \quad \beta \equiv \frac{\beta_2}{\beta_1} \quad \text{and} \quad \alpha \equiv -\left( \alpha_1 - \alpha_2 \frac{\beta_1}{\beta_2} \right) \frac{\beta_2}{\beta_1}
\]

(39)

all the standard formulae [4] can be used. One can easily check for example that \(\gamma = (1 + \alpha^2)/\beta\) still holds. Thus, we can avoid a lot of tedious algebra and quote directly the major and minor axes, \(a\) and \(b\) of the mismatched ellipse,

\[
a = \frac{A}{\sqrt{2}} \left( \sqrt{H+1} + \sqrt{H-1} \right) \quad \text{and} \quad b = \frac{A}{\sqrt{2}} \left( \sqrt{H+1} - \sqrt{H-1} \right)
\]

(40)
where
\[
H = \frac{1}{2} \left[ \frac{\beta_1}{\beta_2} + \left( \alpha_1 - \alpha_2 \frac{\beta_1}{\beta_2} \right)^2 \frac{\beta_2}{\beta_1} + \frac{\beta_2}{\beta_1} \right].
\]  
(41)

As in Section 5.2, the circle of the matched beam can be converted to the ellipse of the mismatched beam by the application of a diagonal matrix of the form
\[
\begin{pmatrix}
\lambda & 0 \\
0 & 1/\lambda
\end{pmatrix}
\]
after a suitable rotation. The rotation has no significant influence since the original distribution is rotationally symmetric. From (40) we see that
\[
\lambda = \frac{1}{\sqrt{2}} \left( \sqrt{H + 1} + \sqrt{H - 1} \right) \quad \text{and} \quad \frac{1}{\lambda} = \frac{1}{\sqrt{2}} \left( \sqrt{H + 1} - \sqrt{H - 1} \right).
\]  
(42)

It is quickly verified that the two equations in (42) are consistent. Thus the square of the distance of a particle from the origin is
\[
\begin{aligned}
Y^2 + Y'^2 &= \lambda^2 A^2 \sin^2(\phi + B) + \lambda^{-2} A^2 \cos^2(\phi + B).
\end{aligned}
\]  
(43)

Averaging over all phases simplifies (43) to
\[
\langle Y^2 + Y'^2 \rangle = \frac{1}{2} \left( \lambda^2 + \lambda^{-2} \right) A^2.
\]  
(44)

Since the factor \((\lambda^2 + \lambda^{-2})\) is independent of radius and orientation, (44) applies to the whole beam independent of its distribution. Thus, we can express the effective increase in emittance as,
\[
\epsilon_{\text{diluted}} = \frac{1}{2} \left( \lambda^2 + \lambda^{-2} \right) \epsilon_0.
\]  
(45)

First we note that even for quite large values of \(\lambda\), the effect on the emittance is less than one might intuitively expect. For example, if \(\lambda=1.4\) the circle in normalised phase space is deformed in the ratio 2:1 and yet the emittance is only increased by 23%. Evaluating \((\lambda^2 + \lambda^{-2})/2\) by use of (42), we find that
\[
\epsilon_{\text{diluted}} = H \epsilon_0 = \frac{1}{2} \left[ \frac{\beta_1}{\beta_2} + \left( \alpha_1 - \alpha_2 \frac{\beta_1}{\beta_2} \right)^2 \frac{\beta_2}{\beta_1} + \frac{\beta_2}{\beta_1} \right] \epsilon_0.
\]  
(46)

where the subscript 1 denotes the expected values for the beam ellipse and subscript 2 denotes the mismatched values.

8. EMITTANCE EXCHANGE INSERTION

Beams usually have different emittances in the two transverse planes and it can happen that there is a preference for having the smaller value in a particular plane. For example, in a
collider with a horizontal crossing angle the luminosity is independent of the horizontal emittance and it is therefore an advantage to arrange for the smaller of the two emittances to be in the vertical plane. The exchange of the emittances can be made in the transfer line before injection to the collider. A complete exchange of the transverse phase planes requires a transformation of the form,

\[
\begin{pmatrix}
    x \\
x' \\
z \\
z'
\end{pmatrix}_{2} =
\begin{pmatrix}
    0 & 0 & m_{13} & m_{14} \\
    0 & 0 & m_{23} & m_{24} \\
    m_{31} & m_{32} & 0 & 0 \\
    m_{41} & m_{42} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
    x \\
x' \\
z \\
z'
\end{pmatrix}
\]  \quad (47)

This can be achieved by using skew quadrupole lenses. First we shall derive a thin-lens formulation for a skew quadrupole and then search for the conditions to satisfy the above transformation.

For a rotated co-ordinate system (see Fig. 12), the rotation matrix, \( R \), is given by

\[
\begin{align*}
xx &= x \cos \theta + y \sin \theta \\
yy &= -x \sin \theta + y \sin \theta 
\end{align*}
\]  \quad (48)

so that \( R = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \).

![Fig. 12 Rotated co-ordinate system](image)

The skew quadrupole lens is just a normal lens rotated by 45°. Thus the transfer matrix, \( M_s \), in the thin-lens approximation would be related to the transfer matrix, \( M_q \), of the normal lens by,

\[
M_s = R^{-1} M_q R
\]  \quad (49)

For a pure skew quadrupole \( \theta = \pi/4 \), \( \sin 2\theta = 1 \) and \( \cos 2\theta = 0 \)
Consider now three skew quadrupoles $\delta_1$, $\delta_2$, and $\delta_3$ in a normal lattice (see Fig. 13).

The normal lattice matrix is $\mathbf{C} = \mathbf{BA}$, and the lattice with skews is, $\mathbf{M} = (\delta_3)\mathbf{B}(\delta_2)\mathbf{A}(\delta_1)$. To save you the bother of matrix multiplication we quote the result,

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & \delta_1 & 0 \\
0 & 0 & 1 & 0 \\
\delta_1 & 0 & 0 & 1
\end{pmatrix}.
$$

(50)

Fig. 13 Three skew quadrupoles embedded in a normal lattice

The normal lattice matrix is $\mathbf{C} = \mathbf{BA}$, and the lattice with skews is, $\mathbf{M} = (\delta_3)\mathbf{B}(\delta_2)\mathbf{A}(\delta_1)$. To save you the bother of matrix multiplication we quote the result,

$$
\begin{pmatrix}
(c_{11} + b_{12}a_{34}\delta_1\delta_2) & c_{12} & (c_{12}\delta_1 + b_{12}a_{33}\delta_2) & b_{12}a_{34}\delta_2 \\
(c_{21} + b_{22}a_{34}\delta_1\delta_2) & 0 & (c_{22}\delta_1 + b_{22}a_{33}\delta_2) & 0 \\
\delta_1(c_{34}\delta_1 + b_{34}a_{12}\delta_2) & 0 & \delta_3(c_{33} + b_{34}a_{12}\delta_1\delta_2) & 0 \\
(c_{41} + b_{42}a_{34}\delta_1\delta_2) & 0 & (c_{42}\delta_1 + b_{42}a_{33}\delta_2) & 0 \\
\end{pmatrix}
$$

(51)

In order to force (51) into the form of (47), we need the top left-hand side and bottom right-hand side sub-matrices to be zero. Thus,

$$
\begin{align*}
0 &= c_{11} + b_{12}a_{34}\delta_1\delta_2 \\
0 &= c_{21} + b_{22}a_{34}\delta_1\delta_2 + \delta_1(c_{34}\delta_1 + b_{34}a_{12}\delta_2) \\
0 &= c_{12} \\
0 &= c_{22} + b_{34}a_{12}\delta_2 \\
0 &= c_{34} \quad \text{and} \quad \Delta \phi_z = m\pi \quad n, m \text{ integer}.
\end{align*}
$$

(52)

The most basic requirements are, $c_{12} = c_{34} = 0$. Thus, the basic lattice should give,
For example, an [FDFD] structure with 90° phase advance per cell, or an [FDFDFD] with 60° phase advance per cell would be suitable. Assuming that this phase condition is satisfied, then we can write for the lens strengths $\delta_1$, $\delta_2$ and $\delta_3$

$$
\delta_1 \delta_2 = - \frac{c_{11}}{a_{12} b_{12}} = - \frac{c_{33}}{a_{12} b_{34}} \tag{53}
$$

$$
\delta_2 \delta_3 = - \frac{c_{22}}{a_{12} b_{34}} = - \frac{c_{44}}{a_{34} b_{12}} \tag{54}
$$

Equations (53) and (54) indicate that symmetry is needed between the horizontal and vertical planes, which can be satisfied by the FODO cells mentioned above. Despite the apparent complexity of the coefficients it is possible to find solutions. For example let us choose an [FDFD] structure with 90° per cell and the skew quadrupoles set at symmetric positions as shown in Fig. 14.

By symmetry $A = B$ and the input $\alpha$ and $\beta$ values will equal the output values and also the values at the central skew quadrupole. So,

$$
A \left( \begin{array}{c}
B_1 \\
B_2 \\
B_3
\end{array} \right) = \left( \begin{array}{c}
\cos \Delta \phi_z + \alpha_z \sin \Delta \phi_z \\
\beta_z \sin \Delta \phi_z \\
\cos \Delta \phi_z - \alpha_z \sin \Delta \phi_z
\end{array} \right) \left( \begin{array}{c}
B_1 \\
B_2 \\
B_3
\end{array} \right)
$$

$$
= \left( \begin{array}{c}
\beta_z \sin \Delta \phi_z \\
\cos \Delta \phi_z - \alpha_z \sin \Delta \phi_z
\end{array} \right)
$$

Matrix $C$ will have a similar form with the phase shifts $2\Delta \phi_x$ and $2\Delta \phi_z$. For this case we have chosen $\Delta \phi_x = \Delta \phi_z = \pi/2$, so that
\[ A = B = \begin{pmatrix} \alpha_x & \beta_x & 0 & 0 \\ -(1 + \alpha_x^2)\beta_x^{-1} & -\alpha_x & 0 & 0 \\ 0 & 0 & \alpha_z & \beta_z \\ 0 & 0 & -1 & -\alpha_z \end{pmatrix} \] (55)

\[ C = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \] (56)

The skew quadrupole gradients would then be,

\[ \delta_1 = \delta_2 = \delta_3 = \frac{1}{\sqrt{\beta_x \beta_z}} \] (57)

Referring back to the thin-lens formulation of a FOD0 cell in the lectures by J. Rossbach and P. Schmüser in these proceedings, we see that

\[ \sin \left( \frac{\Delta \phi}{2} \right) = \frac{L}{2} \delta_F = -\frac{L}{2} \delta_D \]

and for \( \Delta \phi = \pi/2 \)

\[ \delta_F = -\delta_D = \frac{\sqrt{2}}{L} \] (58)

where \( L \) is the half-cell length. Other examples can be found in Ref. [5].

* * *

REFERENCES


