A Fluctuation-Dissipation Relation for Semiclassical Cosmology

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March 30, 1994

(umdpp 93-164)

Abstract

Using the concept of open systems where the classical geometry is treated as the system and the quantum matter field as the environment, we derive a fluctuation-dissipation theorem for semiclassical cosmology. This theorem which exists under very general conditions for dissipations in the dynamics of the system, and the noise and fluctuations in the environment, can be traced to the formal mathematical relation between the dissipation and noise kernels of the influence functional depicting the open system, and is ultimately a consequence of the unitarity of the closed system. In particular, for semiclassical gravity, it embodies the backreaction effect of matter fields on the dynamics of spacetime. The backreaction equation derivable from the influence action is in the form of a Einstein-Langevin equation. It contains a dissipative term in the equation of motion for the dynamics of spacetime and a noise term related to the fluctuations of particle creation in the matter field. Using the well-studied model of a quantum scalar field in a Bianchi Type-I universe we illustrate how this Langevin equation and the noise term are derived and show how the creation of particles and the dissipation of anisotropy during the expansion of the universe can be understood as a manifestation of this fluctuation-dissipation relation.

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1 Introduction

An important relation between the dissipation in the dynamics of a system and the fluctuations in a heat bath with which the system interacts is the fluctuation-dissipation relation (FDR) [1]. A first example of its manifestation is the Nyquist noise in an electric circuit [2]. This relation is of practical interest in the design of noisy systems [3]. It is also of theoretical interest in statistical physics because it is a categorical relation which exists between the stochastic behavior of many microscopic particles and the deterministic behavior of a macroscopic system. It is therefore also useful for the description of the interaction of a system with fields, such as effects related to radiation reaction and vacuum fluctuations between atoms and fields in quantum optics [4]. The form of the FDR is usually given under near-equilibrium conditions via linear response theory (LRT) [5]. We will see in this paper that this relation has a much wider scope and a broader implication than has been understood before. In particular we want to address problems involving gravity and quantum fields in black holes and the early universe; and we are interested in seeing this relation validated and implemented under non-equilibrium conditions for quantum fields in curved spacetimes [6]. The problem we choose for illustration is the backreaction of particles created [7, 8, 9, 10, 11] in a cosmological spacetime (without event horizon) [12, 13, 14, 15, 16, 17, 18, 19]. The conceptual framework we adopt is that of a quantum open system [20], the formal scheme is that of the closed-time-path [21] and the influence functional [22, 23] formalisms, and the paradigm we use for comparison is that of quantum Brownian motion [24, 25, 26].

Sciama [27] was the one who had the great insight of proposing a fluctuation-dissipation relation [28, 29] for the depiction of quantum processes in black holes [30], uniformly-accelerated observers [31, 32] and de Sitter universe [33]. Using Einstein’s analysis of the Brownian motion as a guide he showed that the Hawking and Unruh radiations can be seen as excitations of vacuum fluctuations and the detector response as following a dissipation-fluctuation relation. A crucial element for this interpretation to be possible is the existence of an Euclidean section in the Schwarzschild, Rindler and de Sitter metrics, imparting a periodicity in the Green’s function of the matter field, thus turning it into a thermal propagator [34, 35] (in the imaginary-time Matsubara sense), and forging the equivalence of the system with finite-temperature results [36]. The same condition applies to Hawking radiation in de Sitter universe [33], which by virtue of its possession of an event horizon, also admits a FDR interpretation [37]. The derivation of the FDR in this class of spacetime was based on a linear response theory, which hinges on the thermal equilibrium condition set up by the created particles. For spacetimes without an event horizon, or for systems under non-equilibrium conditions, one would not ordinarily think that a FDR could exist [38, 20]. The generalization of this relation to non-equilibrium conditions is a much more difficult problem.

When Sciama first proposed this way of thinking, one of us was involved in the backreaction studies of quantum processes in cosmological spacetimes [19]. The dissipative effect of particle creation on the dynamics of spacetime seems to point to the existence of a fluctuation-dissipation relation, except that one factor (dissipation) is not clear, and the other factor
(noise) is missing. Two important steps had to be taken before this picture began to make better sense. In the calculation of Hartle and Hu [17] on anisotropy damping, the Schwinger-DeWitt effective action (‘in-out’) formalism [39] gives rise to an effective geometry which is complex, making it difficult to interpret what dissipation really is. The adoption of the Schwinger-Keldysh (closed-time-path, CTP, or ‘in-in’) [21] formalism by Calzetta and Hu [40] yields a real and causal equation of motion for the effective geometry, from which one can relate the source of dissipation in spacetime dynamics to the energy density of particles created and explicitly identify the viscosity function associated with anisotropy damping [41].

The adoption of the CTP formalism was an encouraging step in the right direction, but one needs to understand the statistical mechanical meaning of these quantum processes better in order to appraise the validity of adopting the well-established concepts and results in statistical mechanics for their depiction or explication. For the particular task of showing a fluctuation-dissipation relation at work for quantum fields in a general cosmological spacetime not required to possess an event horizon, there is also the noise or fluctuation term missing. These inquiries were summarized in a report written by one of us [42] in which some tentative replies were given in the form of three conjectures:

1) That colored noise associated with quantum field fluctuations is generally expected in gravitation and cosmology;
2) That the backreaction of particle creation in a dynamical spacetime can be viewed as the manifestation of a generalized fluctuation-dissipation relation; and
3) That all effective field theories, including semiclassical gravity or even quantum gravity (to the extent that it could be viewed as an effective field theory), are intrinsically dissipative in nature.

There, it was also suggested that one can use the Caldeira-Leggett model [23] to study the theoretical meaning of dissipation and probe into the relation of noise and dissipation. The next stage of work in this quest concentrated on the properties of quantum open systems [20] and extending the theory to quantum fields and to curved spacetimes.

Using the influence functional formalism of Feynman and Vernon [22], one can identify noise in an environment from the imaginary part of the influence action. The characteristics of noise depends on the spectral density of the environment, the coupling of the system with the environment, and other factors. Using a model of the Brownian particle coupled nonlinearly with a bath of harmonic oscillators, Hu, Paz and Zhang [25] deduced the noise autocorrelation functions and a generalized fluctuation-dissipation relation for systems driven by intrinsic colored and multiplicative noises. This forms the basis for the second stage of investigation. To generalize these results to quantum fields, Hu and Matacz [43, 44] recently analyzed the problem of QBM in a parametric oscillator bath. A parametric oscillator bath enables one to study particle creation in quantum fields, where the Brownian particle can play the role of an Unruh detector, or, in a cosmological backreaction problem, the scale factor of the universe. One can study the detector’s response or the effect on the universe due to the fluctuations of the quantum field. They found that the characteristics of quantum noise vary with the nature of the field, the type of coupling between the field
and the background spacetime, and the time-dependence of the scale factor of the universe. They showed how a uniformly accelerating detector in Minkowski space, a static detector outside a black hole and a comoving observer in a de Sitter universe all observe a thermal spectrum. By writing the influence functional in terms of the Bogolubov coefficients which determine the amount of particles produced, they also identified the origin of noise in this system to particle creation [45, 46]. The influence functional method not only reproduces the known results, but also enables one to look into the hitherto unknown domain of noise, fluctuations, and decoherence.

A program for studying the backreaction of particle creation in semiclassical cosmology in the open system conceptual framework using influence functional methods was recently outlined in [46, 45, 47]. The backreaction of these quantum field processes manifests as dissipation effect, which is described by the dissipation kernel in the influence action. Using a model where the quantum Brownian particle and the oscillator bath are coupled parametrically (the field parameters change in time through the time-dependence of the scale factor of the universe, which is governed by the semiclassical Einstein equation) Hu and Matarcz [46] derived an expression for the influence functional in terms of the Bogolubov coefficients as a function of the scale factor. From the variation of the influence action they obtained an equation of motion describing the dynamics of spacetime in the form of an Einstein-Langevin equation.

After these recent works, it is clear that the influence functional method is the appropriate framework for studying the nature and origin of noise in quantum fields and to explore the statistical mechanical meaning of quantum processes like particle creation and backreaction in the early universe and black holes. Two additional aspects, however, need be considered to complete the story. First, how is it related to the CTP formalism, which gave us, to begin with, the correct dissipation side of the story? This problem was taken up in a recent paper of Calzetta and Hu [45], who showed how noise and fluctuations in semiclassical gravity can also be obtained with the original CTP formalism. They also showed that the CTP and the IF formalisms are indeed intimately related. They derived an expression for the CTP effective action in terms of the Bogolubov coefficients and showed how noise is related to the fluctuations in particle number. From there, they show how an Einstein-Langevin equation naturally arises as the equation of motion for the effective geometry, from which a new, extended theory of semiclassical gravity is obtained. The work of Calzetta, Matarcz and the present authors shows clearly that the old framework of semiclassical gravity is only a mean field theory. This theory based on the Einstein equation with a source driven by the expectation value of the energy momentum tensor should be replaced by one based on an Einstein-Langevin equation which describes also the fluctuations of matter fields and dissipative dynamics of spacetime.

Notice that in moving from the first stage of this investigation based on the CTP formalism to the second stage based on the IF formalism, one has to elevate the treatment of classical spacetimes as external fields to reduced density matrices. In making these transitions and back, several issues need be addressed. The central issue is the quantum to classical transition for the spacetime sector [48]. The important question behind the transition from
quantum gravity to semiclassical gravity is decoherence. This is a subject of much recent interest. We refer the reader to recent work for the exposition of different viewpoints and approaches [49, 50, 51]. Here, for the backreaction problem, we shall adopt the results of Paz and Sinha [52], which is based on a reduced density matrix formalism adapted to quantum cosmology. There, the model of a Bianchi-I universe coupled to a scalar field was used to derive conditions for transition from quantum cosmology to the semiclassical limit via decoherence, and the relationship between decoherence and backreaction was investigated.

After these previous investigations paved the way for the use of open-system concepts applied to the backreaction problem of quantum fields in curved spacetimes, we are finally in a position to look at the full picture and explore the existence of a fluctuation-dissipation relation for semiclassical gravity in general. We shall use the model of particle creation in Bianchi Type I universe to explore this relation. In Sec. 2, we give a summary of the results for the quantum Brownian model, assuming a general nonlinear coupling between the system and the environment, giving rise to colored and multiplicative noise. Readers familiar with the QBM problem can skip over this section. In Sec. 3 we begin with the density matrix of the universe and show how coarse-graining the matter field viewed as an environment produces the reduced density matrix, and how the influence functional defined in the evolutionary operator for the reduced density matrix contains the relevant information we need—the dissipation and noise kernels. In Sec. 4 we analyze the phase and the real components of the influence functional in detail, sorting out the divergent and renormalized terms in the phase. We show that the renormalized phase part provides the dissipative term in the equation of motion, and the real component contributes to decoherence and noise. We show how a colored noise of the quantum field can be identified, and with it the existence of a fluctuation-dissipation relation between these kernels. In Sec. 5, we discuss the physical meaning of this relation. We first show that noise measures the difference in the amounts of particle creation along two histories. Since this is also the condition for decoherence to occur, we see that a relation also exists between decoherence and particle creation. With this noise term, we then derive the Einstein-Langevin equation for the anisotropy tensor. We show that it is identical in form to that derived via the CTP formalism before [40], but with a new stochastic source term from the noise, as anticipated in [42]. Finally, we show how the dissipation in the anisotropy of spacetime can be related to the particles created. Thus noise and dissipation which are connected by a formal relation, are both related to particle creation, and the backreaction of particle creation is an embodiment of the FDR. In Sec. 6 we discuss the physical interpretation of the FDR in a more general context. We show how the changing rate of particle creation and the strength of backreaction effect can be gauged consistently by the fluctuation-dissipation relation valid for time-dependent conditions. We also describe related problems for future investigations.
2 Influence Functional for Quantum Open System

2.1 Quantum Brownian Motion Paradigm

Let us first review a model problem of quantum Brownian motion (QBM) where the role of noise and dissipation are well understood. Subsequently we will draw analogies from this problem to analyze the quantum cosmology problem of our interest.

Consider a Brownian particle interacting with a set of harmonic oscillators. The classical action of the Brownian particle is given by

\[ S[x] = \int_0^t ds \left\{ \frac{1}{2} M \dot{x}^2 - V(x) \right\}. \] (2.1)

The action for the environment is given by

\[ S_e[q_n] = \int_0^t ds \sum_n \left\{ \frac{1}{2} m_n \dot{q}_n^2 - \frac{1}{2} m_n \omega_n^2 q_n^2 \right\}. \] (2.2)

We will assume that the action for the system-environment interaction has the following form

\[ S_{\text{int}}[x, \{q_n\}] = \int_0^t ds \sum_n v_n(x) q_n^k \] (2.3)

where \( v_n(x) = -\lambda c_n f(x) \) and \( \lambda \) is a dimensionless coupling constant. If one is interested only in the averaged effect of the environment on the system the appropriate object to study is the reduced density matrix of the system \( \rho_r \), which is related to the full density matrix \( \rho \) as follows

\[ \rho_r(x, x') = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dq' \rho(x, q; x', q') \delta(q - q'). \] (2.4)

It is propagated in time by the evolution operator \( \mathcal{J}_r \)

\[ \rho_r(x, x', t) = \int_{-\infty}^{+\infty} dx_i \int_{-\infty}^{+\infty} dx_i' \mathcal{J}_r(x, x', t | x_i, x_i', 0) \rho_r(x_i, x_i', 0). \] (2.5)

If we assume that at a given time \( t = 0 \) the system and the environment are uncorrelated

\[ \hat{\rho}(0) = \hat{\rho}_s(0) \times \hat{\rho}_e(0), \] \hspace{1cm} (2.6)

then \( \mathcal{J}_r \) does not depend on the initial state of the system and can be written as

\[ \mathcal{J}_r(x_f, x'_f, t | x_i, x'_i) = \int_{x_i}^{x_f'} dx_i' \int_{x'_i}^{x'_f} dx' \exp \left\{ \frac{i}{\hbar} \left( S[x] - S[x'] \right) \right\} \mathcal{F}[x, x'] \]
\[ = \int_{x_i}^{x_f'} dx_i' \int_{x'_i}^{x'_f} dx' \exp \left\{ \frac{i}{\hbar} S_{\text{eff}}[x, x'] \right\}. \] \hspace{1cm} (2.7)
where the subscripts $i,f$ denote initial and final variables, and $S_{\text{eff}}[x,x']$ is the effective action for the open quantum system. The influence functional $\mathcal{F}[x,x']$ is defined as

$$
\mathcal{F}[x,x'] = \int_{-\infty}^{+\infty} dq_i \int_{-\infty}^{+\infty} dq_i' \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dq' \times \exp \left\{ i \left[ S_e[q] + S_{\text{int}}[x,q] - S_e[q'] - S_{\text{int}}[x',q'] \right] \rho_e(q_i, q_i', 0) \right\} = \exp \left\{ i S_{IF}[x,x'] \right\}
$$

(2.8)

where $S_{IF}[x,x']$ is the influence action. Thus $S_{\text{eff}}[x,x'] = S[x] - S[x'] + S_{IF}[x,x']$.

From its definition it is obvious that if the interaction term is zero, the influence functional is equal to unity and the influence action is zero. In general, the influence functional is a highly non-local object. Not only does it depend on the time history, but also this is the most important property— it also irreducibly mixes the two sets of histories in the path integral of (2.7). Note that the histories $x$ and $x'$ could be interpreted as moving forward and backward in time respectively. Viewed in this way, one can see the similarity of the influence functional and the generating functional in the closed-time-path, or Schwinger-Keldysh [21] integral formalism.

We will assume that initially the bath is in thermal equilibrium at a temperature $T = (k_B \beta)^{-1}$. The $T = 0$ case corresponds to the bath oscillators being in their respective ground states. It can be shown [26] that the influence action for the model given by the interaction in (2.3) to second order in $\lambda$ is given by

$$
S_{IF}[x,x'] = \left\{ \int_0^t ds \left[ -\Delta V(x) \right] - \int_0^t ds \left[ -\Delta V(x') \right] \right\}
$$

$$
- \int_0^t ds_1 \int_0^{s_1} ds_2 \lambda^2 \left[ f(x(s_1)) - f(x'(s_1)) \right] \mu^{(k)}(s_1 - s_2) \left[ f(x(s_2)) + f(x'(s_2)) \right]
$$

$$
+ i \int_0^t ds_1 \int_0^{s_1} ds_2 \lambda^2 \left[ f(x(s_1)) - f(x'(s_1)) \right] \mu^{(k)}(s_1 - s_2) \left[ f(x(s_2)) - f(x'(s_2)) \right]
$$

(2.9)

where $\Delta V(x)$ is a renormalization of the potential that arises from the contribution of the bath. It appears only for even $k$ couplings. For the case $k = 1$ the above result is exact. This is a generalization of the result obtained in [22] where it was shown that the non-local kernel $\mu^{(k)}(s_1 - s_2)$ is associated with dissipation or the generalized viscosity function that appears in the corresponding Langevin equation and $\mu^{(k)}(s_1 - s_2)$ is associated with the time correlation function of the stochastic noise term. The dissipation part has been studied in detail by Calzetta, Hu, Paz, Sinha and others [40, 41, 53, 78, 52, 45] in cosmological backreaction problems. We shall elaborate somewhat on the nature of the noise part of the problem and then analyze their connection. In general $\nu$ is nonlocal, which gives rise to colored noises. Only at high temperatures would the noise kernel become a delta function, which corresponds to a white noise source. Let us first see the meaning of the noise kernel.
2.2 Noise

The noise part of the influence functional is given by

\[
\exp\left\{-\frac{1}{\hbar} \int_0^t ds_1 \int_0^t ds_2 \left[f(x(s_1)) - f(x'(s_1))\right] \nu^{(k)}(s_1 - s_2) \left[f(x(s_2)) - f(x'(s_2))\right]\right\} \tag{2.10}
\]

where \(\nu^{(k)}\) is redefined by absorbing the \(\lambda^2\). This term can be rewritten using the following functional Gaussian identity [22] which states that the above expression is equal to

\[
\int \mathcal{D}\xi^{(k)}(t) \mathcal{P}[\xi^{(k)}] \exp\left\{-\frac{i}{\hbar} \int_0^t ds \left[f(x(s)) - f(x'(s))\right]\right\} \tag{2.11}
\]

where

\[
\mathcal{P}[\xi^{(k)}] = P^{(k)} \exp\left\{-\frac{1}{\hbar} \int_0^t ds_1 \int_0^t ds_2 \frac{1}{2} \xi^{(k)}(s_1) [\nu^{(k)}(s_1 - s_2)]^{-1} \xi^{(k)}(s_2)\right\} \tag{2.12}
\]

is the functional distribution of \(\xi^{(k)}(s)\) and \(P^{(k)}\) is a normalization factor given by

\[
[P^{(k)}]^{-1} = \int \mathcal{D}\xi^{(k)}(s) \exp\left\{-\frac{1}{\hbar} \int_0^t ds_1 \int_0^t ds_2 \xi^{(k)}(s_1) [\nu^{(k)}(s_1 - s_2)]^{-1} \xi^{(k)}(s_2)\right\}. \tag{2.13}
\]

The influence functional can then be rewritten as

\[
\mathcal{F}[x, x'] = \int \mathcal{D}\xi^{(k)}(s) \mathcal{P}[\xi^{(k)}] \exp\left\{\frac{i}{\hbar} \hat{S}_{IF}[x, x', \xi^{(k)}]\right\}
\equiv \left\langle \exp\left\{\frac{i}{\hbar} \hat{S}_{IF}[x, x', \xi^{(k)}]\right\} \right\rangle_{\xi} \tag{2.14}
\]

where

\[
\hat{S}_{IF}[x, x', \xi^{(k)}] = \int ds \left\{-\Delta V(x) \right\} - \int ds \left\{-\Delta V(x') \right\}
- \int_0^t ds_1 \int_0^t ds_2 \left[f(x(s_1)) - f(x'(s_1))\right] \nu^{(k)}(s_1 - s_2) \left[f(x(s_2)) + f(x'(s_2))\right]
- \int ds \xi^{(k)}(s) f(x(s)) + \int ds \xi^{(k)}(s) f(x'(s)) \tag{2.15}
\]

so that the reduced density matrix can be rewritten as

\[
\rho_r(x, x') = \int \mathcal{D}\xi^{(k)}(s) \mathcal{P}[\xi^{(k)}] \rho_r(x, x', [\xi^{(k)}]). \tag{2.16}
\]
The full effective action can be written as

\[
S_{\text{eff}}[x, x', \xi] = \{S[x] + \int_0^t ds \xi(s)f(x(s))\} - \{S[x'] + \int_0^t ds \xi(s)f(x'(s))\} - \int ds_1 \int ds_2 \left[ f(x(s_1)) - f(x'(s_1)) \right] \mu^{(k)}(s_1 - s_2) \left[ f(x(s_2)) + f(x'(s_2)) \right].
\]

From equation (2.15) we can view \(\xi^{(k)}(s)\) as a nonlinear external stochastic force and the reduced density matrix is calculated by taking a stochastic average over the distribution \(P[\xi^{(k)}]\) of this source.

From (2.12), we can see that the distribution functional is Gaussian. The Gaussian noise is therefore completely characterized by

\[
\langle \xi^{(k)}(s) \xi^{(k)}(s') \rangle = 0, \quad \langle \xi^{(k)}(s) \xi^{(k)}(s') \rangle = \hbar \nu^{(k)}(s_1 - s_2).
\]

We see that the non-local kernel \(\nu^{(k)}(s_1 - s_2)\) is just the two-point time correlation function of the external stochastic source \(\xi^{(k)}(s)\) multiplied by \(\hbar\).

In this framework, the expectation value of any quantum mechanical variable \(Q(x)\) is given by [54]

\[
\langle Q(x) \rangle = \langle \mathcal{D}_s \xi^{(k)}(s) \mathcal{P}[\xi^{(k)}] \int_{-\infty}^{+\infty} dx \rho_r(x, x, [\xi^{(k)}])Q(x) \rangle = \langle \langle Q(x) \rangle_{\text{quantum}} \rangle_{\text{noise}}.
\]

This summarizes the interpretation of \(\nu^{(k)}(s_1 - s_2)\) as a noise or fluctuation kernel.

### 2.3 Langevin Equation

We now derive the semiclassical equation of motion generated by the influence action (2.9). This will allow us to see why the kernel \(\nu^{(k)}(s_1 - s_2)\) should be associated with dissipation. Define a “center-of-mass” coordinate \(\bar{x}\) and a “relative” coordinate \(\Delta\) as follows

\[
\bar{x}(s) = \frac{1}{2}[x(s) + x'(s)] \quad \Delta(s) = x'(s) - x(s).
\]

The semiclassical equation of motion for \(\bar{x}\) is derived by demanding (cf. [40])

\[
\frac{\delta}{\delta \Delta} \left[ S[x] - S[x'] + S_I[x', x] \right]_{\Delta=0} = 0.
\]
Using the sum and difference coordinates (2.20) and the influence action (2.9) we find that (2.21) leads to

\[
\frac{\partial L_r}{\partial \dot{\bar{x}}} - \frac{d}{dt} \frac{\partial L_r}{\partial \dot{\bar{x}}} - 2 \frac{\partial f(\bar{x})}{\partial \dot{\bar{x}}} \int_0^t ds \ \gamma^{(k)}(t - s) \frac{\partial f(\bar{x}(s))}{\partial s} = F^{(k)}(t)
\]

where \( \frac{d}{ds} \gamma^{(k)}(t - s) = \mu^{(k)}(t - s) \). We see that this is in the form of a classical Langevin equation with a nonlinear stochastic force \( F^{(k)}(s) = -\xi^{(k)}(s) \frac{\partial f(\bar{x})}{\partial x} \). This corresponds to a multiplicative noise unless \( f(\bar{x}) = \bar{x} \) in which case it is additive. \( L_r \) denotes a renormalized system Lagrangian. This is obtained by absorbing a surface term and the potential renormalization in the influence action into the system action. The nonlocal kernel \( \gamma^{(k)}(t - s) \) is responsible for non-local dissipation. In special cases like a high temperature ohmic environment, this kernel becomes a delta function and hence the dissipation is local.

### 2.4 Fluctuation-Dissipation Relation

Recall that the label \( k \) is the order of the bath variable to which the system variable is coupled. \( \gamma^{(k)}(s) \) can be written as a sum of various contributions

\[
\gamma^{(k)}(s) = \sum_l \gamma_l^{(k)}(s)
\]

where the sum is over even (odd) values of \( l \) when \( k \) is even (odd). To derive the explicit forms of each dissipation kernel, it is useful to define first the spectral density functions

\[
I^{(k)}(\omega) = \sum_n \delta(\omega - \omega_n) \ k \ \pi \hbar^{k-2} \frac{\lambda^2 \tilde{c}^2(\omega_n)}{(2m_n \omega_n)^k}.
\]

It contains the information about the environmental mode density and coupling strength as a function of frequency. Different environments are classified according to the functional form of the spectral density \( I(\omega) \).

In terms of these functions, the dissipation kernels can be written as

\[
\gamma_l^{(k)}(s) = \int_0^\infty \frac{d\omega}{\pi} \frac{1}{\omega} I^{(k)}(\omega) \ M_l^{(k)}(z) \ \cos l\omega s
\]

where \( M_l^{(k)}(z) \) are temperature dependent factors derived in [26] and \( z = \text{coth} \frac{1}{2} \beta \hbar \omega \). Analogously, the noise kernels \( \nu^{(k)}(s) \) can also be written as a sum of various contributions

\[
\nu_l^{(k)}(s) = \sum_l \nu_l^{(k)}(s)
\]

where the sum runs again over even (odd) values of \( l \) for \( k \) even (odd). The kernels \( \nu_l^{(k)}(s) \) can be written as

\[
\nu_l^{(k)} = \hbar \int_0^\infty \frac{d\omega}{\pi} I^{(k)}(\omega) \ N_l^{(k)}(z) \ \cos l\omega s
\]
where $N_i^{(k)}(z)$ is another set of temperature-dependent factors given by [26].

To understand the physical meaning of the noise kernels of different orders, we can think of them as being associated with $l$ independent stochastic sources that are coupled to the Brownian particle through interaction terms of the form (2.15)

$$
\int_0^t ds \sum_i \xi_i^{(k)}(s) f(x). \tag{2.28}
$$

This type of coupling generates a stochastic force in the associated Langevin equation

$$
F_{\xi^{(k)}}(s) = -\xi_i^{(k)}(s) \frac{\partial f(x)}{\partial x}, \tag{2.29}
$$

which corresponds to multiplicative noise. The stochastic sources $\xi_i^{(k)}$ have a probability distribution given by (2.12) which generates the correlation functions (2.18) for each $k$ and $l$.

To every stochastic source we can associate a dissipative term that is present in the real part of the influence action. The dissipative and the noise kernels are related by generalized fluctuation-dissipation relations of the following form

$$
\nu_i^{(k)}(t) = \int_{-\infty}^{+\infty} ds \ K_i^{(k)}(t-s) \gamma_i^{(k)}(s), \tag{2.30}
$$

where the kernel $K_i^{(k)}(s)$ is

$$
K_i^{(k)}(s) = \int_0^{+\infty} \frac{d\omega}{\pi} L_i^{(k)}(z) l \omega \cos \omega s. \tag{2.31}
$$

and the temperature-dependent factor $L_i^{(k)}(z) = N_i^{(k)}(z)/M_i^{(k)}(z)$.

A fluctuation dissipation relation of the form (2.30) exists for the linear case where the temperature dependent factor appearing in (2.31) is simply $L^{(1)} = z$. The fluctuation-dissipation kernels $K_i^{(k)}$ have rather complicated forms except in some special cases. In the high temperature limit, which is characterized by the condition $k_B T \gg \hbar \Lambda$, where $\Lambda$ is the cutoff frequency of the environment, $z = \text{coth} \beta \hbar \omega/2 \rightarrow 2/\beta \hbar \omega$ we obtain

$$
L_i^{(k)}(z) \rightarrow \frac{2k_B T}{\hbar \omega}. \tag{2.32}
$$

In the limit $\Lambda \rightarrow +\infty$, we get the general result

$$
K_i^{(k)}(s) = \frac{2k_B T}{\hbar} \delta(s). \tag{2.33}
$$
which tells us that at high temperature there is only one form of fluctuation-dissipation relation, the Green-Kubo relation [1]

$$\nu_i^{(k)}(s) = \frac{2k_BT}{\hbar} n_i^{(k)}(s).$$  \hfill (2.34)

In the zero temperature limit, characterized by $z \to 1$, we have

$$L_i^{(k)}(z) \to l.$$  \hfill (2.35)

The fluctuation-dissipation kernel becomes $k$-independent and hence identical to the one for the linearly-coupled case

$$K(s) = \int_0^{+\infty} \frac{d\omega}{\pi} \omega \cos \omega s.$$  \hfill (2.36)

It is interesting to note that the fluctuation-dissipation relations for the linear and the nonlinear dissipation models are exactly identical both in the high temperature and in the zero temperature limits. In other words, they are not very sensitive to the different system-bath couplings at both high and zero temperature limits. The fluctuation-dissipation relation reflects a categorical relation (backreaction) between the stochastic stimulation (fluctuation-noise) of the environment and the averaged response of a system (dissipation) which has a much deeper and universal meaning than that manifested in specific cases or under special conditions.

Our aim in the next section would be to consider a model consisting of quantum fields coupled to a cosmological background metric and cast it into the system-environment form as discussed here. Consequently we shall see that one can construct an influence functional of a form very similar to (2.8) and hence derive a fluctuation-dissipation relation of the form (2.30).

### 3 Influence Functional for Quantum Cosmology

#### 3.1 Reduced Density Matrix of the Universe

The model we will analyze here is the same as that used in [52] from which we will quote results relevant to our study. Our "system" will consist of a minisuperspace model with $D$ degrees of freedom denoted by coordinates $r^m \ (\text{with} \ m = 1, \ldots, D)$. The minisuperspace modes will be coupled to "environment" degrees of freedom that we schematically represent by $\Phi$ (they will be later associated with the modes of a scalar field). The quantum mechanical description of this Universe will be given by the wave function of the Universe $\Psi = \Psi(r^m, \Phi)$ which, as a consequence of the existence of a classical Hamiltonian constraint, satisfies the Wheeler-DeWitt equation:

$$H\Psi = (H_e + H_\Phi)\Psi = 0$$  \hfill (3.1)
In the class of models we consider, the Hamiltonian corresponding to the minisuperspace variables can be written as

\[
H_r = \frac{1}{2M} G^{mm'} p_m p_{m'} + MV(r^m) \quad (3.2)
\]

The matrix \( G^{mm'} \) determines the metric in the minisuperspace (the supermetric) and the quantity \( M \) is proportional to the square of the Planck mass. In the following we will set \( \hbar = 1 \) throughout. In the above Wheeler-DeWitt equation we assume that the momenta are replaced by operators according to a covariant factor ordering prescription. The Hamiltonian constraint represents an important distinction from the quantum Brownian motion case discussed previously, because it implies that there is no preferred notion of time in this case and the wavefunction satisfies (3.1) rather than the Schrödinger equation. The Hamiltonian associated with the environment degrees of freedom is some function \( H_\Phi(\Phi, \pi_\Phi, r^m, p_m) \) that we will specify later.

We will be interested in making predictions concerning only the behavior of the minisuperspace variables \( r^m \) which we consider the “relevant” part of the universe. To achieve such a coarse-grained description we will work with the reduced density matrix of the system which is defined as:

\[
\rho_{\text{red}}(r', r) = \int d\Phi \Psi^* (r, \Phi) \Psi (r', \Phi) \quad (3.3)
\]

For some region of the minisuperspace, (3.1) admits solutions that are oscillatory functions of \( r^m \) of the following WKB form:

\[
\Psi (r, \Phi) = e^{iS(r)/4} C(r) \psi (r, \Phi) \quad (3.4)
\]

In this regime, the system variables \( r \) and the environment variables \( \Phi \) behave as heavy and light modes respectively (the Planck mass plays the role of a large mass parameter) in analogy with the Born-Oppenheimer approximation. This also provides some justification of the system-environment split akin to the Brownian motion case. Thus if one assumes that all the functions \( S, C, \psi \) can be expanded in powers of \( M^{-1} \) and substitutes these expansions into (3.1), one gets, to leading order (i.e., \( M^0 \)):

\[
\frac{1}{2} G^{mm'} \frac{\partial S_0}{\partial r^m} \frac{\partial S_0}{\partial r^{m'}} + V(r) = 0 \quad (3.5)
\]

which is essentially the minisuperspace version of the Hamilton-Jacobi equation. To the next order in \( M \) one obtains,

\[
iG^{mm'} \frac{\partial S_0}{\partial r^{m'}} \frac{\partial}{\partial r^m} \psi_0 = H_\Phi(\Phi, \pi_\Phi, r^m, p_m = \frac{\partial S_0}{\partial p_m}) \psi_0 \quad (3.6)
\]

This last equation is obtained provided we choose the prefactor \( C_0 \) identical to the \( H_\Phi = 0 \) case. Thus, if we define the WKB time \( t \) as

\[
\frac{d}{dt} = G^{mm'} \frac{\partial S_0}{\partial r^{m'}} \frac{\partial}{\partial r^m} \quad (3.7)
\]
the equation (3.6) reduces to the familiar Schrödinger equation that reads:

$$\frac{i}{\hbar} \frac{d \psi}{dt} = H_0 \psi$$

(3.8)

From now on we will drop all the $0$-subindices which should be considered as implicit in all the equations where $S, C$ and $\psi$ appear. The Hamilton-Jacobi equation (3.5) will have a $D - 1$ parameter family of solutions and for each one of these solutions we can build a wave function like (3.4). In general one can assume that the wave function of the Universe is a superposition of these terms, each of which will be called a WKB branch:

$$\Psi(r, \Phi) = \sum_n e^{iM_{S(n)}(r)} C_{(n)}(r) \psi_{(n)}(r, \Phi)$$

(3.9)

Here the subindex $(n)$ labels the WKB branch characterized by a set of parameters $(n)$ that uniquely defines the particular solution to the Hamilton-Jacobi equation. However, in the rest of our analysis, we will consider the wavefunction to be represented by a single term of the above sum, i.e., by a particular WKB branch. We will drop the subscript $n$ from now on with this understanding.

The reduced density matrix associated with the wave function (3.4) is:

$$\rho_{\text{red}}(r', r) = e^{iM[S(r) - S(r')]} C(r) C(r') \mathcal{I}(r', r)$$

(3.10)

where

$$\mathcal{I}(r', r) = \int \psi^*(r', \Phi) \psi(r, \Phi) d\Phi$$

(3.11)

The influence of the environment on the system is summarized by the above function $\mathcal{I}$ and it will be the basic object of our interest. It has been shown in references [52, 55] that this is the object that is exactly analogous to the Feynman-Vernon influence functional $\mathcal{F}(x, x')$ in the case where the environment is in a pure state. We will therefore call $\mathcal{I}(r', r)$ the influence functional and analyze the fluctuation and dissipation phenomena in analogy to the QBM problem. To facilitate making these connections, we write the influence functional in the form

$$\mathcal{I}(r, r') = \exp \{i \Gamma(r, r') \}$$

(3.12)

where the influence action can be written as

$$\Gamma(r, r') = \Theta(r, r') + i \tilde{\Gamma}(r, r')$$

(3.13)

The phase $\Theta$ and the real exponent $\tilde{\Gamma}$ which constitute the influence functional will be the basic objects of our interest (note that $\tilde{\Gamma}$ is positive since the overlap is bounded by unity).

### 3.2 Bianchi-I Minisuperspace with a Conformal Scalar Field

We now specialize our model to a minisuperspace of Bianchi I universe coupled to a massless conformal scalar field. The line element is given by [56]

$$ds^2 = a^2 d\eta^2 - a^2 \epsilon_{ij}^{2\beta} dx^i dx^j,$$

(3.14)
where $\eta$ is the conformal time. The traceless $3 \times 3$ matrix $\beta$ measures the anisotropy, its time rate of change gives the shear. For Type-I universe, it can always be parametrized by the principal eigenvalues

$$\beta = \text{diag}(\beta_1, \beta_2, \beta_3)$$

(3.15)

or, equivalently by $\beta_\pm$ defined by

$$\beta_1 = \beta_+ + \sqrt{3}\beta_-, \quad \beta_2 = \beta_+ - \sqrt{3}\beta_-, \quad \beta_3 = -2\beta_+$$

(3.16)

Rewriting the scale factor as $a = e^\eta$, the Einstein Hilbert action can be written as

$$S_g = 6M \int d\eta \{ e^{2\alpha} (\dot{\alpha}^2 + \dot{\beta}_+^2 + \dot{\beta}_-^2) \}

(3.17)

where $M = M_{\text{pl}}$, and a dot denotes taking a derivative with respect to the conformal time $\eta$. We normalize the spatial volume to 1 assuming $T^3$ spatial topology.

The action for the scalar field is given by

$$S_f = \frac{1}{2} \int d^4x \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{6} R \phi^2 \right)$$

(3.18)

which, after integrating by parts and defining the conformal field $X = a\phi$, can be written as:

$$S_f = \frac{1}{2} \int d^4x \left\{ \dot{X}^2 + X \nabla^{(3)} X - \left( \dot{\beta}_+^2 + \dot{\beta}_-^2 \right) X^2 \right\}$$

(3.19)

where the spatial Laplacian is given by $\nabla^{(3)} = \epsilon^{ijk} \partial_i \partial_j$.

As usual, we expand the field $X$ in an orthonormal basis of eigenfunctions of $\nabla^{(3)}$. As the spatial sections are flat, the eigenfunctions are simple trigonometric functions and the momenta are quantized due to the periodic boundary conditions associated with the $T^3$. We will denote the basis as $\{ Q_{k\sigma}(\vec{x}), k = (k_x, k_y, k_z), k_j = 2\pi n_j, \sigma = \pm \}$. The index $\sigma$ labels the functions according to their parity. The expansion of the field $X$ reads:

$$X(\vec{x}, \eta) = \sum_{k\sigma} Q_{k\sigma}(\vec{x}) \chi_{k\sigma}(\eta)$$

(3.20)

The variables of the minisuperspace constituting our open system are $r^m = (\alpha, \beta_+, \beta_-)$ or $(\alpha, \beta_{ij})$ and the ‘environment’ variables are the collection of field amplitudes $\chi_{k\sigma}, \ k_j = 2\pi n_j, \ \sigma = \pm$. Using our previous expressions it is easy to show that the Hamiltonian can be written in the form of (3.2), where the gravitational part has the supermetric

$$G^{mn'} = \frac{1}{a^4} \text{diag}(-1, +1, +1).$$

(3.21)

On the other hand the matter Hamiltonian can be written as

$$H_X = \sum_{k\sigma} H_{k\sigma} = \sum_{k\sigma} \frac{1}{2} \left( \pi_{k\sigma}^2 + \Omega_k^2 \chi_{k\sigma}^2 \right)$$

$$\Omega_k^2 = \epsilon^{ij}_{\sigma=\pm} k^i k^j + \frac{1}{144M^2a^4} (p_{\beta+}^2 + p_{\beta-}^2)$$

(3.22)
We will assume the wave function of the universe can be written as (3.4), where the function $S$ obeys the Hamilton–Jacobi equation (3.5) which in this case is given by:

$$\frac{e^{-2a}}{2M} \left( -(\partial_{\alpha} S)^2 + (\partial_{\beta_+} S)^2 + (\partial_{\beta_-} S)^2 \right) = 0 \tag{3.23}$$

This equation can be separated and solved as

$$S(\alpha, \beta_{\pm}) = \tilde{S}_{\gamma}(\alpha) + b_+ \beta_+ + b_- \beta_- \tag{3.24}$$

with

$$\partial_{\alpha} \tilde{S}_{\gamma}(\alpha) = \pm |\tilde{b}| \tag{3.25}$$

where we use $\tilde{b}$ to denote the two dimensional constant vector $(b_+, b_-)$.

As we can see, a particular solution to the Hamilton–Jacobi equation is parametrized by two integration constants $(b_+ \text{ and } b_-)$ and by the sign that defines $\tilde{S}(a)$ in equation (3.24). Therefore, the label $(n)$ that characterizes a solution of (3.23) stands for the set of constants $(\tilde{b}, \pm)$. Every function $S_{(n)}$ generates a 2-parameter family of trajectories in the three dimensional minisuperspace (these are the curves orthogonal to the $S_{(n)} =$constant hypersurfaces). These trajectories are exact solutions to the Einstein’s equations, and if we restrict our considerations to the plane $(\alpha, \beta_+)$, the trajectories are defined by the equation

$$\frac{\partial \beta_+}{\partial \alpha} = -\frac{b_+}{\partial_{\alpha} S} \tag{3.26}$$

The minisuperspace trajectories can be found by integrating the above equation and are straight lines (with slope given by $\pm b_+ / |\tilde{b}|$) corresponding to the well known Kasner’s solutions. In that case, for the “expanding” (i.e. $\dot{\alpha} > 0$) branch, we have $\beta_+ = \frac{b_+}{|\tilde{b}|} \alpha + \beta_{+0}$, where $\beta_{+0}$ is an integration constant.

### 3.3 Influence Action

We have to compute the influence functional (3.13) according to the strategy described in the beginning of this section and for that we have to solve the Schrödinger equation (3.6). It is possible to make the following ansatz for the matter wave function

$$\psi(r, X) = \psi(r, \{\chi_k\}) = \prod_k \psi_k(r, \chi_k) \tag{3.27}$$

Thus, the influence functional is expressed as an infinite product while the phase $\Theta$ and the real exponent $\Gamma$ can be written as a sum of contributions from each mode.

Each component of the wave function satisfies the following Schrödinger equation:

$$i \frac{\partial \psi_k}{\partial \eta} = H_k \psi_k. \tag{3.28}$$
with a Hamiltonian given by:

\[
H_k = -\frac{1}{2} \frac{d^2}{d\chi_k^2} + \frac{1}{2} (\Omega_k^2 \chi_k^2) = -\frac{1}{2} \frac{d^2}{d\chi_k^2} + \frac{1}{2} \left( e^{2\beta_0} k^ik^j + \beta_+^2 + \beta_-^2 \right) \chi_k^2
\]

(3.29)

where as before, we have used a dot to denote the derivative with respect to the conformal time, which also happens to coincide with the WKB time as can be seen from applying the definition (3.7) to the model of sect. 3.2.

Let us now describe how we compute the influence functional. We will make a Gaussian ansatz for the wave function \( \psi_k \) that corresponds to assuming that the state for the scalar perturbations is a particular vacuum. Thus, we write each component of the wave function as (for simplicity we will omit the index \( k \)):

\[
\psi(r, f) = \left( \frac{\pi}{\omega_i} \right)^{\frac{1}{4}} e^{-\frac{\pi}{4} \int w_i dt} e^{\frac{i}{\pi} f^2 w}
\]

(3.30)

where \( w \equiv \dot{u}/u \equiv w_r + iw_i \), and \( w_r, w_i \) are the real and imaginary parts of \( w \). The equation satisfied by the function \( u \) is easily derived from the Schrödinger equation and can be written as:

\[
\ddot{u} + \Omega_k^2 u = 0
\]

(3.31)

The computation of the overlap factor involves solving the above equations. In our model this can be done using a perturbative scheme if we assume that the anisotropy coordinates are small. In that case, we can write (up to second order in the anisotropy):

\[
\Omega_k^2 = \omega_k^2 - \lambda_1 - \lambda_2
\]

(3.32)

where

\[
\omega_k = |k|^2, \quad \lambda_1 = -2\beta_0 k^i k^j \quad \text{and} \quad \lambda_2 = -2\beta_0^2 k^i k^j - (\beta_+^2 + \beta_-^2)
\]

(3.33)

Then, the equation for \( u \) can be solved by a standard iteration procedure [52, 31]. Assuming that the anisotropy is "switched off" at early and late times, and taking the initial state as the conformal vacuum, the expressions for \( \bar{\Gamma} \) and \( \Theta \) of the exponent of the influence functional defined in (3.13) are given respectively by

\[
\bar{\Gamma}(r, r') = \omega_k^2 \cos(2\omega_k(\eta_1 - \eta_2)) \left[ \frac{1}{16} \int_{r'}^{r_a} \int_{\eta_1}^{\eta_2} d\eta_1 d\eta_2 \frac{\lambda_1(\eta_1) \lambda_1(\eta_2)}{\omega_k^2} \cos(2\omega_k(\eta_1 - \eta_2)) \right] - \frac{1}{16} \int_r^{r'} \int_{\eta_1}^{\eta_2} d\eta_1 d\eta_2 \frac{\lambda_1(\eta_1) \lambda_1(\eta_2)}{\omega_k^2} \cos(2\omega_k(\eta - \eta + \eta_1 - \eta_2))
\]

(3.34)
and

\[
\Theta(r, r') = \frac{1}{2} \omega_{\mathbf{k}} (\eta - \eta') + \frac{1}{4\omega_{\mathbf{k}}} \int_{\eta_1}^{\eta_2} d\eta_1 \lambda_2(\eta_1) - \frac{1}{4\omega_{\mathbf{k}}} \int_{\eta_1'}^{\eta_2'} d\eta_1' \lambda_2(\eta_1') + \\
+ \frac{1}{8} \int_{\eta_1}^{\eta_1'} d\eta_1 \int_{\eta_2}^{\eta_2'} d\eta_2 \frac{\lambda_1(\eta_1)\lambda_1(\eta_2)}{\omega_{\mathbf{k}}^2} \sin(2\omega_{\mathbf{k}}(\eta_1 - \eta_2)) - \\
- \frac{1}{8} \int_{\eta_1}^{\eta_1'} d\eta_1 \int_{\eta_2}^{\eta_2'} d\eta_2 \frac{\lambda_1(\eta_1)\lambda_1(\eta_2)}{\omega_{\mathbf{k}}^2} \sin(2\omega_{\mathbf{k}}(\eta_1 - \eta_2)) + \\
+ \frac{1}{8} \int_{\eta_1}^{\eta_1'} d\eta_1 \int_{\eta_2}^{\eta_2'} d\eta_2 \frac{\lambda_1(\eta_1)\lambda_1(\eta_2)}{\omega_{\mathbf{k}}^2} \sin(2\omega_{\mathbf{k}}(\eta' - \eta + \eta_1 - \eta_2))
\] (3.35)

up to second order in anisotropy. The total phase \( \Theta \) and the total real exponent \( \Gamma \) of the influence functional are obtained by summing over \( \mathbf{k} \) of (3.35) and (3.34) respectively. In performing these sums, divergent expressions will arise which will have to be regularized and renormalized.

The above equations clearly show the history dependence of the influence functional since they are written in terms of time integrals of functions that depend on \( \beta_\pm(\eta_1) \). Therefore, the phase and the real exponent are functionals of the zero order WKB histories.

Notice that since in this model we have more than one minisuperspace degree of freedom, even within a WKB branch, we have a whole family of trajectories rather than a single trajectory. So as far as the solutions of the Hamilton-Jacobi equation is concerned, this implies restricting ourselves to the family of trajectories given by the solution of (3.26) with a fixed value of \( b_+/|\tilde{b}| \). These are a family of parallel straight lines with the slope fixed by \( n \) and different \( \beta \) intercepts. We note that in the configuration space of the \( \alpha - \beta_+ \) plane, one and only one trajectory passes through each point. Hence each point in configuration space can be associated with an entire history, and thus \( \mathcal{I} \) is a functional of two histories as in the Brownian motion example.

As it stands, \( \mathcal{I}(r, r') \) is still not in a form that can be put in one-to-one correspondence with the \( \mathcal{F}(x, x') \) of the QBK problem, because the latter is an explicit function of time, whereas in the former, the WKB time is defined through (3.7) as a function of the coordinates \( r \). The definition of \( \eta= \) constant surfaces depends on the choice of the hypersurface in minisuperspace on which the initial condition of the wave function is specified. In our case the initial conditions were specified on a \( \alpha= \) constant hypersurface. Thus our constant WKB time hypersurfaces are those with \( \alpha= \) constant. Now, let us specialize to the situation where \( \mathcal{I}(r, r') \) is evaluated on two points such that \( \alpha = \alpha' \). From the above discussion then we know that this implies that \( \eta = \eta' \). The two histories, \( \beta_\pm(\eta_1) \) and \( \beta_\pm'(\eta_1) \) that enter into the calculation of the influence functional are the parallel lines (with slope determined by \( (n) \) , passing through the points \( (\alpha, \beta_+) \) and \( (\alpha, \beta'_+) \) respectively. Now, the influence functional can be written as \( \mathcal{I}(\beta_\pm, \beta_\pm', \eta) \) and can finally be compared with that of the QBK problem.
4 Fluctuations in Quantum Fields and Dissipation of Spacetime Anisotropy

4.1 Regularized Influence Action

It has been pointed out in [52] that the influence action \( \Gamma \) is identical to the Schwinger–Keldysh (or Closed Time Path) effective action which is a functional of two histories and can be computed using diagrammatic techniques. Thus \( \Gamma \) is essentially the same as the quantity given by (3.11) in [40], with \( \beta \) and \( \beta' \) corresponding to \( \beta_{ij}^{\pm}, \beta_{ij}^{-} \) in the CTP context, where the + and - superscripts refer to the positive and negative contour branches respectively. This identification is useful as it connects with the well-known results in semiclassical gravity [40]. This connection provides both conceptual and technical advantages as it offers clearer physical interpretations of the results in quantum cosmology and makes available many results obtained previously in the application of the CTP formalism in quantum field theory in curved spacetimes.

We now proceed to evaluate \( \tilde{\Gamma} \) and \( \Theta \) by summing the equations (3.34) and (3.35) over all modes \( \{ k \} \) subject to the restriction \( \alpha = \alpha' \). Some of the mode sums appearing in these expressions are divergent and finite need to be regularized. The regularized influence action for this problem can be calculated using Feynman diagram [40] or dimensional regularization techniques [52]. The phase of the influence functional can be written as

\[
\Theta = \Gamma_{\text{div}} + \Gamma_{\text{ren}} \tag{4.1}
\]

where \( \Gamma_{\text{div}} \) and \( \Gamma_{\text{ren}} \) represent the divergent and finite contribution to the phase respectively. \( \Gamma_{\text{div}} \) (obtained as terms containing the \( 1/\epsilon \) factor in dimensional regularization, where \( \epsilon = n - 4 \) and \( n \) is the dimension of spacetime) is given by [52]

\[
\Gamma_{\text{div}} = \int d\eta_1 d\eta_2 (\beta_{ij} - \beta'_{ij})(\eta_1) \gamma_{\text{div}}(\eta_1 - \eta_2)(\beta^{ij} + \beta'^{ij})(\eta_2) \tag{4.2}
\]

where

\[
\gamma_{\text{div}}(\eta_1 - \eta_2) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{i\omega(\eta_1 - \eta_2)} \frac{-\omega^4}{4(4\pi)^2(n^2 - 1)\epsilon} \tag{4.3}
\]

\( \Gamma_{\text{div}} \) can be rewritten as

\[
\Gamma_{\text{div}} = \frac{1}{4(4\pi)^2(n^2 - 1)\epsilon} \int d\eta [\tilde{\beta}_{i}^{2} - \tilde{\beta}'_{i}^{2}] + \text{surface terms}, \tag{4.4}
\]

where the surface terms can be written as integrals of total derivatives of functions of \( \beta \) and \( \beta' \) and can be discarded. As it stands this explicitly divergent term cannot be absorbed by renormalization of the bare coupling constants present in the original action since from (3.17) we see that no term of this higher derivative form appears there. Hence we follow the usual procedure used in quantum field theory in curved spacetime of first dimensionally regularizing the effective action, modifying the original classical action by adding appropriate
counterterms to cancel the divergence, and finally taking the limit \( \epsilon \to 0 \). The modified classical action including the counterterms up to second order in \( \beta \) is given by \[ 17 \]

\[
\bar{S} = \int d\eta \left[ -6Ma^2 + \frac{1}{180(4\pi)^2} \left\{ \left( \frac{\dot{a}}{a} \right)^4 - 3 \left( \frac{\ddot{a}}{a} \right)^2 \right\} \right] \\
+ \int d\eta \left( M\dot{\beta}^2 a^2 + \frac{1}{180(4\pi)^2} \left[ 3\epsilon^{-1}\dot{\beta}^2 + 3\ln(\mu a)\dot{\beta}^2 - \left\{ \left( \frac{\dot{a}}{a} \right)^2 + \left( \frac{\ddot{a}}{a} \right)^2 - \dot{\beta}^2 \right\} \right] \right) \\
\] (1.5)

where \( \mu \) has dimensions of mass and sets the renormalization scale. The total phase of the density matrix is now given by

\[
\bar{S}(a, \beta) = \bar{S}(a, \beta') + \Theta(a, \beta, \beta')
\] (4.6)

Inserting (4.5) for \( \bar{S} \) in the above expression we notice that the pole term in \( \epsilon \) cancels exactly.\(^1\)

The rest of the exponent, \( \Gamma_{\text{ren}} \) and \( \tilde{\Gamma} \), is finite:

\[
\Gamma_{\text{ren}} = \int d\eta_1 d\eta_2 (\beta_{ij} - \beta'_{ij})(\eta_1) \gamma_{\text{ren}}(\eta_1 - \eta_2)(\beta^{ij} + \beta'^{ij})(\eta_2)
\] (4.7)

and

\[
\tilde{\Gamma} = \int d\eta_1 d\eta_2 (\beta_{ij} - \beta'_{ij})(\eta_1) \tilde{\gamma}(\eta_1 - \eta_2)(\beta^{ij} - \beta'^{ij})(\eta_2),
\] (4.8)

where the kernels \( \gamma_{\text{ren}} \) and \( \tilde{\gamma} \) are given by

\[
\gamma_{\text{ren}}(\eta) = \frac{1}{60(4\pi)^2} \int_{-\infty}^{+\infty} d\omega \frac{\omega^4}{2\pi} \log(i \frac{\omega - i \epsilon}{\mu})
\] (4.9)

and

\[
\tilde{\gamma}(\eta) = \frac{1}{60(4\pi)^2} \int_{0}^{+\infty} d\omega \frac{\omega^4}{2\pi} \cos \omega \eta.
\] (4.10)

Notice that the kernel \( \tilde{\gamma}(\eta) \) is even whereas \( \gamma_{\text{ren}}(\eta) \) contains an odd and even part given by

\[
\gamma_{\text{odd}}(\eta) = \frac{1}{60(4\pi)^2} \int_{0}^{+\infty} d\omega \frac{\omega^4}{2\pi} \sin \omega \eta
\] (4.11)

and

\[
\gamma_{\text{even}}(\eta) = \frac{1}{60(4\pi)^2} \int_{-\infty}^{+\infty} d\omega \frac{\omega^4}{2\pi} \cos \omega \eta \ln \left| \frac{\omega}{\mu} \right|
\] (4.12)

The kernel \( \gamma_{\text{ren}}(\eta) \) is manifestly real and can also be seen to be causal \cite{40}.

Note that \( \Gamma \) and \( \Gamma_{\text{ren}} \) play distinct roles here. \( \tilde{\Gamma} \) is responsible for the decoherence between alternative histories \( \beta \) and \( \beta' \) in the sense that it suppresses the contribution of widely

\(^1\)However, we would like to add a cautionary note at this point. We are assuming without proof that the \( R^2 \) type terms can be added as counterterms at this level after making the WKB ansatz. Since addition of such terms at the level of the quantum cosmology Hamiltonian which was our starting point involves the introduction of new canonical degrees of freedom, the validity of this assumption is not entirely clear.
differing histories to the influence functional, and hence suppresses the off diagonal terms of the reduced density matrix. This feature and its connection to particle production was explored before in [52, 45]. On the other hand, when we attempt to derive the effective equation of motion for $\beta$ by varying the effective action $S_{\text{eff}}$, only $\Gamma_{\text{ren}}$ contributes to generating the equation of motion. The equation of motion obtained under such variation is identical to the real, causal dissipative equation for $\beta$ obtained by Calzetta and Hu in [40]. In fact, as we will show more explicitly later, $\Gamma_{\text{ren}}$ provides the dissipative contribution to the equation of motion. Thus in the present form of the influence functional $\Gamma_{\text{ren}}$ contributes only to the equation of motion and not to decoherence, and $\Gamma$ contributes only to decoherence, and not to the equation of motion. However, in the following we will show how $\tilde{\Gamma}$ also plays the dual role of generating noise and will indeed contribute to the effective equations of motion with a stochastic source.

### 4.2 Correspondence with QBM

Now that we have the complete form of the influence functional, we can proceed to compare its exponent given by (4.7) and (4.8) with that of (2.9) of the QBM problem. We can see that it corresponds to the $k = 2$, $f(x) = x$ case in (2.9) with the identification $\beta_k \equiv x$ and $q_k \equiv \chi_k$. It is by no means obvious that our cosmological example should correspond to $f(x) = x$, i.e., the linear coupling case, because in our approximation we had retained up to quadratic terms in the anisotropy. In fact, from (3.32) we see that the system-environment coupling contains terms quadratic in $\beta$ as well as a quadratic coupling in velocities, which is not even covered by our Brownian motion model. However, though these terms are originally present, when correctly dimensionally regularized, the terms proportional to $\lambda_2$ that contain the non-linear coupling vanish. Hence we are left with only an effective linear coupling in the anisotropy. The local potential renormalization terms $\Delta V$'s can be identified with $\Gamma_{\text{div}}$ in the cosmological case and we have already dealt with the renormalization. Using the time reflection symmetry of the kernel $\tilde{\gamma}$ we obtain

$$
\tilde{\Gamma} = 2 \int_0^\eta d\eta_1 \int_0^{\eta_1} d\eta_2 \tilde{\gamma}(\eta_1 - \eta_2) \left( \beta^{\tilde{ij}} - \beta^{\tilde{ij}} \right)(\eta_2)
$$

(4.13)

and for the phase $\Gamma_{\text{ren}}$, using the time reflection properties of $\gamma_{\text{odd}}(\eta)$ and $\gamma_{\text{even}}(\eta)$ we can rewrite it as

$$
\Gamma_{\text{ren}} = \int_0^\eta \int_0^\eta d\eta_1 d\eta_2 \beta_{ij}^+(\eta_1) \tilde{\gamma}(\eta_1 - \eta_2) \beta^{\tilde{ij}}(\eta_2) \\
- \int_0^\eta \int_0^\eta d\eta_1 d\eta_2 \beta_{ij}^-(\eta_1) \tilde{\gamma}(\eta_1 - \eta_2) \beta^{\tilde{ij}}(\eta_2) \\
+ 2 \int_0^\eta \int_0^{\eta_1} d\eta_2 \left( \beta_{ij} - \beta_{ij}^\prime \right)(\eta_1) \gamma_{\text{odd}}(\eta_1 - \eta_2) \left( \beta^{\tilde{ij}} + \beta^{\tilde{ij}} \right)(\eta_2)
$$

(4.14)

where

$$
\tilde{\gamma}(\eta_1 - \eta_2) = \gamma_{\text{even}}(\eta_1 - \eta_2) - \gamma_{\text{odd}}(\eta_1 - \eta_2) \text{sgn}(\eta_1 - \eta_2)
$$

(4.15)
is an even kernel. Now we must compare the expressions (4.13) for the real exponent and the phase (4.14) with the corresponding expressions in the influence action (2.9) for the Brownian motion case in order to properly identify the noise and dissipation contributions. Comparing the real exponents we see that the noise kernel for the anisotropy in this case is given by

$$\nu(\eta) = 2\tilde{\gamma}(\eta) = \frac{1}{30(4\pi)^2} \int_0^{+\infty} \frac{d\omega}{2\pi} \omega^4 \cos \omega \eta$$  \hfill (4.16)

In trying to compare the phase terms we notice that the third term in (4.14) is indeed of the form of that in (2.9) and we can identify the dissipation kernel \(\mu(\eta)\) for the cosmology case as

$$\mu(\eta) = -2\gamma_{odd}(\eta)$$  \hfill (4.17)

and it is manifestly odd in time.

The regularized influence action can therefore be written as

$$\Gamma(\beta, \beta') = \int_0^{\eta} \int_0^{\eta} d\eta_1 d\eta_2 \beta_{ij}(\eta_1) \tilde{\gamma}(\eta_1 - \eta_2) \beta_{ij}(\eta_2)$$

$$- \int_0^{\eta} \int_0^{\eta} d\eta_1 d\eta_2 \beta_{ij}'(\eta_1) \gamma(\eta_1 - \eta_2) \beta_{ij}'(\eta_2)$$

$$- \int_0^{\eta} d\eta_1 \int_0^{\eta} d\eta_2 \mu(\eta_1 - \eta_2) (\beta_{ij}' + \beta_{ij})(\eta_2)$$

$$+ i \int_0^{\eta} d\eta_1 \int_0^{\eta} d\eta_2 \nu(\eta_1 - \eta_2) (\beta_{ij}' - \beta_{ij})(\eta_2)$$  \hfill (4.18)

The first two terms contribute a non-local potential to the effective action but do not contribute to the mixing of \(\beta\) and \(\beta'\) histories like the third and fourth terms. We will now show in some greater detail that the third term with the kernel \(\mu\) that is odd in the time domain contributes to the dissipation and the last term containing \(\nu\) is associated with noise.

### 4.3 Noise

Let us first concentrate on the fourth term. Its contribution to the influence functional is given by

$$\exp[- \int_0^{\eta} d\eta_1 \int_0^{\eta} d\eta_2 \nu(\eta_1 - \eta_2) (\beta_{ij}' - \beta_{ij})(\eta_2)]$$  \hfill (4.19)

We will proceed in exact analogy with the analysis of noise in the case of QBM described in Sec. 2.2. The term in (4.19) can be rewritten using functional Gaussian identity (2.11) which in this case states that the above expression is equal to

$$\int D\xi(\eta) P[\xi] \exp[i \int_0^{\eta} d\eta' \xi_{ij}(\eta') (\beta_{ij}' - \beta_{ij})(\eta')]$$  \hfill (4.20)
where
\[ \mathcal{P}[\xi] = P_0 \exp \left[ - \int_0^{\eta} \int_0^{\eta} d\eta_2 \frac{1}{2} \xi_{ij}(\eta_1) \nu^{-1}(\eta_1 - \eta_2) \xi_{ij}(\eta_2) \right] \]
(4.21)
is the functional distribution of \( \xi(\eta) \) and \( P_0 \) is a normalization factor given by
\[ P_0^{-1} = \int D\xi(\eta) \exp \left[ - \int_0^{\eta} \int_0^{\eta} d\eta_2 \xi_{ij}(\eta_1) \nu^{-1}(\eta_1 - \eta_2) \xi_{ij}(\eta_2) \right]. \]
(4.22)
The influence functional can then be written as
\[ e^{ixr} = \int D\xi(\eta) \mathcal{P}[\xi] \exp i \hat{\Gamma}[\beta, \beta', \xi] \equiv <\exp i \hat{\Gamma}[\beta, \beta', \xi]>_{\xi} \]
(4.23)
where the angled brackets denote an average with respect to the stochastic distribution \( \mathcal{P}[\xi] \).
The modified influence action \( \hat{\Gamma}[\beta, \beta', \xi] \) is given by
\[
\hat{\Gamma}[\beta, \beta', \xi] = \int_0^{\eta} \int_0^{\eta} d\eta_1 d\eta_2 \beta_{ij}(\eta_1) \beta_{ij}(\eta_2) \\
- \int_0^{\eta} \int_0^{\eta} d\eta_1 d\eta_2 \beta'_{ij}(\eta_1) \beta'_{ij}(\eta_2) \\
- \int_0^{\eta} d\eta_1 \int_0^{\eta_1} d\eta_2 \lambda(\eta_1 - \eta_2) (\beta_{ij} + \beta'_{ij})(\eta_2) \\
- \int d\eta_1 \xi_{ij}(\eta_1) \beta_{ij} + \int d\eta_1 \xi_{ij}(\eta_1) \beta'_{ij} \]
(4.24)
The term coupling a stochastic source \( \xi \) to \( \beta \) will manifest itself as the noise in the equation of motion derived from this effective action. We see that the influence action \( \Gamma \) can be written as an average of \( \hat{\Gamma} \) over this stochastic distribution function.

The reduced density matrix can thus also be written as a stochastic average
\[ \rho_{\text{red}}[\beta, \beta'] = <e^{i\hat{S}_{\text{eff}}(\beta, \beta'; \xi)}>_{\xi} \]
(4.25)
where the full effective action \( \hat{S}_{\text{eff}} \) is given by
\[
\hat{S}_{\text{eff}} = \hat{S}[a, \beta] + \int d\eta' \xi_{ij}(\eta') \beta_{ij} - \left\{ \hat{S}[a, \beta'] + \int d\eta' \xi_{ij}(\eta') \beta'_{ij} \right\} \\
- \int_0^{\eta} d\eta_1 \int_0^{\eta_1} d\eta_2 \lambda(\eta_1 - \eta_2) (\beta_{ij} + \beta'_{ij})(\eta_2) \]
(4.26)
and \( \hat{S} \) is given by (4.5). Our relevant equations of motion will be derived by varying \( \hat{S}_{\text{eff}} \). From this equation we can view \( \xi(\eta) \) as an external stochastic force linearly coupled to \( \beta \), though the linearity is a feature specific to truncation of the perturbation series at quadratic order in the effective action. In general we will have non-linear coupling.

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Since the distribution functional (4.21) is Gaussian, this is a Gaussian type noise, which as in (2.18), is completely characterized by

\[
< \xi(\eta) >_{\xi} = 0 \\
< \xi(\eta_1) \xi(\eta_2) >_{\xi} = \nu(\eta_1 - \eta_2)
\] (4.27)

Therefore the non-local kernel \(\nu(\eta_1 - \eta_2)\) is just the two-point time-correlation function of the external stochastic source \(\xi(\eta)\). Since this correlation function is non-local, this noise is colored. As suggested in [42, 44] we believe this is a rather general feature of noise of cosmological origin.

4.4 Fluctuation-Dissipation Relation

Now that we have identified the noise and dissipation kernels \(\nu(\eta)\) and \(\gamma_{\text{odd}}(\eta)\) respectively, we can go ahead and write down the fluctuation-dissipation relation in analogy with the quantum Brownian model [26, 57]. Defining

\[
\mu(\eta) = -2 \gamma_{\text{odd}}(\eta) = \frac{d}{d\eta} \gamma(\eta)
\] (4.28)

The fluctuation-dissipation relation has the familiar form given by (2.30)

\[
\nu(\eta) = \int_{0}^{\infty} d\eta' K(\eta - \eta') \gamma(\eta')
\] (4.29)

where the FD kernel \(K(\eta)\) is given by

\[
K(\eta) = \int_{0}^{\infty} \frac{d\omega}{\pi} \omega \cos \omega \eta
\] (4.30)

This supports the conjecture of [42] that there exists a fluctuation-dissipation relation for the description of the backreaction effect of particle creation in cosmological spacetimes. We see that the FD kernel is identical with that given by (2.36), which is given for more general system-bath couplings of the form (2.3), but with the bath at \(T = 0\). Hence this also vindicates the previous observation [26, 54, 58] that the zero temperature fluctuation-dissipation relation is insensitive to the nature of the system-bath coupling. Since we have not taken the bath at a finite temperature, thermal fluctuations play no role in the above relation and it summarizes the effect solely of quantum fluctuations. Effect of thermal fluctuations can be included easily and we expect a FDR to hold for finite temperature particle creation and backreaction as well.
5 Particle Creation, Noise and Backreaction

5.1 Particle Creation

In this section we would like to examine in some detail the relationship between the noise and dissipation kernels and particle creation from the vacuum. We would also be interested in comparing this approach to that in [40] and [42] where the relationship between particle production and anisotropy dissipation was discussed in some depth.

Let us first concentrate on the noise term. Since we know that the noise term comes from the real part \( \tilde{\Gamma} \) of the exponent of the influence functional, we will analyze this part and try to rewrite in a form such that it is easy to identify the part associated with particle production. It can be shown [52] that \( \tilde{\Gamma}(\beta, \beta') \) can be rewritten as

\[
\tilde{\Gamma}(\beta, \beta') = \int \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega_k^2} |B_k(\beta) - B_k(\beta')|^2
\]

(5.1)

where

\[
B_k(\beta) = -\frac{i}{\omega_k} \int d\eta_1 \frac{1}{\omega_k} [2\beta_{ij}(\eta_1)k^i k^j]
\]

(5.2)

and \( \omega_k = |\vec{k}| = (\sum_i k_i^2)^\frac{1}{2} \). As we may recall from (3.33), the term \( 2\beta_{ij}(\eta_1)k^i k^j = \lambda_1 \) is the expansion of the natural frequency to the first anisotropy order. One can of course go to higher orders.

Now we can show that a close relation exists between the \( B(\beta) \) function and the Bogoliubov coefficients associated with the particle creation that takes place as a consequence of the anisotropy evolution [17, 15]. This can be seen as follows.

The conformally related massless scalar field \( X = a\Phi \) in our model can be decomposed into modes as

\[
X = \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot\vec{x}} \chi_k(\eta)
\]

(5.3)

\( \chi_k \) satisfies the following equation to first order in anisotropy

\[
\frac{d^2\chi_k}{d\eta^2} + (\omega_k^2 + 2\beta_{ij}(\eta_1)k^i k^j)\chi_k = 0
\]

(5.4)

The solution to the above equation ( again to first order in \( \beta \) ) is given by

\[
\chi_k(\eta) = \chi_k^{in}(\eta) \left[ 1 + \int^{\eta} d\eta_1 \right. \\
- \left. \chi_k^{in*}(\eta) \int^{\eta} d\eta_1 \frac{1}{2i\omega_k} [2\beta_{ij}(\eta_1)k^i k^j] e^{2i\omega_k\eta_1} \right]
\]

(5.5)

where

\[
\chi_k^{in}(\eta) = \frac{1}{\sqrt{2\omega_k}} e^{i\omega_k\eta}
\]

(5.6)
is the solution to (5.4) with $\beta_{ij} = 0$ and corresponds to the ‘in’ conformal vacuum in the far past. Assuming that the anisotropy is switched off at time $\eta$ the term on the left hand side of (5.5) can be associated with $\chi_k^{\text{out}}(\eta)$ the ‘out’ vacuum. As is well known, the “in” and “out” basis can be related in terms of Bogolubov coefficients $\alpha_k$ and $\hat{\beta}_k$ as

$$\chi_k^{\text{out}}(\eta) = \alpha_k \chi_k^{\text{in}}(\eta) + \hat{\beta}_k \chi_k^{\text{in}*}(\eta)$$  \hspace{1cm} (5.7)

Comparing (5.5) and (5.7) we can identify the $\hat{\beta}_k$ Bogolubov coefficient as

$$\hat{\beta}_k = \int_0^\eta d\eta' \frac{1}{2i\omega_k} (2\beta_{ij}(\eta') k^i k^j)$$  \hspace{1cm} (5.8)

As we see from its definition, the function $B$ is proportional to this Bogolubov coefficient $\hat{\beta}_k$.

$$B_k(\beta) = \omega_k e^{-2i\omega_k \eta} \hat{\beta}_k$$  \hspace{1cm} (5.9)

Thus

$$\hat{\mathcal{G}}(\beta', \beta) = \frac{1}{(2\pi)^4} \frac{1}{4} |\hat{\beta}_k - \hat{\beta}'_k|^2,$$  \hspace{1cm} (5.10)

where $\hat{\beta}_k$ and $\hat{\beta}'_k$ are the Bogolubov coefficients associated with the anisotropy histories $\beta_{ij}$ and $\beta'_{ij}$ respectively. It is obvious from (5.19) that the noise will be non-zero only provided $\hat{\beta}_k \neq \hat{\beta}'_k$, i.e., if there is different amounts of particle production along the two histories. Since this term is also associated with decoherence this is also a necessary condition for decoherence to occur. This has also been noticed from a slightly different point of view in [45, 59].

This demonstrates a connection between the process of particle production and the noise or fluctuation.

### 5.2 Einstein-Langevin Equation

We will now show how this noise can be incorporated into the equation of motion as a Langevin type equation. In this process we will also demonstrate the role of the kernel $\mu$ in providing dissipation. The key difference from the earlier treatment [40] is that the equation of motion will be derived from the quantity $\dot{S}_{\text{eff}}(\beta, \beta', \xi)$ rather than the “noise averaged” quantity $S_{\text{eff}}(\beta, \beta')$. This has also been discussed in other contexts in [26, 45, 46]. The first step is to write $\dot{S}_{\text{eff}}(\beta, \beta', \xi)$ in terms of the following variables

$$\bar{\beta}_{ij} = \frac{1}{2}(\beta_{ij} + \beta'_{ij})$$

$$\Delta = \beta_{ij} - \beta'_{ij}$$  \hspace{1cm} (5.11)

The equation of motion is then derived as

$$\frac{\delta \dot{S}_{\text{eff}}(\bar{\beta}_{ij}; \Delta)}{\delta \bar{\beta}_{ij}} \bigg|_{\Delta = 0} = 0$$  \hspace{1cm} (5.12)
yielding

\[-2M \frac{d}{d\eta} (a^2 \dot{\beta}_{ij}) + \frac{1}{30(4\pi)^2} \frac{d^2}{d\eta^2} [\omega_{ij} \ln(\bar{\mu}a)] + \frac{1}{90(4\pi)^2} \frac{d}{d\eta} \left\{ \left( \frac{\dot{a}}{a} \right)^2 + \left( \frac{\ddot{a}}{a} \right) \right\} \beta_{ij} \]

\[+ \int d\eta_1 \gamma_{\text{ren}}(\eta - \eta_1) \beta_{ij}(\eta_1) = -j_{ij}(\eta) + \xi_{ij}(\eta) \]

(5.13)

Here \( j_{ij} \) is an external source term added in order to switch on the anisotropy in the distant past [17]. It is worth comparing these results with those in [40] where similar equations were deduced from the CTP effective action. Comparing (5.13) with (3.18) in [40] we find that they are exactly the same except for the stochastic force \( \xi_{ij} \) on the right hand side. The real and causal kernel \( K_4 \) there (including the numerical factor \( 1/[30(4\pi)]^2 \)) is identical to our kernel \( \gamma_{\text{ren}} \). We will show that the odd part of this kernel can be associated with dissipation. One could in fact interpret (3.18) obtained by Calzetta and Hu as (5.13) averaged with respect to the noise distribution. Since this is a Gaussian noise, \( <\xi>=0 \), we obtain (3.18) of [40], where the \( \beta \)'s are also to be interpreted as noise-averaged variables. In this sense, we have gone beyond previous analysis in extracting the underlying stochastic behavior that is lost in the smoothed out average version given in [40].

To make the analogy with a Langevin equation more explicit it is convenient to integrate (5.13) once with respect to \( \eta \). This gives the following equation

\[-2Ma^2 \ddot{\beta}_{ij} + \frac{1}{30(4\pi)^2} \frac{d}{d\eta} [\omega_{ij} \ln(\bar{\mu}a)] + \frac{1}{90(4\pi)^2} \frac{d}{d\eta} \left\{ \left( \frac{\dot{a}}{a} \right)^2 + \left( \frac{\ddot{a}}{a} \right) \right\} \beta_{ij} \]

\[+ \int d\eta_2 \int d\eta_1 \gamma_{\text{ren}}(\eta_2 - \eta_1) \dot{\beta}_{ij}(\eta_1) = c_{ij} + s_{ij} \]

(5.14)

where \( c_{ij}(\eta) = -\int d\eta' j_{ij}(\eta') \) and \( s_{ij}(\eta) = \int d\eta' \xi_{ij}(\eta') \).

Defining the variable \( q_{ij} = d\beta_{ij}/d\eta \) we can write the above equation in the following form

\[\frac{d}{d\eta} \left( \dot{M} \frac{dq_{ij}}{d\eta} \right) + \kappa \frac{dq_{ij}}{d\eta} + k q_{ij} = c_{ij} + s_{ij} \]

(5.15)

where

\[\dot{M} = \frac{1}{30(4\pi)^2} \ln(\bar{\mu}a) \]

(5.16)

\[k = -2Ma^2 + \frac{1}{90(4\pi)^2} \left( \left( \frac{\dot{a}}{a} \right)^2 + \left( \frac{\ddot{a}}{a} \right) \right) \]

(5.17)

\[\kappa q_{ij} = \int d\eta_2 \int d\eta_1 f(\eta_2 - \eta_1) \frac{dq_{ij}}{d\eta_1} \]

(5.18)

and \( d^2 f(\eta)/d\eta^2 = \gamma_{\text{ren}} \). This equation is identical in form to the equation (3.15) in [42] except for the term \( s_{ij} \) on the right hand side, which is indeed the stochastic contribution from the noise anticipated there. This equation is a generalized Einstein equation in the
Langevin form, in that there is a dissipative term in the dynamics and a noise term in the source. It has been conjectured [42] and shown [45] that in a more complete description of semiclassical gravity the semiclassical Einstein equation driven by the expectation values of the energy-momentum tensor should be replaced by an Einstein-Langevin equation, where there is an additional stochastic source arising from the fluctuations of quantum fields. The conventional semiclassical Einstein equation is in this sense, the simplified mean-field theory.

5.3 Dissipation and Backreaction

Equation (5.15) is in the form of a generalized damped harmonic oscillator driven by a stochastic force $s_{ij}$. (Of course the generalized mass $M$ and spring constant $k$ are time dependent, so strictly speaking it has the damped harmonic oscillator analogy only when these quantities are positive, as was also pointed out in [60].)

The second term on the left hand side of (5.15) represents the damping term involving a non-local (velocity dependent) friction force. That this term is associated with dissipation can be quickly seen as follows [41]. In the Fourier transformed version of a damped harmonic oscillator equation the imaginary term is associated with dissipation. Writing (5.15) in terms of the Fourier transform $\hat{q}_{ij}(\omega) = \int d\eta e^{-i\omega\eta}q_{ij}(\eta)$ we notice that the only imaginary contribution comes from the second term on the left hand side, which can be written as

$$F(q) = \int \frac{d\omega}{2\pi} \epsilon^{i\omega\eta} \gamma_{ren}(\omega) q_{ij}(\omega)$$

(5.19)

where $\gamma_{ren}(\omega)$ is the Fourier transform of $\gamma_{ren}(\eta)$ defined in (4.9). Thus we see that the dissipation is associated with the imaginary part of $\gamma_{ren}(\omega)$ or equivalently with the odd part of the kernel $\gamma_{ren}(\eta)$ given by $\gamma_{odd}$ defined in (4.10). This is consistent with our earlier identification of $\gamma_{odd}$ as the dissipation kernel from the form of the influence functional compared with the Brownian motion case. In fact, as in [40, 41] we can isolate the generalized (frequency dependent) viscosity function $\zeta(\omega)$ by writing

$$i\zeta(\omega)\omega q_{ij}(\omega) = i\text{Im} \gamma_{ren}(\omega) q_{ij}(\omega)$$

(5.20)

From (4.9) we can identify $\zeta(\omega)$ as

$$\zeta(\omega) = \frac{|\omega|^3}{60(4\pi)^2}$$

(5.21)

which is identical to that found in [42] and which is not surprising, since our kernel $\gamma(\omega)$ in (5.13) and $K_4$ in (3.18) in [40] are identical up to numerical factors.

Once having made the identification of the velocity dependent viscous force in the equation of motion, we can calculate the dissipated energy density by integrating $\vec{F}.\vec{v}$ (with $\vec{v} = q_{ij}$ acting as velocity) over all frequencies and come up with an expression identical to (3.18) in [40, 41].

$$\rho_{\text{dissipation}} = \int_0^\infty \frac{d\omega}{2\pi} [\omega/\beta_i(\omega)][\zeta(\omega)|\omega/\beta_i(\omega)|].$$

(5.22)
which has been shown there to be identical to the total energy of particle pairs created by a given anisotropy history $\beta$. In this way, we can see the connection between particle production and the dissipation kernel and hence the process of dissipation itself. Earlier in this section we had demonstrated the connection between particle production and the noise or fluctuation term. On the other hand, (4.29), the fluctuation-dissipation relation, embodies a relationship between the processes of fluctuation and dissipation of anisotropy. So this completes the full circle of connections among these processes. In a way, one can say that as a physical process, particle production is contributing to both the noise and dissipation, and of course these are two different manifestations of the loss of information due to integrating over the field modes.

6 Discussion

In closing, we would like to discuss the meaning of the FDR in semiclassical cosmology in a broader context and mention some related problems for future investigation.

1) FDR under Finite Temperature and Non-Equilibrium Conditions

In this paper we have discussed in detail the FDR in semiclassical cosmology under a zero temperature bath. A similar relation between the noise and dissipation kernels exists for baths at finite temperature. The form will be similar to that derived for the QBM problem in Sec. 2 [26]. One can take the finite temperature calculation via the CTP formalism [60] and perform a similar analysis as we have done for the vacuum case and obtain the results explicitly. In reality both vacuum and thermal bath results will enter into the picture, since once particle creation commences, given sufficient time and assuming some (collisional) interaction amongst the created particles, the bath will soon acquire a finite temperature character. This heat-up process is expected to happen quickly near the Planck time, especially so for anisotropic universes, as particles are created profusely there [12, 10], generating a large amount of entropy [68, 69, 62, 63, 70, 71]. The copious creation of particles near the Planck time is accompanied by large fluctuations and noise, and it induces a strong backreaction on the spacetime dynamics, dissipating the anisotropy rapidly [12, 15, 17]. The weaker anisotropy in the universe’s expansion induces lesser particle creation. The lower particle creation rate is accompanied by a smaller fluctuation and noise, which in turn gives weaker dissipation of spacetime anisotropy. The surviving anisotropy would continue to sustain particle creation, albeit in much smaller amounts. And this goes on. (The backreaction

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2 As has been discussed earlier [61], the energy density of the quantum field at any moment will contain two parts. There is a zero temperature component and a finite temperature component, the former corresponds to spontaneous creation from the vacuum, and the latter is of the nature of stimulated creation from particles already present [7, 62, 63, 64].

3 A finite temperature bath at every moment is only an idealization. To use a finite temperature description one has to discriminate the conditions for the bath to thermalize, and for the system to be equilibrated with it. These vary with the nature of the bath (massive or massless, linear or nonlinear interactions, spectral density) and the form of interaction between the system and the bath. See the analysis of [65, 66, 67]
follows a Lenz law behavior which was expounded in earlier studies [7, 72, 69]. At each stage we expect to see a balance between the rate of particle creation and the strength of fluctuations and dissipation.

In reality the spacetime-field combined system involving particle creation exists in a highly non-equilibrium state. To give a quantitative description of the above processes one needs to describe the dynamics of the actual statistical state of both the system and the environment in a self-consistent manner, which is a highly non-trivial problem. What we have described in this paper is only the first step, which depicts the effect of particle production from a vacuum (zero-temperature) bath. The interaction of created particles and how they alter the environment (e.g., thermalization) is not accounted for. In the second step one needs to also examine the evolution of the environment (quantum field) taking into consideration the effects of spontaneous and induced particle creation, their interaction and the entropy generation processes, all in the context of a changing background spacetime whose dynamics at each moment affects and is also affected by the activities of this environment.

Despite all these complexities, even in highly nonequilibrium conditions we expect that a generalized FDR (in the form given in this paper) will still hold and be useful to guide us on understanding the complex physical processes in the system and the environment. From the above depiction of the physical scenario and from previous studies of the statistical mechanics of quantum field processes in cosmology, one can see that there exists a balance between particle creation (in the field) and its backreaction (on spacetime), which can be attributed to the interlocked relation between fluctuations and dissipation. There is also a mathematical justification: it is a relation between the real and imaginary parts of the effective action for the open system. Similar in nature to the optical theorem in scattering theory or the Kramers-Krönig relation in many body theory [45], these relations describe the dissipative and reactive parts of the response function of an open system to influences from the environment. They are of a categorical nature because they originate ultimately from the unitarity condition of the dynamics of the combined closed system. They only take the form of dissipation in the open system because we have identified a certain subsystem as the system of interest and decided to follow its effective dynamics; and they take the form of fluctuations in the environment because we refer to them in reference to the mean value of the environment variables, the remaining information is downgraded in the form of fluctuations. Had we decided not to coarse-grain the environment, or choose to observe the two subsystems with equal interest and accuracy, such a relation governing the mutual reaction would still exist, except that the concepts of dissipation and fluctuations will no longer be appropriate. (Both subsystems will be governed by equations of motion in the form of an integral differential equation, and treated in a nondiscriminate and balanced way. See, e.g., [73]).

In the context of semiclassical cosmology, the open system is the spacetime sector, whose dynamics is influenced by the matter fields. The expansion of the universe amplifies the vacuum fluctuations of the matter field into particles, which act as the source in the Einstein equation driving the universe. The averaged effect of particles created imparts a dissipative component in the spacetime dynamics, and the fluctuations in particle creation constitute
the noise. The particular forms of the dissipation and noise kernels and their effects may vary under different conditions—zero or finite temperature, equilibrium or non-equilibrium—but the existence of such a relation between the fluctuations in the matter field and the dissipative effect on the spacetime dynamics should remain. We will have opportunities later to explore related problems which can shed more light on these issues.

2) Relation with FDR in Spacetimes with Event Horizons

As we mentioned in the Introduction, our search for a FDR in cosmological spacetimes without event horizons was inspired by Sciama’s proposal to view the Hawking and Unruh effects as manifestations of a fluctuation-dissipation relation between the field quanta and the detector response. De Sitter universe is an important class of cosmological spacetimes with event horizons. For this one can use the thermal property of the field to perform a linear response theory (LRT) analysis for the derivation of the FDR [37]. Our derivation here based on the influence functional formalism attacks the problem at a more basic level, where equilibrium condition between the system and the environment is not necessarily present at every stage. It is of interest to compare the results between the equilibrium limit of the IF or the CTP formalisms and that of LRT. This can be done explicitly by carrying out an analysis similar to this paper on the de Sitter universe and see how the FDR obtained from the IF compare with that from the LRT. Formally this would render explicit the relation between the IF formalism to (non-equilibrium) statistical field theory and perturbative thermal (finite-temperature) field theory.

More meaningfully, as was originally conceived by one of us [74, 42], this would provide a channel to generalize the conventional way of treating Hawking effect associated with black holes and accelerated observers based on thermal propagators and event horizons to non-stationary conditions. This involves cases like non-uniformly accelerating observers and realistic collapse dynamics, where an event horizon or Euclidean section does not always exist but is dynamically generated. Our motivation for finding a way to treat these more general conditions is to seek a deeper meaning to the Hawking effect, and through it to explore the subtle connection between quantum field theory, relativity theory and statistical mechanics. In our view, the open system concept explicated by the influence functional formalism provides a more solid basis to understand its statistical mechanical meanings and a broader framework to tackle the less unique situations which cannot easily be treated by purely geometric means, powerful and elegant as they are. It also brings the effects of quantum fields on observer kinematics and spacetime dynamics in line with the more common statistical mechanical phenomena involving ordinary matter. These problems are currently under investigation.

3) Related Problems in Semiclassical Cosmology and Inflationary Universe

For particle creation-backreaction problems similar to the Bianchi-I model studied here, one can obtain similar results for other matter fields in other types of spacetimes of astrophysical or cosmological interest. An example is a massless minimally-coupled scalar field in a Robertson-Walker or de Sitter universe. It mimics the linearized graviton modes and has
practical use for the description of primordial stochastic gravitons. The particle production problem was first studied by Grishchuk [11], the backreaction by Grishchuk [13] and Hu and Parker [14] via canonical quantization methods, and by Hartle [18] and Calzetta and Hu [40] via the in-out and in-in effective action method respectively. The influence functional approach expounded here would enable one to get from first principle the entropy generation from graviton production [70, 71], and the noise associated with them, which is related to the fluctuations in graviton number [64, 45]. On the aspect of backreaction in semiclassical cosmology, one can also derive the Einstein-Langevin equation for the study of graviton production and metric fluctuations. This problem is currently pursued by Calzetta and Hu [75]

A related problem of interest is the evolution of the homogeneous mode of the inflaton which describes the inflation mechanism [76] and the inhomogeneous modes as progenator of structures in the early universe. The influence functional method was used by Hu, Paz and Zhang [57] Laflamme and Matacz [77] and others to discuss the decoherence of the long-wavelength sectors of the inflaton, and the origin of quantum fluctuations as noise for the galaxy formation problem. Our result here provides an example for the consistent treatment of the evolution of these modes, their interaction, and their backreaction on the spacetime, which can offer some physical insight into the no-hair type of theorems in inflationary universe. These problems are under study by Matacz, Raval and the authors.

4) Minisuperspace in Quantum Cosmology as an Open System: Geometrodynamical Noise and Gravitational Entropy

We have discussed the question of the validity of the minisuperspace approximation [56] in quantum cosmology [78, 79, 80], wherein only the homogeneous cosmologies are quantized and the inhomogeneous cosmologies ignored [81]. We used an interacting quantum field model and calculated the effect of the inhomogeneous modes on the homogeneous mode via the CTP effective action. This effect manifests in the effective equation of motion for the system as a dissipative term. For quantum cosmology, this backreaction turns the Wheeler-De Witt [82] equation for the full superspace into an effective equation for the minisuperspace with dissipation. Extending the CTP to the IF formalism as is done here, one can derive the noise associated with the truncated inhomogeneous cosmological modes. One can also define an entropy function from the reduced density matrices, which measures the information loss in the minisuperspace truncation. These can perhaps be called geometrodynamical noise and gravitational entropy. It would be interesting to compare this statistical mechanical definition with the definition suggested by Penrose [83] in classical general relativity and by one of us in the semiclassical context [72]. Some initial thoughts on this problem are described in [84], while details are to be found in [85].

Acknowledgements We thank Esteban Calzetta and Juan Pablo Paz for interesting discussions. Research is supported in part by the National Science Foundation under grant PHY91-19726.
References


[70] e.g., R. H. Brandenberger, V. Mukhanov and T. Prokopec, Phys. Rev. Lett. 69, 3606 (1992)


[75] E. Calzetta, B. L. Hu and A. Matacz, “Quantum Fluctuations and Galaxy Formation” (in preparation)


