Equivalence of Chern-Simons Matter Models

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Abstract

Not only does Chern-Simons coupling characterize statistics, but also spin and scaling dimension of matter fields. We demonstrate transmutation of spins in Chern-Simons matter theory, and thus show equivalence of several models. We study Chern-Simons vector model in some details, which provides a consistent check to the assertion of equivalence.
Introducing Chern-Simons interaction in three spacetime dimensions is equivalent to attaching to particles 'magnetic' flux tubes. Considerable interests focus recently on strong Chern-Simons interactions, so that each electron carries two flux tubes or more, for instance. This is a key point in a recent successful theory of the half-filled Landau level to understand the quantum Hall effects with the filling factors \( \nu = 1/2 \), and \( \nu = p/(2p + 1) \) (\( p \) an integer) [1, 2]. However, as the interaction is getting strong, perturbation theory which has being used in many cases can not be reliable, in general. In this Letter, we suggest alternatives of an extensively used model, the Dirac field coupled to Chern-Simons, in the relativistic field theory formalism. We shall show the following three quantum field theory models, which look apparently so different, are equivalent one to the others. The first is

\[
\mathcal{L}_F = \bar{\psi} \left[ -i \gamma_\mu (\partial_\mu + ia_\mu + i C_\mu) + M \right] \psi - \frac{i}{8\pi \alpha} \epsilon_{\mu \nu \lambda} a_\mu \partial_\nu a_\lambda ,
\]

(1)

where \( \psi \) is a two-component Dirac field, the Dirac matrices in three (Euclidean) dimensions \( \gamma_\mu = \sigma_\mu \) with \( \sigma_\mu \) (\( \mu = 1, 2, 3 \)) Pauli matrices, and \( C_\mu \) an external gauge field. And the second,

\[
\mathcal{L}_B = \frac{1}{2} B_\mu^\dagger \left[ -i \epsilon^{\mu \nu \lambda} (\partial_\mu + ia_\mu + i C_\mu) + M \delta^{\nu \lambda} \right] B_\lambda - \frac{i}{8\pi (\alpha - 1/2)} \epsilon_{\mu \nu \lambda} a_\mu \partial_\nu a_\lambda ,
\]

(2)

where \( B_\mu \) is a complex vector field. Finally, the third,

\[
\mathcal{L}_{FF} = \bar{\Psi} \left[ -i \frac{2}{3} \tau_\mu (\partial_\mu + ia_\mu + i C_\mu) + M \right] \Psi - \frac{i}{8\pi (\alpha - 1)} \epsilon_{\mu \nu \lambda} a_\mu \partial_\nu a_\lambda ,
\]

(3)

where \( \Psi \) is a spin-\( \frac{3}{2} \) field with four components and \( L_\mu \) are \( 4 \times 4 \) matrices. While it is of the most interest to explore intrinsic relations among different models, obviously (1), (2), and (3) can be good perturbation theories only around \( \alpha = 0, 1/2, \) and 1, respectively.

Our expectation of the equivalence is based on the observation that the Chern-Simons coefficient characterizes not only the statistics, but also the spins of the coupled matter fields. Therefore, it is possible to trade Chern-Simons interaction for higher spins, and vice versa. To understand the mechanism of the spin transmutation precisely, we shall calculate the partition functions of the three models, by using the particle path integral method [3, 4]. And we obtain

\[
Z_F[C] = Z_B[C] = Z_{FF}[C] ,
\]

(4)

which founds the equivalence.

The Chern-Simons coupling characterizes as well the scaling dimensions of matter fields and composite operators, which have been calculated by using the renormalization group method, and other physical quantities [5, 6, 7, 8]. The equivalence shown here makes it possible for one to perturbatively expand a quantum field theory in a
proper version with weak Chern-Simons interaction but higher spin. In the second half of the Letter, we shall consider the model (2) in some details, while leave further investigation of (3) to a separate publication except a comment: the model (3) around $\alpha = 1$ might have something to do with the half-filled Landau level, as it describes fermion-like particles carrying about two flux tubes. Some aspects of model (2) in a slightly different form (where the $B_\mu$ field was real) were discussed in literature [9, 10, 11], here we look into some others, which provide a consistent check to the assertion of equivalence. In particular, combining with the known results about model (1) [5], we shall discuss the change of scaling dimension of the matter field against the statistical parameter $\alpha$.

Deriving (4), we start from the partition function $Z_F(C)$,

$$Z_F[C] = \int \mathcal{D}a_\nu \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-S_F} .$$

(5)

The path-integral over fermion fields is of Gaussian type, then it is readily to obtain

$$Z_F[C] = \int \mathcal{D}a_\mu \sum_{n=0}^\infty \frac{1}{n!} (-W_F)^n \exp\left\{ \frac{i}{8\pi \alpha} \int d^3x \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \right\} ,$$

(6)

$$W_F = Tr \log \left[ -i(\partial_\mu + ia_\mu + iC_\mu)\gamma_\mu + M \right] .$$

(7)

To perform the path integral over $a_\mu$ field, we use a particle path integral representation of $W_F$ [3, 4],

$$W_F = \int \mathcal{D}X e^{\int_{0}^{1} dt [M\sqrt{X^2} + ia \cdot \dot{X} + iC \cdot \dot{X}]} + \frac{i}{2} \Phi[\frac{\dot{X}}{|X|}],$$

(8)

where

$$\Phi[e] = \int_D \int d\nu e \cdot [\partial_\nu e \times \partial_{\nu e}] ,$$

(9)

is defined as spin factor. Geometrically, this is the area enclosed by a closed path $e(t)$ ($0 \leq t \leq 1$) on a two dimensional unit sphere. The numerical coefficient of the spin factor in (8) is just the spin of the primary matter field, which is apparently $1/2$ for the Dirac field. Now the integral over $a_\mu$ is readily to perform, it gives

$$Z_F[C] = \sum_{n=1}^\infty \int \mathcal{D}X e^{\int_{0}^{1} dt \left[ M\sqrt{X^2} + ia \cdot \dot{X} + iC \cdot \dot{X} \right] - \frac{i}{2} \Phi[\frac{\dot{X}}{|X|}] - i\alpha \Theta_{ii} + i\alpha \sum_{i<j} \Theta_{ij}} ,$$

(10)

where,

$$\Theta_{ij} = \int_{0}^{1} dt \int_{0}^{1} ds \frac{dX^\mu}{dt} \frac{dX^\nu}{ds} <a_\mu(x_i)a_\nu(x_j)> ,$$

(11)

$$<a_\mu(x)a_\nu(y)> = 8\pi \epsilon_{\mu\nu\lambda} \frac{x^\lambda - y^\lambda}{|x - y|^3} .$$

(12)
(12) is the propagator of Chern-Simons gauge field in the Landau gauge. Introduce a notation of unit vector \( e(s, t) = \frac{x(s,t) - x(t)}{|x(s,t) - x(t)|} \in S^2 \), \( \Theta_{ij} \) takes a compact form

\[
\Theta_{ij} = \int_0^1 ds \int_0^1 dt e \cdot [\partial_s e \times \partial_t e].
\]  

Now, we come to the key point for spin transmutation. It is shown [3, 4] the self-energy \( \Theta_{ij} \) relating to the spin factor \( \Phi \) (see (9)) via

\[
\Theta_{ij} - 2\Phi = \frac{4\pi}{\text{(mod } 8\pi)};
\]  

while the relative energy \( \Theta_{ij} \) \((i \neq j)\) is the Gaussian linking number, which is quantized

\[
\Theta_{ij} \in 4\pi \mathbb{Z}.
\]

By using (14) and (15), we shift \( \alpha \) by \(-1/2\), and so rewrite (10) as

\[
Z_F[C] = \sum_{n=1}^{\infty} \int \prod_{i=1}^{n} \mathcal{D}X_i \exp \{ -\int \sum_{i=1}^{n} \int dt(M\sqrt{\mathbf{X}_i^2} + i C_i \cdot \mathbf{X}_i) - i\Phi[\frac{\mathbf{X}_i}{|\mathbf{X}_i|}] - i \frac{\alpha - \frac{1}{2}}{2} \Theta_{ii} + i(\alpha - 1) \sum_{i<j} \Theta_{ij} \}.
\]  

Notice now the spin factor \( \Phi \) in (16) has a numerical coefficient 1, replacing 1/2 in (10). (16) can be written as

\[
Z_F[C] = \int \mathcal{D}a_\nu \sum_{n=0}^{\infty} \frac{1}{n!} (-W_B)^n \exp \left[ \frac{i}{8\pi(\alpha - 1/2)} \int d^3x \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \right],
\]  

with

\[
W_B = \text{Tr} \log \left[ -(\partial_\mu + i a_\mu + i C_\mu) L_\mu + M \right],
\]  

where \( L_\mu^{\nu} \), which may be chosen as \(-i\epsilon_{\mu\nu\lambda}\), is the spin matrix with the eigenvalue 1. The right hand side of (17) is actually the partition function of (2). This completes the derivation of the first equation in (4). To derive the second equation in (4), we use (14) and (15) once again, and shift \((\alpha - 1/2)\) in (16) by \(-1/2\) to \((\alpha - 1)\), so that

\[
Z_F[C] = \sum_{n=1}^{\infty} \int \prod_{i=1}^{n} \mathcal{D}X_i \exp \{ -\int \sum_{i=1}^{n} \int dt(M\sqrt{\mathbf{X}_i^2} + i C_i \cdot \mathbf{X}_i) - i \frac{3}{2} \Phi[\frac{\mathbf{X}_i}{|\mathbf{X}_i|}] - i \frac{\alpha - 1}{2} \Theta_{ii} + i(\alpha - 1) \sum_{i<j} \Theta_{ij} \}.
\]  

And we have

\[
Z_F[C] = \int \mathcal{D}a_\nu \sum_{n=0}^{\infty} \frac{1}{n!} (-W_{FF})^n \exp \left[ \frac{i}{8\pi(\alpha - 1)} \int d^3x \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \right],
\]  

\[
W_{FF} = \text{Tr} \log \left[ -(\partial_\mu + i a_\mu + i C_\mu) L_\mu + M \right],
\]

where the eigenvalue of \( L_z \) is 3/2, that of \( L_z^2 \) is \((3/2)(3/2 + 1)\). Obviously, the right hand side of (20) is the partition function of (3). To conclude, Lagrangians (1), (2),
and (3) are just different versions of the same Chern-Simons quantum field theory. It is straightforward to use the above process and work out other versions, where the spin matrices with higher eigenvalues are involved. As a special case, the Chern-Simons theory turns out to be a free theory for particles with spin \(1/2 + \alpha\), if \(\alpha\) is an integer or half integer.

Next, we discuss the model (2) (the external gauge field \(C_\mu\) is ignored hereafter). It is not difficult to check the vector field model (2) and the Dirac field model (1) have a same set of symmetries [13]. In particular, (2) is invariant too under the \(U(1)\) gauge transformation:

\[
a_\mu \rightarrow a_\mu - i \partial_\mu \Lambda, \quad B_\mu \rightarrow e^{i\Lambda} B_\mu.
\]  

(22)

The equations of motion which follow from (2) are

\[
\epsilon_{\mu\nu\lambda} (\partial_\nu + i a_\nu) B_\lambda + i M B_\mu = 0,
\]  

(23)

\[
\epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda = 4\pi (\alpha - 1) i j_\mu,
\]  

(24)

where the current is \(j_\mu = -\epsilon_{\mu\nu\lambda} B_\nu B_\lambda\) (the equation for \(B_\mu^\ast\), similar to (23), is omitted). From (23) and (24), one readily obtains the current conservation and a constraint for the \(B_\mu\) field:

\[
\partial_\mu j_\mu = 0,
\]  

(25)

\[
(\partial_\mu + i a_\mu) B_\mu = 0.
\]  

(26)

Since \(a_\mu\) has no independent degree of freedom (see (24)), the constraint (26) reduces the independent degrees of freedom of \(B_\mu\) field to 2, which is the same with that of the Dirac field in three dimensions.

Let us consider the perturbation expansion of model (2). Notice no gauge fixing for the vector field \(B_\mu\), its inverse propagator and propagator are \([12]\]

\[
D_{\mu\nu}^{-1} = -\epsilon_{\mu\nu\lambda} p_\lambda + M \delta_{\mu\nu}, \quad D_{\mu\nu} = \frac{1}{p^2 + M^2} (\epsilon_{\mu\nu\lambda} p_\lambda + \frac{p_\mu p_\nu}{M} + M \delta_{\mu\nu}).
\]  

(27)

Taken into account the gauge and Lorentz symmetries, the polarization tensor of the Chern-Simons field in the Landau gauge takes a form

\[
\Pi_{\mu\nu} = \Pi_e (p, M) (\delta_{\mu\nu} p^2 - p_\mu p_\nu) + \Pi_\phi (p, M) \epsilon_{\mu\nu\lambda} p_\lambda.
\]  

(28)

By the regularization by dimensional reduction [5], it is readily to calculate the polarization tensor contributed from one vector field loop, which is finite:

\[
\Pi_e (p, M) = \frac{1}{12\pi} \frac{1}{\sqrt{M^2}} [1 + \mathcal{O}(\frac{p^2}{M^2})],
\]  

(29)

\[
\Pi_\phi (p, M) = -\frac{1}{2\pi} \text{sign}(M) [1 + \mathcal{O}(\frac{p^2}{M^2})].
\]  

(30)
In the low energy (or large matter mass) limit, we see that the Chern-Simons gauge field behaves like a dynamical, topologically massive gauge field. This is exactly what happens in the model of Dirac field coupled to Chern-Simons gauge field, (1). We argue that, at two-loop and beyond, the Chern-Simons mass receives no further corrections, which extends the no-renormalization theorem [14, 15] to the vector model [11]. However, like in the Chern-Simons Dirac field model, the two-loop vector matter field self-energy does have a simple pole, or in other words, it is logarithmic divergent,

$$\Sigma_{\mu\nu}(p, M) = -\epsilon_{\mu\nu\lambda}p_\lambda \left( \frac{\alpha - 1/2)^2}{4} \left( \frac{1}{\epsilon} + \frac{1}{2} \ln \frac{\mu^2}{p^2 + M^2} + \text{finite} \right) \right), \quad (31)$$

therefore, the field $B_\mu$ needs infinite wave-function renormalization, and it obtains an anomalous dimension. Therefore, the scaling dimension of $B_\mu$ is

$$[B_\mu] = 1 - \frac{(\alpha - 1/2)^2}{8} + \mathcal{O}((\alpha - 1/2)^4). \quad (32)$$

Recall the scaling dimension of the Dirac field in model (1) is given [5] as

$$[\psi] = 1 - \frac{\alpha^2}{3} + \mathcal{O}(\alpha^4). \quad (33)$$

As (1) and (2) are just different versions of the same theory as we have seen, $\psi$ and $B_\mu$ describe the same matter field for a given $\alpha$. On the other hand, it is natural to assume that the scaling dimension of the matter field is a continuous function of $\alpha$ in the region, for instance, $(0, 1/2)$. Then, since as $\alpha$ varies away from both the fermion ($\alpha = 0$) and boson ($\alpha = 1/2$) points quantum fluctuation has the same tendency decreasing the dimension of the matter field, there must exist at least one local minimum on the curve of the scaling dimension of matter versus $\alpha$ between $\alpha = 0$ and $1/2$.

It is interesting to notice that when the matter mass is taken to be zero, $M = 0$, the vector model (2) possesses $SU(2)$ gauge symmetry, which replaces $U(1)$ with the massive theory. To see this, we set

$$a_\mu = a_\mu^1, \quad \text{and} \quad B_\mu = a_\mu^2 + ia_\mu^3, \quad (34)$$

substitute into (2), then have

$$\mathcal{L} = \frac{i}{2} \epsilon_{\mu\nu\lambda}(a_\mu^a \partial_\nu a_\lambda^a + e\varepsilon^{abc}a_\mu^a a_\nu^b a_\rho^c). \quad (35)$$

This is the well-known $SU(2)$ Chern-Simons topological field theory. But, the equivalence discussed above cannot be simply extended to the zero mass cases, as applying the particle path integral method to massless (Dirac) fields is ambiguous.

To conclude the Letter, we remark the equivalence, however, can be generalized to the self-interactions in the Chern-Simons models. For instance, if one adds the
four-fermion interaction \((\bar{\psi}\psi)^2\) in (1), it can be seen this is equivalent to adding a four-boson interaction \((B_\mu^\ast B_\nu)^2\) in the vector version of the Chern-Simons theory, (2).

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