Scattering of Fermions off Dilaton Black Holes

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Abstract

We discuss how various properties of dilaton black holes depend on the dilaton coupling constant \(a\). In particular we investigate the \(a\)-dependence of certain mass parameters both outside and in the extremal limit and discuss their relation to thermodynamical quantities. To further illuminate the role of the coupling constant \(a\) we look at a massless point particle in a dilaton black hole geometry as well as the scattering of (neutral) fermions. In this latter case we find that the scattering potential vanishes for the zero angular momentum mode which seems to indicate a catastrophic deradiation when \(a > 1\).

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Dilaton black holes are studied because they arise as solutions to the low-energy string effective field theory [1], and because they shed new light on the old question of black holes as elementary particles [2].

The dilaton black holes are solutions to field equations that describe a scalar (dilaton) field and an electromagnetic field coupled to Einstein gravity. It has proven useful to let the dilaton coupling depend on a non-negative parameter $a$. The value $a = 0$ corresponds to the usual Einstein theory whereas the coupling $a = 1$ is the value dictated by string theory. Furthermore, the value $a = \sqrt{3}$ seems to be special. For this value of $a$ rotating solutions [3] as well as a string black hole [4] have been found. It also arises in dimensional reduction of $d = 5$ supergravity studied in the context of supersymmetric solitons [5]. In general many features of the solution (but not all) depend on the value of $a$. The present work corroborates this picture.

The main problem treated in this note is the scattering of spinning particles off of dilaton black holes. We derive the corresponding potential and discuss its behaviour as a function of $a$. In particular we find that it generally diverges at the horizon in the extremal limit for $a > 1$. For $a = \sqrt{3}$ there is an additional pole, interestingly enough, although the relation to the abovementioned rotating solution, string solution or soliton solutions is unclear.

This note also contains a derivation and discussion of certain mass-parameters: the ADM-mass, the Tolman mass and the mass of the black hole. Although we relate the usual definition of the temperature and entropy of the black hole to its ADM mass, we use the vanishing of the black hole mass as the main indication for the vanishing of the black hole in the extremal limit. This is because it has been argued that the thermo-dynamical description breaks down in that limit [6,2].
The action we consider is

\[ S = \int d^4x \sqrt{-g}(-R + 2(\nabla \Psi)^2 + e^{-2\alpha\Psi}F^2) \]  

where \( R \) is the Ricci curvature scalar, \( \Psi \) the dilaton field, \( F^2 \) the square of the electromagnetic field strength and \( a \) a non-negative real parameter. The corresponding field equations are

\[ \nabla_\mu (e^{-2\alpha\Psi}F^{\mu
u}) = 0, \]  

\[ \nabla^2\Psi + \frac{a}{2}e^{-2\alpha\Psi}F^2 = 0, \]  

\[ R_{\mu\nu} = 2\nabla_\mu \Psi \nabla_\nu \Psi + 2e^{-2\alpha\Psi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}g_{\mu\nu}e^{-2\alpha\Psi}F^2. \]  

The line element for an electrically charged black hole solution is \([7,8]\)

\[ ds^2 = \Delta \sigma^{-2}dt^2 - \sigma^2(\Delta^{-1}dr^2 + r^2d\Omega_2^2), \]  

with \( \Delta \) and \( \sigma \) given by

\[ \Delta = (1 - \frac{r_+}{r})(1 - \frac{r_-}{r}), \quad \sigma^2 = (1 - \frac{r_-}{r})^{\frac{2\alpha^2}{r^2+a^2}}. \]  

Here \( d\Omega_2^2 \) is the metric in the unit two-sphere \( S^2 \), \( r_+ \) and \( r_- \) are implicitly given by

\[ r_+r_- = Q^2(1 + a^2) \quad (2M - r_+)(\frac{1 + a^2}{1 - a^2}) = r_-, \]  

\( M \) represents the metric mass and \( Q \) is the electric charge on the hole. The electric field is given by

\[ F = \frac{Q}{r^2}dt \wedge dr, \]  

and the dilaton by

\[ e^{2\alpha\Psi} = \sigma^2. \]  

The scalar charge \( \Sigma \) is defined as \( \Psi \sim \Psi_0 - \Sigma/r \) for large \( r \) which gives

\[ r_- = \frac{(1 + a^2)\Sigma}{a} \Rightarrow r_+ = a\frac{Q^2}{\Sigma}. \]
The metric mass can thus be written

\[ M = \frac{a^2 Q^2 + (1 - a^2) \Sigma^2}{2a \Sigma}. \]  

(11)

Some aspects of these solutions can be better understood when one analytically continues the Schwarzschild coordinates. To this end we define an advanced null coordinate \( v \) by \( v = t + r^* \) where

\[ \frac{dr^*}{dr} = \frac{\sigma^2}{\Delta}. \]  

(12)

In terms of this coordinate we easily obtain the advanced Eddington-Finkelstein form of the metric eq.(5)

\[ ds^2 = \frac{\Delta}{\sigma^2} dv^2 - 2dvdr - r^2 \sigma^2 d\Omega_2^2. \]  

(13)

It is clear that this metric in general is free from singularities when \( a \leq 1 \). Off extremality \((r_- \neq r_+)\) the analytic continuation is singular at \( r = r_- \) when \( a > 1 \). However this singularity generally disappears in the extremal limit, i.e. in this limit the metric is non-singular all the way down to the singularity at \( r = 0 \).

We finally give the Kruskal form of the metric. Defining the \((U, V)\) coordinates in the usual way we get

\[ ds^2 = -32M^3 \frac{d}{dr} e^{-\frac{r^*}{2M}} (dU^2 - dV^2) - r^2 \sigma^2 d\Omega_2^2. \]  

(14)

From this expression one can easily convince oneself that the \((U, V)\) part of the metrics of the \( a = 0 \) and \( a = 1 \) solutions are in fact identical (when \( 2M \) is replaced by \( r_+ \) in the \( a = 1 \) solution). Hence, when \( a = 1 \) the sole effect of the singularity at \( r = r_- \), (which coincides with the event-horizon in the extremal limit), is the vanishing of the angular part of the metric. The structure of the \( a \neq 1 \) extensions is much more complicated and will not be considered here.

Now we want to discuss the energy content of the solutions and to this end we calculate certain mass parameters. Let \( \xi^a \) be a time translation Killing vector field which is timelike
near infinity such that $\xi^a \xi_a = 1$ and which has vanishing norm $\xi^a \xi_a = 0$ at the horizon. Let $S$ denote the region outside the event-horizon. The boundary $\partial S$ of $S$ is taken to be the event-horizon $\partial B$ of the black hole and a two-surface $\partial S_\infty$ at infinity. Due to the asymptotic flatness of the dilaton black hole solutions it follows that the ADM mass of the black hole configuration is given by [9]

$$M^\infty = \frac{1}{8\pi} \int_S (2T^a_{\ b} - T \delta^a_{\ b}) \xi^b d\Sigma_a + \frac{1}{4\pi} \int_{\partial B} \xi^a \Sigma_{ab} \equiv M^M + M^H. \tag{15}$$

The last integral is just the surface gravity $\kappa$ multiplied with the surface area $A^H$ of the event-horizon. $M^M$ and $M^H$ are naturally interpreted as the gravitational mass of the matter distribution outside the event-horizon and the gravitational mass of the black hole, respectively. In the following we will set $\vec{\xi} = \partial_i$ since this vector always vanishes on the event-horizon independent of the value of $a$. It then follows that the first integral $M^M$ is the usual Tolman mass of the matter outside the event-horizon [10]. From the action eq. (1) we derive the energy-momentum tensor $T_{\mu\nu}$ for the dilaton and the electromagnetic field

$$T^t_{\ t} = \frac{\sigma^2}{\Delta} \left( \frac{\Delta Q^2}{\sigma^4 r^4} + \frac{\Sigma^2 \Delta^2}{\sigma^4 r^2 (r - r_-)^2} \right), \tag{16}$$

$$T^r_{\ r} = -\frac{\Delta}{\sigma^2 r^2 (r - r_-)^2} - \frac{Q^2}{\Delta r^4}, \tag{17}$$

$$T^\Omega_{\ \Omega} = -\frac{Q^2}{\sigma^2 r^4} + \frac{\Delta \Sigma^2}{\sigma^2 r^2 (r - r_-)^2}. \tag{18}$$

The Tolman mass density is then given by

$$\mathcal{M}^M = T^t_{\ t} - T^r_{\ r} - 2T^\Omega_{\ \Omega} = \frac{2Q^2}{\sigma^2 r^4}. \tag{19}$$

This expression reduces to the usual one in the Reissner-Nordström geometry when $a$ is set to zero. From the above expression it follows that $\mathcal{M}^M$ generally diverges on $r = r_-$ when $a > 0$ thus giving rise to a curvature singularity there. However, the total energy outside the event-horizon is always positive and finite. It is rather surprising that it is only the electric field that contributes to the gravitational energy. The dilaton contribution cancels due to the large pressures in the angular directions.
It is interesting to relate the temperature and the entropy of the black hole to its ADM mass. Let $\overrightarrow{N}$ denote a second Killing vector field (normalized to unity $N^a N_a = -1$) orthogonal to the event-horizon. The surface gravity $\kappa$ can then be written $\kappa^2 = N_a \nabla^a N^b \nabla_b$ [9].

The area of the event-horizon is $A^H = 4\pi r_+^2 \sigma^2 (r = r_+)$. This implies that the contribution to the total gravitational mass of the system from behind the event-horizon is

$$M^H = 2\pi r_+ (1 - \frac{r_-}{r_+}).$$

(20)

This quantity always vanishes when $r_- = r_+$. In this sense the black hole disappears completely in the extremal limit. The red-shifted temperature $T^H$ measured by a distant observer is formally given by

$$T^H = \frac{\hbar}{2\pi k_B} |\kappa| = \frac{\hbar}{4\pi k_B r_+} (1 - \frac{r_-}{r_+}) \frac{1}{r_+^2} = \frac{\hbar}{4\pi k_B r_+} \left( \frac{M^H}{2\pi r_+} \right) \frac{1}{r_+^2}.$$  

(21)

From this it is clear that $T^H$ vanishes when $a < 1$, it is finite when $a = 1$ and infinite whenever $a > 1$ in the extremal limit. By simply using the event-horizon area of the black hole as a measure of the entropy content $S^H$ of the black hole we get

$$S^H = \frac{1}{4} A^H = \pi (r_+ \left( \frac{M^H}{2\pi r_+} \right) \frac{1}{r_+^2})^2 = \frac{\hbar}{8\pi k_B} \frac{M^H}{T^H}.$$  

(22)

It follows that the entropy vanishes in the extremal state independent of the value of $a$. It has been argued that the thermo-dynamic interpretation breaks down near the extremal limit [6,2]. Nevertheless we feel that the above relations give an indication of the behaviour of the thermodynamical entities in this limit, but we emphasize the vanishing of $M^H$.

The qualitative thermo-dynamical properties of the dilaton holes in the extremal limit depend crucially on the value of the coupling constant $a$. It is interesting to investigate whether this is reflected in the dynamics of particles propagating in these geometries. Since the case of a real massless scalar field is treated in [2] we shall consider a classical photon and the propagation of neutrinos.

Consider a photon propagating in a general dilaton black hole background with fixed $\theta$ coordinate. Define $E = p_t$ and $L = p_\phi$. Then $g_{\mu\nu} p^\mu p^\nu = 0$ implies
\[ r^2 = E^2 - \frac{L^2}{r^2}(1 - \frac{r_+^-}{r})(1 - \frac{r_-^+}{r})^{\frac{3a^2}{r^2 + a^2}} \equiv E^2 - V^2 . \] (23)

From this it follows that

\[ \ddot{r} = -\frac{1}{2} \frac{dV^2}{dr} = \frac{L^2}{r^3}(1 - \frac{r_+^-}{r})^{\frac{3a^2}{r^2 + a^2}} (1 - \frac{3r_+^-}{2r} - \frac{r_-^+}{2r}(\frac{1 - 3a^2}{1 + a^2})(\frac{r - r_+^-}{r})). \] (24)

In the extremal limit the analysis of this expression is particularly simple. Due to the exponent appearing in this expression it is natural to concentrate the attention to three \(a\)-parameter intervals: (I) \(0 \leq a < 1/\sqrt{3}\), (II) \(a = 1/\sqrt{3}\) and (III) \(a > 1/\sqrt{3}\). The second derivative \(\ddot{r}\) may vanish outside \(r_+\) when \(a\) takes values in the parameter intervals (I) and (II) while turning points only exist outside \(r_+\) in region (III) if we have the additional restriction \(a < 1\). When \(a = 1\) the turning point coincides with \(r_+\) and when \(a > 1\) the turning point is behind \(r_+\). Hence, in the extremal limit closed circular photon orbits can only exist outside \(r = r_+\) when \(a < 1\).

Another interesting question is whether \(\ddot{r}\) may diverge. In (I) and (II) this function is finite for all non-zero values of \(r\). When \(1/\sqrt{3} < a < 1\) \(\ddot{r}\) diverges on \(r_+\). With \(a = 1\) the radial acceleration is finite on \(r_+\) while for \(a > 1\) it diverges again for this value of the radial coordinate. The significance of the \(a = 1/\sqrt{3}\) solution is somewhat unclear. It does not seem to have any remarkable thermo-dynamical properties in the extremal limit. However, it is worth noting that only \(a = 1\) and not \(a = 1/\sqrt{3}\) appears as a special value for the scalar field dynamics in the extremal limit [2].

Let us now turn to the scattering of particles with spin. In flat space fermions can be described by two, in general coupled, equations of two 2-component spinors \(P^a\) and \(\overline{Q}_\beta\). Following [11] we write \(P^0 \equiv F_1, P^1 \equiv F_2, \overline{Q}^i \equiv G_1\) and \(\overline{Q}^0 \equiv -G_2\). In the Newman-Penrose formalism the equations governing the dynamics of massless fermions can be written as

\[ (D + \epsilon - \rho)F_1 + (\delta^a + \pi - \alpha)F_2 = 0 \] (25)

\[ (\Delta + \mu - \gamma)F_2 + (\delta + \beta - \tau)F_1 = 0 \] (26)

\[ (D + \epsilon^* - \rho^*)G_2 - (\delta + \pi^* - \alpha^*)G_1 = 0 \] (27)
Using the standard null-frame notation \((\bar{t}, \bar{n}, \bar{m}, \bar{m}^*)\) it follows that \(D = \bar{t}, \Delta = \bar{n}, \delta = \bar{m}\) and \(\delta^* = \bar{m}^*\). It is also assumed that \(l^a n_a = -m^a m^*_a = -1\) and \(l^a m_a = n^a m_a = 0\). We will let \(\bar{t}\) represent the four-velocity of an outwards radially moving massless and electrically uncharged particle with vanishing orbital angular momentum \((L = 0)\)

\[
l^\mu = \frac{\sigma^2}{\Delta} \delta_0^\mu + \delta_1^\mu.
\]

(29)

Further, \(n^\mu\) is identified with the four-velocity of the corresponding inwards moving particle

\[
n^\mu = \frac{1}{2} \delta_0^\mu - \frac{\Delta}{2\sigma^2} \delta_1^\mu.
\]

(30)

We also define the complex vector \(m^\mu\):

\[
m^\mu = \frac{1}{\sqrt{2r\sigma}} (\delta_2^\mu + \frac{i}{\sin \theta} \delta_3^\mu).
\]

(31)

The non-zero spin coefficients are then given by

\[
\rho = \frac{-1}{r\sigma} \partial_r (r\sigma), \quad \mu = \frac{-\Delta}{2r\sigma^3} \partial_r (r\sigma)
\]

\[
\gamma = \frac{1}{4} \partial_r \left( \frac{\Delta}{\sigma^2} \right), \quad \alpha = -\beta = \frac{-\cot \theta}{2\sqrt{2r\sigma}}
\]

(32)

Let \(f_1 = r\sigma F_1, f_2 = F_2, g_1 = G_1\) and \(g_2 = r\sigma G_2\). Also define

\[
\hat{L} = \partial_\theta + \frac{i}{\sin \theta} \partial_\phi + \frac{1}{2} \cot \theta
\]

\[
\hat{V} = \frac{\sigma^2}{\Delta} \partial_t + \partial_r
\]

\[
\hat{W} = \frac{1}{2} \partial_i - \frac{\Delta}{2\sigma^2} \partial_r - \frac{\Delta}{2r\sigma^3} \partial_r (r\sigma) - \frac{1}{4} \partial_r \left( \frac{\Delta}{\sigma^2} \right)
\]

(33)

(34)

(35)

(36)

We concentrate our attention to the F-equations since the G-equations are just the complex conjugate of these. Then eq.(25) becomes

\[
\hat{V} f_1 + 2^{-1/2} \hat{L} f_2 = 0
\]

(37)

and eq.(26) can similarly be written
\[ r^2 \sigma^2 \dot{W} f_2 + 2^{-1/2} \hat{L} f_1 = 0. \] (38)

By utilizing that
\[ \dot{W} = -\frac{\sqrt{\Lambda}}{2r \sigma^2} V^\dagger (r \sqrt{\Delta}) \] (39)

eq(38) can be brought to the more useful form
\[ r \sqrt{\Delta} V^\dagger (r \sqrt{\Delta} f_2) - 2^{-1/2} \hat{L} f_1 = 0. \] (40)

Define
\[ f_1 = R_{1/2}(r) S_{1/2}(\theta) e^{i\omega t} e^{im \phi}, \quad g_1 = R_{-1/2}(r) S_{1/2}(\theta) e^{i\omega t} e^{im \phi} \] (41)
\[ f_2 = R_{-1/2}(r) S_{-1/2}(\theta) e^{i\omega t} e^{im \phi}, \quad g_2 = R_{1/2}(r) S_{-1/2}(\theta) e^{i\omega t} e^{im \phi}. \] (42)

where \( m \) takes on positive and negative integer values. Then the two equations above separate into
\[ \dot{V} R_{1/2} - \lambda R_{-1/2} = 0 \] (43)
\[ r^2 \sigma^2 \dot{W} R_{-1/2} + \lambda R_{1/2} = 0 \] (44)

and
\[ \hat{L} S_{-1/2} + \sqrt{2} \sqrt{\lambda} S_{1/2} = 0 \] (45)
\[ \hat{L}^\dagger S_{+1/2} - \sqrt{2} \sqrt{\lambda} S_{-1/2} = 0 \] (46)

where \( \lambda \) is the separation constant. Define \( P_{-1/2} = \sqrt{2} \sqrt{\Delta} R_{-1/2} \) and \( P_{1/2} = R_{1/2} \) and let \( \sqrt{2} \lambda \to \lambda \). The radial equations then become
\[ r \sqrt{\Delta} V P_{1/2} = \lambda P_{-1/2} \] (47)
\[ r \sqrt{\Delta} V^\dagger P_{-1/2} = \lambda P_{1/2}. \] (48)

Further define two functions \( Z_\pm \) by \( Z_\pm = P_{1/2} \pm P_{-1/2} \). Then a one dimensional wave equation in \( r^* \) follows
\begin{equation}
\left( \frac{d^2}{dr^*} + \omega^2 \right) Z_{\pm} = V_{\pm} Z_{\pm}, \tag{49}
\end{equation}

where

\begin{equation}
V_{\pm} = \lambda^2 \frac{\Delta}{r^2 \sigma^4} \pm \lambda \frac{d}{dr^*} \left( \frac{\sqrt{\Delta}}{r \sigma^2} \right). \tag{50}
\end{equation}

Solving this equation for \( Z_{\pm} \) we can easily obtain the solutions for \( P_{\pm 1/2} \). When \( a = 0 \) and \( Q = 0 \) this expression reduces to the one found in the Schwarzschild geometry. According to standard scattering theory \( V_- \) and \( V_+ \) will give rise to the same reflection and transmission coefficients \cite{11}. It is easily seen that \( V_\pm \) vanishes on \( r = r_+ \) in the extremal limit when \( a < 1 \) and is finite outside. When \( a = 1 \) it is finite on \( r = r_+ \). When \( a > 1 \) the potential is singular at the horizon in the extremal limit. This is exactly what happens in the scalar field case. In addition to this singularity structure the potential above exhibits an additional pole at \( r = r_+ \) for \( a = \sqrt{3} \). This pole did not appear neither in the point particle case nor in the dynamics of the real scalar field. It is interesting to note that the value \( a = \sqrt{3} \) arises in Kaluza-Klein dilaton black hole solutions \cite{3,5,4} and string solutions \cite{4}.

The equations for the \( S_{\pm 1/2} \) functions are independent of \( a \). It follows that these functions are identical to the corresponding functions in the Schwarzschild geometry, i.e. spherical harmonics \( Y_m^l(\theta, \phi) \). The regularity of these functions on the unit sphere implies that \( \lambda = l(l + 1) \). Hence, for the lowest angular momentum mode \( l = 0 \) it follows that \( V_\pm \) vanishes. This feature is also present in the massless point particle case but it is absent in the dynamics of a real massless scalar field. An immediate consequence of the vanishing of \( V_\pm \) is that the outgoing Hawking radiation for \( a > 1 \) holes in the extremal state apparently gives rise to a divergent change of the gravitational mass \( M^H \) relative to infinity

\begin{equation}
\left. \frac{dM^H}{dt} \right|_{l=0} = \int \frac{\Gamma(l = 0, \omega) \omega d\omega}{e^{\omega/T^H} + 1} \to \infty. \tag{51}
\end{equation}

Since \( V_\pm = 0 \) this implies that the transmission coefficient \( \Gamma(l = 0, \omega) \) equals unity in the extremal limit and seems to indicate a catastrophic deradiation of these configurations. We interpret the result to mean that we are outside the domain of validity of the semiclassical
approximation and that we are no longer free to ignore the back reaction on the geometry. However, it would seem that we could retreat sufficiently far away from extremality for the thermodynamical results to be valid and still obtain a very large rate for the outgoing Hawking radiation. Naively this would seem to eventually result in a negative gravitational mass of the black hole when $M^H$ is sufficiently small.\footnote{Note that although the potential (50) vanishes also for a Reissner-Nordström black hole there is no catastrophic mass-loss in that case. This is because the temperature of those holes vanishes in the extremal limit.}

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