\[ N = 2 \] Topological Yang-Mills Theories

and Donaldson’s Polynomials I

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Abstract

The \( N = 2 \) topological Yang-Mills and holomorphic Yang-Mills theories on compact Kähler surfaces are reexamined using a different realization of \( N = 2 \) supersymmetry for auxiliary fields. This new approach allows us to realize the non-algebraic part of Donaldson’s polynomials as well as the algebraic part. We calculate Donaldson’s polynomials on \( H^{2,0}(S, ) \oplus H^{0,2}(S, ) \) for every Kähler surface with \( p_g \geq 1 \).

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1. Introduction

The $N = 2$ super Yang-Mills theory on arbitrary four-manifolds can be twisted to define $N = 1$ topological Yang-Mills (TYM) theory which realize Donaldson’s polynomial invariants of smooth four-manifolds [1] as correlation functions [2].

Very recently, Witten determined the Donaldson invariants of compact Kähler surfaces with $p_g \geq 1$ using some standard properties of $N = 1$ super Yang-Mills theory [3].

Some time ago, one of the author proposed $N = 2$ TYM theory on compact Kähler surfaces [4]. His construction was directly based on the $N = 1$ TYM theory utilizing the complex and Kähler structures of the moduli space of anti-self-dual (ASD) connections. He also proposed $N = 2$ holomorphic Yang-Mills (HYM) theory whose partition function is a generating functional of certain Donaldson invariants [5], adapting the two-dimensional construction of Witten to Kähler surfaces [6]. However, it turns out that the both theories realize the algebraic part of the Donaldson invariants, analogous to the invariants defined by J. Li [7], rather than all the Donaldson invariants.

In this paper, we reexamine $N = 2$ TYM and HYM theories using a different realization of $N = 2$ (global) supersymmetry for auxiliary fields. This allows us to determine the non-algebraic part of Donaldson’s polynomial on $H^{2,0}(S, ) \oplus H^{0,2}(S, )$ and show that all the relevant informations are contained in the algebraic part. The algebraic part of Donaldson’s polynomials will be studied in the forthcoming paper [8].

This paper is organized as follows; in sect. 2, we construct new $N = 2$ TYM theory after brief reviews on the previous approach. In sect. 3, we calculate the non-algebraic part of Donaldson’s polynomials for every Kähler surfaces with $p_g \geq 1$ based on $N = 2$ HYM theory. In sect. 4, we briefly discuss $N = 1$ supersymmetric perturbation analogous to the mass term in [3].

We now turn to the statement of mathematical result of this paper.

Statement of Our Result

Let $M$ be a simple simply connected 4-manifold with $b_2^+(M) \geq 3$. Let $q_d(M)$ denote the $SU(2)$ polynomials on $H_0(M, ) \oplus H_2(M, )$, where $d = 4k - \frac{3}{2}(1 + b_2^+ )$. Kronheimer and Mrowka [9] have announced that the Donaldson series $q(M) = \sum_d q_d(X)/d!$ is given by

$$q(M) = e^{Q/2} \sum_{i=1}^{n} a_i e^{K_i},$$

(1.1)

where $Q$ is the intersection form, regarded as a quadratic function ($Q \in \text{Sim}^2(H^2(M, ))$), of $M$, $K_i \in H_2(M)$ denote the Kronheimer-Morowka (K-M) classes and $a_i$ are non-zero
rational numbers. Since $Q$ is a homeomorphism invariant, any relevant information for smooth structures is contained in $K_i$ and $\alpha_i$.

Recently, Brusse proved that the K-M classes $K_i$ are of the type $(1, 1)$, i.e. $K_i \in H^{1,1}(X)$, for a simple simply connected algebraic surfaces $X$ with $p_g(X) \geq 1$ [10], using the pureness of the Donaldson invariants for simply connected algebraic surfaces [11]. Then, one of his Corollary that for all $\omega^{0,2} \in H^{0,2}(X)$

$$q(\omega^{0,2} + \omega^{2,0}) = q_0 \int \omega^{0,2} \wedge \omega^{2,0},$$

where $q_0$ is Donaldson’s polynomial of degree zero, can be immediately followed from (1.1). This result says that the algebraic part of Donaldson’s polynomials, i.e. the polynomials defined by Jun-Li [7], contains as much information as the full polynomials for a simple simply connected algebraic surface.

Very recently, Witten showed (at the physical level of rigor) that all compact Kähler surfaces with $p_g \geq 1$ are of simple type [3]. His completely explicit formula for the full polynomials also imply that all K-M classes are of the type $(1, 1)$, in fact, they are linear combinations of components of the canonical divisor [12].

Our explicit calculation will show (at the physical level of rigor) that for every compact Kähler surface $S$ with $p_g(S) \geq 1$ and for all $\omega^{0,2} \in H^{0,2}(S)$

$$q(\omega^{0,2} + \omega^{2,0}) = q_0 \int \omega^{0,2} \wedge \omega^{2,0}.$$  

(1.3)

This says the algebraic part of the polynomials determines the full polynomials for all compact Kähler surfaces. This also implies that every Kähler surface with $p_g \geq 1$ is of simple type. Our result, then, supports Witten’s claim.

Our computation is based on the two global symmetries $U$ and $R$ of $N = 2$ TYM theory [4] and the equivalence of $N = 2$ TYM theory to $N = 2$ HYM theory [5]. We would like to emphasize here that our computation uses a very elementary technique of standard quantum field theory.

2. $N = 2$ Supersymmetry

We consider a compact Kähler surface $S$ with Kähler form $\omega$ and $p_g \geq 1$. Let $E$ be a complex vector bundle over $S$ with the restriction of structure group to $SU(2)$. We write $\mathfrak{e}$ for the Lie algebra bundle associated with $E$ by adjoint representation. We introduce a
positive definite quadratic form \((a, b) = -\text{Tr} ab\) on \(\cdot\), where \(\text{Tr}\) denotes the trace in the 2-dimensional representation. Then, the bundle \(E\) is classified by the instanton number;
\[
k = c_2(E), \quad S = \frac{1}{8\pi^2} \int_S F \wedge F \in .
\]

Let \(\mathcal{A}\) denote the space of all connections and \(\mathcal{G}\) be the group of gauge transformations. Picking a complex structure \(J\) on \(S\), one can introduce a complex structure \(J_A\) to \(\mathcal{A}\) as well as to \(\mathcal{A}/\mathcal{G}\),
\[
J_A \delta A = J \delta A, \quad \delta A \in T\mathcal{A},
\]
by identifying \(T^{1,0} \mathcal{A}\) and \(T^{0,1} \mathcal{A}\) in \(T \mathcal{A} = T^{1,0} \mathcal{A} \oplus T^{0,1} \mathcal{A}\) with the \(E\) valued \((1, 0)\)-forms and \((0, 1)\)-forms on \(S\), respectively.

The global supersymmetry operator \(\delta_\psi\) of the \(N = 1\) topological Yang-Mills theory can be interpreted as the exterior (covariant) derivative on \(\mathcal{A}/\mathcal{G}\) \([2][13]\). Using the complex structures \(J\) and \(J_A\), we find the \(N = 2\) transformation laws for the basic multiplet \((A', A'', \psi, \bar{\psi}, \varphi)\) \([4]\);
\[
\begin{align*}
\mathbf{s} A' &= -\psi, & \mathbf{s}\psi &= 0, \\
\bar{\mathbf{s}} A' &= 0, & \bar{\mathbf{s}}\psi &= -i \partial_A \varphi, & \bar{\mathbf{s}}\varphi &= 0, \\
\mathbf{s} A'' &= 0, & \mathbf{s}\bar{\psi} &= -i \bar{\partial}_A \varphi, & \mathbf{s}\varphi &= 0, \\
\bar{\mathbf{s}} A'' &= -\bar{\psi}, & \bar{\mathbf{s}}\bar{\psi} &= 0,
\end{align*}
\]
where \(\psi \in \Omega^{1,0}_E\), \(\bar{\psi} \in \Omega^{0,1}\) and \(\varphi \in \Omega^{0,0}_E\). We introduce two global quantum numbers (or ghost numbers) \((U, R)\), which assign \((1, 1)\) to \(\mathbf{s}\) and \((1, -1)\) to \(\bar{\mathbf{s}}\). The above transformation laws play a central role in constructing \(N = 2\) TYM theories.

The commutation relations of the fermionic symmetry generators \(\mathbf{s}, \bar{\mathbf{s}}\) are
\[
\mathbf{s}^2 = 0, \quad (\mathbf{s}\bar{\mathbf{s}} + \bar{\mathbf{s}}\mathbf{s}) = i d_A \varphi = -i \delta_\varphi, \quad \bar{\mathbf{s}}^2 = 0,
\]
where \(\delta_\varphi\) is the generator of a gauge transformation with infinitesimal parameter \(\varphi\). Thus \(\{\mathbf{s}, \bar{\mathbf{s}}\} = 0\) precisely on the \(\mathcal{G}\) invariant space or if it acts on \(\mathcal{G}\)-invariant functionals of \(A', A'', \psi, \bar{\psi}, \varphi\). Geometrically, \(\bar{\mathbf{s}}\) can be interpreted as the operator of Dolbeault cohomology group on \(\mathcal{A}/\mathcal{G}\). More precisely, \(\bar{\mathbf{s}}\) is the operator of Dolbeault cohomological analogue of \(\mathcal{G}\)-equivariant cohomology\(^1\) of \(\mathcal{A}\). This can be roughly described as follows; we let \(\text{Lie}(\mathcal{G})\)

\(^1\) Note that the transformations (2.2) are slightly different, \textit{in conventions}, from those in \([4]\). Here we follow the usual conventions in physics literatures. The relations between the BRST cohomology of topological Yang-Mills theory and the equivariant cohomology are explained very clearly in \([6]\).
be the Lie algebra of $\mathcal{G}$. The $\mathcal{G}$ action on $\mathcal{A}$ is generated by vector fields $V(\varphi) = \sum a \varphi^a V_a$ where we pick an orthonormal basis $T_a$ of $\text{Lie}(\mathcal{G})$. Note that $s$ and $\bar{s}$ can be represented as

$$s = -\sum_i \psi^i \frac{\partial}{\partial A^i} + i \sum_{i,a} \varphi^a V^i_a \frac{\partial}{\partial \psi^i},$$

$$\bar{s} = -\sum_i \bar{\psi}^i \frac{\partial}{\partial A^i} + i \sum_{i,a} \varphi^a V^i_a \frac{\partial}{\partial \bar{\psi}^i},$$

(2.4)

where $\bar{i}, i$ are the local holomorphic and anti-holomorphic indices of $\mathcal{A}$. Then, we find

$$s \bar{s} + \bar{s} s = -i \mathcal{L}_{V(\varphi)},$$

(2.5)

where $\mathcal{L}_{V(\varphi)}$ is the Lie derivative with respect to $V_a$. Thus, $\{s, \bar{s}\} = 0$ on the $\mathcal{G}$-invariant subspace $\Omega_0^*(\mathcal{A})$ of $\Omega^*(\mathcal{A}) \otimes \text{Fun}(\text{Lie}(\mathcal{G}))$. Physically, $\Omega_0^*(\mathcal{A})$ is the fixed point locus of $s$ and $\bar{s}$ symmetry or the BRST invariant configuration space.

2.1. The Old Construction

In the previous construction, we introduced an anti-ghost $B$, a self-dual two form $B = B^{2,0} + B^{0,2} + B^0 \omega \in \Omega^2(E)$ in the adjoint representation, with $(U, R) = (-2, 0)$. Then (2.3) naturally leads us to multiplet $(B, \chi, -\bar{\chi}, H)$ with transformation laws

$$sB = -i \chi, \quad s\chi = 0,$$

$$s\bar{B} = i \bar{\chi}, \quad s\bar{\chi} = 0,$$

$$s\bar{\chi} = H - \frac{i}{2} [\varphi, B], \quad sH = -\frac{i}{2} [\varphi, \chi],$$

$$\bar{s} \chi = H + \frac{i}{2} [\varphi, B], \quad \bar{s} H = -\frac{i}{2} [\varphi, \bar{\chi}].$$

(2.6)

The ghost numbers of the various fields are given by

$$\begin{array}{cccccccc}
\text{Fields} & A' & A'' & \psi & \bar{\psi} & \varphi & B & \chi & \bar{\chi} & H \\
\text{U Number} & 0 & 0 & 1 & 1 & 2 & -2 & -1 & -1 & 0 \\
\text{R Number} & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0
\end{array}$$

(2.7)

The action of $N = 2$ TYM theory can be written in the form \footnote{The action of $N = 1$ TYM theory can be written as $S = -i \bar{\psi} W [2][14]$. The relation between $N = 1$ and $N = 2$ theories can be most conveniently understood with an analogy to the exterior derivative $d = \partial + \bar{\partial}$. An exact real $(p, p)$-form $\alpha = d\beta$ on a compact Kähler manifold can be written as $\alpha = \frac{i}{2} \bar{\partial} \bar{\partial} \gamma = i \bar{\partial} \partial \gamma$ for some $(p - 1, p - 1)$-form $\rho$. The $U$ and $R$ numbers, for example, correspond to $(r + s)$ and $(r - s)$, respectively, for an $(r, s)$-form.}

$$S_{\text{Old}} = \frac{s \bar{s} - \bar{s} s}{2} \mathcal{B}_T = \frac{s \bar{s} - \bar{s} s}{2} \left(- \frac{1}{\hbar^2} \int \text{Tr} B \wedge \ast F - \frac{1}{\hbar^2} \int \text{Tr} \chi \wedge \ast \bar{\chi} \right).$$

(2.8)
Note that $V$ has $(U, R) = (-2, 0)$, such that the action has $(U, R) = (0, 0)$. We find that
\[
S_{old} = \frac{1}{\hbar^2} \int_S \text{Tr} \left[ -H^{2,0} \wedge * (H^{0,2} + iF^{0,2}) - H^{0,2} \wedge * (H^{2,0} + iF^{2,0}) + i\chi^{2,0} \wedge * \bar{\partial}_A \psi \\
+ i\bar{\chi}^{0,2} \wedge \partial_A \chi + i[\varphi, \chi^{2,0}] \wedge * \bar{\chi}^{0,2} + i[\varphi, \chi^{0,2}] \wedge * \bar{\chi}^{2,0} - \frac{i}{2} B^{2,0} \wedge * \bar{\partial}_A \partial_A \varphi \\
+ \frac{i}{2} B^{0,2} \wedge * \partial_A \partial_A \varphi + \frac{i}{2}[\varphi, B^{2,0}] \wedge * [\varphi, B^{0,2}] - \left( 2H^0 (H^0 + i f) \right) \right] (2.9)
\]
where $f = \frac{i}{2} \Lambda F$ and $\Lambda$ is adjoint to the wedge multiplication of $\omega$.

We can integrate out $H^{2,0}$, $H^{0,2}$ and $H^0$ from the action by setting $H = -\frac{i}{2} F$ or by the Gaussian integral, which leads to modified transformation laws
\[
s\bar{\chi}^{2,0} = -\frac{i}{2} F^{2,0} - \frac{1}{2}[\varphi, B^{2,0}], \quad s\chi^{2,0} = -\frac{i}{2} F^{2,0} + \frac{1}{2}[\varphi, B^{2,0}], \\
s\bar{\chi}^{0,2} = -\frac{i}{2} F^{0,2} - \frac{1}{2}[\varphi, B^{0,2}], \quad s\chi^{0,2} = -\frac{i}{2} F^{0,2} + \frac{1}{2}[\varphi, B^{0,2}], \\
s\chi^0 = -\frac{i}{2} [\varphi, B^0], \quad s\chi^0 = -\frac{i}{2} f - \frac{1}{2}[\varphi, B^0].
\] (2.10)

One can see that the locus of $s$ and $\bar{s}$ fixed points in the above transformations is precisely the space of ASD connections. By the fixed point theorem of Witten for a quantum field theory with global fermionic symmetry [15], the path integral localizes to an integral over the moduli space $\mathcal{M}$ of ASD connections.

For the details how $N = 2$ TYM theory (or TYM theory in general) realizes the Donaldson invariants, we refer the reader to [4] ([2][14]). We will show in the subsequent section that the old $N = 2$ TYM theory realizes the algebraic part of Donaldson’s polynomials only.

3. New Construction

3.1. $N = 2$ Topological Yang-Mills Theory

In the new construction, we will impose different transformation laws for anti-ghost multiplets. We introduce a commuting anti-ghost $B^0 \in \Omega^0(E)$ in the adjoint representation with $(U, R) = (-2, 0)$. Then (2.3) leads us to multiplet $(B^0, i\chi^0, -i\bar{\chi}^0, H^0)$ with transformation laws
\[
sB^0 = -i\chi^0, \quad s\chi^0 = 0, \\
s\bar{B}^0 = i\bar{\chi}^0, \quad s\bar{\chi}^0 = 0, \\
s\chi^0 = H^0 - \frac{1}{2}[\varphi, B^0], \quad sH^0 = -\frac{i}{2}[\varphi, \chi^0], \\
s\bar{\chi}^0 = H^0 + \frac{1}{2}[\varphi, B^0], \quad s\bar{H}^0 = -\frac{i}{2}[\varphi, \bar{\chi}^0].
\] (3.1)
We also introduce an anti-commuting anti-ghost $\chi^{2,0} \in \Omega^{2,0}_{(E)}$ with $(U, R) = (-1, 1)$ and an anti-commuting anti-ghost $\bar{\chi}^{0,2} \in \Omega^{0,2}_{(E)}$ with $(U, R) = (-1, -1)$ and the transformation laws

$$
\begin{align*}
\mathbf{s}\chi^{2,0} &= 0, \\
\mathbf{s}H^{2,0} &= -i[\varphi, \chi^{2,0}], \\
\mathbf{\bar{s}}\chi^{2,0} &= H^{2,0}, \\
\mathbf{\bar{s}}H^{2,0} &= 0, \\
\mathbf{s}\bar{\chi}^{0,2} &= H^{0,2}, \\
\mathbf{s}H^{0,2} &= 0, \\
\mathbf{\bar{s}}\bar{\chi}^{0,2} &= 0, \\
\mathbf{\bar{s}}H^{0,2} &= -i[\varphi, \bar{\chi}^{0,2}].
\end{align*}
$$

(3.2)

One can easily check that these satisfy the commutation relations (2.3).

**Action Functional**

The most general form of new $N = 2$ supersymmetric action is

$$
S = i\mathbf{s} \bar{\nabla} + i\mathbf{\bar{s}} V + \frac{(\mathbf{ss} - \mathbf{\bar{s}s})}{2} B,
$$

(3.3)

where $V$ and $\bar{\nabla}$ should be $\mathbf{\bar{s}}$ and $\mathbf{s}$ closed quantities with $(U, R)$ numbers $(-1, 1)$ and $(-1, -1)$, respectively. One finds the following unique choices

$$
\begin{align*}
\bar{\nabla} &= -\frac{1}{\hbar^2} \int_{S} \text{Tr} \bar{\chi}^{0,2} \wedge *F^{2,0}, \\
V &= -\frac{1}{\hbar^2} \int_{S} \text{Tr} \chi^{2,0} \wedge *F^{0,2}, \\
B &= -\frac{1}{\hbar^2} \int_{S} \text{Tr} (B^0 f + \alpha \chi^0 \bar{\chi}^0) \omega^2 - \frac{2\beta}{\hbar^2} \int_{S} \text{Tr} \chi^{2,0} \wedge *\chi^{0,2},
\end{align*}
$$

(3.4)

where $\alpha, \beta = 0, 1, 0$. For $\alpha = \beta = 1$, we find

$$
\begin{align*}
S &= \frac{1}{\hbar^2} \int_{S} \text{Tr} \left[ -2H^{2,0} \wedge *H^{0,2} - iH^{2,0} \wedge *F^{0,2} - iH^{0,2} \wedge *F^{2,0} + 2i[\varphi, \chi^{2,0}] \wedge *\chi^{0,2} \\
&+ i\chi^{2,0} \wedge *\partial A \bar{\psi} + i\chi^{0,2} \wedge *\partial A \psi - \left( 2H^0 H^0 + 2iH^0 f - 2i[\varphi, \chi^0] \right) \chi^0 \\
&+ \frac{1}{2} B^0 \Lambda ((i\partial A \partial A - i\partial A \partial A) \varphi - 2[\psi, \bar{\psi}]) - \frac{1}{2}[\varphi, B^0][\varphi, B^0] \\
&- i\chi^0 \Lambda \partial A \psi - i\chi^0 \Lambda \partial A \bar{\psi} \right] \frac{\omega^2}{2!} \right].
\end{align*}
$$

(3.5)

We can integrate out $H^{2,0}$, $H^{0,2}$ and $H^0$ from the action by setting $H^{2,0} = -\frac{i}{2} F^{2,0}$, $H^{0,2} = -\frac{i}{2} F^{0,2}$ and $H^0 = -\frac{i}{2} f^0$, or by the Gaussian integral, which leads to modified transformation laws

$$
\begin{align*}
\mathbf{s}\bar{\chi}^{0,2} &= -\frac{i}{2} F^{0,2}, \\
\mathbf{s}\chi^{2,0} &= -\frac{i}{2} F^{2,0}, \\
\mathbf{s}\chi^0 &= -\frac{i}{2} f - \frac{1}{2} [\varphi, B^0], \\
\mathbf{\bar{s}}\chi^0 &= -\frac{i}{2} f + \frac{1}{2} [\varphi, B^0].
\end{align*}
$$

(3.6)
One see that the locus of $s$ and $\bar{s}$ fixed points in the above transformations is precisely the space of ASD connections. Now we can rewrite the action as

$$S = \frac{1}{\hbar^2} \int_S \left[ -\frac{1}{2} F_{\mu\nu} \wedge * F^{\mu\nu} + i \chi^{2,0} \wedge * \bar{\partial}_A \bar{\psi} + i \chi^{0,2} \wedge * \partial_A \psi - 2 i [\varphi, \chi^{2,0}] \wedge * \bar{\chi}^{0,2} \\
- \left( \frac{1}{2} f^2 - 2 i [\varphi, \chi^{0}] \bar{\chi}^{0} - i \bar{\chi}^{0} \Lambda \bar{\partial}_A \bar{\psi} - i \chi^{0} \Lambda \partial_A \psi - \frac{1}{2} [\varphi, B^0] [\varphi, B^0] \right)
+ \frac{1}{2} B_0^0 \Lambda ((i \partial_A \bar{A} - i \bar{\partial}_A \partial_A) \varphi - 2 [\psi, \bar{\psi}]) \right] \frac{\omega^2}{2!}.$$  

(3.7)

One can easily check that this new theory shares almost all the properties with the old theory studied in [4]. The only important difference between the two theories is that $\chi^{2,0}$ ($\chi^{0,2}$) is no longer $s$-exact ($\bar{s}$-exact) in the new construction.

**Observables**

Geometrically, the $s$ and $\bar{s}$ correspond to the holomorphic and anti-holomorphic part of the exterior derivative on $\mathcal{A}/\mathcal{G}$ if they act to the gauge invariant quantities. In particular, $s$ operator is the Dolbeault cohomology operator. Since $\mathcal{A}/\mathcal{G}$ does not have the Kähler structure in general, the Hodge decomposition theorem cannot be applied. However, if we restrict $\mathcal{A}/\mathcal{G}$ to the moduli space $\mathcal{M}$ of ASD connections, an $\bar{s}$ closed quantity is automatically $s$ closed since $\mathcal{M}$ has the Kähler structure.

The candidates to the nontrivial topological observables depending on $H^2(S, \cdot)$ are

$$\omega^{2,0} = \frac{1}{8\pi^2} \text{Tr} \left( \psi \wedge \bar{\psi} \right) \wedge \omega^{0,2},$$
$$\bar{\omega}^{0,2} = \frac{1}{8\pi^2} \text{Tr} \left( \bar{\psi} \wedge \bar{\psi} \right) \wedge \omega^{2,0},$$
$$\omega^{1,1} = \frac{1}{4\pi^2} \text{Tr} \left( \varphi F^{1,1} + \psi \wedge \bar{\psi} \right) \wedge \omega^{1,1},$$

(3.8)

where $\omega^{p,q} \in H^{p,q}(S, \cdot)$ and we generally denote $\bar{\omega}^{r,s}$ as an $(r, s)$-form on $\mathcal{A}/\mathcal{G}$. Note that $\bar{\omega}^{r,s}$ carries $(U, R) = (r + s, r - s)$.

The only quantity which is both $s$ and $\bar{s}$ invariant is $\omega^{1,1}$. Note that $\omega^{0,2}$ is $s$ invariant while $s$ invariant modulo the equation of motion $\bar{\partial}_A \bar{\psi} = 0$ of $\chi^{2,0}$. On the other hand $\omega^{2,0}$ is $s$ invariant while $\bar{s}$ invariant modulo the equation of motion $\partial_A \psi = 0$ of $\bar{\chi}^{0,2}$. Thus one can not use $\omega^{2,0}$ and $\omega^{0,2}$ as observables. However, it is important to note that $\omega^{2,0}$ and $\bar{\omega}^{0,2}$ are both $s$ and $\bar{s}$ invariant if they are restricted to the moduli space $\mathcal{M}$ to where the path integral will be eventually localized. Since the Donaldson invariants can be viewed as the cohomology ring of $\mathcal{M}$, we should include $\omega^{2,0}$ and $\bar{\omega}^{0,2}$ to realize the full invariants.
There is a nice method to deal with on-shell invariant quantities, explained in [16]. To use \( \tilde{\omega}^{0,2} \) and \( \tilde{\omega}^{2,0} \), we should change the transformation laws as

\[
\bar{s} \chi^{0,2} = \frac{\hbar^2}{4\pi^2} \varphi \omega^{0,2}, \quad s \chi^{2,0} = \frac{\hbar^2}{4\pi^2} \varphi \omega^{2,0}.
\]

(3.9)

and add the terms

\[-\frac{1}{8\pi^2} \text{Tr} (\psi \wedge \psi) \wedge \omega^{0,2} - \frac{1}{8\pi^2} \text{Tr} (\bar{\psi} \wedge \bar{\psi}) \wedge \omega^{2,0},\]

(3.10)

to the action (3.7). Then the action is both \( s \) and \( \bar{s} \) invariant with the modified transformation laws of (3.9). Since \( \varphi \) is both \( s \) and \( \bar{s} \) closed, the commutation relations (2.3) between \( s \) and \( \bar{s} \) remain unchanged.

Although we can do the same thing in the old theory, it does violate \( s^2 = \bar{s}^2 = 0 \). This is why the old theory realizes only the algebraic part of Donaldson’s polynomials, defined by algebraic cycles which are Poincaré dual to elements of \( H^{1,1}(S) \).

At this point, it is sufficient to consider only \( N = 1 \) part of the supersymmetry. We choose \( \bar{s} \) symmetry. Since \( \tilde{\omega}^{0,2} \) is \( \bar{s} \) invariant and \( \tilde{\omega}^{2,0} \) is \( \bar{s} \) invariant modulo \( \chi^{0,2} \) equation of motion, it is sufficient to change \( \chi^{0,2} \) transformation law only

\[
\bar{s} \chi^{0,2} = \frac{\hbar^2}{4\pi^2} \varphi \omega^{0,2},
\]

(3.11)

and add

\[-\frac{1}{8\pi^2} \int_S \text{Tr} (\psi \wedge \psi) \wedge \omega^{0,2} \equiv -\tilde{\omega}^{2,0},\]

(3.12)

to the action (3.7);

\[
S' = \frac{1}{\hbar^2} \int_S \left[ -\frac{1}{2} F^{2,0} \wedge * F^{0,2} + i \chi^{2,0} \wedge * \partial_A \bar{\psi} + i \chi^{0,2} \wedge * \partial_A \psi - 2i [\varphi, \chi^{2,0}] \wedge * \chi^{0,2} \\
- \left( \frac{1}{2} f^2 - 2i [\varphi, \chi^0] \chi^0 - i \chi^0 \Lambda \partial_A \psi - i \chi^0 \Lambda \partial_A \bar{\psi} - \frac{1}{2} [\varphi, B^0][\varphi, B^0] \right) \right] - \frac{1}{8\pi^2} \int_S \text{Tr} (\psi \wedge \psi) \wedge \omega^{0,2}.
\]

(3.13)

**Fermionic Zero Modes**

Important properties common to both old and new \( N = 2 \) topological Yang-Mills theories are the roles of fermionic zero-modes. It can be conveniently interpreted using the language of holomorphic vector bundles. It is well known that an ASD connection \( A \) endows \( E \) with a holomorphic structure \( \mathcal{E}_A \) of given topological type. Let \( \text{End}_0(\mathcal{E}_A) \) be
the trace-free endomorphism bundle of $\mathcal{E}_A$. It turns out that zero-modes of $\bar{\chi}_0$, $\bar{\psi}$ and $\bar{\chi}^{0,2}$ define elements of $H^0(\text{End}_0(\mathcal{E}_A))$, $H^1(\text{End}_0(\mathcal{E}_A))$ and $H^2(\text{End}_0(\mathcal{E}_A))$, respectively. The formal complex dimension of the moduli space $\mathcal{M}$ is $(-h^{0,0} + h^{0,1} - h^{0,2})$, where $h^{0,p} = \dim H^p(\text{End}_0(\mathcal{E}_A))$.

Since the fermionic zero-modes of $(\bar{\chi}_0, \bar{\psi}, \bar{\chi}^{0,2})$ carry the $U$-charge $(-1, 1, -1)$, the half of the net violation $\Delta U/2$ of the $U$-number in the path integral measure is equal to the formal complex dimension. It is important to note that there is no net $R$ number violation in the path integral measure [4]. We assume, throughout this paper, that there exist the zero-modes pairs of $\psi$ and $\bar{\psi}$ only. Then the moduli space is a smooth Kähler manifold with complex dimension $d = 4k - 3(1 - h^{0,1} + h^{0,2})$, identical to the number of $\bar{\psi}$ zero-modes.

It is convenient to introduce quantum operators $\hat{U}$ and $\hat{R}$ such that

\begin{align}
\hat{U} \chi^0 &= u^{-1} \chi^0, & \hat{U} \psi &= u \psi, & \hat{U} \chi^{2,0} &= u^{-1} \chi^{2,0}, & \hat{U} B^0 &= u^{-2} B^0, \\
\hat{U} \bar{\chi}^0 &= u^{-1} \bar{\chi}^0, & \hat{U} \bar{\psi} &= u \bar{\psi}, & \hat{U} \bar{\chi}^{0,2} &= u^{-1} \bar{\chi}^{0,2}, & \hat{U} B^0 &= B^0, \\
\hat{R} \chi^0 &= r \chi^0, & \hat{R} \psi &= r \psi, & \hat{R} \chi^{2,0} &= r \chi^{2,0}, & \hat{R} \varphi &= u^2 \varphi, \\
\hat{R} \bar{\chi}^0 &= r^{-1} \bar{\chi}^0, & \hat{R} \bar{\psi} &= r^{-1} \bar{\psi}, & \hat{R} \bar{\chi}^{0,2} &= r^{-1} \bar{\chi}^{0,2}, & \hat{R} \varphi &= \varphi.
\end{align}

(3.14)

Then the action $S$ is invariant under the transformations generated by $\hat{U}$ and $\hat{R}$. Now the fermionic part $\mathcal{D}X_f$ of path integral measure, after integrating out every non-zero modes, reduces to

$$\mathcal{D}\hat{X}_f = \prod_i^d \psi_i \bar{\psi}_i,$$

(3.15)

which transforms, under $\hat{U}$ and $\hat{R}$, as

$$\mathcal{D}\hat{X}_f \to \mathcal{D}\hat{X}_f u^{-2d}.$$  

(3.16)

Thus, the expectation value of topological observables

$$\left\langle \prod_i^n \tilde{\omega}^{r_i, s_i} \right\rangle = \frac{1}{\text{vol}(\mathcal{G})} \int \mathcal{D}X e^{-S} \prod_i^r \tilde{\omega}^{r_i, s_i},$$

(3.17)

evaluated with the action $S$ vanishes unless (see [3] for related analysis)

$$\sum_{i=1}^n (r_i + s_i) = 2d \quad \text{and} \quad \sum_{i=1}^n (r_i - s_i) \quad \Rightarrow \quad \sum_{i=1}^r (r_i, s_i) = (d, d).$$

(3.18)

This selection rule is, more or less, identical to the statement that the Donaldson invariants are pure Hodge type of $(d, d)$ [11].
Clearly, the action \( S' = S - \tilde{\omega}^{2,0} \) is not invariant under the transformations generated by \( \hat{U} \) and \( \hat{R} \). However, the path integral measure, after integrating out every non-zero-modes, is identical to the one defined by the action \( S \), since the additional term does not change the equations for zero-modes. Therefore the partition function \( <1'> \) for the action \( S' \) can be interpreted as the following expectation value;

\[
<1'> = \left( \sum_{n=0}^{\infty} \frac{1}{n!} (\tilde{\omega}^{2,0})^n \right),
\]

evaluated in the theory with the action \( S \). Clearly, this is non-zero only for \( d = 0 \) and identical to \( <1> \).

### 3.2. \( N = 2 \) Holomorphic Yang-Mills Theory

We now turn to \( N = 2 \) HYM theory. Since the terms which are proportional to the Kähler form are identical in the old and new actions \( S_{old} \) and \( S \), we can repeat the procedure in [5] to obtain \( N = 2 \) HYM theory. It is convenient to choose delta function gauge by setting \( \alpha = \beta = 0 \) in (3.4). Now the action for \( N = 2 \) TYM theory is

\[
S = \frac{1}{\hbar^2} \int_s \text{Tr} \left[ -iH^{2,0} \wedge \ast F^{0,2} - iH^{0,2} \wedge \ast F^{2,0} + i\chi^{2,0} \wedge \ast \bar{\partial}_A \bar{\psi} + i\tilde{\chi}^{0,2} \wedge \ast \partial_A \psi \\
- \left( 2iH^0 \phi - i\chi^0 \Lambda \partial_A \psi - i\tilde{\chi}^0 \Lambda \partial_A \bar{\psi} + \frac{1}{2} B_0 \Lambda \left( (i\partial_A \bar{\partial}_A - i\partial_A \partial_A) \varphi - 2[\psi, \bar{\psi}] \right) \right) \right].
\]

Then, the action of \( N = 2 \) HYM theory becomes

\[
S_H = \frac{1}{\hbar^2} \int_s \text{Tr} \left[ -iH^{2,0} \wedge \ast F^{0,2} - iH^{0,2} \wedge \ast F^{2,0} + i\chi^{2,0} \wedge \ast \bar{\partial}_A \bar{\psi} + i\tilde{\chi}^{0,2} \wedge \ast \partial_A \psi \\
- \frac{1}{8\pi^2} \int_s \text{Tr} (i\phi F + \psi \wedge \bar{\psi}) \wedge \omega - \frac{\varepsilon}{8\pi^2} \int_s \text{Tr} (\phi^2) \frac{\omega^2}{2!}.
\]

This is equivalent to the action studied in [5]. The difference is that \( \chi^{2,0} \) and \( \tilde{\chi}^{0,2} \) are no longer BRST exact in this new setting.

Since \( N = 2 \) HYM theory has the same \( N = 2 \) supersymmetry and the same topological observables as those of \( N = 2 \) TYM theory, we can repeat the same procedure to deal with the on-shell invariant quantities. We, once again, consider the \( \bar{s} \) symmetry only. Adding (3.12) to the action, we have

\[
S'_H = \frac{1}{\hbar^2} \int_s \text{Tr} \left[ -iH^{2,0} \wedge \ast F^{0,2} - iH^{0,2} \wedge \ast F^{2,0} + i\chi^{2,0} \wedge \ast \bar{\partial}_A \bar{\psi} + i\tilde{\chi}^{0,2} \wedge \ast \partial_A \psi \\
- \frac{1}{4\pi^2} \int_s \text{Tr} (i\phi F + \psi \wedge \bar{\psi}) \wedge \omega - \frac{\varepsilon}{8\pi^2} \int_s \text{Tr} (\phi^2) \frac{\omega^2}{2!}
\]

\[
- \frac{1}{8\pi^2} \int_s \text{Tr} (\psi \wedge \psi) \wedge \omega^{0,2},
\]

10
where the change of the transformation of $\tilde{s}$ as (3.11) is understood. Of course, we can add (3.12) to the action $S$ given by (3.20) in the beginning and then define the mapping to $N = 2$ HYM theory. Both procedures give identical results.

The partition function $Z(\varepsilon)_d$ of the HYM theory with action $S_H$ is a generating functional

$$
Z(\varepsilon)_d = \sum_{r,s} \frac{\varepsilon^s}{r!s!} (\tilde{\omega}^r \Theta^s) + \text{exponentially small terms},
$$

where

$$
\tilde{\omega} = \frac{1}{4\pi^2} \int_S \text{Tr} \left( i\varphi F + \psi \wedge \bar{\psi} \right) \wedge \omega,
$$

$$
\Theta = \frac{1}{8\pi^2} \int_S \text{Tr} \left( \varphi^2 \right) \frac{\omega^2}{2!}.
$$

Note that the partition function $Z'(\varepsilon)_d$ with action $S'_H$ is identical to $Z(\varepsilon)_d$;

$$
Z'(\varepsilon)_d = \left\langle \sum_{n=0}^{\infty} \frac{1}{n!} (\tilde{\omega}^{2,0})^n \right\rangle_H = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r,s} \frac{\varepsilon^s}{r!s!} (\tilde{\omega}^{2,0})^n (\tilde{\omega}^r \Theta^s) + \ldots
$$

$$
= \sum_{r,s} \frac{\varepsilon^s}{r!s!} (\tilde{\omega}^r \Theta^s) + \ldots
$$

$$
= Z(\varepsilon)_d.
$$

**Simple Calculation**

Now we will determine the Donaldson polynomial invariants on $H^{0,2} \oplus H^{2,0}$.

We consider the partition function $Z'(\varepsilon)_d$ of HYM theory with action $S'_H$. It is convenient to integrate $H^{2,0}, H^{0,2}, \lambda^{2,0}, \bar{\lambda}^{2,0}$ out, which leads to the delta function constraints $\prod_x \delta(F^{2,0}(x)) \delta(F^{0,2}(x)) \delta(\partial_A \psi(x)) \delta(\bar{\partial}_A \bar{\psi}(x))$. Due to the $N = 2$ supersymmetry, there are no loop corrections. Thus the partition function $Z'(\varepsilon)_d$ becomes

$$
Z'(\varepsilon)_d = \frac{1}{\text{vol}(G)} \int_{T^A.1} DA' \ DA'' \ D\psi \ D\bar{\psi} \ D\varphi
$$

$$
\times \exp \left( \frac{1}{4\pi^2} \int_S \text{Tr} \left( i\varphi F^{1,1} + \psi \wedge \bar{\psi} \right) \wedge \omega + \frac{\varepsilon}{8\pi^2} \int_S \frac{\omega^2}{2!} \text{Tr} \varphi^2 \right)
$$

$$
\times \exp \left( \frac{1}{8\pi^2} \int_S \text{Tr} \left( \psi \wedge \bar{\psi} \right) \wedge \omega^{0,2} \right).
$$

It is more convenient to represent $\tilde{\omega}^{0,2}$ and $\tilde{\omega}^{2,0}$ by

$$
\tilde{\omega}^{0,2} = \frac{1}{8\pi^2} \int_{\Gamma} \text{Tr} \left( \bar{\psi} \wedge \bar{\psi} \right), \quad \tilde{\omega}^{2,0} = \frac{1}{8\pi^2} \int_{\Gamma} \text{Tr} \left( \psi \wedge \psi \right),
$$

(3.27)
where $\Gamma$ and $\bar{\Gamma}$ denote homology cycles Poincaré dual to $\omega^{2,0}$ and $\omega^{0,2}$, respectively. Now we want to determine the expectation value $\langle (\tilde{\omega}^{0,2})^m \rangle_H$ evaluated in the $N = 2$ HYM theory with action $S'_H$

$$
\langle (\tilde{\omega}^{0,2})^m \rangle_H = \sum_{n=0}^{\infty} \frac{1}{n!} \langle (\tilde{\omega}^{0,2})^n (\tilde{\omega}^{2,0})^n \rangle_H \\
= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{r,s} \frac{r^s}{r!s!} \langle (\tilde{\omega}^{0,2})^m (\tilde{\omega}^{2,0})^n \omega^r \Theta^s \rangle + \cdots \\
= \frac{1}{m!} \sum_{r,s} \frac{r^s}{r!s!} \langle (\tilde{\omega}^{0,2})^m (\tilde{\omega}^{2,0})^n \omega^r \Theta^s \rangle + \cdots \\
= \frac{1}{m!} \langle (\tilde{\omega}^{0,2})^m (\tilde{\omega}^{2,0})^m \rangle_H .
$$

Thus, we consider

$$
\langle (\tilde{\omega}^{0,2} \tilde{\omega}^{2,0})^m \rangle_H = \frac{1}{\text{vol}(G)} \int_{T_{A^{1,1}}} DA' DA'' D\psi D\bar{\psi} D\varphi \\
\times \exp \left( \frac{1}{4\pi^2} \int_S \text{Tr} (i\varphi F^{1,1} + \psi \wedge \bar{\psi}) \wedge \omega + \frac{\varepsilon}{8\pi^2} \int_S \frac{\omega^2}{2} \text{Tr} \varphi^2 \right) \\
\times \left( \frac{1}{8\pi^2} \int_{\Gamma} \text{Tr} \psi \wedge \psi + \frac{1}{8\pi^2} \int_{\Gamma} \text{Tr} \bar{\psi} \wedge \bar{\psi} \right)^m .
$$

Since $\psi$ and $\bar{\psi}$ are coupled as free fields, we have the trivial propagator

$$
< \psi_i^a(x) \bar{\psi}_j^b(y) > = -i4\pi^2 \varepsilon_{ij} \delta^{ab} \delta^4(x - y).
$$

Upon performing the $\psi$ and $\bar{\psi}$ integral, we see (3.29) is equivalent to

$$
\langle (\tilde{\omega}^{0,2} \tilde{\omega}^{2,0}) \rangle_H = \frac{1}{\text{vol}(G)} \int_{T_{A^{1,1}}} DA' DA'' D\psi D\bar{\psi} D\varphi \\
\times \exp \left( \frac{1}{4\pi^2} \int_S \text{Tr} (i\varphi F^{1,1} + \psi \wedge \bar{\psi}) \wedge \omega + \frac{\varepsilon}{8\pi^2} \int_S \frac{\omega^2}{2} \text{Tr} \varphi^2 \right) \times m!(\Gamma \cdot \bar{\Gamma})^m
$$

where $\Gamma \cdot \bar{\Gamma} = \int_S \omega^{0,2} \wedge \omega^{2,0}$ denotes the intersection number. Thus we have the following factorization;

$$
\langle (\tilde{\omega}^{0,2} \tilde{\omega}^{2,0})^m \rangle_H = \sum_{r,s} \frac{r^{s+d-2m}}{r!s!} \langle (\tilde{\omega}^{0,2} \tilde{\omega}^{2,0})^m \omega^r \Theta^s \rangle + \cdots \\
= m! \sum_{r,s} \frac{r^{s+d-2m}}{r!s!} \langle \omega^r \Theta^s \rangle (\Gamma \cdot \bar{\Gamma})^m + \cdots ,
$$

---

3 Actually, the above equation (3.31) should contain a group theoretical factor due to the trace. Since we are dealing with $SU(2)$ case, the omitted factor is $\text{dim}(SU(2))^m = 3^m$. It seems to us that mathematicians usually omit this term.
that is,

$$\sum_{r,s} \langle (\tilde{\omega}^{0,2} \tilde{\omega}^{2,0})^m \tilde{\omega}^r \Theta^s \rangle = m! \sum_{r,s} \langle \tilde{\omega}^r \Theta^s \rangle (\tilde{\Gamma} \cdot \Gamma)^m. \quad (3.33)$$

If \( d = 2m \), we have

$$\langle (\tilde{\omega}^{0,2} \tilde{\omega}^{2,0})^m \rangle = m!(\tilde{\Gamma} \cdot \Gamma)^m < 1 >. \quad (3.34)$$

Then

$$\langle (\tilde{\omega}^{0,2} + \tilde{\omega}^{2,0})^d \rangle = \frac{2m!}{m!m!} \langle (\tilde{\omega}^{0,2} \tilde{\omega}^{2,0})^m \rangle = \frac{2m!}{m!}(\tilde{\Gamma} \cdot \Gamma)^m < 1 >. \quad (3.35)$$

Equivalently

$$q_d(\omega^{2,0} + \omega^{0,2}) = q_0 \frac{2m!}{m!} \left( \int_S \omega^{2,0} \wedge \omega^{0,2} \right), \quad (3.36)$$

where \( q_0 = < 1 > \). Thus we have

$$q(\omega^{2,0} + \omega^{0,2}) = q_0 \int_S \omega^{2,0} \wedge \omega^{0,2}. \quad (3.37)$$

4. Introducing More Fields

Now we turn back to \( N = 2 \) TYM theory discussed in sect. 2. There is yet another possibility of introducing more fields \( B^{2,0} \) and \( B^{0,2} \) with transformation laws

$$sB^{0,2} = -i\eta^{0,2}, \quad s\eta^{0,2} = 0,$$

$$\tilde{s}B^{0,2} = 0, \quad \tilde{s}\eta^{0,2} = [\varphi, B^{0,2}],$$

$$sB^{2,0} = 0, \quad s\eta^{2,0} = -[\varphi, B^{2,0}],$$

$$\tilde{s}B^{2,0} = i\eta^{2,0}, \quad \tilde{s}\eta^{2,0} = 0. \quad (4.1)$$

Now we can add the following terms

$$-\frac{1}{\hbar^2} \int_S \text{Tr} \left( B^{2,0} \wedge *F^{0,2} + B^{0,2} \wedge *F^{2,0} + \eta^{0,2} \wedge *\eta^{2,0} \right) \quad (4.2)$$

to \( B \) in (3.4). Then, the new \( B \) becomes

$$B = -\frac{1}{\hbar^2} \int_S \text{Tr} \left( B \wedge *F + \alpha \chi \wedge *\chi \right), \quad (4.3)$$

where we combine various fields into self-dual two-forms

$$B = \sqrt{2} B^{2,0} + \sqrt{2} B^{0,2} + B^0 \omega,$$

$$\chi = \sqrt{2} \chi^{2,0} + \sqrt{2} \eta^{0,2} + \chi^0 \omega,$$

$$\bar{\chi} = \sqrt{2} \bar{\eta}^{2,0} + \sqrt{2} \bar{\chi}^{0,2} + \bar{\chi}^0 \omega. \quad (4.4)$$
Using
\[
\frac{(s\bar{s} - \bar{s}s)}{2} \text{Tr} \left( B^{2,0} \wedge *F^{0,2} + B^{0,2} \wedge *F^{2,0} \right) = 0,
\]
and
\[
\frac{(s\bar{s} - \bar{s}s)}{2} \text{Tr} \chi^{0,2} \wedge *\chi^{2,0} = \text{Tr} \left( -2i[\varphi, \chi^{0,2}] \wedge *\chi^{2,0} - 2[\varphi, B^{2,0}] \wedge *[\varphi, B^{0,2}] \right),
\]
the new action becomes, for \( \alpha = 1 \),
\[
S = \frac{1}{\hbar^2} \int_S \left[ -H^{2,0} \wedge * (H^{0,2} + iF^{0,2}) - H^{0,2} \wedge * (H^{2,0} + iF^{2,0}) + i\chi^{2,0} \wedge *\partial_A \bar{\psi} 
\right.
+ i\chi^{0,2} \wedge *\partial_A \psi + 2i[\varphi, \chi^{2,0}] \wedge *\chi^{0,2} + 2i[\varphi, \eta^{0,2}] \wedge *\bar{\eta}^{2,0} + 2[\varphi, B^{2,0}] \wedge *[\varphi, B^{0,2}]
\left. - \left( 2H^0 (H^0 + i f) - i\chi^0 \Lambda \partial_A \bar{\psi} - i\chi^0 \Lambda \partial_A \psi - 2i[\varphi, \chi^0] \chi^0 - \frac{1}{2} [\varphi, B^0]^2 \right) + \frac{1}{2} B^0 \Lambda \left( (i\partial_A \bar{\partial}_A - i\partial_A \partial_A) \varphi - 2[\psi, \bar{\psi}] \right) \right] \frac{\omega^n}{2!}.
\]
(4.7)

This looks almost identical to the old action \( S_{old} \). Note that the newly introduced fields \( B^{2,0}, B^{0,2} \) and \( \bar{\eta}^{2,0}, \eta^{0,2} \) are completely decoupled from other fields.

We can change \( \chi^{0,2} \) transformation law
\[
\bar{s} \chi^{0,2} = \frac{\hbar^2}{4\pi^2} \varphi \omega^{0,2},
\]
(4.8)
and add
\[
- \frac{1}{8\pi^2} \int_S \text{Tr} (\psi \wedge \bar{\psi}) \wedge \omega^{0,2}
\]
(4.9)
to the action (4.7). One can also add to the action the following term to the action \( S \) without breaking the \( \bar{s} \) invariance;
\[
- \bar{s} \frac{i}{2\hbar^2} \int_S \text{Tr} B^{2,0} \wedge *\chi^{0,2},
\]
(4.10)
which becomes
\[
\int_S \text{Tr} \left[ \frac{1}{\hbar^2} \bar{\eta}^{2,0} \wedge *\chi^{0,2} - \frac{i}{4\hbar^2 \pi^2} \varphi B^{2,0} \wedge *\omega^{2,0} \right].
\]
(4.11)

Note that adding above terms to the action do not invariant under the transformations generated by \( \hat{U} \) and \( \hat{R} \). We would just like to mention that the added terms (4.9) and (4.11) are analogous to the Mass term of the \( N = 1 \) chiral multiplet discussed in [3].
Then the corresponding action functional of HYM theory is

\[
S_H = \frac{1}{\hbar^2} \int_S \left[ -\frac{1}{2} F^{2,0} \wedge * F^{0,2} + i \chi^{2,0} \wedge * \partial_A \psi + i \chi^{0,2} \wedge * \partial_A \psi + 2i [\varphi, \chi^{2,0}] \wedge * \chi^{0,2} \\
+ \eta^{2,0} \wedge * \chi^{0,2} + 2i [\varphi, \eta^{0,2}] \wedge * \eta^{2,0} + 2 [\varphi, B^{2,0}] \wedge * [\varphi, B^{0,2}] \right] - \frac{i}{4\pi^2} \int_S \varphi B^{2,0} \wedge * \omega^{2,0} \\
- \frac{1}{8\pi^2} \int_S \text{Tr} (\psi \wedge \psi) \wedge \omega^{0,2} - \frac{1}{4\pi^2} \int_S \text{Tr} (i \varphi F + \psi \wedge \psi) \wedge \omega - \frac{\varepsilon}{8\pi^2} \int_S \text{Tr} (\varphi^2) \omega^2 \frac{\omega^2}{2!}.
\]

It remains to see whether this kind of $N = 1$ supersymmetric perturbation will be useful in our formalism.

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References