Ultra violet-Renormalon Calculus

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Abstract

We consider large-order perturbative expansions in QED and QCD. The coefficients of the expansions are known to be dominated by the so called ultraviolet (UV) renormalons which arise from inserting a chain of vacuum-polarization graphs into photonic (gluonic) lines. In large orders the contribution is associated with virtual momenta $k^2$ of order $Q^2 e^n$ where $Q$ is external momentum, $e$ is the base of natural logs and $n$ is the order of perturbation theory considered. To evaluate the UV renormalon we develop formalism of operator product expansion (OPE) which utilizes the observation that $k^2 \gg Q^2$. When applied to the simplest graphs the formalism reproduces the known results in a compact form. In more generality, the formalism reveals the fact that the class of the renormalon-type graphs is not well defined. In particular, graphs with extra vacuum-polarization chains are not suppressed. The reason is that while inclusion of extra chains lowers the power of $\ln k^2$ their contribution is enhanced by combinatorial factors.

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1 Introduction

Large-order behaviour of perturbative expansions has been studied for about forty years starting from the seminal paper by Dyson [1] (for a review and collection of papers see Ref. [2]). Although a lot of insight has been gained some practical aspects of the issue in case of QCD have not been clarified so far and attracted for this reason a renewed attention recently [3, 4, 5, 6, 7, 8]. In this note we address ourselves to the problems of calculation and of calculability of the so called ultraviolet renormalon [9, 10, 11, 12].

The importance of the ultraviolet renormalon rests on the observation that it dominates large orders both in QCD and QED [10]. To be more specific we shall have in mind perturbative calculations of the famous ratio $R$:

$$R(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$$

as function of energy $\sqrt{s}$ and its Euclidean counterpart $\Pi(Q^2)$,

$$Q^2 \frac{d\Pi(Q^2)}{dQ^2} = \frac{Q^2}{12\pi^2} \int_0^\infty \frac{R(s) \, ds}{(s + Q^2)^2}$$

as function of Euclidean momentum $Q$. The quantity $Q^2 d\Pi(Q^2)/dQ^2$ is represented as an expansion in QCD coupling constant $\alpha_s(Q^2)$:

$$Q^2 \frac{d\Pi(Q^2)}{dQ^2} = (\text{parton model}) \times \sum_{n=0}^{n=\infty} a_n \alpha_s^n(Q^2),$$

where first three coefficients have been calculated explicitly [13].

Now, the generic behaviour of the expansion coefficients at large $n$ looks as

$$a_n \xrightarrow{n \to \infty} Kn^\gamma \frac{n!}{S^n}$$

where $K, \gamma$ and $S$ are constants and actually there exists a variety of sources for factorial growth resulting in different $S$. The constant $S$ takes the smallest absolute value in case of ultraviolet renormalon:

$$S_{UV \, \text{renorm}} = -\frac{1}{b_0}$$
where $b_0$ is the first coefficient in the corresponding $\beta$-function:

$$Q^2 \frac{d}{dQ^2} \left( \alpha(Q^2) \right) = -b_0 \alpha^2(Q^2) - b_1 \alpha^3(Q^2) + \ldots$$

(6)

Note that in QCD $b_0$ is positive and the series (4) is sign alternating while in $U(1)$ case $b_0$ is negative. The sign alternation allows for the Borel summation of the series and from this point of view there exists an important difference between QCD and $U(1)$. In this paper we are concerned however with calculating of $a_n$ at large $n$, with no attempt of summation of the perturbative series being made.

The main technical tool we are using is the operator product expansion, or expansion in inverse powers of $k^2$ where $k$ is the virtual momentum flowing through the gluonic line (see Fig. 1). The idea of such calculations is outlined first by Parisi [11] and is based on the observation that effectively $k^2 \gg Q^2$ in case of the ultraviolet renormalon. More specifically, we shall see later that

$$(k^2)_{\text{eff}} \sim Q^2 \epsilon^n.$$  

(7)

where $n$ is the order of perturbative expansion and $\epsilon$ is the base of natural logs. We will elaborate the idea of the expansion in $Q^2/k^2$ on explicit examples.

Our primary objective in this paper is to work out a scheme for explicit evaluation of renormalon contributions. In sect 2 we will address ourselves to computation of the simplest renormalon-type graphs represented on Fig. 1. In $U(1)$ case such calculations have been performed in a number of papers [14, 15, 16]. In particular we reproduce the results of Beneke [16] who evaluated the contribution of the renormalon chain directly, without use of OPE. What we would like to add here is a simplification of the scheme. Moreover, the generalization to the QCD case is imminent. As the next step we generalize the procedure to the case of two renormalon chains, i.e. three-loop skeleton graphs (sect 3). We will demonstrate that this three-loop contribution dominates over the two-loop one.

The consideration of graphs with one or two renormalon chains naturally brings us to the problem of calculability of the renormalon contribution in general (sect 4). By this we mean the problem of identifying the graphs which control the coefficient in front of the factorial (see eq. (4)). In particular, one may increase the number of renormalon chains and the question is whether this contribution is suppressed or not.  

Superficially, the more complicated graphs can be neglected since at least one extra factor of $\alpha$ is not accompanied in this case by a log. The use of the operator product

1In case of QCD because of the complexity of the gauge-fixing problems even the set of the graphs delegated to each type is not so well defined a priori. We shall see, however, that the technique developed allows to circumvent this problem as well.
expansion mentioned above allows to analyse the problem in a general way. The result turns unexpected: the contribution of the graphs with extra chains is not suppressed and the class of the “renormalon-type” graphs is not well defined at all.

The reason is that we have in fact two large parameters, namely, \( \ln k^2/Q^2 \) and the order of the perturbative expansion, \( n \). Moreover, effectively \( \ln k^2/Q^2 \sim n \), as is implied by eq. (31). While the coefficient in front of the highest power of the log can be found in a straightforward way – and this is the essence of the renormalization group analysis – it turns to be not enough to evaluate the asymptotic of \( a_n \). The other contributions lose the log factors but gain extra factors of \( n \) because of combinatorics.

2 Operator product expansion. Two-loop example.

Let us consider polarization operator \( \Pi_{\mu\nu} \),

\[
\Pi_{\mu\nu} = i \int dx e^{ix\phi} \langle 0 | T \{ j_\mu(x), j_\nu(0) \} | 0 \rangle = (q_\nu q_\mu - g_{\mu\nu} q^2) \Pi(Q^2), \quad Q^2 = -q^2
\]

of electromagnetic current \( j_\mu \) in the simplified model with \( N_f \) fermionic fields

\[ j_\mu = \sum_q Q_q \bar{q} \gamma_\mu q \]

where \( Q_q \) are corresponding electric charges. Strong interactions are mediated by \( U(1) \) gluonic field \( B_\mu \) and the Lagrangian of the model is

\[ L = -\frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \sum_q \bar{q} \gamma_\mu i \partial_\mu q + g B^{\mu} j_\mu^1 + e A^\mu j_\mu \]

where \( g \) and \( e \) are the strong and electromagnetic couplings, respectively, and

\[ j_\mu^1 = \sum_q \bar{q} \gamma_\mu q. \]

The sum electric charges of fermions is equal to zero to avoid a mixing between fields \( A_\mu \) and \( B_\mu \) due to fermionic loops,

\[ \sum_q Q_q = 0 \]

The simplest renormalon-type graphs are depicted in Fig. 1 and we will explain the basic features of the technique on this example.

The sum of these graphs can be cast into the following form:

\[ \epsilon^2 \Pi_{\mu\nu}(q) \epsilon^{\mu}_{(1)} \epsilon^{\nu}_{(2)} = -\int \frac{d^4k}{(2\pi)^4} \frac{g^2(k^2)}{k^2} \langle T | \gamma^* | T | \gamma^* \rangle \]
where we have made the Euclidian rotation in the integration over a gluon momentum \( k_\mu \) and have introduced polarization vectors \( e^{\mu}_{(1,2)} \) for initial and final virtual photons. The running coupling \( g^2(k^2) \) sums up vacuum bubble insertions and has the form

\[
g^2(k^2) = g^2(Q^2) \left( 1 + b_0 \frac{g^2(Q^2)}{4\pi} \ln \frac{k^2}{k_0^2} \right)^{-1}
\]  

(14)

with first \( \beta \)-function coefficient

\[
b_0 = -\frac{N_f}{3\pi}.
\]

(15)

The matrix element \( \langle \gamma^*|T|\gamma^* \rangle \) represents the forward amplitude of gluon-photon scattering and the operator \( T \) is

\[
T = \int dx \ e^{ikx} T \{ j^1_\mu(x), j^{1\mu}(0) \}.
\]

(16)

By assumption – to be checked a posteriori – the momentum \( k \) flowing through the gluon line is much larger than external momentum \( Q \). Then it is logical to start by expanding in inverse powers of \( k \). Thus we come to consider OPE for \( T \)-product of two gluonic currents \( j^1_\mu \)

\[
T = \int dx \ e^{ikx} T \{ j^1_\mu(x), j^{1\mu}(0) \} = \sum c_i(k) O_i(0),
\]

(17)

where \( O_i \) are local operators. We will discuss this operator expansion in more detail in the next section. Here we find coefficients \( c_i \) in the tree approximation using the Schwinger background field technique [17] (for a review, see Ref. [18]). In this technique lines in the graph of Fig. 2 are understood as propagators in external electromagnetic and gluonic fields and the operator \( T \) takes the form:

\[
T = -\sum g \int dx \langle x|\bar{q}(X)\gamma^\mu \frac{1}{p^2 + m^2} \gamma^\mu q(X)|0 \rangle + (k \rightarrow -k)
\]

(18)

Note that within the framework considered operators of the coordinate \( X_\mu \) and of momentum \( p_\mu \) are introduced. Moreover

\[
[X_\mu, p_\nu] = -ig_{\mu\nu}, \quad [X_\mu, X_\nu] = 0, \quad [p_\mu, p_\nu] = 0
\]

(19)

The operator \( P_\mu \) corresponds to

\[
P_\mu = p_\mu + gB_\mu + \epsilon A_\mu Q_y
\]

(20)

and the matrix element (18) is taken over eigenstates of operator \( X_\mu \),

\[
X_\mu |x\rangle = x_\mu |x\rangle.
\]

(21)
The construction of OPE reduces to expansion of eq. (18) in powers of $P_\mu$. The first and second terms of the expansion vanish upon averaging over directions of 4-vector $k_\mu$ and use of equations of motion, $i\not\!{\partial}q = 0$. Thus, the expansion starts from operators of dimension six,

$$T = \sum_q \frac{2}{3k^4} \not\!q \gamma^\mu [P^\nu [P_\nu P_\mu]] q + O(k^{-6}).$$

(22)

Substituting the commutator

$$[P_\mu, P_\nu] = ig G_{\mu\nu} + i\epsilon F_{\mu\nu} Q_q$$

we get

$$T = - \frac{2}{3k^4} \left( \epsilon \partial^\nu F_{\nu\mu} \sum_q Q_q \not\!q \gamma^\mu q + g D^\nu G_{\nu\sigma} \sum_q \not\!q \gamma^\sigma q \right) + O(k^{-6}).$$

(24)

The next step is to evaluate the matrix element $\langle \gamma^* | T | \gamma^* \rangle$. The part of $T$ containing $D^\nu G_{\nu\mu}$ will contribute only on three-loop level and will be considered in the next section. As for the part of $T$ containing $\partial^\nu F_{\nu\mu}$ it immediately factorizes into

$$\langle \gamma^* | T | \gamma^* \rangle = - \frac{2}{3k^4} \left( \langle \gamma^* | \epsilon \partial^\nu F_{\nu\mu} | 0 \rangle \langle 0 | j^\mu | \gamma^* \rangle + \langle \gamma^* | j^\mu | 0 \rangle \langle 0 | \epsilon \partial^\nu F_{\nu\mu} | \gamma^* \rangle \right)$$

(25)

where $j^\mu$ is the electromagnetic current (see eq. (9)). The matrix element of $\partial^\nu F_{\nu\mu}$ is trivial:

$$\langle 0 | \epsilon \partial^\nu F_{\nu\mu} | \gamma^* \rangle = - \epsilon \left( q^2 \epsilon^{(1)}_{\mu} - q_\mu (q e^{(1)}) \right).$$

(26)

The matrix element $\langle \gamma^* | j_\mu | 0 \rangle$ is given by the well-known one-loop graph (see Fig. 3) and is equal to

$$\langle \gamma^* | j_\mu | 0 \rangle = \frac{4eN_f}{3} \frac{1}{Q_q} \left( q^2 \epsilon^{(2)}_{\mu} - q_\mu (q e^{(2)}) \right) \int \frac{d^4 p}{(2\pi)^4} p^\mu \frac{k^2}{Q^2} \epsilon^{(2)}_{\mu} - q_\mu (q e^{(2)})$$

(27)

where $\langle Q_q^2 \rangle$ is the averaged square of electric charge,

$$\langle Q^2_q \rangle = \frac{1}{N_f} \sum_q Q^2_q.$$

(28)

Note that the integral over the fermionic loop has been evaluated with logarithmic accuracy. The upper limit of integration, $p^2 \sim k^2$, is implied by our OPE construction while the lower bound, $p^2 \sim Q^2$, arises from account in the integrand for the external momentum $Q$.

Substituting the result (25) for $\langle \gamma^* | T | \gamma^* \rangle$ into (13) we come to

$$\Pi(Q^2) = \frac{N_f \langle Q^2_q \rangle}{144\pi^4} Q^2 \int_{k^2/Q^2}^{\infty} \frac{dk^2}{k^4} \ln \frac{k^2}{Q^2} \cdot g^2(k^2)$$

(29)
Here $Q^2 = -q^2$ and the integration over Euclidean $k^2$ runs over $k^2 > Q^2$. We are interested in the expansion of $\Pi(Q^2)$ in $\alpha_1(Q^2) = g^2(Q^2)/4\pi$. Expanding eq. (14) we get

$$g^2(k^2) = 4\pi \alpha_1(Q^2) \sum_{n=0}^{\infty} \left(-b_0 \alpha_1(Q^2)\right)^n \ln^n \frac{k^2}{Q^2}$$  \hspace{1cm} (30)$$

where $b_0$ is the first coefficient in the $\beta$-function. Performing the integration over $k^2$ in the r.h.s. of eq. (29),

$$Q^2 \int_{k^2 < Q^2} \frac{dk^2}{k^4} \ln^n \frac{k^2}{Q^2} = n!,$$  \hspace{1cm} (31)

we arrive at the final expression for the UV renormalon contribution:

$$\Pi(Q^2) = -\frac{N_f \langle Q^2 \rangle}{36\pi^3 b_0} \sum_{n=1}^{\infty} \left(-b_0 \alpha_1(Q^2)\right)^n n!.$$  \hspace{1cm} (32)

Differentiating this expression we get

$$Q^2 \frac{d\Pi(Q^2)}{dQ^2} = \frac{N_f \langle Q^2 \rangle}{12\pi^2} \left(-\frac{1}{4\pi b_0}\right) \sum_{n=2}^{\infty} \frac{n-1}{n} n! (-b_0 \alpha_1)^n.$$  \hspace{1cm} (33)

In other words, at large $n$ the coefficients in $\alpha_1$ expansion (for the definition see eq. (3)) are given by

$$a_n \underset{n \to \infty}{\sim} -\frac{1}{4\pi b_0} (-b_0)^n \cdot n!.$$  \hspace{1cm} (34)

This result coincides with that of [16].

To summarize, in this section we utilized the operator product expansion to evaluate the results for the graphs in Fig. 1 which were found earlier in $U(1)$ case. The advantage of the operator expansion is not only the compactness of the calculation but also ensuring automatically the gauge invariance so that the generalization to QCD case is straightforward and reduces to the change in $b_0$. Indeed since the operator product expansion is based on a set of gauge invariant operators the only dependence on $\ln k^2/Q^2$ arises through the use of eq. (30) (or its two-loop generalization) and we do not need even to specify the gauge fixing or the class of graphs involved explicitly. This technical point is especially important in case of non-abelian gauge theories.

### 3 Comments on operator product expansion.

In this section we combine a few simple remarks on the use of OPE exemplified in the preceding section.
First of all, let us note that our calculation justifies the assumption about dominance of large $k^2$. Indeed, the integrals (31) over $k^2$ are saturated at large $n$ by a saddle point at $k^2_{eff} = Q^2 e^n$ (see eq. (7)). The width of the range of $k^2$ which contribute to the saddle point integration is

$$\frac{k^2 - k^2_{eff}}{k^2_{eff}} \sim \frac{1}{\sqrt{n}}$$

(35)

It means, in particular, that precise value of the lower limit in integrals (29), (31) is of no importance at large $n$.

Thus, expansion in $Q^2/k^2$ is fully justified. Moreover, it is clear from the calculations above that it is the dimension of operators $O_i$ (see eq. (17)) which is most important. Namely, for an operator of dimension $d$ the contribution to $a_n$ is proportional to

$$Q^{d-4} \int_{k^2 \sim Q^2} \frac{dk^2}{k^{d-2}} \ln^n \frac{k^2}{Q^2} = \frac{n!}{((d-2)/2)^{n+1}}.$$  

(36)

In the preceding section we dealt with operators of dimension $d = 6$. Operators of higher dimensions give rise contributions which are suppressed by powers of $(d-4)/2$ and can be safely neglected.

However, the question naturally arises on the role of operators of dimension four. Although such operators did not appear in the tree approximation of the preceding section they show up via loop corrections. It is worth emphasizing therefore that these operators are not relevant to asymptotic of $a_n$.

Indeed, matrix elements of $d = 4$ operators over virtual photons have the form

$$\langle \gamma^* | O_i^{d=4} | \gamma^* \rangle = A_i e^2 \left( q^2 (e^{(1)} e^{(2)}) - (qe^{(1)})(qe^{(2)}) \right)$$

(37)

where $A_i$ are dimensionless and the corresponding contributions to $\Pi(Q^2)$ are

$$\Delta_i \Pi \sim \int dk^2 A_i c_i^{d=4}(k^2) g^2(k^2)$$

(38)

where $c_i^{d=4}$ are OPE coefficients, $c_i^{d=4} \sim 1/k^2$. To get rid of the apparent ultraviolet divergence in (38) one needs to consider instead of $\Pi(Q^2)$ the quantity $Q^2 \cdot d\Pi(Q^2)/dQ^2$ which contains no ultraviolet uncertainty.

The finiteness of $Q^2 \cdot d\Pi(Q^2)/dQ^2$ implies that the parameters $A_i$ are independent of $Q^2$. In other words, the $d = 4$ part of OPE has zero anomalous dimension. It is remarkable that this conclusion is valid to any order in perturbation theory. This observation implies that contribution of large $k^2$, $k^2 \gg Q^2$, to the integrals (38) drops off from $Q^2 \cdot d\Pi(Q^2)/dQ^2$. 

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Thus $d = 4$ operators do not contribute to UV renormalons and the UV renormalon calculus is defined by $d = 6$ operators. In the preceding section we had one such operator explicitly:

$$O_{Fj} = e \partial^\mu F_{\mu j}$$

(39)

where $j_\mu$ is defined by eq. (9). The calculation of matrix element $\langle \gamma^*|O_{Fj}|\gamma^*\rangle$ (see eq. (25)) can be interpreted as the mixing of $O_{Fj}$ with operator

$$O_{F2} = (e \partial^\mu F_{\mu j})^2,$$

(40)

i.e. with logarithmic accuracy we have

$$O_{Fj}|_{k^2} = O_{Fj}|_{Q^2} + \frac{e N_f (Q^2)}{12\pi^2} \ln \frac{k^2}{Q^2} \cdot O_{F2}|_{Q^2}$$

(41)

where we marked the normalization points of the operators. Since the diagonal anomalous dimensions of the operators $O_{Fj}, O_{F2}$ are vanishing eq. (41) specifies the matrix of the anomalous dimensions completely.

Note also that the log in the r.h.s. of eq. (41) results in enhancement of $a_n$ by factor of $n$ (see eq. (31)) and this enhancement is crucial for the consistency of our calculation. The point is that in one-loop order the operator $O_{F2}$ appears in the OPE (17) not only through mixing with $O_{Fj}$ as in the tree approximation but directly as well. In that case large momentum $k$ flows through all fermionic lines in graph of Fig. 1. However then there is no log factor similar to the one in eq. (41) so that the direct $O_{F2}$ coefficient can be neglected with $1/n$ accuracy.

There are also $d = 6$ four-fermionic operators which contribute to $\Pi(Q^2)$ on three-loop level and these will be discussed in the next section. Here we would like only to note that their logarithmic mixing with the operators $O_{Fj}$ and $O_{F2}$ results in an enhancement of the three-loop skeleton graph.

Finally we note that so far we considered for simplicity one-loop $\beta$-function for $\alpha_1$. Actually, one has to account for the first two coefficients in $\beta$-function. The procedure is rather standard and leads to a non-vanishing constant $\gamma$ in eq. (4). We shall give examples in the next section.

To summarize, the operator product expansion provides with systematic means of calculating asymptotic of the perturbative expansion coefficients $a_n$ associated with renormalon-type graphs.
4 Three-loop example.

In this section we evaluate the renormalon contribution associated with the graphs with two chains of vacuum-polarization insertions, or three-loop skeleton graphs, see examples in Fig. 4. In fact we have already mentioned that the contribution of the operator $D^\mu G_{\nu\mu}$ (see eq. (27)) arises only on a three loop level and now we come to consider this contribution in more detail.

Note that $D^\mu G_{\nu\mu}$ can be substituted by $(-g)\sum \bar{q}\gamma_\mu q$ via equations of motion so that we get for the corresponding part $T_1$ of $T$

$$ T_1 = \frac{2}{3} g^2 \mu_0 O_2, \quad O_1 = \left( \sum_q \bar{q} \gamma_\mu q \right)^2. \tag{42} $$

Four-fermion local operators of dimension $d = 6$ appear in $T$ also due to to the graphs of the type represented in Fig. 5. As far as explicit calculations of these graphs are concerned they are very similar to those used in derivation of QCD sum rules [19] since OPE is exploited in the both cases. The difference is that in the former case it is the momentum carried by the electromagnetic current which is considered to be large while now the large momentum is brought in by the gluonic current.

Omitting details we present the results for four-fermion operators generated by graphs of the type considered (see Fig. 5):

$$ T_2 = -\frac{3g^2}{k^4} \mu_0 O_2, \quad O_2 = \left( \sum_q \bar{q} \gamma_\mu \gamma_5 q \right)^2. \tag{43} $$

Graphs for $T_1$ and $T_2$ are eventually parts of three-loop graphs, see dotted boxes in Fig. 4.

Unlike the operator $O_{Fj}$ (see eq. (39)), operators $O_1, O_2$ have nonzero anomalous dimension and to extract the matrix element $\langle \gamma^a | T | \gamma^a \rangle$ we need to apply the standard machinery of renormalization group. The operator basis of the problem consists of operators $O_1, O_2, O_{Fj}, O_{F2}$ (see eqs. (42,43,39,40)) Their coefficients $c_1, c_2, c_{Fj}, c_{F2}$ are functions of $\mu^2$ where $\mu$ is the normalization point.

The set of renormalization group equations in one-loop approximation looks as

$$ \mu^2 \frac{d}{d\mu^2} c_{F2} = -\frac{N_f}{12\pi^2} \mu_0 c_{Fj}, $$

$$ \mu^2 \frac{d}{d\mu^2} c_{Fj} = -\frac{1}{12\pi^2} \left( c_1 + c_2 \right), $$

$$ \mu^2 \frac{d}{d\mu^2} c_1 = \frac{\alpha_1}{\pi} \left( \frac{2N_f + 1}{3} c_1 + \frac{11}{6} c_2 \right), $$

$$ \mu^2 \frac{d}{d\mu^2} c_2 = \frac{3\alpha_1}{2\pi} c_1. $$
The solution for $c_1, c_2$ has the form

$$c_1(\mu^2) = \frac{1}{1 + (11/4 \pi^2 \gamma_1^2)} \left\{ \left[ c_1(\mu_0^2) + \frac{11}{6 \pi \gamma_1} c_2(\mu_0^2) \right] \left[ \frac{\alpha_1(\mu^2)}{\alpha_1(\mu_0^2)} \right]^{\gamma_1} + \left[ \frac{11}{4 \pi^2 \gamma_1^2} c_1(\mu_0^2) - \frac{11}{6 \pi \gamma_1} c_2(\mu_0^2) \right] \left[ \frac{\alpha_1(\mu^2)}{\alpha_1(\mu_0^2)} \right]^{\gamma_2} \right\},$$

and

$$c_2(\mu^2) = \frac{1}{1 + (11/4 \pi^2 \gamma_1^2)} \left\{ \left[ \frac{3}{2 \pi \gamma_1} c_1(\mu_0^2) + \frac{11}{4 \pi^2 \gamma_1^2} c_2(\mu_0^2) \right] \left[ \frac{\alpha_1(\mu^2)}{\alpha_1(\mu_0^2)} \right]^{\gamma_1} + \left[ -\frac{3}{2 \pi \gamma_1} c_1(\mu_0^2) + c_2(\mu_0^2) \right] \left[ \frac{\alpha_1(\mu^2)}{\alpha_1(\mu_0^2)} \right]^{\gamma_2} \right\},$$

where anomalous dimensions $\gamma_{1,2}$ are

$$\gamma_{1,2} = \frac{1}{\pi b_0} \left( \frac{2N_f + 1}{6} \pm \sqrt{\left( \frac{2N_f + 1}{6} \right)^2 + \frac{11}{4}} \right).$$

Coefficients $c_{Fj}$ and $c_{Fz}$ are obtained then by a simple integration. Initial conditions for $c_i(\mu^2)$ are set by eqs. (42, 43) at $\mu^2 = k^2$,

$$c_1(k^2) = \frac{2}{3} \frac{g^2(k^2)}{k^4}, \quad c_2(k^2) = -3 \frac{g^2(k^2)}{k^4}, \quad c_{Fj}(k^2) = c_{Fz}(k^2) = 0.$$

We need to calculate the value of $c_{Fz}(\mu^2 = Q^2)$ because with logarithmic accuracy the matrix element $\langle \gamma^* | T | \gamma^* \rangle$ is given by

$$\langle \gamma^* | T | \gamma^* \rangle = 2e^2c_{Fz}(\mu^2 = Q^2) q^2 \left[ q^2(e^{(1)}e^{(2)}) - (qe^{(1)})(qe^{(2)}) \right].$$

The result for $c_{Fz}(\mu^2 = Q^2)$ is rather lengthy expression,

$$c_{Fz}(\mu^2 = Q^2) = \frac{N_f (Q^2)}{9} \frac{1}{k^4} \frac{1}{g^2(k^2)} \frac{1}{\pi^2 b_0^2 + (11/4 \gamma_1^2)} \times$$

$$\left\{ \left( \frac{2}{3} - \frac{9}{2 \pi b_0 \gamma_1} - \frac{33}{4 \pi^2 b_0^2 \gamma_1^2} \right) \frac{1}{2 - \gamma_1} \left[ \ln \kappa - \frac{1}{2 - \gamma_1} (1 - \kappa^{(\gamma_1 - 2)}) \right] \right\}$$

$$+ \left\{ -3 + \frac{9}{2 \pi b_0 \gamma_1} + \frac{11}{6 \pi^2 b_0^2 \gamma_1^2} \right\} \frac{1}{2 - \gamma_2} \left[ \ln \kappa - \frac{1}{2 - \gamma_2} (1 - \kappa^{(\gamma_2 - 2)}) \right],$$

where

$$\kappa = \frac{\alpha_1(Q^2)}{\alpha_1(k^2)} = 1 + \alpha_1(Q^2) \ln \frac{k^2}{Q^2}.$$

The polarization operator $\Pi$ is given then by integration over $k$,

$$\Pi = \frac{Q^2}{8 \pi^2} \int_{Q^2}^{\infty} dk^2 g^2(k^2) c_{Fz}(\mu^2 = Q^2).$$
Coefficients of expansion in $\alpha_1$ are defined by integrals of the type

$$Q^2 \int_{Q^2}^{\infty} \frac{dk^2}{k^4} \left[ \alpha_1(k^2) \right]^\varepsilon = \left[ \alpha_1(Q^2) \right]^\varepsilon \sum_{n=0}^{\infty} \frac{\Gamma(n + \delta + \delta_1)}{\Gamma(\delta)} \left[ -b_0 \alpha_1(Q^2) \right]^n \tag{53}$$

where $\delta_1$ arises from account of the second coefficient in the $\beta$-function $b_1$ (see, e.g., [20]) and equals to

$$\delta_1 = -\frac{b_1}{b_0^2}. \tag{54}$$

The final formula for the expansion of $\Pi$ in powers of $\alpha_1(Q^2)$ has the form:

$$\Pi = \frac{N_f \langle Q_s^2 \rangle}{72\pi^2} \frac{1}{\pi^2 b_0^2 + (11/4\gamma_1^2)} \sum_{n=2}^{\infty} \left[ -b_0 \alpha_1(Q^2) \right]^n \times$$

$$\left\{ \left( \frac{2}{3} - \frac{9}{2\pi b_0 \gamma_1} + \frac{33}{4\pi^2 b_0^2 \gamma_1^2} \right) \frac{1}{2 - \gamma_1} \left[ \frac{\Gamma(n + 2 - \gamma_1 + \delta_1)}{\Gamma(3 - \gamma_1)} - \Gamma(n + \delta_1) \right] + \right.$$  

$$\left. \left( -3 + \frac{9}{2\pi b_0 \gamma_1} + \frac{11}{6\pi^2 b_0^2 \gamma_1^2} \right) \frac{1}{2 - \gamma_2} \left[ \frac{\Gamma(n + 2 - \gamma_2 + \delta_1)}{\Gamma(3 - \gamma_2)} - \Gamma(n + \delta_1) \right] \right\} \tag{55}$$

Assuming for the moment $\gamma_1 = \gamma_2 = \delta_1 = 0$, i.e. omitting effects of anomalous dimensions and two-loop $\beta$-function, we get for coefficients $a_n$ defined by eq. (3)

$$a_n \xrightarrow{n \to \infty} = -\frac{7}{72\pi^2 b_0^2} (-b_0)^n (n + 1)! \tag{56}$$

Comparing this result with the eq. (31) we see that the contribution of the four-fermion operators, or of the three-loop graphs, dominates at large $n$ over that of the two-loop graph, it contains an extra factor $n$. Moreover, this conclusion is not modified if nonzero $\gamma_i$ and $\delta_1$ are accounted for. Indeed, the dependence on $\delta_1$ is universal for two-loop and three-loop contributions and drops off from the ratio of $a_n$. As for the anomalous dimensions $\gamma_i$ of the four-fermionic operators (see eq. (47) $\gamma_1$ is negative, $\gamma_2$ is positive. Notice that in the realistic case of QCD one also gets quite a few different diagonal operators both with positive and negative anomalous dimensions[19, 21]. In the large $n$ limit the operator with the most negative $\gamma$ dominates so that account for the anomalous dimensions only strengthens the conclusion on the dominance of three-loop graphs.

As a particular illustration let us consider the case when the number of flavors $N_f$ is large, $N_f \gg 1$. Then $\gamma_1 = -2$, $\gamma_2 = 0$, $\delta_1 = 0$ and eq. (55) leads to

$$a_n \xrightarrow{n \to \infty} = \frac{1}{96N_f} (-b_0)^n (n + 3)! \tag{57}$$
We got an extra $n^3$ as compared to the two-loop contribution (34). On the other hand let us note that in large $N_f$ limit the three-loop contribution is suppressed by an extra factor $1/N_f$.

To summarize, we have demonstrated in this section that three-loop graphs generate new, four-fermionic operators. The technique of the operator product expansion works the same simple as in the two-loop case. Although the final expressions accounting for the anomalous dimensions are somewhat cumbersome they are straightforward. An interesting conclusion is that in the contribution of four fermionic operators dominates for high order coefficients.

5 Calculability of ultraviolet renormalon. Conclusions.

In this section we comment on the definition of the ultraviolet renormalon. The basic observation is that if one tries to define the ultraviolet renormalon as a set of graphs producing a certain asymptotic behaviour, $a_n \sim n!(\bar{b}_n)^n$, then this set of graphs is in fact ill-defined.

The demonstration of this point is straightforward. Let us go back to our operator product expansion (17). The coefficient functions are represented as series in $\alpha_1(k^2)$

$$c_i(k^2) = h_0 + h_1 \alpha_1(k^2) + h_2 \alpha_1^2(k^2) + \ldots + h_l \alpha_1^l(k^2) + \ldots \quad (58)$$

The tacit assumption is that one can approximate $c_i(k)$ by, say, first term in this expansion as far as $\alpha_1(k^2)$ is small. This logic does not work however if we use the operator product expansion (17) to evaluate the asymptotic of the coefficients $a_n$ of expansion in our "original" $\alpha_s(Q^2)$. Indeed, from eq. (53) we conclude that for any finite $l$ the contribution of $h_l$ to the asymptotic of $a_n$ has the same $n$ dependence. This means, in turn, that all the terms in the expansion (58) are the same important.

Technically this is because we have in fact two large parameters, $n$ and $ln(k^2/Q^2)$, and only the logs are controlled by renormalization group. However as a result of integration over $k^2$ powers of logs are converted into powers of $n$ (see, e.g., eq. (31)). What we demonstrated above is that lower powers of logs have larger statistical weight. As a result the two large parameters, large combinatorial $n$ and large log, get mixed up and we need to look into extremum with respect to the both. This problem looks much harder than fixing leading logs. It is not ruled out, in particular, that the contributions of various $\bar{a}_l$ in eq. (58) cancel between themselves and the true asymptotic is very different from (4),(5).

It is worth emphasizing that this difficulty with obtaining the asymptotic of the expansion coefficients is specific for the ultraviolet renormalon and does not plague the calculation
of the infrared renormalon. Since renormalons are reflections of the Landau poles in perturbative expansions (see, e.g., [2]) it means that if indeed the true asymptotic of the UV renormalon can be different from (4),(5) then the Landau pole in ultraviolet does not necessarily manifests itself in, say, polarization operator $\Pi(Q^2)$. This difference in the status of the infrared and ultraviolet renormalons looks indeed surprising.

As another illustration of the uncertainties in the evaluation of the ultraviolet region in perturbation theory let us consider a tower of renormalons which means that we introduce another scale, $k'$ such that

\[(k')^2 \gg k^2 \gg Q^2; \quad (k')^2 \sim k^2 \exp(l)\]  

and $l$ is large enough. Then coefficients $h_l$ are given by

\[h_l \sim (-b_0)^l.\]  

The corresponding asymptotic of $a_n$ is again

\[a_n \xrightarrow{n \to \infty} \text{const} \cdot n!(-b_0)^n.\]  

It shows that no matter how large virtual momenta are they do not detach from asymptotic of $a_n$.

Thus, we conclude that there are technical means to evaluate the true asymptotic of the UV renormalon.

As for the most interesting question on possible phenomenological implications of the results obtained we have unfortunately little to say. Pessimistically, one may argue that all the problems with evaluating the UV asymptotic are not physical and the reason why we fail is simply because we want to expand $\alpha_s(k^2)$ which is much smaller than $\alpha_s(Q^2)$ at $k^2 \gg Q^2$ in terms of the latter. To this end we need to keep many terms absolute value of each of which is much larger than $\alpha_s(k^2)$ itself. If we do not endeavor the expansion, however, the contribution is small and uninteresting.

On the optimistic side, one might hope that the failure to evaluate the UV renormalon might signify that $1/Q^2$ terms related to the renormalon are important. To clarify the point, let us turn for a moment to infrared renormalon (see, e.g., [10]). The corresponding series is

\[(a_n)_{IR} \sim n!(b_0/2)^n.\]  

This series is non Borel summable and if the perturbative expansion still represents an asymptotical expansion the corresponding uncertainty is $Q^{-1}$. Introduction of $Q^{-4}$ terms
phenomenologically [19] did turn successful. Let us emphasize that the lack of Borel summa-

bility in no way proves that $Q^{-4}$ terms are large numerically. However, there is some comfort 
in that the perturbative expansion might signal the importance of the power like corrections.

Similarly, one might take the failure to evaluate the asymptotic of the UV renormalon as 
a signal that the corresponding power-like corrections, this time $Q^{-2}$ terms, are important 
phenomenologically. The main problem is not so much that this assumption would be too 
bizarre but rather lack of framework which would allow to develop phenomenology basing 
on this assumption.

Although we do not have constructive ways to introduce this phenomenology let us 
note that the natural direction would be Nambu-Jona-Lasinio type models [22] since in 
these models one introduces four-fermion interaction as a low-energy effective interaction 
in QCD. We have seen that UV renormalon is naturally related to four-fermion interaction 
as well. One could hope to match $1/Q^2$ corrections due to the renormalon with NJL type 
phenomenology.

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References


**Figure captions.**

**Fig 1.** Building up the simplest renormalon-type graph. Dashed line denotes gluon while solid lines refer to fermions. One starts with an exchange of a vector particle of momentum $k$ and inserts vacuum polarization bubbles $n$ times.

**Fig 2.** The graph used to evaluate coefficients of operator expansion associated with the simplest renormalon graphs of Fig. 1. Momentum $k$ carried by the gluonic line is considered to be large. The fermion is understood to propagate in external electromagnetic and gluonic fields so that the graph is in fact a subgraph of the one-gluon exchange depicted in Fig. 1.

**Fig 3.** One-loop graph describing transition of electromagnetic current $j_\mu$ into a virtual photon with momentum $q$.

**Fig 4.** Three-loop skeleton graphs giving rise to four-fermionic operators. Momentum $k$ flowing through the gluonic lines is considered to be large so that the operator product expansion is an expansion in inverse powers of $k^{-2}$. The dotted boxes mark subgraphs producing four fermionic operators $O_2$ and $O_1$ (see eqs. (43) and (42), respectively).

**Fig 5.** The graph giving rise to the operator $O_2$ in the limit of large $k$. It is a subgraph of the first graph in Fig. 4 where it is marked by a dotted box.