Josephson Phenomena and Quantum Ferromagnetism in Double-Layer Quantum Hall Systems

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We present an effective theory for double-layer quantum Hall systems in which various Josephson phenomena occur except for the Meissner effect. The effective Hamiltonian consists of a kinetic term of the Goldstone mode and a tunneling term of electrons, and corresponds to that of the quantum ferromagnet model. By treating the electromagnetic field dynamically it is shown that the so-called commensurate-incommensurate phase transition does not exist in this model.

It is intriguing that Josephson phenomena may occur in certain double-layer quantum Hall (DLQH) systems [1–4]. This possibility has been pointed out based on the Chern-Simons (CS) theory of the planar electrons. A recent experiment [5] has revealed an anomalous behavior in the activation energy versus the parallel magnetic field in systems where the Josephson effect had been predicted. It may well be that this experiment presents the first indication of the Josephson behavior in DLQH systems, as discussed in Ref. [2].

However, the problem seems to be still controversial. Recently, Yang et al. [6] proposed a quantum ferromagnet model of DLQH systems, in which they argue that the Meissner effect does not exist. They also argue that the activation energy anomaly observed [5] is to be understood as a result of a commensurate-incommensurate phase transition induced by the parallel magnetic field. It is urgent problems to analyze a detailed mechanism of the absence of the Meissner effect and to examine the fate of other Josephson phenomena in such a model. It is also important to clarify the relation between the CS scheme [1] and the ferromagnet model [6].

Our CS formalism possesses one peculiar property in which the kinetic term of the Goldstone mode is absent when the lowest Landau level (LLL) projection is made. On the contrary, the term exists in other approaches [7]. At least at the effective Hamiltonian level, as we shall show, the CS scheme [1] and the ferromagnet model [6] are identical except for this kinetic term. We wish to examine how various predictions [1,2] of Josephson phenomena are modified by the existence of this term. In this paper we do not pursue the problem whether the kinetic term should exist or not after the LLL projection. We rather explore the consequences when it exists.

Our results on the quantum ferromagnet model are as follows. (A) The Josephson effect occurs. (B) There is no commensurate-incommensurate phase transition as a function of $B_\perp$. We have obtained a different result from Ref. [6] since we have treated the electromagnetic (EM) fields dynamically. (C) The complete screening (Meissner effect) does not occur, as is consistent with a result in Ref. [6]. (D) All the results of our CS scheme are reproduced as the kinetic term is gradually suppressed. In this paper we use units such that $\hbar = 1$.

The microscopic Hamiltonian of the double-layer electron system is given by

$$
\mathcal{H} = \sum_\alpha \left( \frac{1}{2M} |iD_k^\alpha \psi_\alpha|^2 + e A_\perp^\alpha |\psi_\alpha|^2 \right) - \lambda (\psi_1^\dagger e^{-i\pi d A_\perp} \psi_2 + \psi_2^\dagger e^{i\pi d A_\perp} \psi_1),
$$

(1)

with $iD_k^\alpha = i\partial_k - eA_k^\alpha - eA_\perp^\alpha$, $k = x, y$. Here, $M$ is the effective mass of the electron; $-e$ the charge of the electron; $\psi_\alpha$ is the electron field and $A_\perp^\alpha$ the electric potential at the layer $\alpha$. External magnetic field $B_\perp$ is applied perpendicular to the layers with $A_k^\alpha = -\frac{1}{2} \varepsilon_{kj} x_j B_\perp$. We consider additionally magnetic field $B_k$ parallel to the layers described by the potentials $A_\perp^\alpha$ and $A_k^\alpha$. We have taken the layers parallel to the $xy$ plane with the interlayer distance $d$. Electrons tunnel across the layers with strength $\lambda$, which is assumed to be much smaller than the Coulomb energy between electrons ($\lambda \ll e^2/\varepsilon_l B$); here, $\varepsilon_l$ is the dielectric constant and $t_B = 1/\sqrt{\varepsilon_l B}$ is the magnetic length. The Coulomb interactions between electrons are generated by treating the gauge fields dynamically. The Hamiltonian (1) is invariant under the EM gauge transformation.

The existence of Josephson behaviors in certain DLQH systems is physically understood based on the Halperin wave function. At a generic filling factor $\nu = 2/(m + n)$ with $m$ and $n$ integers ($m$ is odd), it is well known that
the DLQH state is described by the wave function [8]:
\[ \Psi(N_0, N_0) = \prod (z_{i}^{r} - z_{i}^{s})^{m} \prod (s_{i}^{r} - s_{i}^{s})^{m} \prod (z_{i}^{r} - z_{i}^{s})^{n} \times \exp \left\{ -\frac{1}{4} \psi_{B} \left( \sum |z_{i}^{r}|^2 + \sum |s_{i}^{r}|^2 \right) \right\}, \] (2)
where the complex coordinate \( z_{r}^{\alpha} \) is used for the \( r \)-th electron position in the layer \( \alpha \), and \( N_0 \) denotes the number of electrons in each layer. It is clear that integers \( m \) and \( n \) describe the intralayer and interlayer correlations of electrons, respectively. A remarkable feature appears in this wave function when \( m = n \): At the specific filling factor \( \nu = 1/m \) with \( m = n \), one cannot tell whether two electrons belong to the same layer or the different layers in the Halperin wave function (2), because the intralayer and interlayer electron correlations are identical. Namely, at the same filling factor, there are many quantum Hall (QH) states described by the wave functions \( \Psi(N_0 + \Delta N, N_0 - \Delta N) \), where \( \Delta N \) electrons have been transferred from one layer to the other. Corresponding to the electron number \( \Delta N \), the phase \( \theta \) is defined as the conjugate variable. With a weak tunneling interaction included, these QH states are mixed among themselves, and the resulting states are specified by \( \theta \):
\[ \Psi(\theta) = \sum_{\Delta N} e^{i\theta \Delta N} \Psi(N_0 + \Delta N, N_0 - \Delta N). \] (3)
Recall that the wave function of the superconductor Josephson junction [9] is given by
\[ \Psi_{\text{super}}(\theta) = \sum_{\Delta N} e^{i\theta \Delta N} \Psi_1(N_0 + \Delta N) \Psi_2(N_0 - \Delta N), \] (4)
where \( \Psi_\alpha(N_\alpha) \) describes the state in the bulk \( \alpha \) with the electron number \( N_\alpha \). Because of the similarity of the wave functions we are naturally led to the anticipation for the Josephson behaviors in the present DLQH system.
We would like to construct an effective theory describing the low energy dynamics of the collective modes \( \theta \) and \( \Delta N \). When the tunneling term acts on these QH states, its component would be expanded as
\[ \psi_1 \psi_2 \approx \sqrt{\rho_1 \rho_2} e^{-i\theta} \approx \rho_0 \left( 1 - \frac{(\Delta \rho)^2}{2 \rho_0^2} \right) e^{-i\theta}, \] (5)
as far as these collective modes are concerned, where \( \rho_1 = \rho_0 + \Delta \rho \) and \( \rho_2 = \rho_0 - \Delta \rho \) are the electron densities on each layer. By keeping only the kinetic term of the phase field \( \theta \) and the tunneling term, and by requiring the gauge invariance, the effective Hamiltonian for \( \theta \) and \( \Delta \rho \) would be given by
\[ \mathcal{H} = \frac{1}{2} \rho_0 (\partial_\theta + \epsilon \Delta A_z)^2 \]
\[ -2\lambda \rho_0 \left( 1 - \frac{(\Delta \rho)^2}{2 \rho_0^2} \right) \cos(\theta + \epsilon A_z). \] (6)
Since \( \theta \) and \( \Delta \rho \) are canonical conjugate variables, the effective Lagrangian reads \( \mathcal{L} = \Delta \rho (\partial_\theta + \epsilon \Delta A_z) - \lambda \). Here, \( \Delta A_\mu = A_\mu^1 - A_\mu^2 \) is the EM potential difference across the layers; \( \mu = 0, x, y \).
This effective Hamiltonian is the same one used in the quantum ferromagnet model of DLQH systems [6] where \( \rho_0 \approx \epsilon^2 / e B \), and the same one used in the CS theory of DLQH systems [1,2] where the limit \( \rho_0 \to \infty \) is taken, as implies the LLL projection in the mean-field approximation of this scheme. Thus, at least at the effective Hamiltonian level, the similarity and the difference between these two formalisms are clear. In this paper we treat \( \rho_0 \) as a free parameter.
It is helpful to make an intuitive picture of the system by using the language of quantum ferromagnetism [6,7] with pseudospin \( S_K \) \((K = X, Y, Z)\). Without the EM fields and the tunneling interaction \((\lambda = 0)\) the Hamiltonian reads \( H = (\rho_0 / 2)(\partial_\theta \theta)^2 \) with \( S_K \propto \sin \theta \) and \( S_Y \propto \cos \theta \). Then, the system resembles the XY model, where the ground state is a ferromagnet with a spontaneous magnetization: \( \Theta \) the Goldstone mode \( \theta \) designates the orientation of this magnetization. Now, the tunneling interaction \((\propto \Delta S_Y)\) is equivalent to applying the pseudomagnetic field in the \( Y \) direction. Its effect is to align the axis of the magnetization with the \( Y \) axis \((\theta = 0)\). This axis modulates locally in the presence of the external parallel magnetic field \( B_y \) as we shall see, \( \theta \propto B_y \) for \( B_y < B_y^c \) and \( \theta = 0 \) for \( B_y \gg B_y^c \) with a critical value \( B_y^c \). The pseudospin \( S_K \) oscillates around the axis of the magnetization in the SU(2) pseudospaces, which is observed as a plasma oscillation since \( \Delta \rho \propto S_Z \propto \sin(\omega \rho \epsilon) \) with \( \omega \rho \) the plasma frequency. It would be easy and instructive for readers to interpret the following results in terms of quantum ferromagnetism.
We consider the case with no parallel electric field \((E_z = 0\) and hence no Hall current. We take the parallel magnetic field in the \( y \) direction and regard the system to be uniform in the \( y \) and \( z \) directions, which allows us to set \( \partial_y \theta = 0 \). In accord with Ref. [6] we choose the gauge [10] such that \( B_y = -\partial_x A_z \) and \( E_z = \partial_t A_z - \Delta A_0 / d \). We treat only these EM fields dynamically. The equations of motion are derived from the effective Lagrangian as
\[ \partial_t \theta + \epsilon \Delta A_z = \frac{2\lambda \Delta \rho}{\rho_0} \cos(\theta + \epsilon A_z), \] (7)
\[ \partial_t \Delta A_0 = \frac{2\lambda \Delta \rho}{\rho_0} \sin(\theta + \epsilon A_z). \] (8)
Here, \( \Delta A_0 \) and \( A_z \) are dynamical fields subject to the Maxwell equations. When no voltage is applied externally between the two layers, the electric field \( E_z \) between the two layers is generated by the charging \( \Delta \rho \) on the layer and it is given by \( E_z = 4\pi \epsilon \Delta \rho / \epsilon \), or
\[ \Delta A_0 = \frac{-4\pi \rho \Delta \rho d}{\epsilon} + d \partial_t A_z. \] (9)
We relate $A_z$ to the phase $\theta$ and the field $B_\parallel$ later.

Let us study the physical meaning of basic equations (7) and (8). First, (7) describes that the time evolution of the phase $\theta$ is controlled by the electrochemical potential induced by the movement of electrons from one layer to the other layer: The term $(2\lambda \Delta \rho/\rho_0 \cos(\theta + edA_z))$ is a kind of the chemical potential associated with the tunneling interaction. In order to see the physical meaning of (8), we note that the charge $J^0_\parallel$ and the drift current $J^2_\parallel$ on each layer are given by

$$ J^1_0 = -J^2_0 = -\frac{\delta C}{\delta A_0} = -e \Delta \rho, $$

$$ J^1_\parallel = -J^2_\parallel = \frac{\delta C}{\delta A_\parallel} = -e \rho_0 \partial_\parallel \theta, $$

and that the tunneling current is given by

$$ J_\parallel = \frac{\delta C}{\delta A_\parallel} = -2e \lambda \rho_0 \sin(\theta + edA_z). $$

It is now clear that (8) describes simply the charge conservation on each layer: $\partial_\parallel J^1_\parallel = \partial_\parallel J^2_\parallel = J_\parallel$.

From (7) and (9) it follows that there is no charging ($\Delta \rho = 0$) in the static configuration. The static Hamiltonian reads from (6) as

$$ \mathcal{H}_{\text{matter}} = \frac{1}{2} \rho_0 (\partial_\parallel \theta)^2 - 2\lambda \rho_0 \cos(\theta + edA_z). $$

This is precisely the Hamiltonian used in Ref. [6] to discuss the ground state structure of the DLQH system. By setting that $A_z = -x B_\parallel$, they argue that the system has a commensurate-incommensurate phase transition as a function of $B_\parallel$. However, their conclusion is incorrect, as we shall show, because they have neglected the fact that $A_z$ is a dynamical field subject to the Maxwell equation.

The relevant Maxwell equation is $\partial_\parallel B_\parallel = -4\pi J^2_\parallel$, with the drift current being $J_\parallel = J^3_\parallel \delta(z - z_1) + J^3_\parallel \delta(z - z_2)$, where $z_\parallel$ denotes the $z$ coordinate of each layer ($z_1 > z_2$).

Integrating this Maxwell equation, we obtain

$$ B_\parallel = 4\pi \rho_0 \partial_\parallel \theta [\theta(z - z_1) - \theta(z - z_2)] + B_\parallel(x, t), $$

where $\theta(z - z_\parallel)$ is a step function; $B_\parallel$ is an "integration constant" which represents the parallel magnetic field outside the layers, that is, for $z > z_1$ or $z < z_2$. Hence, we identify $B_\parallel$ as the external parallel magnetic field. Within the junction ($z_1 > z > z_2$) we have

$$ B_\parallel = -\partial_\parallel A_z = -4\pi \rho_0 \partial_\parallel \theta + B_\parallel. $$

The term $4\pi \rho_0 \partial_\parallel \theta$ represents the screening effect which has been neglected in Ref. [6]. When $B_\parallel$ is a constant field, we obtain $A_z = 4\pi \rho_0 \theta - x B_\parallel$, choosing an integration constant to be zero without loss of generality. This is the desired equation relating $A_z$ to $\theta$ and $B_\parallel$.

It is convenient to define [10]

$$ \delta \equiv \theta + edA_z = (1 + 4\pi \rho_0 d)\theta - edx B_\parallel, $$

and rewrite (15) as

$$ B_\parallel = -\frac{\kappa}{ed} \partial_\parallel \delta + \frac{\kappa}{4\pi \rho_0 d} B_\parallel, $$

with $\kappa = 4\pi \rho_0 d/(1 + 4\pi \rho_0 d)$. Substituting (9) and (16) into (7) and (8) we obtain [10]

$$ \partial_\parallel \delta = \frac{4\pi \rho_0 d \Delta \rho}{\varepsilon} + 2\lambda \rho_0 \cos \delta, $$

$$ \partial_\parallel \Delta \rho = \frac{\kappa}{4\pi \rho_0 d} \partial_\parallel^2 \delta - 2\lambda \rho_0 \sin \delta, $$

which govern the dynamics of the collective modes.

We first study the Meissner effect, which is a static phenomenon with $\Delta \rho = 0$. The total Hamiltonian of the system is given by adding the Maxwell term of the parallel field $B_\parallel$ to the matter part (13):

$$ \mathcal{H}_i = \frac{d}{8\pi} B_\parallel^2 + \frac{1}{2} \rho_0 (\partial_\parallel \theta)^2 - 2\lambda \rho_0 \cos(\theta + edA_z). $$

We search for the configuration of $\theta$ and $A_z$ which minimizes this Hamiltonian. As a saddle point equation we derive the relation (16) relating $A_z$ to $\theta$ and $B_\parallel$. Substituting this relation into (20), we obtain

$$ \mathcal{H}_i = \frac{\kappa}{8\pi \rho_0 d} (\partial_\parallel \delta)^2 - 2\lambda \rho_0 \cos \delta + \frac{d}{8\pi} B_\parallel^2, $$

from which it follows that $\kappa \partial_\parallel^2 \delta - 8\pi \rho_0 \cos \delta = 0$. This is the sine-Gordon equation, which we can also get from equations (18) and (19) in the static case.

The ground state of the system (21) is given by $\delta = 0$, where $\theta = (\kappa/4\pi \rho_0 d)x B_\parallel$ and $B_\parallel = B_\parallel/(1 + 4\pi \rho_0 d)$ from (16) and (17). The parallel magnetic field is partially screened, and its definite fraction penetrates between the two layers. Hence, the complete screening (Meissner effect) does not occur. It should be remarked that the complete screening is realized in the limit $\rho_0 \rightarrow \infty$, which is the ground state of the CS scheme [1].

In the actual system we solve the equation of motion by imposing the boundary condition at the edge of the junction, which reads $B_\parallel = B_\parallel$, or $\partial_\parallel \delta = -edB_\parallel$ from (17). The field $B_\parallel$ is equal to $B_\parallel$ at the edge and gradually decreases to the ground-state value inside the junction. The Josephson penetration depth is $\lambda_J = \sqrt{\kappa/8\pi \rho_0 d \rho_0}$. However, this ground-state configuration is possible only when $B_\parallel$ is less than a critical value $B^c_\parallel = (2\kappa/ed \lambda_J)$. For $B_\parallel \geq B^c_\parallel$ the magnetic flux penetrates into the junction as sine-Gordon vortices. Eventually for $B_\parallel \gg B^c_\parallel$ the magnetic flux penetrates into the junction freely. Then,
also inside the junction we have \( B_y = B_{\|1} \), \( \delta = -ze \Delta B_{\|} \) and \( \theta = 0 \). In the CS scheme where \( \kappa = 1 \) we estimate \( B_{\|} \approx 100 \text{ Gauss} \) by using typical sample parameters given in Ref. [5]. These features are well known in superconductor Josephson junction [9], which is also described by the sine-Gordon equation.

It is true that \( \theta \propto zB_{\|} \) for \( B_{\|} < B_{\|1}^* \) and \( \theta = 0 \) for \( B_{\|} \gg B_{\|1}^* \). From this fact, one might think that the system is in a commensurate phase for small \( B_{\|} \) and in an incommensurate phase for large \( B_{\|} \), as suggested in Ref. [6]. However, \( B_{\|1}^* \) is the point at which the magnetic flux begins to penetrate into the junction as sine-Gordon vortices and not a phase transition point. There is no reason to identify this with the observed critical point [5] in the activation energy anomaly. The main feature of the anomaly is that with increasing \( B_{\|} \) the activation energy drops rapidly up to the critical point and then it becomes almost constant. In the CS scheme [2] we have explained how this anomalous behavior is induced by plasmon excitations thermally activated. Let us study if the same idea works in the ferromagnet model as well.

Experimentally, the external field \( B_{\|} \) is strong enough to penetrate into the junction freely. Thus, in (7) and (8) we set \( A_z = -ze B_{\|} \) and treat \( \theta \) as a uniform fluctuation. Then, taking a spatial average [11] over the junction with size \( l \), we find that \( \omega_p^2 \Delta \rho + \omega_p^2 \Delta \rho = 0 \), where

\[
\omega_p^2 = \omega_p^2 \left| \frac{\sin(e \Delta B_{\|}/2)}{e \Delta B_{\|}/2} \right| + 4 \lambda^2,
\]

with \( \omega_p^2 = \frac{8 \pi e^2 d \lambda_0}{\epsilon} \). This is the plasmon frequency, which does not depend on the parameter \( \rho_1 \), as should be, since only uniform oscillations are considered. Plasmons are neutral, but they are oscillation modes of the charging \( e \Delta \rho \) in each layer, \( \Delta \rho \propto \sin(\omega_p t) \), and scatter electrons. Therefore, plasmon excitations affect transport phenomena. In particular, we expect that the activation energy determined by making the resistivity \( \rho_{xx} \) should detect their excitations. Then, as a function of \( B_{\|} \) the activation energy drops rapidly up to the critical value \( B_{\|1}^* = \frac{2 \pi}{e \Delta l} \), following the plasmon formula (22). Furthermore, it can be argued that the activation energy is a constant \( 2 \lambda \) for \( B_{\|} \geq B_{\|1}^* \) based on a certain physical assumption [2]. The resulting theoretical formula fits the reported data quite well [2].

The system shows an entirely different aspect when external leads are attached and external voltage \( V_{\text{ext}} \) is applied between the two layers. This is because the tunneling current flows (\( \partial_t \Delta \rho \neq 0 \)) without charge accumulation (\( \Delta \rho = 0 \)). In this case the electrochemical potential energies due to the charging \( \Delta \rho \) vanish, and \( \Delta A_0 \) is simply given by \( V_{\text{ext}} \). Thus, in the absence of the parallel magnetic field (\( A_z = 0 \) and \( \partial_z \theta = 0 \)), we obtain from basic equations (7) and (8) with (12) that

\[
\partial_t \theta = -e \Delta A_0 = e V_{\text{ext}},
\]

\[
J_z = e \partial_t \Delta \rho = -2 e \lambda_0 \sin \theta,
\]

which are the familiar equations describing the Josephson effect in superconductor [9]. We conclude that the Josephson current (24) flows between the two layers with the phase \( \theta \) being determined by (23).

In particular, in a QH system with DC-voltage feed, we find \( \theta = e V_{\text{ext}} \) from (23), and the Josephson current (24) oscillates in time. Because electrons tunnel in the presence of voltage (\( V_{\text{ext}} \neq 0 \)), they acquire energy and emit EM radiation. The frequency is \( e V_{\text{ext}} \), which is one half of that in superconductor Josephson junction. This is the characteristic feature of the Josephson tunneling in the DLQH system [1]. On the other hand, in a QH system with DC-current feed (\( |J_z| = \text{constant} < 2 e \lambda_0 \)), the phase \( \theta \) is fixed to be a constant (\( \Delta = \theta_0 \)) by (24). Then, from (23) it follows that \( V_{\text{ext}} = 0 \). Therefore, the tunneling current is a superconducting current.

In this paper we have shown that also in the quantum ferromagnet model [6] various interesting Josephson phenomena are predicted except for the Meissner effect. The difference between this model and the CS scheme exists in the drift current and not in the tunneling current, which originates in the kinetic term of the Goldstone mode on the layers. As is well known, the essence of the Josephson phenomena exists in the coherent tunneling of electrons. We would like to examine elsewhere if the term is generated also in our CS scheme by making an analysis beyond the mean-field approximation. Even if it is generated, we have demonstrated that the essential predictions of the Josephson behaviors in DLQH systems are not modified.

It is very interesting that the Meissner effect does not exist although the Josephson effect does exist in the present model. The reason is that the junction is made of the layers. In case of superconductor it is made between two superconducting bulks into which the parallel magnetic field cannot penetrate. In this case we need to set \( B_{\|} = 0 \) in (15) as the boundary condition imposed outside the junction (or inside the bulk): thus, the Meissner effect follows necessarily. This explains why we can have the Josephson current without being accompanied by the Meissner effect in the QH-state Josephson junction.

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REFERENCES


[10] Note that another gauge, $A_2 = 0$, has been chosen in our CS scheme [2]. The field $\delta$ defined by (16) is physically identical to the phase $\theta$ itself in Ref. [2].
