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Gravitational Mass in Asymptotically de Sitter Space-Times

KEN-ICHI NAKAO, TETSUYA SHIROMIZU AND KEI-ICHI MAEDA[†]

*Department of Physics, Kyoto University,
Sakyo-ku, Kyoto 606-01, Japan*

and

[†]Department of Physics, Waseda University, Tokyo 169-50, Japan

ABSTRACT

The Abbott-Deser (AD) mass in the asymptotically de Sitter space-time is a conserved Killing “energy”. We show that the AD mass can be negative under some situation but may deserve a title of the “gravitational mass”. In order to see the property of the negative AD mass, we consider the Tolman-Bondi’s dust sphere solution. The negative AD mass prevents the dust sphere from gravitational collapse. Rather it causes the cosmic expansion, resulting in the isotropic and homogeneous de Sitter space-time locally. As a result, the cosmic no hair conjecture holds although the AD mass can be negative.

1. Introduction

About ten years ago, Abbott and Deser presented a reasonable definition of the conserved Killing “energy” in the space-time with a cosmological constant Λ , i.e., in the de Sitter and anti-de Sitter space-times (more strictly speaking, the asymptotically de Sitter and anti-de Sitter space-times).^[1] This is the so-called Abbott-Deser (AD) mass and in the limit of $\Lambda \rightarrow 0$, it agrees with the ADM mass^[2], which is the conserved energy of the asymptotically flat space-time.

The positivity of the AD mass as well as the Arnowitt-Deser-Misner (ADM) mass is one of the most important issues of general relativity in connection with the stability of the maximally symmetric space-times, i.e., of the Minkowski, de Sitter and anti-de Sitter space-times. Hence, in the case of the space-time with $\Lambda > 0$, the positivity of the AD mass seems to be closely related with the *cosmic no hair conjecture*, which states that all space-times with a positive cosmological constant approach the de Sitter space-time asymptotically.^[3] By Shoen and Yau, the positivity of the ADM mass was proven^[4] and later Witten presented a simpler proof by spinor representation.^[5] While the positivity of the AD mass in the case of $\Lambda < 0$ was discussed by Gibbons et al.^[6] and that in the asymptotically de Sitter space-time ($\Lambda > 0$) has been recently investigated and proven by one of the present authors^[7] under some ansatz corresponding intuitively to “zero-momentum” condition.

In this paper, we study the sign of the AD mass M_{AD} in the asymptotically de Sitter space-time under more generic situations than those investigated by the previous researches and we show that AD mass can be negative without any violation of the energy condition (apart from Λ). As we will see later, the condition of $M_{AD} \lesssim 0$ is roughly given as $L \gtrsim H^{-1} \equiv \sqrt{3/\Lambda}$, where L is the length scale of a source of M_{AD} . This means that the source of M_{AD} should extend over the cosmological horizon scale in order that the AD mass is negative. The fact that the “timelike” Killing vector of the de Sitter space-time is no longer timelike outside the cosmological horizon suggests that a distribution of

matter field over the cosmological horizon scale makes the space-time necessarily dynamical. If it is true, conversely we would expect that when the space-time is “stationary or static”, M_{AD} is non-negative. Furthermore we may conjecture that in the case of stationary space-times, the uniqueness theorem of Kerr-de Sitter space-time holds and a space-time will be the de Sitter one if and only if M_{AD} vanishes.

We also show that M_{AD} has a meaning of the *gravitational mass* in the case of the spherically symmetric space-time. In order to see the effect of the negativity of M_{AD} on the motion of matter, we investigate the Tolman-Bondi space-time with positive Λ , which describes the motion of an inhomogeneous dust sphere. In this model, we shall exactly see that the circumference radius Y of the negative mass region should satisfy $Y > H^{-1}$. We also see that in this model the negative M_{AD} produces the repulsive gravitational field and hence the dust fluid elements in the region which suffers this repulsive force cannot collapse into singularity but such a region approaches locally the de Sitter space-time asymptotically. Hence, this result supports the cosmic no hair conjecture.

This paper is organized as follows. In Sec.2, we shall summarize M_{AD} and rewrite the original expression given by Abbott and Deser by use of the physical quantities of 3-dimensional spacelike hypersurface. In Sec.3, we show that M_{AD} can be negative in the case of a conformally flat 3-space with a constant mean curvature condition. By investigating spherically symmetric cases, we see the physical meaning of negativity of M_{AD} in Sec.4. Some discussions are given in Sec.5. In this paper, we adopt the units $c = G = 1$ and follow MTW^[8] for sign convention of metric and Riemann tensor, etc. The Latin indices take 1, 2, 3 while the Greek indices take 0, 1, 2, 3.

2. Abbott-Deser Mass in the Asymptotically de Sitter Space-Time

We consider the asymptotically de Sitter space-time which satisfies the Einstein equations with a positive cosmological constant Λ ,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \quad (2.1)$$

where $T_{\mu\nu}$ is the energy-momentum tensor of the matter and radiation fields which vanish at spatial infinity.¹

First, we divide the 4-dimensional metric $g_{\mu\nu}$ into the metric of the background de Sitter space-time $\bar{g}_{\mu\nu}$ and the deviation $h_{\mu\nu}$ from that as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \quad (2.2)$$

Here $h_{\mu\nu}$ is not necessarily small everywhere but we assume that it vanishes at infinity. Substituting Eq.(2.2) into Eq.(2.1), we rewrite the Einstein equations as

$$R_L^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu}R_L - \Lambda h^{\mu\nu} = 8\pi(-\bar{g})^{-1/2}T^{\mu\nu}, \quad (2.3)$$

where $h_L^\mu = \bar{g}^{\mu\alpha}h_{\alpha\nu}$ and $\bar{g}$ is the determinant of $\bar{g}_{\mu\nu}$. Notice that the zeroth order terms with respect to $h_{\mu\nu}$ in the left hand side of Eq.(2.1) vanishes. The left hand side of Eq.(2.3) is composed only by the first order terms with respect to h , i.e.,

$$\begin{aligned} R_L^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu}R_L - \Lambda h^{\mu\nu} = & \frac{1}{2}(2\bar{\nabla}_\rho\bar{\nabla}^{(\mu}h^{\nu)\rho} - \bar{\nabla}_\rho\bar{\nabla}^\rho h^{\mu\nu} - \bar{\nabla}^\mu\bar{\nabla}^\nu h^\rho_\rho \\ & - \bar{g}^{\mu\nu}\bar{\nabla}_\rho\bar{\nabla}_\sigma h^{\rho\sigma} + \bar{g}^{\mu\nu}\bar{\nabla}_\rho\bar{\nabla}^\rho h^\sigma_\sigma) - \Lambda\left(h^{\mu\nu} - \frac{1}{2}\bar{g}^{\mu\nu}h^\rho_\rho\right), \end{aligned} \quad (2.4)$$

where $\bar{\nabla}_\mu$ is the covariant derivative with respect to $\bar{g}_{\mu\nu}$. All the higher terms of $\mathcal{O}(h^2)$ and contributions from $T_{\mu\nu}$ are included in the right hand side of Eq.(2.3).

¹ It is worthy to note that there does not always exist a spatial infinity under an arbitrary time slicing condition. For example, in a complete chart of de Sitter space-time, time-slices defined by the cosmic time do not have infinities. But, one can construct “asymptotically flat” slices in asymptotically de Sitter space-times and obtain spatial infinities, e.g., by a constant-mean-curvature condition. This infinities are contained in future conformal infinities I^+ .

The left hand side of Eq.(2.3) obeys the “Bianchi identity”,

$$\bar{\nabla}_\mu \left(R_L^{\mu\nu} - \frac{1}{2} \bar{g}^{\mu\nu} R_L - \Lambda h^{\mu\nu} \right) = 0, \quad (2.5)$$

Hence we obtain

$$\bar{\nabla}_\mu T^{\mu\nu} = 0. \quad (2.6)$$

Here by introducing the Killing vector $\bar{\xi}_\mu$ associated with the background de Sitter space-time, we can construct a conserved current $\mathcal{J}^\mu$ as

$$\mathcal{J}^\mu = T^{\mu\nu} \bar{\xi}_\nu, \quad (2.7)$$

which satisfies the conservation law, $\partial_\mu \mathcal{J}^\mu = 0$, by virtue of the background Killing equation $\bar{\nabla}_{(\mu} \bar{\xi}_{\nu)} = 0$, where ∂_μ is a usual partial derivative. From $\mathcal{J}^\mu$, we can construct a conserved quantity of this system as

$$E(\bar{\xi}) = \int \mathcal{J}^0 d^3x = \int \left(R^{0\nu} - \frac{1}{2} \bar{g}^{0\nu} R_L - \Lambda h^{0\nu} \right) \bar{\xi}_\nu (-\bar{g})^{1/2} d^3x. \quad (2.8)$$

In case that $\bar{\xi}^\mu$ is timelike Killing vector, E is regarded as the Killing energy, the so-called Abbott-Deser (AD) mass.

As for a background 4-metric, we adopt the following form,

$$d\bar{s}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) f_{ab} dx^a dx^b, \quad (2.9)$$

where

$$a(t) = e^{Ht}, \quad \text{with} \quad H \equiv \sqrt{\frac{\Lambda}{3}}, \quad (2.10)$$

and f_{ab} is the metric of the flat 3-space.²

² As one needs asymptotic regions in order to define AD mass, our consideration may be restricted on future development of slices with asymptotic regions and then into a subregion of space-times in terms of *time*. For example, in de Sitter space-time the region which we can treat is one which is foliated by expanding flat 3-spaces.

Then the “timelike” Killing vector of the background de Sitter space-time is given by

$$\bar{\xi}^\mu = (-1, Hx^a), \quad (2.11)$$

where $(x^a) = (x, y, z)$ in the Cartesian coordinates.

Then from Eq.(2.8), the AD mass M_{AD} is given by

$$M_{AD} \equiv E(\bar{\xi}) = a(t)(M + \Delta M), \quad (2.12)$$

where

$$M = \frac{1}{16\pi} \oint d\bar{S}_a (\partial_b h^{ab} - \bar{g}^{ab} \partial_b h_c^c), \quad (2.13)$$

and

$$\begin{aligned} \Delta M = & \frac{1}{8\pi} \oint d\bar{S}_a \left[+\frac{1}{2} \bar{\xi}_b (\bar{g}^{ab} \partial_0 h_c^c + \bar{g}^{ab} \partial_c h^{0c} - \bar{g}^{ac} \partial_c h^{0b} - \partial_0 h^{ab}) \right. \\ & \left. + H(h^{0a} + h^{00} \bar{\xi}^a - h^{ab} \bar{\xi}_b) \right]. \end{aligned} \quad (2.14)$$

$d\bar{S}_a$ is infinitesimal surface element of flat slices in de Sitter space-time. As pointed out by Abbott and Deser, $\bar{\xi}^\mu$ is timelike with respect to $\bar{g}_{\mu\nu}$ only within the background cosmological horizon, $a(t)r \equiv \sqrt{\bar{g}_{ab} x^a x^b} < H^{-1}$, and therefore the usual meaning of “energy” is lost on super-horizon scale.^[1] However, the exact conservation, i.e., $\partial_t M_{AD} = 0$ is satisfied only when we consider whole region including super-horizon scale. Furthermore, as will be shown later, even if the source of M_{AD} extends over the horizon scale, M_{AD} has a meaning of the “gravitational mass”. Hence, in this paper, we consider the surface integral of Eqs.(2.13) and (2.14) is performed over a 2-sphere with radius r in the limit of $r \rightarrow \infty$. This consideration is supported from the fact that the AD mass of Schwarzschild-de Sitter space-times is equal to its “mass parameter” (see Sec.4).

It should be noted that the expression of M agrees with the ADM mass and is non-negative as proven by Shiromizu^[7]. However, as will be shown in the

next section, M itself is not conserved and ΔM can be negative. To see this, we rewrite Eq.(2.14) by the quantities of the 3-dimensional spacelike hypersurface, i.e., the intrinsic metric γ_{ab} and extrinsic curvature K_{ab} . Since $h_{\mu\nu}$ becomes small for $r \rightarrow \infty$, the extrinsic curvature is written as

$$K_b^a \equiv -\frac{1}{2}\gamma^{ac}\mathcal{L}_n\gamma_{cb} \longrightarrow -\frac{1}{2}(\partial_0 h_b^a + \partial_b h^{0a} + \bar{g}^{ca}\partial_c h_b^0) - H\delta_b^a\left(1 + \frac{1}{2}h^{00}\right), \quad (2.15)$$

where $\mathcal{L}_n$ is the Lie derivative along the unit normal n_ν of the hypersurface. By the use of Eq.(2.15), we obtain

$$\Delta M = \frac{1}{8\pi} \oint d\bar{S}_a \bar{\xi}^b (K_b^a - K\delta_b^a - 2H\delta_b^a) + \delta M, \quad (2.16)$$

where

$$\delta M \equiv \frac{1}{8\pi} \oint d\bar{S}_a \left[\frac{1}{2}\bar{\xi}^b (\partial_b h^{a0} - \delta_b^a \partial_c h^{c0}) + H h^{a0} \right], \quad (2.17)$$

and K is the trace part of the extrinsic curvature. We can see that δM vanishes identically from Eq.(2.11) as follows;

$$\begin{aligned} \delta M &= \frac{H}{16\pi} \oint d\bar{S}_a (x^b \partial_b h^{a0} - x^a \partial_b h^{b0} + 2h^{a0}) \\ &= \frac{H}{16\pi} \oint d\bar{S}_a \partial_b (h^{0a} x^b - h^{0b} x^a) = \frac{H a^2(t)}{8\pi} \int \partial_a \partial_b (h^{0[a} x^{b]}) d^3 x = 0, \end{aligned} \quad (2.18)$$

where in the last equality we use the fact that the ordinary partial derivatives are commutable. Hence, concerning $\gamma_{ab} \rightarrow \bar{g}_{ab}$ for $r \rightarrow \infty$, we obtain the final expression for ΔM as

$$\begin{aligned} \Delta M &= \frac{H}{8\pi a^3(t)} \oint d\bar{S}_a \gamma^{1/2} x^b (K_b^a - K\delta_b^a - 2H\delta_b^a) \\ &= \frac{H}{8\pi a(t)} \int D_a [x^b (K_b^a - K\delta_b^a - 2H\delta_b^a)] \gamma^{1/2} d^3 x, \end{aligned} \quad (2.19)$$

where γ is the determinant of γ_{ab} and D_a is the covariant derivative within the 3-space.

3. Sign of Abbott-Deser Mass

As we state in preface of Sec.2, one prepare asymptotically flat slices in order to define the AD mass. Since each slices must be imbeded in 4-dimensional space-times satisfying Einstein equations, the intrinsic metric and the extrinsic curvature must be related with each other by the constraint equations,

$${}^{(3)}R - K_b^a K_a^b + K^2 = 16\pi\rho_H + 2\Lambda, \quad (3.1)$$

$$D_a(K_b^a - \delta_b^a K) = 8\pi J_b, \quad (3.2)$$

where ${}^{(3)}R$, ρ_H and J_b are the Ricci scalar of 3-space, the energy and momentum densities for the normal line observer, respectively.

In this section, we shall restrict our attention to the family of space-times, which admits at least one foliation Σ with the constant-mean-curvature condition,

$$K = -3H. \quad (3.3)$$

Then, one notice that, using Eq.(3.2), ΔM is written down as follow;

$$\Delta M = \frac{H}{8\pi a(t)} \int_\Sigma \left(D_{(a} x_{b)} - \frac{1}{3} \gamma_{ab} D_c x^c \right) \tilde{K}^{ab} \gamma^{1/2} d^3 x + \frac{H}{a(t)} \int_\Sigma x^a J_a \gamma^{1/2} d^3 x, \quad (3.4)$$

where $\tilde{K}^{ab}$ is the traceless part of K^{ab} and $x_a \equiv \gamma_{ab} x^b$. What one can see easily from this expression is that, in the case of “zero-momentum”, i.e., $J_a = 0$, ΔM vanishes if, for example, x^a is the conformal Killing vector (or Killing vector) or $\tilde{K}^{ab} = 0$, on the slice (There may exist the cases of $\Delta M = 0$ under different conditions^[7]). As we stated in Sec.2, when ΔM vanishes, the AD mass is non-negative and the future development of this slice is de Sitter space-time if and only if the AD mass vanishes^[7].

Next, we consider cases in which ΔM does not vanish. For simplicity, we concentrate on the conformally flat 3-space in which the intrinsic metric is given

by

$$dt^2 = \gamma_{ab} dx^a dx^b = a^2(t) \psi^4(t, x^c) f_{ab} dx^a dx^b. \quad (3.5)$$

Then the constraint equations (3.1) and (3.2) become as

$$\Delta \psi = -\left(\frac{1}{8} \tilde{K}_b^a \tilde{K}_a^b + 2\pi \rho_H\right) \psi^5 a^2(t), \quad (3.6)$$

$$D_a \tilde{K}_b^a = 8\pi J_b, \quad (3.7)$$

where Δ is the flat Laplacian operator. Here it should be noted that, in the case of $h_{\mu\nu} = 0$, i.e., the de Sitter space-time, the condition (3.3) agrees with the Friedmann equation $\dot{a}/a = H$ and therefore the asymptotically de Sitter condition agrees with the asymptotic flatness for the intrinsic metric, i.e., $\psi \rightarrow 1$, which is consistent with our claim $h_{\mu\nu} \rightarrow 0$ for $r \rightarrow \infty$ in the previous section.

Now we can rewrite M_{AD} into physically more transparent form. By Eq.(3.5), M becomes

$$M = -\frac{1}{2\pi} \oint dS_a f^{ab} \partial_b \psi. \quad (3.8)$$

Hence, ψ has the following asymptotic form

$$\psi \longrightarrow 1 + \frac{M}{2r}, \quad (3.9)$$

and, by Eq.(3.6), M is given by

$$M = \int_{\Sigma} \left(\rho_H + \frac{1}{16\pi} \tilde{K}_b^a \tilde{K}_a^b \right) a^2(t) \psi^5 d^3x. \quad (3.10)$$

Imposing the weak energy condition $\rho_H \geq 0$, M is clearly non-negative.

In present case, as x^a is conformal Killing vector automatically, the first term of right hand side of Eq.(3.4) vanishes. Thus one gets

$$\Delta M = \frac{H}{a(t)} \int_{\Sigma} x^a J_a \gamma^{1/2} d^3x. \quad (3.11)$$

From Eq.(3.11), it is obvious that when $x^b J_b$ is negative, ΔM is also negative. Hence, M_{AD} is reduced by the “imploding” motion of matter and radiation fields.

Furthermore, it is obvious that there exist cases in which M_{AD} is negative since we can consider arbitrary large H so that $|\Delta M| > M$.

Here we shall roughly estimate the condition that M_{AD} is negative. Because of the dominant energy condition,

$$\rho_H^2 \geq J_a J^a \sim \left(\frac{x^a J_a}{r} \right)^2, \quad (3.12)$$

we obtain

$$|\Delta M| \simeq H L \times |x^a J_a / r| a^2 V \lesssim H L \times \rho_H a^2 V, \quad (3.13)$$

where L and V are recognized, respectively, as the “co-moving” length and volume scales of the region in which the sources exist. On the other hand, from Eqs.(3.10), we obtain

$$M = \int \left(\rho_H + \frac{1}{16\pi} \tilde{K}_b^a \tilde{K}_a^b \right) a^2(t) \psi^5 d^3x \gtrsim \rho_H a^2 V. \quad (3.14)$$

Hence we get the condition of $M < |\Delta M|$ as

$$L \gtrsim H^{-1}. \quad (3.15)$$

The above equation means that when M_{AD} is negative, the size of the matter or radiation source is comparable to or larger than the cosmological horizon scale. This condition is consistent with that of the linear analysis by Abbott and Deser in which the contribution to M_{AD} from the perturbation outside the cosmological horizon is negative.^[1]

4. Spherically Symmetric Source

In the case of the asymptotically flat space-time without Λ , “mass” is recognized not only as conserved energy but also as a source of the gravity. Hence there is a technical terminology such as *gravitational mass* for the ADM mass. The AD mass is the conserved quantity and also has a meaning of energy in a sense. However, here, a question arises: “Does the AD mass produce the gravitational field by the same manner as the asymptotically flat space-time?” In order to get the answer for this question, we shall investigate the space-time with a spherically symmetric source.

4.1 TIDAL FORCE OF THE SPHERICALLY SYMMETRIC SOURCE

In Newtonian theory, the tidal force is given by $\sim M_N/r^3$ with Newtonian gravitational mass M_N . Also in general relativity, the tidal force in the asymptotically flat space-time is given by the same form as in the Newtonian theory but now the Newtonian mass M_N should be replaced by the ADM mass M_{ADM} . In this sense, M_{ADM} deserves the title of the “gravitational mass”. More concretely speaking, a gravitational mass in general relativity is a source of the tidal force which produces a shear of timelike or null geodesic congruence and therefore it is given by the electric part E_b^a of the Weyl tensor $C_{\mu\nu\rho\sigma}$.

E_b^a in the vacuum region ($\rho_H = J_a = 0$) is given by

$$E_b^a \equiv C^a{}_{\mu b\nu} n^\mu n^\nu = {}^{(3)}R_b^a - H \tilde{K}_b^a - \tilde{K}_c^a \tilde{K}_{cb}, \quad (4.1)$$

where we have imposed the condition (3.3) to derive the above expressions. In the case of the spherically symmetric space-time, without loss of generality, the intrinsic metric is expressed in the form of Eq.(3.5), i.e.,

$$d\ell^2 = a^2(t)\psi^4(r,t)(dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2), \quad (4.2)$$

where we use the spherical polar coordinate. Assuming the constant-mean-curvature time slicing condition (3.3) and using the first equality of Eq.(2.8),

the AD mass is written as

$$M_{AD} = a(t)M + \lim_{r \rightarrow \infty} \frac{H}{2} a^3(t) r^3 \tilde{K}_r^r. \quad (4.3)$$

Hence the asymptotic form of E_b^a is given by

$$E_\theta^\theta = E_\varphi^\varphi = -\frac{1}{2} E_r^r \longrightarrow \frac{M}{a^2(t)r^3} + \frac{1}{2} H \tilde{K}_r^r = \frac{M_{AD}}{a^3(t)r^3}. \quad (4.4)$$

where we have used Eq.(3.8) and assumed $\tilde{K}_r^r \rightarrow \mathcal{O}(r^{-3})$ in order that M_{AD} is finite. The above equation shows that AD mass may deserve the title of the gravitational mass.³

4.2 MOTION OF AN INHOMOGENEOUS DUST SPHERE WITH NEGATIVE MASS

In order to see the effect of the sign of the AD mass on the evolution of matter, we consider here the Tolman-Bondi space-time with Λ , which describes the motion of an inhomogeneous dust sphere. The metric of this space-time can be written as following form by using the co-moving coordinates,

$$ds^2 = -dt^2 + X^2(r,t)dr^2 + Y^2(r,t)(d\theta^2 + \sin^2 \theta d\varphi^2). \quad (4.5)$$

Then from the Einstein equations, we obtain,

$$X = \frac{Y'}{W(r)}, \quad (4.6)$$

³ In the case of non-spherically symmetric case, the asymptotic behavior of E_b^a can be different from the spherically symmetric one. For example, the existence of linear momentum causes the dipole component of the extrinsic curvature as $\tilde{K}_r^r \sim P \cos \theta / (ar)^2$ by Eq.(3.7), where $P \sim \int J_z dv$ is the momentum of the system of z-direction, and hence $H \tilde{K}_r^r$ becomes leading term of E_r^r in the asymptotic region $r \rightarrow \infty$ since ${}^{(3)}R_r^r \sim M/(ar)^3$. However it should be noted that the isotropic component of the tidal force produced by M_{AD} is same as the spherically symmetric case, $M_{AD}/(ar)^3$. H. Sato has pointed out that this effect of the linear momentum may correspond to a kind of the “dragging of inertia”. Discussion about this problem will be published elsewhere.

$$\dot{Y}^2 = -V(Y; r) \equiv W^2 - 1 + \frac{2m(r)}{Y} + H^2 Y^2, \quad (4.7)$$

where $\cdot$ and $'$ are the derivative with respect to t and r , respectively. $W(r)$ is an arbitrary function which corresponds to the binding energy. $m(r)$ is also an arbitrary function which is related with the energy density $\rho(r, t)$ of the dust sphere by

$$m(r) = 4\pi \int_0^r \rho W(r) X Y^2 dr. \quad (4.8)$$

Hence $m(r)$ is regarded as the effective mass within $[0, r]$. Eq.(4.7) is regarded as the energy equation of a zero-energy particle in the potential $V(Y; r)$ in which the radial coordinate r is merely a parameter labeling a dust fluid element. Hence we can see the motion of a dust fluid element at r by investigating the potential $V(Y; r)$.

Here we assume for the density distribution as

$$\rho = \begin{cases} > 0, & 0 \leq r \leq r_s, \\ = 0, & r > r_s. \end{cases} \quad (4.9)$$

Assuming the 3-space is everywhere regular at $t = t_0$, the behavior of X , Y and W near the origin is given by the regularity

$$\begin{aligned} X(r, t_0) &\longrightarrow a(t_0)[1 + \mathcal{O}(r^3)], \\ Y(r, t_0) &\longrightarrow a(t_0)r[1 + \mathcal{O}(r^2)], \quad \text{for } r \longrightarrow 0 \\ W(r) &\longrightarrow 1 + \mathcal{O}(r), \end{aligned} \quad (4.10)$$

where $a(t)$ is given by Eq.(2.9). On the other hand, in order that the space-time is asymptotically de Sitter, the behavior of X , Y and W for $r \rightarrow \infty$ should be,

$$\begin{aligned} X(r, t_0) &\longrightarrow a(t_0), \\ Y(r, t_0) &\longrightarrow a(t_0)r, \quad \text{for } r \longrightarrow \infty \\ W(r) &\longrightarrow 1. \end{aligned} \quad (4.11)$$

Here we shall calculate the AD mass of this system. By Eq.(4.11), for $r \rightarrow \infty$,

$m = \text{constant}$ and Eq.(4.7) becomes as

$$\dot{Y}^2 = \frac{2m}{Y} + H^2 Y^2. \quad (4.12)$$

This equation is easily integrated and the solution consistent with Eq.(4.11) is obtained as

$$\begin{aligned} X &= a(t) \left(1 - \frac{m}{2H^2 a^3(t) r^3}\right)^{-1/3} \left(1 + \frac{m}{2H^2 a^3(t) r^3}\right), \\ Y &= a(t) r \left(1 - \frac{m}{2H^2 a^3(t) r^3}\right)^{2/3}. \end{aligned} \quad (4.13)$$

The extrinsic curvature K_a^b is then obtained as

$$\begin{aligned} K_r^r &= -H \left(1 - \frac{2m}{H^2 a^3(t) r^3} + \frac{m^2}{4H^4 a^6(t) r^6}\right) \left(1 - \frac{m^2}{4H^4 a^6(t) r^6}\right)^{-1}, \\ K_\theta^\theta = K_\varphi^\varphi &= -H \left(1 + \frac{m}{2H^2 a^3(t) r^3}\right) \left(1 - \frac{m}{2H^2 a^3(t) r^3}\right)^{-1}, \end{aligned} \quad (4.14)$$

and other components vanish. Using Eqs.(4.13) and (4.14), we can calculate the AD mass of this system by Eqs.(2.12), (2.13) and (2.19). From Eq.(4.13), we see that $h_{ab} \rightarrow \mathcal{O}(r^{-3})$ for $r \rightarrow \infty$ in the Cartesian coordinates and hence $M = 0$ by Eq.(2.13). On the other hand, ΔM becomes as

$$\Delta M = \frac{H}{8\pi} \lim_{r \rightarrow \infty} \int (K_r^r - K - 2H) a^2(t) r^3 \sin \theta d\theta d\varphi = \frac{m}{a(t)}. \quad (4.15)$$

Then, from Eq.(2.12), we obtain

$$M_{AD} = m(r \rightarrow \infty) = m(r \geq r_s). \quad (4.16)$$

Here it is worthy to note that the region $r \geq r_s$ of this space-time necessarily agrees with the Schwarzschild-de Sitter space-time by Birkhoff's theorem.

Assuming $W = 1$, we perform the following coordinate transformation;

$$\begin{aligned} dt &= dT + \frac{\dot{Y} dY}{1 - \dot{Y}^2}, \\ dr &= \frac{1}{Y'} \left(\dot{Y} dT + \frac{dY}{1 - \dot{Y}^2} \right). \end{aligned} \quad (4.17)$$

Then, using Eq.(4.6), we obtain the familiar static form for the metric of the Schwarzschild-de Sitter space-time as

$$\begin{aligned} ds^2 &= -(1 - \dot{Y}^2) dT^2 + \frac{dY^2}{1 - \dot{Y}^2} + Y^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \\ &= - \left(1 - \frac{2M_{AD}}{Y} - H^2 Y^2 \right) dT^2 + \left(1 - \frac{2M_{AD}}{Y} - H^2 Y^2 \right)^{-1} dY^2 \\ &\quad + Y^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \end{aligned} \quad (4.18)$$

where we have used Eqs.(4.12) and (4.16) in the second equality. Now it is obvious from the above equation that the mass parameter of the Schwarzschild-de Sitter space-time is just the AD mass.

Since the local motion of the fluid element depends only on the effective mass m , the qualitative feature of the dust motion is essentially the same as that of a homogeneous dust ball, which has been investigated in Ref.[12].⁴ However, $m < 0$ does never happen in the case of homogeneous dust sphere. Therefore in this paper we shall focus only on the case of $m < 0$ and investigate its effect on the motion of the dust fluid element.

First, we show in what situation m and M_{AD} become negative. In order that m is negative, W should be negative from Eq.(4.8) in some regions since ρ , X and Y are all non-negative quantities. Hence, for example, if we choose W and ρ

⁴ It should be noted that the inhomogeneity of the density distribution may change drastically the global structure of space-time. We can see that the naked singularity of the type discovered by Eardley and Smarr^[10], which does not appear in the homogeneous case, is realized even if there is a positive cosmological constant Λ . However, in this paper, we restrict the local motion of the dust fluid element and the discussion with respect to this problem will be appear elsewhere.^[11]

as in the Fig.1, $m(r)$ can become negative at some point within $(r_-, r_+]$, where $W = 0$ at $r = r_\pm$. Here it should be noted that only the combination of Y and m (or ρ) which makes $V(Y; r)$ non-positive is allowed (see Eq.(4.7)) and, by this constraint, we get an important relation. Assuming $m(r_+) \leq 0$, because of $V \leq 0$ at $r = r_+$, we obtain

$$|m(r_+)| \leq \frac{1}{2} Y(r_+, t_0) [H^2 Y^2(r_+, t_0) - 1]. \quad (4.19)$$

It is worthy to note that when $H = 0$, the above equation cannot be satisfied. This supports the fact that the gravitational mass cannot be negative in the asymptotically flat space-time. From Eq.(4.19), we can see that $Y(r_+, t_0) \geq H^{-1}$, where the equality can be satisfied only if $m = 0$. Since, $W < 0$, i.e., $Y' < 0$ in the region $(r_-, r_+]$, we obtain

$$Y > H^{-1} \quad \text{for the region of } m(r) < 0. \quad (4.20)$$

The above inequality should be compared with Eq.(3.15).

We can easily see that, in the case of $m < 0$, there is only one positive root $Y = Y_B$ for $V = 0$ and

$$V = \begin{cases} > 0 & Y < Y_B, \\ \leq 0 & Y \geq Y_B, \end{cases} \quad (4.21)$$

Hence the fluid element with $m < 0$ cannot reach the singularity $Y = 0$. Instead, it goes to $Y = \infty$ and the infinitesimal region occupied by this fluid element approaches to the de Sitter space-time ($\rho \rightarrow 0$) locally. The conformal diagram of this space-time is depicted in Fig.2 in which the central region of the dust sphere with $m > 0$ is omitted. As already mentioned, in this spherically symmetric model, the vacuum region ($\rho = 0$) agrees with the Schwarzschild-de Sitter space-time by Birkhoff's theorem and therefore in the case of $M_{AD} = m(r \rightarrow \infty) < 0$ the naked singularity exists as can be seen from Fig.2 (right vertical line, $Y = 0$). However, even if such a naked singularity exists, $t = \text{constant}$ hypersurface which

is denoted by dashed lines in Fig.2 does not intersect with this singularity. In other words, there is a regular Cauchy surface whose time development is also regular in this space-time. Hence one might regard this naked singularity as a kind of the initial singularity in the Friedmann universe model, which does not cause a serious problem in the evolution of the universe. Further, it is worthy to note that ADM mass in the asymptotically flat space-time is non-negative under the situation as the above in which there is a regular Cauchy surface and its time development is also regular.

From this result, we see that the negative mass produces just the repulsive force and makes the space-time to be de Sitter asymptotically and hence it seems that the negative AD mass is not obstacle to the cosmic no hair conjecture.

5. Summary and Discussion

In this paper, we have shown that AD mass in the de Sitter space-time can be negative and it has a meaning of the *gravitational mass* by investigating the spherically symmetric and asymptotically de Sitter space-time. Intuitively, the AD mass is reduced by the imploding motion of matter and radiation fields. When the AD mass is negative, the extent of the matter and radiation causing AD mass beyond over the cosmological horizon scale H^{-1} . Since the “timelike” Killing vector of the background de Sitter space-time is no longer timelike outside the cosmological horizon $a(t)r > H$, the configuration with $M_{AD} < 0$ seems to be non-stationary. Conversely the AD mass of the stationary space-time seems to be necessarily non-negative.

From the analysis of the inhomogeneous dust sphere with $M_{AD} < 0$, the dust fluid in the region suffered by the negative effective mass m necessarily expands to $Y \rightarrow \infty$ and such a region approaches to the de Sitter space-time asymptotically in local sense. Hence, although the AD mass can be negative, cosmic no hair conjecture holds.

From our investigation, one gets also important geometrical fact. We have proved positivity of AD mass of the family of the asymptotically de Sitter space-time in which there is at least one $K = -3H$ slice with conformal Killing vector x^a or with vanishing $\tilde{K}_b^a$. Conversely, this means that such a slice cannot exist in the space-time with negative AD mass.

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REFERENCES

1. L. Abbott and S. Deser, Nucl. Phys. **B195**, 76(1982).
2. R. Arnowitt, S. Deser and C.W. Misner, in *Gravitation an introduction current research*, ed. by L. Witten, N.Y., Wiley (1962).
3. G.W. Gibbons and S.W. Hawking, Phys. Rev. **D15** 2738 (1977);
S.W. Hawking and I.G. Moss, Phys. Lett. **110B**, 35 (1982);
as for a review, see also e.g. K. Maeda, “*Inflation and Cosmic No Hair Conjecture*”, in the Proceedings of 5th Marcel Grossmann Meeting, ed. by D. G. Blair and M. J. Buckingham (World Scientific, 1989) p.145.
4. P. Shoen and S.T. Yau, Comm. Math. Phys. **65**, 45 (1979); *ibid*, **79**, 231(1981).
5. E. Witten, Comm. Math. Phys. **80**, 381 (1981);
J.M. Nester, Phys. Lett. **83**, 241 (1981).
6. G.W. Gibbons, S.W. Hawking, G. Horowitz and M. Perry, Comm. Math. Phys. **88**, 295 (1983).
7. T. Shiromizu, Kyoto University preprint, KUNS-1231 (1993), submitted to Phys. Rev. **D**.

8. C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation* (Freeman,1973).
9. K. Nakao, GRG **24**, 1069(1992).
10. D.M. Eardley and L. Smarr, Phys. Rev. **D19**, 2239 (1979).
11. K. Maeda and K. Nakao, in preparation.
12. B. Carter, Commun. Math. Phys. **17**, 1067 (1970);
 B. Carter, in *Black Holes* ed. C. DeWitt and B.S. DeWitt (Gordon and Breach, New York, 1972).
 G.W. Gibbons and S.W. Hawking (Ref.1).

FIGURE CAPTIONS

1. The schematic diagram of representative configuration with negative AD mass. A peak of the density exists within the region $(r_-, r_+]$ in which $W < 0$. $r = r_-$ is the maximal surface while $r = r_+$ corresponds to the minimal surface.
2. The conformal diagram of the Tolman-Bondi space-time with $M_{AD} < 0$. The central region of $m > 0$ is omitted. The dashed lines correspond to $t = \text{constant}$ hypersurfaces which do not intersect the singularity $Y = 0$.

Fig. 1

Fig.2