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Towards a classification of finite field theories

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Abstract

We consider the finiteness-conditions on the Yukawa couplings of a general quantum field theory for gauge groups $SU(n)$ ($n > 6$) and a rather general particle content. It is shown that in the class of theories considered (149 different particle contents), only two models are able to fulfill the finiteness-conditions. Only one of these is supersymmetric. For the nonsupersymmetric one the appropriate Yukawa couplings are constructed explicitly.

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1 Introduction

In two recent papers [1][2] we analyzed the finiteness conditions on the Yukawa couplings (YC) in a most general quantum field theory (QFT). We considered not the original system of equations for the YC [3], which is cubic, but a system which emerges from this, if one requires a further restriction on the particle content of a theory. This restriction is called the 'F=1' condition, where F designates a quantity completely determined by the particle content [4]. Although $F=1$ has not been strictly proved to represent a necessary condition for finiteness, it seems to be worthwhile to us to analyze this class of theories, because of the large symmetry of the emerging equations, and also, because it is a class of theories which includes all supersymmetric (SUSY) ones [5]-[13], but contains a lot more theories than the SUSY class. In fact $F \leq 1$ has been proved [4]. It is also worth noting that the one loop finite $F=1$ theories automatically are two loop finite with respect to the gauge coupling [4]. The following general result was found in [1][2]: *If the gauge group is $SU(n)$, and if we restrict ourselves to the seven lowest dimensional representations, then at least one adjoint representation has to appear either in the fermionic or in the bosonic sector, or roughly speaking: The class of finite theories, which contains only the fundamental, the symmetric and the antisymmetric representations with their complex conjugates (c.c.) is empty.*

Naturally one can ask what happens, if the adjoint representation is permitted. Therefore the class of theories we consider in the present paper is defined as follows: In the fermionic representation R_F we allow the fundamental, the symmetric and the antisymmetric representations with their c.c. and the adjoint representation. In the bosonic sector we limit ourselves to the first six of the above irreducible representation (irreps). The gauge group will be $SU(n)$ ($n > 6$). We do not consider the other n-s by the following reasons: First, the groups $SU(3)$, $SU(5)$ and $SU(6)$ admit invariant tensors, which do not appear in the general $SU(n)$ [1][15]. The resulting equations for the YC become more complicated and need further investigation. Second, in $SU(4)$ the exception arises that the antisymmetric irrep is equivalent to its c.c., and our proofs will not apply in this case. Third, $SU(2)$ is isomorphic to $SO(3)$ and does not contain unitary irreps, and should therefore be considered in connection with $SO(n)$.

This class of theories is very large, and it is surprising that we are able to show that only two theories 'survive', after imposing the finiteness conditions on the YC. Only one of these two is a supersymmetric one.

We consider the most general gauge theories for simple Lie groups, containing 2-component Weyl-spinors ψ_i and real scalars ϕ^α in (reducible) representations $(R_F)_{ij}$ and $(R_B)^{\alpha\beta}$, respectively. The vector bosons A_μ^a are in the adjoint representation R_{ad} and a covariant R_ξ -gauge is used. The Lagrangian is [16]

$$\mathcal{L} = \mathcal{L}^{\text{Y.M.}} + \mathcal{L}^{\text{gauge br.}} + \mathcal{L}^{\text{masses}} + \mathcal{L}^{\phi^3} + \mathcal{L}' \quad (1)$$

$$\mathcal{L}' = -\frac{1}{2}\psi_i\psi_j Y_{ij}^\alpha \phi^\alpha - \frac{1}{2}\bar{\psi}_i\bar{\psi}_j Y_{ij}^{\alpha*} \phi^\alpha - \frac{1}{24}V^{\alpha\beta\gamma\delta}\phi^\alpha\phi^\beta\phi^\gamma\phi^\delta. \quad (2)$$

All couplings have to be gauge-invariant and completely symmetric tensors. In the case of the YC, which will be considered exclusively in this paper, this means

$$(T_F^a)_{ik}Y_{kj}^\alpha + (T_F^a)_{jk}Y_{ik}^\alpha + (T_B^a)^{\alpha\beta}Y_{ij}^\beta = 0, \quad Y_{ij}^\alpha = Y_{ji}^\alpha. \quad (3)$$

T_F^a and T_B^a denote the generators of the gauge-transformation in the fermionic and bosonic representation, respectively.

A necessary condition for finiteness is, of course, the vanishing of the gauge coupling β -function to first and second loop order requiring [3]

$$22c_g - 4S_F - S_B = 0, \quad (4)$$

respectively

$$3Tr(C_F Y^{\beta\dagger} Y^\beta) - g^2 d_g (6Q_F + 6Q_B + c_g(10S_F + S_B - 34c_g)) = 0. \quad (5)$$

We define the Dynkin indices S_F , S_B and the Casimir operators C_F , C_B by

$$Tr(T^a T^b) = \delta^{ab} S \text{ and } C = T^a T^a. \quad (6)$$

d_g and c_g denote the dimension of the group and the Casimir eigenvalue of the adjoint representation, respectively. Finally Q_F and Q_B are defined by $Q = \frac{1}{d_g} Tr C^2$. From the F=1 condition for the particle content

$$Q_B - 2Q_F + c_g(S_F - 2c_g) = 0, \quad (7)$$

for the YC the conditions

$$Y_{ij}^\beta Y_{kl}^\beta + Y_{ik}^\beta Y_{jl}^\beta + Y_{il}^\beta Y_{jk}^\beta = 0 \quad \forall i, j, k, l \quad (8)$$

$$Tr(Y^{\alpha\dagger} Y^\beta) = \delta^{\alpha\beta} y^\alpha \quad \forall \alpha, \beta \quad (9)$$

$$(Y^{\beta\dagger} Y^\beta)_{ij} = \delta_{ij} c_i \quad \forall i, j \quad (10)$$

follow [4], where $c_i \delta_{ij} = 6g^2 (C_F)_{ij}$ holds and where y^α are real numbers obeying $y^\alpha \geq 0$. All results are obtained by dimensional regularisation [17].

Gauge-invariance of the YC forces the ansatz

$$Y_{ij}^\alpha = Z_{I_s J_t}^{A_r} J_{i_s j_t}^{\alpha_r}. \quad (11)$$

As in [1] r,s,t denote the irreducible representations (irreps). In the fermionic sector these indices take the values 1, ..., 7 corresponding to: the fundamental representation, its complex conjugate (c.c), the antisymmetric representation, its c.c, the symmetric representation, its c.c. and finally the adjoint representation. In the bosonic sector they take the values 1, ..., 3 corresponding to the direct sum of the fundamental irrep and its c.c., antisymmetric irrep and its c.c. and finally to the direct sum of the symmetric irrep with its c.c. This duplication is needed in order to get real bosons. J is an invariant tensor corresponding to the subindices r,s,t, and Z denotes a 'three-dimensional' matrix of complex numbers. The range of indices for A,I,J is determined by the number of occurrences of the several irreps in the bosonic, respectively fermionic sector. A list of all invariants needed can be found in [15] or [1].

In the second section we prove that only two theories are able to fulfill the modified finiteness conditions for the YC (8)-(10). In the third section we explicitly construct these couplings for the nonsupersymmetric model.

2 Proof of the Theorem

First of all a general lemma is given, which will be used in the sequel. We already used this lemma in a similar form in [2], but because of its importance and in order to be selfcontained, we give here a brief sketch of the proof.

Lemma 2.1 : Let $a_{i_1}^\alpha, b_{i_2}^\alpha, c_{i_3}^\alpha$ and $d_{i_4}^\alpha$ be four complex matrices of dimensions $n_1 \times m, n_2 \times m, n_3 \times m$ and $n_4 \times m$. These matrices are assumed to fulfill the set of equations

$$a_{i_1}^{\alpha*} a_{i_1}^\beta + k b_{i_2}^{\alpha*} b_{i_2}^\beta = \text{diag}(x^\alpha) \quad (12)$$

$$c_{i_3}^{\alpha*} c_{i_3}^\beta + k d_{i_4}^{\alpha*} d_{i_4}^\beta = \text{diag}(x^\alpha) \quad (13)$$

$$a_{i_1}^\alpha c_{j_2}^\alpha = b_{i_2}^\alpha d_{j_4}^\alpha = 0, \quad (14)$$

where k is a real constant, and $diag(x^\alpha)$ denotes a diagonal matrix with the diagonal elements x^α . If we define $m \times m$ matrices

$$X^{\alpha\beta} = a_{i_1}^{\beta*} a_{i_1}^\alpha \quad Y^{\beta\alpha} = c_{i_3}^{\beta*} c_{i_3}^\alpha \quad (15)$$

$$U^{\alpha\beta} = b_{i_2}^{\beta*} b_{i_2}^\alpha \quad V^{\beta\alpha} = c_{i_4}^{\beta*} d_{i_4}^\alpha, \quad (16)$$

then the following assertion is true : There is a unitary matrix T , so that the matrices X', Y', U' and V' , defined by $X' = T X T^\dagger, Y' = T Y T^\dagger, U' = T U T^\dagger, V' = T V T^\dagger$ have the following form

$$X' = diag(x_1, \dots, x_s, 0, \dots, 0) \quad Y' = diag(0, \dots, 0, x_{s+1}, \dots, x_m) \quad (17)$$

$$V' = \frac{1}{k} X' \quad U' = \frac{1}{k} Y', \quad (18)$$

where $0 \leq s \leq m$ holds.

Proof: We can conclude from (14) that $XY = 0$ holds and this implies that we can simultaneously diagonalize X and Y by a unitary matrix, because X and Y are positive semidefinite hermitian matrices. The same is true for U and V .

Using now (12) and (13) we arrive at

$$X - Y = k(V - U).$$

Diagonalizing X and Y yields

$$X' - Y' = bdiag(x_1 1, \dots, x_s 1, -y_1 1, \dots, -y_t 1) = -kU' + kV',$$

where $bdiag$ denotes a blockdiagonal matrix, 1 denotes the corresponding unit matrix and

$$x_1, \dots, x_s, y_1, \dots, y_t > 0$$

are the eigenvalues of X , respectively Y . Now we change the basis so that U'' and V'' are diagonal and $X' - Y'$ remains invariant, i.e.

$$X'' - Y'' = X' - Y' = k(V'' - U'').$$

Obviously a unitary matrix T which performs this has to be of the form [2]

$$T = bdiag(K_1, \dots, K_{s+t})$$

with the several K -s unitary. Such a matrix T , in turn, leaves X' and Y' separately invariant, proving our assertion.

What we actually need in the subsequence is the

Corollary 2.1 : For the above defined matrices the relations

$$Tr(V) = \frac{1}{k} Tr(X) \quad Tr(U) = \frac{1}{k} Tr(Y) \quad (19)$$

hold.

Now we return to the finiteness conditions (8)-(10) on the YC. Inserting the ansatz (11) into (8), contracting the invariant tensors J and comparing coefficients, we obtain the following lengthy set of equations for the Z -s [15]

$$Z_{I_1 J_4}^{A_1} Z_{K_2 L_3}^{A_1} = Z_{I_1 J_6}^{A_1} Z_{K_2 L_3}^{A_1} = 0 \quad (20)$$

$$Z_{I_1 J_4}^{A_1} Z_{K_2 L_5}^{A_1} = Z_{I_1 J_6}^{A_1} Z_{K_2 L_5}^{A_1} = 0 \quad (21)$$

$$Z_{I_1 J_7}^{A_1} Z_{K_2 L_7}^{A_1} = 0 \quad (22)$$

$$Z_{I_3 K_7}^{A_2} Z_{J_4 L_7}^{A_2} + Z_{I_3 L_7}^{A_2} Z_{J_4 K_7}^{A_2} = 0 \quad (23)$$

$$Z_{I_5 K_7}^{A_2} Z_{J_6 L_7}^{A_2} + Z_{I_5 L_7}^{A_2} Z_{J_6 K_7}^{A_2} = 0 \quad (24)$$

$$Z_{I_3 K_7}^{A_2} Z_{J_6 L_7}^{A_2} - Z_{I_3 L_7}^{A_2} Z_{J_6 K_7}^{A_2} = 0 \quad (25)$$

$$Z_{I_4 K_7}^{A_2} Z_{J_5 L_7}^{A_2} - Z_{I_4 L_7}^{A_2} Z_{J_5 K_7}^{A_2} = 0 \quad (26)$$

$$Z_{I_1 K_1}^{A_2} Z_{J_2 L_2}^{A_2} = 0 \quad (27)$$

$$Z_{I_3 K_7}^{A_3} Z_{J_4 L_7}^{A_3} + Z_{I_3 L_7}^{A_3} Z_{J_4 K_7}^{A_3} = 0 \quad (28)$$

$$Z_{I_5 K_7}^{A_3} Z_{J_6 L_7}^{A_3} + Z_{I_5 L_7}^{A_3} Z_{J_6 K_7}^{A_3} = 0 \quad (29)$$

$$Z_{I_3 K_7}^{A_3} Z_{J_6 L_7}^{A_3} - Z_{I_3 L_7}^{A_3} Z_{J_6 K_7}^{A_3} = 0 \quad (30)$$

$$Z_{I_4 K_7}^{A_3} Z_{J_5 L_7}^{A_3} - Z_{I_4 L_7}^{A_3} Z_{J_5 K_7}^{A_3} = 0 \quad (31)$$

$$Z_{I_1 K_1}^{A_3} Z_{J_2 L_2}^{A_3} = 0 \quad (32)$$

together with

$$Z_{I_1 L_7}^{A_1} Z_{J_1 K_4}^{A_1} + Z_{I_1 J_1}^{A_2} Z_{K_4 L_7}^{A_2} + Z_{I_1 J_1}^{A_3} Z_{K_4 L_7}^{A_3} = 0 \quad (33)$$

$$Z_{I_1 L_7}^{A_1} Z_{J_1 K_6}^{A_1} + Z_{I_1 J_1}^{A_2} Z_{K_6 L_7}^{A_2} + Z_{I_1 J_1}^{A_3} Z_{K_6 L_7}^{A_3} = 0 \quad (34)$$

$$Z_{I_2 L_7}^{A_1} Z_{J_2 K_3}^{A_1} - Z_{I_2 J_2}^{A_2} Z_{K_3 L_7}^{A_2} + Z_{I_2 J_2}^{A_3} Z_{K_3 L_7}^{A_3} = 0 \quad (35)$$

$$Z_{I_2 L_7}^{A_1} Z_{J_2 K_5}^{A_1} - Z_{I_2 J_2}^{A_2} Z_{K_5 L_7}^{A_2} + Z_{I_2 J_2}^{A_3} Z_{K_5 L_7}^{A_3} = 0. \quad (36)$$

From (9) the system

$$(2n^2 - 2n)Z_{I_1 K_4}^{A_1*} Z_{I_1 K_4}^{B_1} + (2n^2 + 2n)Z_{I_1 K_6}^{A_1*} Z_{I_1 K_6}^{B_1} + (2n^2 - 2)Z_{I_2 K_7}^{A_1*} Z_{I_2 K_7}^{B_1} = \text{diag}(\kappa^{A_1}) \quad (37)$$

$$(2n^2 - 2n)Z_{I_2 K_3}^{A_1*} Z_{I_2 K_3}^{B_1} + (2n^2 + 2n)Z_{I_2 K_5}^{A_1*} Z_{I_2 K_5}^{B_1} + (2n^2 - 2)Z_{I_1 K_7}^{A_1*} Z_{I_1 K_7}^{B_1} = \text{diag}(\kappa^{A_1}) \quad (38)$$

$$(n^2 - n)Z_{I_1 K_1}^{A_2*} Z_{I_1 K_1}^{B_2} + (2n^3 - 4n^2 - 2n + 4)Z_{I_3 K_7}^{A_2*} Z_{I_3 K_7}^{B_2} + (2n^3 - 2n)Z_{I_5 K_7}^{A_2*} Z_{I_5 K_7}^{B_2} = \text{diag}(\lambda^{A_2}) \quad (39)$$

$$(n^2 - n)Z_{I_2 K_2}^{A_2*} Z_{I_2 K_2}^{B_2} + (2n^3 - 4n^2 - 2n + 4)Z_{I_4 K_7}^{A_2*} Z_{I_4 K_7}^{B_2} + (2n^3 - 2n)Z_{I_6 K_7}^{A_2*} Z_{I_6 K_7}^{B_2} = \text{diag}(\lambda^{A_2}) \quad (40)$$

$$(n^2 + n)Z_{I_1 K_1}^{A_3*} Z_{I_1 K_1}^{B_3} + (2n^3 - 2n)Z_{I_3 K_7}^{A_3*} Z_{I_3 K_7}^{B_3} + (2n^3 + 4n^2 - 2n - 4)Z_{I_5 K_7}^{A_3*} Z_{I_5 K_7}^{B_3} = \text{diag}(\mu^{A_3}) \quad (41)$$

$$(n^2 + n)Z_{I_2 K_2}^{A_3*} Z_{I_2 K_2}^{B_3} + (2n^3 - 2n)Z_{I_4 K_7}^{A_3*} Z_{I_4 K_7}^{B_3} + (2n^3 + 4n^2 - 2n - 4)Z_{I_6 K_7}^{A_3*} Z_{I_6 K_7}^{B_3} = \text{diag}(\mu^{A_3}) \quad (42)$$

emerges. Finally we obtain from (10)

$$(2n - 2)Z_{I_1 K_4}^{A_1*} Z_{J_1 K_4}^{A_1} + (2n + 2)Z_{I_1 K_6}^{A_1*} Z_{J_1 K_6}^{A_1} + (2n - \frac{2}{n})Z_{I_1 K_7}^{A_1*} Z_{J_1 K_7}^{A_1} + \\ (2n - 2)Z_{I_1 K_1}^{A_2*} Z_{J_1 K_1}^{A_2} + (2n + 2)Z_{I_1 K_1}^{A_3*} Z_{J_1 K_1}^{A_3} = c_1 \delta_{I_1 J_1} \quad (43)$$

$$(2n - 2)Z_{I_2 K_3}^{A_1*} Z_{J_2 K_3}^{A_1} + (2n + 2)Z_{I_2 K_5}^{A_1*} Z_{J_2 K_5}^{A_1} + (2n - \frac{2}{n})Z_{I_2 K_7}^{A_1*} Z_{J_2 K_7}^{A_1} + \\ (2n - 2)Z_{I_2 K_2}^{A_2*} Z_{J_2 K_2}^{A_2} + (2n + 2)Z_{I_2 K_2}^{A_3*} Z_{J_2 K_2}^{A_3} = c_1 \delta_{I_2 J_2} \quad (44)$$

$$4Z_{K_2 I_3}^{A_1*} Z_{K_2 J_3}^{A_1} + (4n - 4 - \frac{8}{n})Z_{I_3 K_7}^{A_2*} Z_{J_3 K_7}^{A_2} + (4n + 4)Z_{I_3 K_7}^{A_3*} Z_{J_3 K_7}^{A_3} = c_2 \delta_{I_3 J_3} \quad (45)$$

$$4Z_{K_1 I_4}^{A_1*} Z_{K_1 J_4}^{A_1} + (4n - 4 - \frac{8}{n})Z_{I_4 K_7}^{A_2*} Z_{J_4 K_7}^{A_2} + (4n + 4)Z_{I_4 K_7}^{A_3*} Z_{J_4 K_7}^{A_3} = c_2 \delta_{I_4 J_4} \quad (46)$$

$$4Z_{K_2 I_5}^{A_1*} Z_{K_2 J_5}^{A_1} + (4n - 4)Z_{I_5 K_7}^{A_2*} Z_{J_5 K_7}^{A_2} + (4n + 4 - \frac{8}{n})Z_{I_5 K_7}^{A_3*} Z_{J_5 K_7}^{A_3} = c_3 \delta_{I_5 J_5} \quad (47)$$

$$4Z_{K_1 I_6}^{A_1*} Z_{K_1 J_6}^{A_1} + (4n - 4)Z_{I_6 K_7}^{A_2*} Z_{J_6 K_7}^{A_2} + (4n + 4 - \frac{8}{n})Z_{I_6 K_7}^{A_3*} Z_{J_6 K_7}^{A_3} = c_3 \delta_{I_6 J_6} \quad (48)$$

$$\begin{aligned}
& 2Z_{K_1 I_7}^{A_1*} Z_{K_1 J_7}^{A_1} + 2Z_{K_2 I_7}^{A_1*} Z_{K_2 J_7}^{A_1} + (2n-4)Z_{K_3 I_7}^{A_2*} Z_{K_3 J_7}^{A_2} + (2n-4)Z_{K_4 I_7}^{A_2*} Z_{K_4 J_7}^{A_2} + \\
& 2nZ_{K_5 I_7}^{A_2*} Z_{K_5 J_7}^{A_2} + 2nZ_{K_6 I_7}^{A_2*} Z_{K_6 J_7}^{A_2} + 2nZ_{K_3 I_7}^{A_3*} Z_{K_3 J_7}^{A_3} + 2nZ_{K_4 I_7}^{A_3*} Z_{K_4 J_7}^{A_3} + \\
& (2n+4)Z_{K_5 I_7}^{A_3*} Z_{K_5 J_7}^{A_3} + (2n+4)Z_{K_6 I_7}^{A_3*} Z_{K_6 J_7}^{A_3} = c_4 \delta_{I_7 J_7}.
\end{aligned} \tag{49}$$

Taking the trace of eqs. (37) - (49) and using the definitions

$$A = (4n^2 - 4n)Z_{I_1 K_4}^{A_1*} Z_{I_1 K_4}^{A_1} \quad F = (4n^2 - 4n)Z_{I_2 K_3}^{A_1*} Z_{I_2 K_3}^{A_1} \tag{50}$$

$$B = (4n^2 + 4n)Z_{I_1 K_6}^{A_1*} Z_{I_1 K_6}^{A_1} \quad G = (4n^2 + 4n)Z_{I_2 K_5}^{A_1*} Z_{I_2 K_5}^{A_1} \tag{51}$$

$$C = (4n^2 - 4)Z_{I_1 K_7}^{A_1*} Z_{I_1 K_7}^{A_1} \quad H = (4n^2 - 4)Z_{I_2 K_7}^{A_1*} Z_{I_2 K_7}^{A_1} \tag{52}$$

$$D = (2n^2 - 2n)Z_{I_1 K_1}^{A_2*} Z_{I_1 K_1}^{A_2} \quad I = (2n^2 - 2n)Z_{I_2 K_2}^{A_2*} Z_{I_2 K_2}^{A_2} \tag{53}$$

$$E = (2n^2 + 2n)Z_{I_1 K_1}^{A_3*} Z_{I_1 K_1}^{A_3} \quad J = (2n^2 + 2n)Z_{I_2 K_2}^{A_3*} Z_{I_2 K_2}^{A_3} \tag{54}$$

$$K = (4n^2 - 4n - 8)(n-1)Z_{I_3 K_7}^{A_2*} Z_{I_3 K_7}^{A_2} \quad L = 4(n^3 - n)Z_{I_3 K_7}^{A_3*} Z_{I_3 K_7}^{A_3} \tag{55}$$

$$M = (4n^2 - 4n - 8)(n-1)Z_{I_4 K_7}^{A_2*} Z_{I_4 K_7}^{A_2} \quad N = 4(n^3 - n)Z_{I_4 K_7}^{A_3*} Z_{I_4 K_7}^{A_3} \tag{56}$$

$$P = 4(n^3 - n)Z_{I_5 K_7}^{A_2*} Z_{I_5 K_7}^{A_2} \quad Q = (4n^2 + 4n - 8)(n+1)Z_{I_5 K_7}^{A_3*} Z_{I_5 K_7}^{A_3} \tag{57}$$

$$R = 4(n^3 - n)Z_{I_6 K_7}^{A_2*} Z_{I_6 K_7}^{A_2} \quad S = (4n^2 + 4n - 8)(n+1)Z_{I_6 K_7}^{A_3*} Z_{I_6 K_7}^{A_3}, \tag{58}$$

we get the system of equations

$$A + B + H = C + F + G \tag{59}$$

$$D + K + P = I + M + R \tag{60}$$

$$E + L + Q = J + N + S \tag{61}$$

$$\frac{1}{2}(A + B + C) + D + E = c_1 i_1 n \tag{62}$$

$$\frac{1}{2}(F + G + H) + I + J = c_1 i_2 n \tag{63}$$

$$\frac{1}{2}(F + K + L) = c_2 i_3 \frac{n^2 - n}{2} \tag{64}$$

$$\frac{1}{2}(A + M + N) = c_2 i_4 \frac{n^2 - n}{2} \tag{65}$$

$$\frac{1}{2}(G + P + Q) = c_3 i_5 \frac{n^2 + n}{2} \tag{66}$$

$$\frac{1}{2}(B + R + S) = c_3 i_6 \frac{n^2 + n}{2} \tag{67}$$

$$\frac{1}{2}(C + H + K + M + P + R + L + N + Q + S) = c_4 i_7 (n^2 - 1), \tag{68}$$

where i_1 denotes the number of occurrences of R_1 in R_F , i_2 the number of occurrences of $\bar{R}_1$, and so on. Combining two lower indices into one and two Z-s into one

$$Z_{i_1}^{A_1} = (\sqrt{2n^2 - 2n}Z_{I_1 K_4}^{A_1}, \sqrt{2n^2 + 2n}Z_{I_1 K_6}^{A_1}) \tag{69}$$

$$Z_{i_2}^{A_1} = (\sqrt{2n^2 - 2n}Z_{I_2 K_3}^{A_1}, \sqrt{2n^2 + 2n}Z_{I_2 K_5}^{A_1}) \tag{70}$$

$$Z_{i_3}^{A_1} = \sqrt{2n^2 - 2}Z_{I_1 K_7}^{A_1} \tag{71}$$

$$Z_{i_4}^{A_1} = \sqrt{2n^2 - 2}Z_{I_2 K_7}^{A_1}, \tag{72}$$

we can infer from (37),(38) respectively (20)-(22)

$$Z_{i_1}^{A_1*} Z_{i_1}^{B_1} + Z_{i_4}^{A_1*} Z_{i_4}^{B_1} = \text{diag}(\kappa^{A_1}) \tag{73}$$

$$Z_{i_2}^{A_1*} Z_{i_2}^{B_1} + Z_{i_3}^{A_1*} Z_{i_3}^{B_1} = \text{diag}(\kappa^{A_1}) \tag{74}$$

respectively

$$Z_{i_1}^{A_1} Z_{j_2}^{A_1} = 0 \quad (75)$$

$$Z_{i_3}^{A_1} Z_{j_4}^{A_1} = 0. \quad (76)$$

We use now corollary 2.1 to conclude that

$$A + B = C \quad (77)$$

$$F + G = H \quad (78)$$

have to hold. Inserting this into eqs. (59)-(68), we see that the resulting matrix does not have full rank. Therefore the right hand side of the system has to obey the consistency relation

$$c_2(i_3 + i_4) \frac{n^2 - n}{2} + c_3(i_5 + i_6) \frac{n^2 + n}{2} = c_4 i_7 (n^2 - 1) \quad (79)$$

Using now the following Casimir eigenvalues of the groups $SU(n)$

$$c_1 = \frac{(n-1)(n+1)}{n} \quad (80)$$

$$c_2 = \frac{2(n-2)(n+1)}{n} \quad (81)$$

$$c_3 = \frac{2(n-1)(n+2)}{n} \quad (82)$$

$$c_4 = 2n, \quad (83)$$

and the abbreviations

$$n_1 = i_1 + i_2 \quad n_2 = i_3 + i_4 \quad n_3 = i_5 + i_6 \quad n_4 = i_7, \quad (84)$$

we end up with the relation

$$(n-2)n_2 + (n+2)n_3 = 2nn_4 \quad (85)$$

There are 149 different models fulfilling the eqs. (4),(7) and obeying the restriction that we do not allow the adjoint irrep in R_B . Demanding in addition eq. (85) gives the following result: There is no solution of (85),(4) and (7) for $n > 6$, which is not in the set

$$\{(n_2, n_3, n_4) = (1, 1, 1), (2, 2, 2)\}. \quad (86)$$

It should be emphasized that these solutions are valid for all $SU(n)$. If we do not, for the moment, distinguish between an irrep and its c.c., then the first of the above solution incorporates 3 possible models, in the sequel denoted by l_1, l_2, l_3 , and the second one only one denoted by $\tilde{l}$. The corresponding values for n_1 are $4n-4, 4n, 4n+4$ and 0 for the model $\tilde{l}$. For later use we also give the particle content in the bosonic sector of the models l_i . They are $R_B = (4n+12)(R_1 + \bar{R}_1) + 2(R_2 + \bar{R}_2)$, $R_B = 4n(R_1 + \bar{R}_1) + (R_2 + \bar{R}_2) + (R_3 + \bar{R}_3)$, $R_B = (4n-12)(R_1 + \bar{R}_1) + 2(R_3 + \bar{R}_3)$, respectively. For l_1, l_2, l_3 we shall prove now that only l_2 , which corresponds to the supersymmetric particle content given already in [6], is able to fulfill the finiteness conditions for the YC. On the other hand we will explicitly construct YC for the model $\tilde{l}$, which is definitely not supersymmetric in section 3.

In our discussion of the models l_1, l_2, l_3 we distinguish two cases:

1. $R_2, R_3 \notin R_F, \bar{R}_2, \bar{R}_3 \in R_F$
(The case $R_2, R_3 \in R_F, \bar{R}_2, \bar{R}_3 \notin R_F$ can be settled in the same manner.)

2. $R_2, \bar{R}_3 \in R_F, \bar{R}_2, R_3 \notin R_F$

As the corresponding field theories have to be free of anomalies, we have the additional condition

$$2i_1 + (2n - 8)i_3 + (2n + 8)i_5 = n_1 + (n - 4)n_2 + (n + 4)n_3. \quad (87)$$

In the first case and for the theory l_1 the remaining variables in (59)-(68) are A,B,C,D,H,I,M,R. The resulting system does not have full rank and we get the consistency relation for the right hand side

$$i_1 - i_2 = 4n. \quad (88)$$

This is an obvious contradiction to

$$i_1 - i_2 = 2n, \quad (89)$$

which follows from (87). For l_3 the proof is completely analogous and so we remain with case 2.

In this case the remaining variables are B,C,D,K,R,F,H and I. Again we will consider only theory l_1

The eqs. (20),(22),(25),(27),(37)-(40) imply that the system

$$Z_{I_1 J_6}^{A_1} Z_{K_2 L_3}^{A_1} = 0 \quad (90)$$

$$Z_{I_1 J_7}^{A_1} Z_{K_2 L_7}^{A_1} = 0 \quad (91)$$

$$Z_{I_3 L_7}^{A_2} Z_{J_6 K_7}^{A_2} = 0 \quad (92)$$

$$Z_{I_1 K_1}^{A_2} Z_{J_2 L_2}^{A_2} = 0 \quad (93)$$

$$(2n^2 + 2n)Z_{I_1 K_6}^{A_1*} Z_{I_1 K_6}^{B_1} + (2n^2 - 2)Z_{I_2 K_7}^{A_1*} Z_{I_2 K_7}^{B_1} = \text{diag}(\kappa^{A_1}) \quad (94)$$

$$(2n^2 - 2n)Z_{I_2 K_3}^{A_1*} Z_{I_2 K_3}^{B_1} + (2n^2 - 2)Z_{I_1 K_7}^{A_1*} Z_{I_1 K_7}^{B_1} = \text{diag}(\kappa^{A_1}) \quad (95)$$

$$(n^2 - n)Z_{I_1 K_1}^{A_2*} Z_{I_1 K_1}^{B_2} + (2n^3 - 4n^2 - 2n + 4)Z_{I_3 K_7}^{A_2*} Z_{I_3 K_7}^{B_2} = \text{diag}(\lambda^{A_2}) \quad (96)$$

$$(n^2 - n)Z_{I_2 K_2}^{A_2*} Z_{I_2 K_2}^{B_2} + (2n^3 - 2n)Z_{I_6 K_7}^{A_2*} Z_{I_6 K_7}^{B_2} = \text{diag}(\lambda^{A_2}) \quad (97)$$

has to be fulfilled. It should be noted that eq. (92) holds instead of (25) because $n_3 = 0$ is valid.

We can use now for the eqs. (90),(91),(94),(95), respectively for (92),(93),(96),(97) Corollary 2.1, implying

$$B = C \quad H = F \quad D = R \quad K = I \quad (98)$$

Inserting this in the system (59)-(68), we get, amongst others, the following consistency relation for the right hand side

$$c_1 i_1 n = 2c_3 i_6 \frac{n(n+1)}{2} \quad (99)$$

and finally $i_1 = 2n + 4$. But because of the anomaly condition (87) $i_1 = 2n + 2$ has to hold, and we have the desired contradiction. As the same method works in l_3 , we remain with the supersymmetric solution l_2 .

3 Explicit construction of the YC for a non supersymmetric model

In this section we are going to construct explicitly the YC for the model $\tilde{l}$, mentioned in section 2. Using the "anomaly equation" (87), its particle content is

$$R_F = R_2 + \bar{R}_2 + R_3 + \bar{R}_3 + 2R_4 \quad (100)$$

$$R_B = 3(R_2 + \bar{R}_2) + 3(R_3 + \bar{R}_3). \quad (101)$$

The indices I_3, I_4, I_5, I_6 take only one value, and therefore it is suggestive to define the following abbreviations for the Z-s

$$a_{K_7 A_2} = Z_{I_3 K_7}^{A_2} \quad \alpha_{K_7 A_3} = Z_{I_3 K_7}^{A_3} \quad (102)$$

$$b_{K_7 A_2} = Z_{I_4 K_7}^{A_2} \quad \beta_{K_7 A_3} = Z_{I_4 K_7}^{A_3} \quad (103)$$

$$c_{K_7 A_2} = Z_{I_5 K_7}^{A_2} \quad \gamma_{K_7 A_3} = Z_{I_5 K_7}^{A_3} \quad (104)$$

$$d_{K_7 A_2} = Z_{I_6 K_7}^{A_2} \quad \delta_{K_7 A_3} = Z_{I_6 K_7}^{A_3} \quad (105)$$

The remaining eqs. of the system (20)-(49) can now be written as

$$ab^T + ba^T = cd^T + dc^T = 0 \quad (106)$$

$$ad^T - da^T = bc^T - cb^T = 0 \quad (107)$$

$$\alpha\beta^T + \beta\alpha^T = \gamma\delta^T + \delta\gamma^T = 0 \quad (108)$$

$$\alpha\delta^T - \delta\alpha^T = \beta\gamma^T - \gamma\beta^T = 0, \quad (109)$$

respectively

$$(2n^3 - 4n^2 - 2n + 4)a^\dagger a + (2n^3 - 2n)c^\dagger c = \text{diag}(\lambda^{A_2}) \quad (110)$$

$$(2n^3 - 4n^2 - 2n + 4)b^\dagger b + (2n^3 - 2n)d^\dagger d = \text{diag}(\lambda^{A_2}) \quad (111)$$

$$(2n^3 - 2n)\alpha^\dagger \alpha + (2n^3 + 4n^2 - 2n - 4)\gamma^\dagger \gamma = \text{diag}(\mu^{A_3}) \quad (112)$$

$$(2n^3 - 2n)\beta^\dagger \beta + (2n^3 + 4n^2 - 2n - 4)\delta^\dagger \delta = \text{diag}(\mu^{A_3}) \quad (113)$$

and finally

$$(4n - 4 - \frac{8}{n})\text{Tr}(a^\dagger a) + (4n + 4)\text{Tr}(\alpha^\dagger \alpha) = c_2 \quad (114)$$

$$(4n - 4 - \frac{8}{n})\text{Tr}(b^\dagger b) + (4n + 4)\text{Tr}(\beta^\dagger \beta) = c_2 \quad (115)$$

$$(4n - 4)\text{Tr}(c^\dagger c) + (4n + 4 - \frac{8}{n})\text{Tr}(\gamma^\dagger \gamma) = c_3 \quad (116)$$

$$(4n - 4)\text{Tr}(d^\dagger d) + (4n + 4 - \frac{8}{n})\text{Tr}(\delta^\dagger \delta) = c_3 \quad (117)$$

$$(2n - 4)aa^\dagger + (2n - 4)bb^\dagger + 2ncc^\dagger + 2n\delta\delta^\dagger + 2n\alpha\alpha^\dagger + 2n\beta\beta^\dagger + (2n + 4)\gamma\gamma^\dagger + (2n + 4)\delta\delta^\dagger = c_4\delta_{I_7 J_7}, \quad (118)$$

where $a^T, a^\dagger$ denote the transposed, respectively the hermitian conjugate matrix of a . A simple computation shows that these eqs. are satisfied by the following set of matrices

$$a = \sqrt{\frac{3}{4}}g \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \alpha = \sqrt{\frac{3(n-2)}{4n}}g \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (119)$$

$$b = \sqrt{\frac{3}{4}}g \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \beta = \sqrt{\frac{3(n-2)}{4n}}g \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad (120)$$

$$c = \sqrt{\frac{3(n+2)}{4n}}g \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \gamma = \sqrt{\frac{3}{4}}g \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (121)$$

$$d = \sqrt{\frac{3(n+2)}{4n}}g \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \delta = \sqrt{\frac{3}{4}}g \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad (122)$$

We have thus discovered a new example for an $F = 1$ nonsupersymmetric solution of the finiteness conditions for YC in one loop approximation. Only one model of this type is known so far [18]. This solution for the Y-s is not unique, and the remaining freedom may be used to make the solution of the finiteness conditions for the scalar couplings easier. Of course, the latter is still to be done for our present example.

4 Conclusion and Outlook

In this work we have analyzed the implications of the finiteness conditions on the YC in the one loop approximation of a general quantum field theory under the following restrictions. First of all we consider a class of theories with a certain particle content ("F=1 theories"). Nevertheless this class is bigger than any other which has been under consideration so far and has the advantage of being two loop finite in the gauge coupling, if it is one loop finite with respect to g and the YC Y . This is a feature not shared by the model recently presented in [21]. Secondly we allow in the fermionic sector representations up to the adjoint one, and in the bosonic sector representations up to the symmetric one. The gauge groups we consider are $SU(n)$, where $n > 6$ has to hold. To study irreps "above" the adjoint one or $SU(n)$ -s with $n \leq 6$ would cause further considerable technical problems.

With computer methods it can be shown [15] that there are 149 theories which fulfill these restrictions and the diophantine eqs. (4) and (7). With analytical methods we prove in this paper that there are only two theories, which obey the finiteness conditions (8)-(10) for the YC, namely one supersymmetric theory, which has been known for a long time [6], and a nonsupersymmetric one, which we introduced in section 3.

The current state in our study of the favoured $F = 1$ class is therefore roughly the following: There is no finite theory in any $SU(n)$, if we allow only the six lowest-dimensional representations [1][2]. If we admit the adjoint representation in the bosonic sector, we could reduce the number of possible finite theories to 16 [2]. For one nonsupersymmetric of these the YC have been constructed explicitly [18]. Allowing the adjoint representation in the fermionic sector, we reduced the number of possible finite theories to 2, of which one is supersymmetric. For both the appropriate YC have been constructed. Both assertions, concerning the addition of the adjoint representation, are valid for $SU(n)$ with $n > 6$. Some further directions of research are possible. One is to include the adjoint representations in both sectors. Another possibility is to use the conditions on the quartic scalar couplings to prove further no go theorems in the same manner as above. But this is technically considerably more difficult, because of the larger number of invariant tensors of order 4. For example, there are 9 independent invariant tensors of order 4, if each index transforms according to the adjoint representation. Last but not least there is the chance of furnishing the two nonsupersymmetric theories with one-loop finite scalar couplings. One could try then to extend these results to higher loops in a way similar to the one, which has been proposed in [10] and in [11] for $N = 1$ supersymmetric theories.

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