WODZICKI RESIDUE AND ANOMALIES OF CURRENT ALGEBRAS

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ABSTRACT The commutator anomalies (Schwinger terms) of current algebras in 3+1 dimensions are computed in terms of the Wodzicki residue of pseudodifferential operators; the result can be written as a (twisted) Radul 2-cocycle for the Lie algebra of PSDO’s. The construction of the (second quantized) current algebra is closely related to a geometric renormalization of the interaction Hamiltonian $H_I = j_{\mu} A^\mu$ in gauge theory.

1. INTRODUCTION

One of the problems one meets all the time in quantum field theory is that products of field operators at equal times in the same points in the physical space are ill-defined. The first renormalization which one always performs is the normal ordering of the operator products, i.e. a shift of the energy lowering operators to the right, those giving a vanishing contribution when acting on the Dirac vacuum. In fact, in many cases in 1+1 dimensional models the normal ordering is sufficient to produce well-defined second quantized operators (more precisely, operator valued distributions). One such a case is the algebra of local charges formed as certain quadratic expressions in field operators. In one space dimension the construction leads to affine Kac-Moody and related algebras.

In higher dimensions one needs further renormalizations in addition to the normal ordering. The aim of this talk is to explain the renormalizations needed for local charges in 3+1 space-time dimensions. More specifically, we shall study (chiral) fermions minimally coupled to external gauge fields. The basic idea in the present renormalization scheme is to conjugate the Gauss law generators by unitary operators, which are functions of the external gauge field, in the one-particle space such that the resulting conjugated operators

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can be quantized using the standard normal ordering prescription. As a by-product, we obtain a new geometric renormalization of the Dirac-Yang-Mills interaction hamiltonian.

In one space dimension there is a nontrivial 2-cocycle which defines a central extension of the Lie algebra of pseudodifferential operators, [KK]. This algebra can be identified as the quantum $W_\infty$ algebra. It has as a subalgebra (when the coefficients of the PSDO’s are taken in a simple Lie algebra) an affine algebra. The Kravchenko-Khesin cocycle has been generalized by Radul to all dimensions, [R]. We shall show that a twisted form of the Radul cocycle, when applied to the renormalized local charges, gives an extension of the naive current algebra which is equal to the Mickelsson-Faddeev algebra, [M, F-Sh], [M2]. Thus the MF algebra is closely related to a multidimensional version of the $W_\infty$ algebra.

2. CENTRAL EXTENSION OF ALGEBRAS OF PSDO’s

Let us first consider pseudodifferential operators in one dimension, on a circle. Asymptotically, a PSDO is a defined by a Laurent series

\begin{equation}
    a(x, p) = \sum_{k \leq n} a_k(x) p^k
\end{equation}

where $n$ is some integer and the $a_k$’s are smooth functions on the circle. The momentum $p$ is the symbol of the operator $-i\partial_x$. The product is defined as

\begin{equation}
    a \ast b = \sum_{k=0,1,2,...} \frac{(-i)^k}{k!} \partial_p^k a(x, p) \partial_x^k b(x, p).
\end{equation}

Note that each $(a \ast b)_j$ is a finite sum of products of derivatives in the coefficients $a_i, b_i$.

The algebra of PSDO’s becomes a Lie algebra $\mathcal{B}$ under the commutator $[a, b] = a \ast b - b \ast a$.

The Adler-Manin residue of a PSDO is defined as

\begin{equation}
    \text{Res}(a) = \frac{1}{2\pi} \int \text{tr} \ a_{-1} \ dx
\end{equation}

where we have included the trace in order to allow a generalization to matrix valued PSDO’s. The residue behaves like a trace on the algebra of PSDO’s. It is obviously a linear functional and furthermore it satisfies

\begin{equation}
    \text{Res}([a, b]) = 0.
\end{equation}
The function $\log(p)$ is not a PSDO (since its expansion contains arbitrarily high powers of $p$) but nevertheless one can define a Lie algebra 2-cocycle, the KK cocycle, by the formula

$$c(a, b) = \text{Res}[\log(p), a] \ast b.$$  
(2.5)

This is because the commutator $[\log(p), a]$ is a PSDO,

$$[\log(p), a] = \sum_{k=1, 2, \ldots} -\frac{i^k}{k} \partial_k a(x, p)p^{-k}$$  
(2.6)

The 2-cocycle property

$$c(a, [b, c]) + \text{cycl. permutations} = 0$$

is a simple consequence of (2.4).

In the case when $a, b$ are zeroth order PSDO's, i.e. they are multiplication operators, the value of the KK cocycle is

$$c(a, b) = \frac{i}{2\pi} \int \text{tr} \ a(x)b'(x)dx.$$  
(2.7)

This is exactly the central term of an affine Lie algebra (when $a, b$ take values in a simple Lie algebra).

The KK cocycle can be defined in any number of space dimensions, [R]. Thus one might wonder whether the higher dimensional Radul cocycles have anything to do with anomalies of current algebras. The physically relevant extensions of current algebras in higher dimensions are generally not central extensions but extensions by some abelian ideal. For this reason it is not immediately obvious what is the relevance of the Radul cocycle in higher dimensions. We shall clarify this matter in section 5.

3. THE WODZICKI RESIDUE

Let $M$ be a compact manifold of dimension $n$. A PSDO on $M$, with coefficients in a vector bundle $V$ over $M$, is locally given by a matrix valued symbol $a(x, p)$. Here $x$ is
a local coordinate on $M$ and $p$ is a fiber coordinate in the cotangent bundle $T^*M$. The product rule for symbols is determined from the definition of the operator $A$ acting on sections of $V$.

\begin{equation}
(A\psi)(x) = \frac{1}{(2\pi)^n/2} \int e^{-ix\cdot p} a(x, p) \hat{\psi}(p) d^n p
\end{equation}

where $\hat{\psi}$ is the Fourier transform of the section $\psi$. Thus the symbol of the product $AB$ is

\begin{equation}
(a \ast b)(x, p) = \frac{1}{(2\pi)^n} \int e^{i(x-y)\cdot (p-q)} a(x, q) b(y, p) d^n y d^n q.
\end{equation}

The adjoint of $A$ (in the Hilbert space of square-integrable sections, the measure defined by a Riemannian metric on $M$) is in general a complicated expression in terms of the symbol $a$. We shall give the formula only in the euclidean case:

\begin{equation}
A^* \sim a^* + \Omega a^* + \frac{1}{2!} \Omega^2 a^* + \ldots
\end{equation}

where

$$\Omega = -i \sum_j \partial_{x_j} \partial_{p_j}$$

and $a^*$ is the matrix adjoint of the matrix valued symbol $a$.

Let $a_{-k}$ be a symbol of integral order $-k$ in $n$ space dimensions. Then the trace is asymptotically

$$\text{tr} a_{-k} = \int d^n x \int d^n p \text{tr} a_{-k}(x, p) \sim \int d^n x \int |p|^{-k} |p|^{n-1} d|p|$$

from which follows that $a_{-k}$ is of trace-class if and only if $-k \leq -n - 1$. Similarly, $p_{-k}$ is Hilbert-Schmidt if and only if the integer $-2k \leq -n - 1$.

Most of the time we are interested only on the asymptotic behaviour of the symbols $a$ for large momenta $p$. We assume that a PSDO has an asymptotic expansion of the form

\begin{equation}
a = a_k + a_{k-1} + a_{k-2} \ldots
\end{equation}

where each $a_k = a_k(x, p)$ is a smooth function of $x$ and of $p \neq 0$, $a_k$ is homogeneous of degree $k$ in the momenta, $\frac{\partial^{k}}{|p|^{k+1}} \to 0$ as $|p| \to \infty$. One can show that the asymptotic expansion of the symbol $a \ast b$ can be written as (compare with (2.2))

\begin{equation}
a \ast b = \sum_m \frac{(-i)^{|m|}}{m!} \left( \partial_{p_1}^{m_1} \ldots \partial_{p_n}^{m_n} a(x, p) \right) \left( \partial_{x_1}^{m_1} \ldots \partial_{x_n}^{m_n} b(x, p) \right)
\end{equation}
where the sum is over all multi-indices \( m = (m_1, \ldots, m_n), \ |m| = m_1 + \cdots + m_n, \) and \( m! = m_1! \cdots m_n! \), [H].

The Wodzicki residue [W] of a PSDO \( a \) is defined as a linear functional which depends only on the component \( a_{-n} \),

\[
\operatorname{Res}(a) = \frac{1}{(2\pi)^n} \int \operatorname{tr} a_{-n}(x, p) \eta(d\eta)^{n-1}
\]

where \( \eta = \sum p_k dx_k \) and \( d\eta = \sum dp_k dx_k \) is the symplectic 2-form on the cotangent bundle \( T^* M, \ n > 1 \). In the case \( n = 1 \) this is almost the Adler-Manin residue, [Ad], [Ma]. The difference is the following. If \( a_1 = \frac{a(x)}{p} \) then the residue is zero, because the unit sphere in momentum space consists of two points \( \pm 1 \) and the momentum space integral is

\[
\int p \frac{a(x)}{p} p dx = (a(x)|_{p=+1} - a(x)|_{p=-1}) dx = 0.
\]

However, if \( a_{-1} = \frac{a(x)}{|p|} \) then the integral becomes the sum \( a(x) + a(x) = 2a(x) \). Thus we can write

\[
\operatorname{Res}_{AM}(a) = \frac{1}{2} \operatorname{Res}_{W}(\epsilon a)
\]

where \( \epsilon = p/|p| \). Usually one redefines the Wodzicki residue in one dimension so that it agrees precisely with the Adler-Manin residue, [W].

The Radul cocycle in \( n \) dimensions is defined as

\[
c(a, b) = \operatorname{Res}([\log |p|, a] * b).
\]

For multiplication operators the Radul cocycle vanishes in dimensions higher than one. The structure of the Lie group defined by the Radul cocycle has recently been studied in [KV].

4. RENORMALIZED CURRENTS AS PSDO’S

Consider a system of quantized fermions in external vector potentials \( A \) in \( 3 + 1 \) space-time dimensions. \( A \) takes values in \( \mathfrak{g} \), a finite-dimensional Lie algebra. We set up a hamiltonian formalism for second quantization: we consider field (anti)commutation relations at a fixed time \( t = 0 \).
In Schrödinger picture the wave functions are functions $\phi(A)$ of the potential, with values in a fermionic Fock space $\mathcal{F}$.

We use the temporal gauge $A_0 = 0$ and we consider only time-independent gauge transformations.

The (free) Dirac field $\psi$ satisfies the CAR algebra

$$\psi_{ia}^*(x)\psi_{jb}(y) + \psi_{jb}(y)\psi_{ia}^*(x) = \delta_{ij}\delta_{ab}\delta(x - y)$$

where $i, j$ are space-time indices and $a, b$ are internal symmetry indices; the latter refer to a unitary representation of $g$.

The free vacuum is characterized by the property

$$\psi(u)|0\rangle = 0 = \psi^*(v)|v\rangle \text{ for } u \in \mathcal{H}_+, v \in \mathcal{H}_-$$

where $\mathcal{H}_\pm$ are the positive and negative energy subspaces of the one particle Hilbert space $\mathcal{H}$ and

$$\psi(u) = \int \psi_{ia}(x)u_{ia}(x)d^3x.$$  

In the perturbation theory based on Dyson expansion one writes the scattering amplitudes in terms of expressions like

$$<0|H_I(t_1)H_I(t_2)\ldots H_I(t_n)|0>.$$ 

These are integrated over the times $t_1 > t_2 \cdots > t_n$. Divergencies for the scattering amplitudes occur since the interaction hamiltonian

$$H_I(x) = \psi^*(x)\gamma_0\gamma^kT_a\psi(x)A^a_k(x)$$

involves products of field operators at the same point and does not lead to a well-defined operator distribution in Fock space. We need a renormalization even after normal ordering. (In $1 + 1$ dimensions a normal ordering is sufficient!)

More precisely, the technical reason for divergencies is the following. Let $\epsilon$ be the sign of the free hamiltonian $H_0 = \gamma_0\gamma^k\nabla_k$ in the one-particle representation. Then the off-diagonal blocks $[\epsilon, H_I]$ are Hilbert-Schmidt only when the space-time dimension is at most
2. As discussed in [Ar], the Hilbert-Schmidt property is both necessary and sufficient for quantization of operators of the type

\[ H_I = \sum X_{nm} a_n^* a_m \]

where the \( a_n \)'s satisfy the CAR relations

\[ a_n^* a_m + a_m a_n^* = \delta_{nm} \]

The \( X_{nm} \)'s are the matrix elements of \( H_I \) in a basis of eigenvectors of \( H_0 \) (compactify the physical space to get a discrete basis). Note that the norm of the state \( |H_I|0 > \) is given by

\[ ||H_I|0 > ||^2 = \sum_{E_n > 0, E_m < 0} |X_{nm}|^2, \]

where \( E_n \)'s are the energy levels for \( H_0 \). The finiteness of this norm is just the Hilbert-Schmidt property for one of the off-diagonal blocks of the one-particle operator.

The badly behaving interaction hamiltonian is renormalized as follows. For each \( A \) we construct a unitary operator \( T_A \) in the one-particle space such that

\[ [\epsilon, T_A^{-1}(H_0 + H_I)T_A] \]

is Hilbert-Schmidt.

The strategy is to obtain \( T_A = 1 + t_{-1} + t_{-2} \ldots \) as an expansion in homogeneous pseudodifferential operators \( t_k \) of degree \( k = 0, -1, -2, \ldots \). We shall consider separately the left and right handed sectors in the space of \( 4 \)-component fermions.

Let us consider the following operator acting on two-component fermions:

\[
T_A = 1 - \frac{i}{4|p|^2}[p, A] - \frac{1}{32|p|^4}[p, A]^2 - \frac{1}{8} \left[ \frac{\sigma_k}{|p|^2} - 2 \frac{p p_k}{|p|^4} \partial_k A \right] \\
- \frac{1}{8} \frac{1}{|p|^4}[p, A](A \cdot p) - \frac{1}{8} \frac{1}{|p|^4}(A \cdot p)[p, A] + O(-3)
\]

(4.5)

We have used the following notation: \( p = p_k \sigma_k, A = A_k \sigma_k \) The commutators in (4.5) are all ordinary commutators of Pauli matrices \( \sigma_k \) and of Lie algebra elements in \( g \). The Pauli matrices are normalized such that \( \sigma_1 \sigma_2 = i \sigma_3 \) and \( \sigma_k^2 = 1 \).
Since the physical space is assumed to be compact, there can be no infrared divergencies in the theory, so it sufficient to work with the asymptotic expansions of the operators in momentum space; the asymptotic expansion gives a complete picture of the ultraviolet properties of the theory. If one feels uneasy about the 'infrared singularity' at \( p = 0 \) of the terms in the asymptotic expansion one can always replace the inverse powers \( 1/|p|^n \) by some smooth functions which agree with the original for large values of \( |p| \).

We can write \( T^{-1}_A(H_0 + H_I)T_A = H_0 + W_A \). After a tedious computation we obtain

\[
W_A = T^*_A(p + iA)T_A - p = \frac{ip}{|p|^2} A \cdot p - \frac{1}{8} \left[ p, \left[ \frac{\sigma_k}{|p|^2} - 2 \frac{pp_k}{|p|^4 \cdot \partial_k A} \right] \right] - \frac{\sigma_k}{4|p|^2} [p, \partial_k A]
\]

\[
+ \frac{i}{2|p|^2} \epsilon_{ijk} p_j [A_i, A_k] - \frac{p}{|p|^2} A_m A_m + \frac{p}{|p|^4} (A \cdot p)^2 + O(-2).
\]

(4.6)

It is then a simple computation to show that \([\epsilon, W_A]\) is of degree \(-2\), and therefore the operator is Hilbert-Schmidt. There is no magic in the derivation of the formula (4.5) for \( T_A \). It is a simple recursive procedure. Writing

\[
T_A = 1 + t_{-1} + t_{-2} + \ldots
\]

in the asymptotic expansion, one gets

\[
T^*_A(p + \alpha_0 + \alpha_{-1} + \ldots)T_A = p + \alpha'_0 + \alpha'_{-1} + \ldots,
\]

where

\[
\alpha'_0 = \alpha_0 + [p, t_{-1}]
\]

\[
\alpha'_{-1} = \alpha_{-1} + [\alpha_0, t_{-1}] + pt_{-2} + (T^*_A)_{-2} p - i \sigma_k \partial_k \alpha'_0 - 1.
\]

(4.7)

Again, the commutators above are ordinary matrix commutators (and not *-commutators).

The Hilbert-Schmidt condition on \([\epsilon, W_A]\) is equivalent to the pair of equations

\[
[\epsilon, \alpha'_0] = 0 \text{ and } [\epsilon, \alpha'_{-1}] - i(\partial_{p_1} \epsilon)(\partial_{x_k} \alpha'_0) = 0.
\]
Inserting from (4.7), the first equation is just a linear algebraic equation for the symbol \( t_{-1} \). But then we can solve \( t_{-2} \) from the second equation above.

In the one-particle representation the infinitesimal gauge transformations \( X \) are acting on Schrödinger wave functions \( \phi(A) \) as the operators \( \mathcal{L}_X + X \), where the first part (Lie derivative) is the gauge action on vector potential \( A \) and the second is a multiplication operator acting on the value \( \phi(A) \in \mathcal{H} \). After the conjugation by the \( A \) dependent operator \( T_A \) these become \( \mathcal{L}_X + \theta(X; A) \) with

\[
\theta(X; A) = T_A^{-1}XT_A + T_A^{-1}(\mathcal{L}_X T_A).
\]

By construction,

\[
[\theta(X; A) + \mathcal{L}_X, \theta(Y; A) + \mathcal{L}_Y] = [\theta(X; A), \theta(Y; A)] + \mathcal{L}_X \theta(Y; A) - \mathcal{L}_Y \theta(X; A) + [\mathcal{L}_X, \mathcal{L}_Y]
\]

(4.9)

That is, the functions \( \theta(X; \cdot) \) form a 1-cocycle for the gauge action of \( \text{Map}(M, g) \). If the function \( T_A \) is constructed as above, then \( \theta(X; A) \) is in the Lie algebra of the restricted unitary group \( U_{res}, [M1] \). The latter is the subgroup of \( U(\mathcal{H}) \) consisting of operators \( g \) such that \([e, g]\) is Hilbert-Schmidt.

**Proof:** Since \( H'(A) = T_A^{-1}(H_0 + H_f(A))T_A = H_0 + W_A \) and \([e, W_A]\) is HS, we observe that the sign \( e(A) \) of the Hamiltonian \( H'(A) \) differs from \( e \) by a HS operator. Let \( g \) be a finite gauge transformation. It acts on Schrödinger wave functions by

\[
(R(g)\phi)(A) = g \cdot \phi(g^{-1}Ag + g^{-1}dg)
\]

But after the conjugation by \( T_A \):

\[
(R^g(A)\phi)(A) = (T_{g^{-1}Ag}^{-1} g T_A^{-1} \phi(g^{-1} \cdot A) \equiv \omega(g; A) \phi(g^{-1} \cdot A).
\]

Thus

\[
[e, \omega(g; A)] = eT_{g^{-1}Ag}^{-1}gT_A - T_{g^{-1}Ag}^{-1}gT_Ae
\]

\[
= (T_{g^{-1}Ag}^{-1}gT_A^{-1}gT_A - T_{g^{-1}Ag}^{-1}g(T_AeT_A^{-1})T_A
\]

\[
= T_{g^{-1}A}^{-1}((T_{g^{-1}Ag}^{-1}g) - g(T_AeT_A^{-1}))T_A
\]

\[
\equiv T_{g^{-1}A}^{-1}(e(g \cdot A)g - g(e(A)))T_A
\]
where we have used the equivariantness of the family of Dirac operators,

\[ gH(A)g^{-1} = H(g \cdot A). \]

Thus finite gauge transformations satisfy the HS condition; considering one-parameter subgroups one proves the HS condition for the generators.

The asymptotic expansion for (4.8) is

\[(4.10) \quad \theta(X; A) = X + \frac{i}{4} \left[ p, dX \right] + \theta_{-2} + O(-3) \]

with

\[ \theta_{-2} = -\frac{1}{4} \left[ \sigma_k, A \right] \frac{\partial_k X}{|p|^2} + \frac{1}{2} \frac{\left[ p, A \right]}{|p|^4} p_k \partial_k X \]

\[ + \frac{1}{16} \frac{\left[ p, A \right]}{|p|^4} \left[ p, dX \right]. \]

If \( X \) is any bounded bilinear quantity in the fermion creation and annihilation operators such that its off-diagonal blocks in the one-particle representation with respect to the energy polarization are HS, then the second quantized operator \( \hat{X} \) is well-defined (after normal ordering) and

\[(4.11) \quad [\hat{X}, \hat{Y}] = [\hat{X}, \hat{Y}] + c(X, Y), \]

where \( c \) is a Schwinger term, [L],

\[(4.12) \quad c(Y, Y) = \frac{1}{4} \text{tr} \epsilon, X \text{tr} \epsilon, Y. \]

**Example** Multiplication operators in \( 1 + 1 \) dimensions. Multiplication operators are PSDO’s \( X \) of order zero with a \( p \) independent symbol \( a(x) \). \( \epsilon = \frac{p}{|p|} \) and

\[(4.13) \quad [\epsilon, X] = -2i \delta(p)a'(x). \]

Now the trace (4.12) can be written as the conditionally converging trace

\[(4.14) \quad c(X, Y) = \frac{1}{2} X[\epsilon, Y] = \frac{1}{2\pi i} \int \text{tr} \ a(x)b'(x) dx. \]
This is the central term of an affine algebra.

In the $3 + 1$ dimensional case one just inserts from (4.8) to (4.12):

$$c(X, Y; A) = \frac{1}{4} \text{tr} [\epsilon, \theta(X; A)] [\epsilon, \theta(Y; A)].$$

Actually, we can write

$$c(X, Y; A) = \frac{1}{2} \text{tr} [\epsilon, \theta(X; A)] \theta(Y; A)$$

as a conditionally convergent trace: Compute first the traces for the finite-dimensional matrices, perform the momentum space integration over the spherical angles, next the integration over $|p|$, and finally integrate the star product of symbols in configuration space.

5. COMPUTATION OF THE COMMUTATOR ANOMALY

The result of the computation starting from (4.16) is rather complicated expression involving terms of all orders in $A$. However, there is a great simplification if we are interested only on the cohomology class of the cocycle. A change in the renormalization $T_A \leftrightarrow T'_{A} = T_A g_A$ of the gauge currents (with $g_A \in U_{ren}$) leads to a modification

$$\theta(X; A) \leftrightarrow \theta'(X; A) = g_A^{-1} \theta(X; A) g_A + g_A^{-1} \mathcal{L}_X g_A.$$ 

It is easy to check that

$$c' - c = (\delta \lambda)(X, Y; A) = \lambda([X, Y]; A) - \mathcal{L}_X \lambda(Y; A) + \mathcal{L}_Y \lambda(X; A),$$

where $\lambda$ is the 1-cochain

$$\lambda(X; A) = \frac{1}{4} \text{tr} \left[ \epsilon, g_A^{-1} [X, g_A] + g_A^{-1} \mathcal{L}_X g_A - (\mathcal{L}_X g_A) g_A^{-1} \right]_+,$$

with $[A, B]_+ = AB + BA$.

We want to show that if we choose $\lambda$ in a suitable way then the new cocycle $c'$ takes the form

$$c'(X, Y; A) = \frac{i}{24\pi^2} \int \text{tr} A [dX, dY].$$
Let us first define the regularized trace, with $\Lambda > 0$, 
\[
\text{tr}_\Lambda R = \frac{1}{(2\pi)^3} \int_{|p| \leq \Lambda} \text{tr} r(x, p) d^3 x d^3 p + \frac{1}{(2\pi)^3} \int_{|p| > \Lambda} \text{tr} \left( r - \sum_{k=0}^{3} r_{-k} \right) d^3 x d^3 p
\]
for a PSDO $R$ of degree zero. Define 
\[
\lambda(X; A) = \frac{1}{2} \text{tr}_\Lambda \epsilon \theta(X; A).
\]
Then 
\[
(\delta \lambda)(X, Y; A) = \frac{1}{2} \text{tr}_\Lambda \epsilon [\theta(X; A), \theta(Y; A)].
\]

If the regularize trace were symmetric, this would be equal to the cocycle $c$. In order to show that $\text{tr}(a * b) = \text{tr}(b * a)$ for a pair of symbols one has to perform partial integration both in the momentum and coordinate variables. The coordinate integration does not cause any problems, since we assumed that the manifold is compact and without boundary. However, there can be boundary terms arising from the momentum space integration. Integrating a term of degree $-3$ leads to finite boundary contribution in partial integration: the integrand behaves like $|p|^{-2}$ at infinity, cancelling the factor $|p|^2$ coming from the integration measure. The boundary contributions from terms of degree less than 3 vanish at $|p| \to \infty$.

As we have seen, the terms in $c$ which are not coboundaries of some 1-cochains must be of order $-3$. The terms of order greater than $-3$ vanish in (4.15) by the algebra of Pauli matrices and by the fact that the momentum space integration over spherical angles of a symbol of odd degree gives zero. According to the multiplication rule (3.5), with $X_k \equiv \theta(X; A)_k$, 
\[
S \equiv (\epsilon[\theta(X; A), \theta(Y; A)])_{-3} = \epsilon ([X, Y_{-3}] + [X_{-1}, Y_{-2}]
+ \{X, Y_{-2}\} + \{X_{-1}, Y_{-1}\} + \{X_{-2}, Y\} - (X \leftrightarrow Y)),
\]
where we have denoted $\{a, b\} = -i \sum \partial_{p_i} a \partial_{x_i} b$, 'half Poisson bracket'. The commutators on the right are matrix commutators. The cut-off trace of the $\{X_{-1}, Y_{-1}\}$ term is seen to vanish after performing the spherical part of the momentum space integration. By partial
integration we obtain
\[
\text{tr}_A S = \text{tr}_A \left( [\epsilon, X_{-1}] Y_{-2} + [\epsilon, X_{-2}] Y_{-1} - \{\epsilon, Y\} X_{-2} + \{\epsilon, X\} Y_{-2} \right) \\
+ \frac{i}{(2\pi)^3} \int_{|p|=1} \text{tr} \, \epsilon Y_{-2} (p_k \partial_{x^k} X) \eta(d\eta)^2 - \frac{i}{(2\pi)^3} \int_{|p|=1} \text{tr} \, \epsilon X_{-2} (p_k \partial_{x^k} Y) \eta(d\eta)^2 \\
= \frac{1}{2} \text{tr}_A \left( ([\epsilon, \theta(X; A)] * \theta(Y; A))_{-3} - ([\epsilon, \theta(Y; A)] * \theta(X; A))_{-3} \right) \\
(5.1)
+ \text{Res} \, \epsilon [\log(|p|), \theta(X; A)] * \theta(Y; A).
\]

It follows that
\[
(5.2) \quad c_A(X, Y; A) = \text{Res} \, \epsilon [\log(|p|), \theta(X; A)] \theta(Y; A).
\]

(Here star product should be used). The residue is easily computed (as the spherical integrals in (5.1)). The result is
\[
(5.3) \quad c_A(X, Y; A) = \frac{i}{24\pi^2} \int_x \text{tr} \, A[dX, dY],
\]
which is the cocycle derived in [M, F-Sh] in a different context. Note that the final results (5.2) and (5.3) do not depend on the cut-off parameter $\Lambda$. This method of computing cocycles can be generalized to various directions, [LM].

The action functional in Connes noncommutative geometry model of Yang-Mills theory is also defined in terms of the Wodzicki residue, [C]. It would be interesting to see more precisely the relation between the present hamiltonian approach and the Lagrangian method of Connes.

Actually the formula (5.2) can be applied to the algebra $W'$ of all PSDO’s $P$ (in three dimensions) obeying the condition
\[
(5.4) \quad \text{deg} [\epsilon, P] \leq -2
\]
giving a twisted form of the Radul cocycle on the Lie algebra $W'$,
\[
(5.5) \quad c(P, Q) = \text{Res} \, \epsilon [\log(|p|), P] Q
\]
So we get a generalization of the $W_\infty$ algebra as the central extension (defined by the 2-cocycle (5.5)) of the Lie algebra of restricted spinorial PSDO’s in three dimensions. In
n dimensions the condition (5.4) should be replaced by the requirement that the degree of the commutator is strictly less than $-\frac{1}{2} n$. We shall end the discussion by giving the proof that (5.5) is a 2-cocycle for the restricted Lie algebra $W'$ is $n$ dimensions. Denote

$$Res' A = Res \epsilon A.$$ 

Since $Res[A, B] = 0$ for any pair of PSDO’s $A, B$ we have

$$Res \epsilon[\epsilon, A][\epsilon, B] = Res(A\epsilon B - \epsilon AB - AB \epsilon + \epsilon A \epsilon B \epsilon)$$

$$= 2Res(A \epsilon B - \epsilon AB) = -2Res[\epsilon A]B = -2Res[\epsilon A, B].$$

If now $deg[\epsilon, A] < -n/2$ and $deg[\epsilon, B] < -n/2$ then $deg[\epsilon, A][\epsilon, B] < -n$ and it follows that the residue vanishes. Thus

$$Res'[A, B] = 0$$

for any $A, B \in W'$. Denote $\ell = \log(|p|)$. By (3.5) and performing partial integration in momentum space one gets $Res[\ell, A] = 0$ for any PSDO $A$. Since $\ell$ commutes with $\epsilon$, we have also

$$Res'[\ell, A] = 0$$

although $\ell$ is not a PSDO. Define

$$\omega(A, B, C) = c(A, [B, C]) + c(B, [A, C]) + c(C, [A, B]).$$

Then


Using (5.6) and (5.7) we get

$$\omega(A, B, C) = Res' (A[[B, C], \ell] + A[[\ell, B], C] + A[[\ell, C], B]) = 0,$$

by Jacobi’s identity, proving that $c$ is indeed a 2-cocycle.
The formula (5.3) can be generalized to higher (odd) space dimensions in various ways. One approach is to use the cohomological descent equations starting from a Chern class in dimension \( n + 3 \), \([M, F-Sh], [M2]\). There exists another rather natural and much simpler generalization, which was found recently when studying p-brane symmetries, \([CFNW]\). The realization of that algebra as an algebra of PSDO’s (of type \( W' \)) is now under preparation.

REFERENCES


[KV] M. Kontsevich and S. Vishik, preprint Max-Planck-Institut für Mathematik, Bonn (April 1994); HEPHT 94 04 46


[LM] E. Langmann and J. Mickelsson, work under preparation


