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DECAY OF A SPINLESS PARTICLE INTO FOUR IDENTICAL SPINLESS BOSONS

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Abstract

The theory of invariants is used to find the best forms of the general amplitude for the decay of a scalar or pseudoscalar particle into four identical spinless bosons. These forms of the amplitude are consistent with the conservation of energy, momentum, and angular momentum, as well as with Bose-Einstein statistics. Kinematical zeros of the pseudoscalar amplitude are located. The differential probability for decay into four identical unpolarized particles of an arbitrary unpolarized boson has symmetry properties of the scalar amplitude. This fact suggests some specific methods of data analysis.

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1 Introduction

Ever since the beginning of meson spectroscopy, three-meson decays of unstable bosons have provided important information both on the quantum numbers of the decaying particle and on two-meson resonances appearing as intermediate states. For the prototype of such decays – the decay into three pions – the consequences of physical symmetries have been exhaustively analyzed by Zemach [1]. He derived general expressions for the decay amplitudes depending on the quantum numbers of the initial state and on the pion charges; these expressions are dictated by the conservation of 4-momentum (i.e. 3-momentum and energy), total angular momentum, parity, and isospin, as well as by the bosonic nature of pions. The amplitudes contain “form factors”, i.e. functions of scalar products of pion 4-momenta. These form factors must fulfill certain symmetry criteria but they also depend on the dynamics of the decay, e.g. on whether or not a two-pion resonance is populated in the decay. An important result of Zemach’s analysis is the location of “kinematical” zeros of the different decay amplitudes: the vanishing of an amplitude along certain lines or at certain points in the Dalitz plot is characteristic of the quantum numbers of the decaying particle and independent of the decay mechanism.

The study of three-pion decays of an unstable boson is a relatively simple task compared with the investigation of decays into larger numbers of pions. In the decay of unpolarized particles, the three-pion final state is specified (apart from trivial Euler angles) by two kinematical variables, e.g. the energies of two of the three pions. Each additional pion requires three additional kinematical variables for the specification of the final state. Thus, the four-pion decay of an unstable particle populates the 5-dimensional mass shell in a 6-dimensional kinematical space, which is not only hard to think about but also experimentally difficult to fill with sufficient density for a meaningful measurement of the differential decay probability. If we decide to divide the range of each relevant kinematical variable into k bins, the total number of bins for a four-pion decay is k^5 , as compared to k^2 for a three-pion decay. At an equal average number of events per bin, the measurement of the differential decay probability requires k^3 times more events for a four-pion than for a three-pion decay. With a reasonable value $k = 10$, the increase in the required data sample is by a factor of 1000!

Until recently, such quantities of data were impossible to obtain and to analyze. The advent of modern accelerators, detectors, and computers is starting to make the measurement of differential probabilities of four-body decays feasible. A typical example is the Crystal Barrel (CB) hermetic detector for neutral and charged particles [2], installed at the Low-Energy Antiproton Ring LEAR [3] at CERN. Within a reasonable period of one year, the CB experimental group is capable of measuring – with good resolution – of the order of a million events due to annihilation of a Coulomb-bound antiproton-proton atom (sometimes called protonium) into each of several specific final channels such as four π^0 . Therefore, a complete experimental determination of the differential probability for a four-body decay is within reach. It is clear, though, that it is still far out of reach for any n -body decay with n larger than four.

Another reason that makes the four-pion decay considerably more difficult to analyze than the three-pion decay is the fact that the symmetric group S_4 , i.e. the group of all permutations of four objects contains 24 elements, while the analogous group for three objects S_3 has only 6 elements. Here, the saving grace turns out to be the solvability of S_4 , which greatly simplifies the search for general forms of the decay amplitude and also permits, in a sense, to reduce the 5-dimensional mass shell to a combination of a 3-dimensional and a 2-dimensional space. As is well known from the Galois theory,

alternating groups A_n (i.e. groups of even permutations of n objects) with $n \geq 5$ are simple. As a consequence, the algebraic work required to reproduce Zemach's analysis for $n \geq 5$ should augment well beyond what would be expected from the increase in the order $N = n!$ of the corresponding groups S_n . In the case where $n = 4$, yet another fortunate circumstance is the equality between the number of particles in the final state and the dimensionality of space-time, which entails the existence of only one basic pseudoscalar function of the particle 4-momenta.

In the past, the consequences of isospin symmetry for many-pion decays of the nucleon-antinucleon system have been thoroughly studied by Pais [4]. The "barrier factors" (i.e. the number of 3-momentum factors in the partial amplitudes of such decays) have been determined, and some of these amplitudes have been calculated by Henley and Jacobsohn [5].

Here, we limit ourselves to the analysis of the decay of a spinless boson into four identical spinless bosons. While the prime example of such a decay is the annihilation of $J = 0$ states of protonium into four π^0 , the only condition on the particles in the final state is that they be identical and spinless. Their internal quantum numbers, such as parity, C-parity, charge, isospin, strangeness, etc., do not matter provided, of course, that there exists an initial particle decaying into the given four identical spinless final particles. Other realistic examples of such initial particles are the $J = 0$ states of ^{16}O of sufficiently high excitation that they may decay into four α -particles. For simplicity, though, we shall most often refer to four π^0 as the particles in the final state. Of course, four π^0 have positive C-parity and even isospin $I \leq 4$, i.e. only the $I = 0$ states of protonium can decay into four π^0 without violating isospin conservation.

The restriction to spinless initial particles and identical spinless final particles focuses attention on the main problem that seems not to have been previously treated in a physical context: the determination of a basis of scalar invariants of the alternating group A_4 . However, methods used to solve this problem are expected to be useful also in the analysis of the decay of a boson with arbitrary spin into, for example, four pions of arbitrary charge.

The present article first reviews the pertinent kinematics of the four-pion decay and introduces a simple linear transformation of variables permitting the partial separation of the 6-dimensional kinematical space into a pair of 3-dimensional ones. Then it explains the method used to find a basis of the set of (absolute) invariants (i.e. completely symmetric polynomials) and anti-invariants (i.e. completely antisymmetric polynomials) formed from the scalar products of the pion 4-momenta. This method represents a variant of Emmy Noether's general construction method [6] applicable to any finite group of linear substitutions. The result is a primary basis of 68 invariants and 48 anti-invariants. The primary basis can be reduced to a final basis of nine invariants and five anti-invariants linked by certain algebraic relations, the so-called syzygies. However, if a unique expansion of any decay amplitude in terms of invariants and anti-invariants is desired, the final basis has to be enlarged by two additional invariants and an additional anti-invariant to form the so-called good basis. The decay amplitude of a pseudoscalar initial particle is a product of the basic pseudoscalar and a linear form in the basic anti-invariants with absolutely invariant polynomial coefficients. Following Zemach, we may replace these coefficients by more general symmetric functions, i.e. form factors. However, the structure of kinematical zeros of the pseudoscalar decay amplitude as determined by the basis of anti-invariants is left intact. As an illustration of amplitudes arising from specific form factors, we discuss decays involving intermediate states with the lowest possible spin values.

A special symmetry property of some invariants permits the introduction of a finite vector space, termed the purity space, and the expansion of any specific scalar or pseudoscalar decay amplitude in terms of orthogonal basic vectors of this space. The realization that the differential *probability* for decay of an unpolarized boson with *arbitrary* spin and parity has the symmetry properties of the decay *amplitude* of a scalar initial particle opens the door to potential applications of this expansion to the analysis of experimental data.

Throughout the present article, physics takes precedence over mathematics as far as possible. Higher “links” of the syzygy chain do not seem to be of prime importance for physics while their explication would require an inordinate amount of tedious algebra. Therefore, we restrict ourselves to illustrative examples from the first two links of the chain. Also, we use the language of physics or of the classical theory of invariants rather than that of the more modern mathematics. Occasional translational hints are given, mainly through references to recent mathematical literature.

A first draft of the present article containing the main ideas (but, unfortunately, also many mistakes) has been privately circulated before [7].

2 Kinematics

Sometimes (for example, in the term “kinematical zeros”) kinematics is considered to comprise the consequences of all exact (or nearly exact) symmetries. However, in this Section, we shall discuss only the consequences of Lorentz (or rather Poincaré) invariance and leave aside questions connected with the identity of particles in the final state.

The 4-momentum of pion i is designated by $p_i = (E_i, \vec{p}_i)$, E_i and $\vec{p}_i$ being its total energy and 3-momentum, respectively. The metric is defined by the scalar product of two such 4-momenta:

$$p_i \cdot p_j = E_i E_j - \vec{p}_i \cdot \vec{p}_j = g_{\alpha\beta} p_i^\alpha p_j^\beta . \quad (1)$$

We have

$$p_i^2 = m^2 , \quad (2)$$

$$p_i \cdot p_j \geq m^2 , \quad (3)$$

m being the mass of π^0 . The metric tensor is designated by the symbol $g_{\alpha\beta}$; we have $g_{00} = 1$, $g_{\alpha\alpha} = -1$ for $\alpha \neq 0$, and $g_{\alpha\beta} = 0$ for $\alpha \neq \beta$. Summation over double greek indices from 0 to 3 is always implied.

With the four 4-momenta of the four pions it is possible to construct one basic 4-pseudoscalar ψ that does not vanish identically:

$$\psi = \epsilon_{\alpha\beta\gamma\delta} p_1^\alpha p_2^\beta p_3^\gamma p_4^\delta . \quad (4)$$

Here, $\epsilon_{\alpha\beta\gamma\delta}$ is the completely antisymmetric Levi-Civita tensor with

$$\epsilon^{0123} = -\epsilon_{0123} = 1 . \quad (5)$$

Furthermore, there exist six basic 4-scalars:

$$\begin{aligned} x_1 &= p_2 \cdot p_3 , & x_6 &= p_1 \cdot p_4 , \\ x_2 &= p_3 \cdot p_1 , & x_5 &= p_2 \cdot p_4 , \\ x_3 &= p_1 \cdot p_2 , & x_4 &= p_3 \cdot p_4 . \end{aligned} \quad (6)$$

The indices of the quantities x_j have been chosen in such a way that, if $j + k = 7$, x_j and x_k form a “conjugate” pair containing each pion 4-momentum once and only once. In the array (6) the three conjugate pairs are listed each in one row. For a proof that no further basic pseudoscalars and scalars exist, see Weyl [8, Ch. II §§9–13, Ch. X §§1–2].

Squaring Eq. (4) we obtain the determinant

$$\Delta_4 = \psi^2 = - \begin{vmatrix} m^2 & x_3 & x_2 & x_6 \\ x_3 & m^2 & x_1 & x_5 \\ x_2 & x_1 & m^2 & x_4 \\ x_6 & x_5 & x_4 & m^2 \end{vmatrix} \quad (7)$$

which is a 4-scalar, of course. The minus sign in front of the determinant is a consequence of Eq. (5). Upon explicit evaluation of the determinant in Eq. (7) we obtain

$$\begin{aligned} \Delta_4 = & 2(x_1x_3x_4x_6 + x_1x_2x_5x_6 + x_2x_3x_4x_5) - (x_1^2x_6^2 + x_2^2x_5^2 + x_3^2x_4^2) \\ & - 2m^2(x_1x_2x_3 + x_1x_4x_5 + x_2x_4x_6 + x_3x_5x_6) + m^4 \sum_{j=1}^6 x_j^2 - m^8. \end{aligned} \quad (8)$$

The kinematics of n -body decays and the geometry of the corresponding kinematical spaces have been studied by Byers and Yang [9]. Their article contains the basic ideas used in this Section. However, we are interested only in the special case of four-body decay and mainly in the physically accessible part of the 6-dimensional x -space. These restrictions permit a considerably more pedestrian discussion of kinematics than the approach of Ref. [9].

According to Byers and Yang, the physically possible four-pion decays correspond to positive values of $x_j \geq m^2$ lying inside a connected domain of the 6-dimensional x -space, called the physical region, where

$$\Delta_4 > 0, \Delta_3 > 0, \Delta_2 > 0, \Delta_1 > 0. \quad (9)$$

Here, Δ_l for $1 \leq l \leq 4$ are defined by

$$\Delta_l = (-1)^l \sum \text{all } (l \times l) \text{ diagonal subdeterminants of } \Delta_4. \quad (10)$$

We observe that

$$\Delta_3 = -\frac{\partial \Delta_4}{\partial m^2}, \quad (11)$$

$$\Delta_2 = -\frac{1}{2} \frac{\partial \Delta_3}{\partial m^2}, \quad (12)$$

$$\Delta_1 = -\frac{1}{3} \frac{\partial \Delta_2}{\partial m^2}, \quad (13)$$

and obtain

$$\Delta_3 = 2(x_1x_2x_3 + x_1x_4x_5 + x_2x_4x_6 + x_3x_5x_6) - 2m^2 \sum_{j=1}^6 x_j^2 + 4m^6, \quad (14)$$

$$\Delta_2 = \sum_{j=1}^6 x_j^2 - 6m^4, \quad (15)$$

$$\Delta_1 = 4m^2. \quad (16)$$

The boundary of the physical region is a connected 5-dimensional domain, characterized by

$$\Delta_4 = 0. \quad (17)$$

From Eqs. (4) and (7) we conclude that $\Delta_4 = 0$ corresponds to the kinematical situation where any three pions are coplanar in the rest frame of the fourth one.

Furthermore, if

$$\Delta_3 = 0, \quad (18)$$

the four pions are essentially collinear as becomes clear from the following argument. The 4-vector V_4^α , defined in the Lorentz frame where $\vec{p}_3 = 0$ by $V_4^0 = 0$ and $\vec{V}_4 = m(\vec{p}_1 \times \vec{p}_2)$, may be written in the covariant form

$$V_4^\alpha = -\epsilon^{\alpha\beta\gamma\delta} p_{1\beta} p_{2\gamma} p_{3\delta}. \quad (19)$$

Clearly, we have $\vec{V}_4^2 = -V_4^\alpha V_{4\alpha}$, which can be evaluated using the identity

$$\epsilon_{\alpha\beta\gamma\delta} \epsilon^{\alpha\beta'\gamma'\delta'} = -(g_{\beta'}^{\beta} g_{\gamma'}^{\gamma} g_{\delta'}^{\delta} + g_{\beta}^{\beta'} g_{\gamma}^{\gamma'} g_{\delta}^{\delta'} + g_{\beta}^{\beta'} g_{\gamma'}^{\gamma} g_{\delta}^{\delta'} - g_{\beta}^{\beta'} g_{\gamma}^{\gamma'} g_{\delta}^{\delta'} - g_{\beta}^{\beta'} g_{\gamma'}^{\gamma} g_{\delta}^{\delta'} - g_{\beta}^{\beta'} g_{\gamma}^{\gamma'} g_{\delta}^{\delta'}) . \quad (20)$$

Here, $g_{\alpha}^{\beta} = g_{\alpha\gamma} g^{\gamma\beta}$ has the properties of the Kronecker symbol: $g_{\alpha}^{\beta} = 1$ for $\alpha = \beta$, $g_{\alpha}^{\beta} = 0$ for $\alpha \neq \beta$. The result is

$$\vec{V}_4^2 = 2x_1 x_2 x_3 - m^2 \sum_{j=1}^3 x_j^2 + m^6. \quad (21)$$

If we define V_k^α for $k = 1, 2, 3$ by a cyclical permutation of the four numerical indices in Eq. (19), it is easy to see that

$$\Delta_3 = \sum_{k=1}^4 \vec{V}_k^2. \quad (22)$$

Any term in this sum is positive or zero, since it is the square of a 3-vector in a specific rest frame. Thus, $\Delta_3 = 0$ corresponds to the kinematical situation where any two pions are collinear in the rest frame of any of the remaining two, i.e. a situation with any three pions being collinear in the rest frame of the fourth. From this we conclude that, physically, $\Delta_3 = 0$ implies $\Delta_4 = 0$ (but not vice versa). As observed from the physical region of the 6-dimensional x -space, the 5-dimensional domain $\Delta_3 = 0$ thus lies "outside" the 5-dimensional boundary $\Delta_4 = 0$ but is tangent to it along a 4-dimensional "rim" characterized by $\Delta_4 = \Delta_3 = 0$. These conclusions, which are mathematically not obvious, are a consequence of the fact that the six variables x_i are scalar products of only four 4-momenta.

Except in the academic case of all x_j simultaneously reaching their minimal value m^2 (decay with zero Q -value), Δ_2 is physically always positive. Clearly, so is Δ_1 .

In the decay, the total 4-momentum is conserved, i.e.

$$\sum_{i=1}^4 p_i = P, \quad (23)$$

$$P^2 = M^2 = 4m^2 + 2 \sum_{j=1}^6 x_j, \quad (24)$$

where P and M are the 4-momentum and rest mass, respectively, of the decaying particle. Equation (24) expresses energy conservation in an invariant way and represents a

5-dimensional hyperplane in the 6-dimensional x -space. Its intersection with the physical region may be called the mass shell; it is a closed 5-dimensional domain with a 4-dimensional boundary defined by Eqs. (17) and (24). (Of course, in reality, any decaying particle has a finite energy width entailing a finite "thickness" of the mass shell.) The physical region (9) is the union of all mass shells corresponding to all possible values of $M \geq 4m$. In the rest frame of the decaying particle, all four pions are coplanar at the 4-dimensional boundary of the mass shell defined by Eqs. (17) and (24). If, in addition, Eq. (18) applies, all four pions are collinear in the rest frame of the decaying particle.

We can define the "center" of a mass shell as the point where

$$x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = \frac{C}{6}, \quad (25)$$

with

$$C = \frac{1}{2}(M^2 - 4m^2). \quad (26)$$

The center of the mass shell corresponds to the kinematical configuration in which the four 3-momenta of the pions point from the center of a regular tetrahedron to its four corners, i.e. where these 3-momenta form a regular tetrapod.

Byers and Yang have expressed the Lorentz invariant phase-space element (also called volume element)

$$d\tau = \left(\prod_{i=1}^4 d^4 p_i \right) \left[\prod_{i=1}^4 \delta(p_i^2 - m^2) \right] \delta^4 \left(\sum_{i=1}^4 p_i - P \right) \quad (27)$$

in terms of the quantities defined above. They obtain the remarkably simple result:

$$d\tau = \frac{16\pi^2 \left(\prod_{j=1}^6 dx_j \right) \delta \left(\sum_{j=1}^6 x_j - C \right)}{M^2 \sqrt{\Delta_4}}. \quad (28)$$

It is in the nature of the above kinematical formalism that events related by space reversal ($\vec{p}_i \rightarrow -\vec{p}_i$) occupy the same volume element $d\tau$. This fact has been taken into account in deriving Eq. (28). It is also in the nature of the formalism that only one δ -function occurs in Eq. (28) rather than eight as in Eq. (27): the formalism is insensitive to boosts which change the value of the total 3-momentum. Notice that the phase-space density diverges at the boundary of the physical region.

3 New variables

It is advantageous to introduce new kinematical variables defined by

$$\begin{aligned} s_1 &= x_1 + x_6, & r_1 &= x_1 - x_6, \\ s_2 &= x_2 + x_5, & r_2 &= x_2 - x_5, \\ s_3 &= x_3 + x_4, & r_3 &= x_3 - x_4. \end{aligned} \quad (29)$$

The absolute value of the Jacobian of the linear transformation (29) is 2^3 . We also introduce variables t_i defined by

$$t_i = r_i^2 \quad (i = 1, 2, 3). \quad (30)$$

Notice that s_i and r_i have the dimension of m^2 while t_i has the dimension of m^4 . It is interesting to consider the substitutional behaviour of these new variables under the

group S_4 of permutations of the four pions. While the latter induce permutations between different s_i and between different t_i , they induce more general substitutions with changes of sign between different r_i . Expressing these substitutions by symbols like $+2|+3|-1$, meaning that r_1 is replaced by r_2 , r_2 by r_3 , and r_3 by $-r_1$, we arrive at Table 1. It shows the substitutions induced in the different variables by all 24 permutations of the four pions. The conventional cyclic notation is used to designate permutations. Horizontal lines separate different classes of original permutations of the four pions. The fact that we only need one column for both s_i and t_i is an obvious consequence of the definitions (29) and (30).

Two remarkable facts emerge from Table 1:

- i) If we permute the four pions, the three s_i permute among themselves (and so do the three t_i), according to the factor group of the normal (or "invariant") subgroup $V = \{1, (12)(34), (13)(24), (23)(14)\}$ of S_4 . This factor group is isomorphic to S_3 , while V is isomorphic to Klein's four group and the dihedral group D_2 . An s_i and a t_i with the same index i are "conjugate" in that the same permutation table is valid for both.
- ii) If we permute the four pions, the three r_i permute among themselves with possible sign changes. The permutational behaviour of r_i is the same as for the conjugate s_i and t_i , i.e. those with the same index i . The number of sign changes accompanying any permutation is even (0 or 2).

Here, a change of perspective proves advantageous. In the set of variables $r_1, s_1, t_1; r_2, s_2, t_2; r_3, s_3, t_3$ a permutation of pions induces a permutation of the three indices 1, 2, 3 according to S_3 and a sign change of the three r_i ; the set U of possible sign changes consists of four elements

$$U = \{(+++), (+--), (-+-), (--+)\} \quad (31)$$

which form a multiplicative group isomorphic to V and D_2 .

The last column of Table 1 shows for comparison the permutations induced in the variables x_j by the permutations of the four pions. Here, the group is a (non-invariant) subgroup of A_6 (all permutations of the variables x_j are even). The greater simplicity of the new variables as compared to the old ones is obvious.

Table 1 may be interpreted from the point of view of the theory of group representations. The space of the variables x_j is the 6-dimensional representation space (also called carrier space or, inexactly but simply, representation) of $\text{Sym}^2(W)/W$, in the notation of Serre [10], W being the space of the natural representation of S_4 . $\text{Sym}^2(W)$ is the symmetric square of W and $\text{Sym}^2(W)/W$ is this square, modulo the original representation W [which is canonically contained in $\text{Sym}^2(W)$]. $\text{Sym}^2(W)/W$ is reducible into the 3-dimensional irreducible representation of S_4 characterized by the partition [3,1] and the character ψ in the notation of Serre [10, p. 43] (which is our space of the variables r_i), and the reducible natural representation of S_3 (which is our space of the variables s_i). We prefer not to explicitly reduce the latter space; partly, its separation into a 1-dimensional and a 2-dimensional irreducible space is achieved by energy conservation (see also Section 8). Of course, the character ψ has nothing to do with the basic pseudoscalar ψ of Eq. (4).

To some extent the kinematical space separates into an r -space and an s -space. The energy conservation (24) operates only in the latter and becomes

$$\sum_{i=1}^3 s_i = C. \quad (32)$$

Since $s_i \geq 2m^2$ for $i = 1, 2, 3$, only the basic octant of the 3-dimensional s -space is physical. Events lie in it on the plane (32) that we shall henceforth call the C -plane. There are six possible orderings of the variables s_1, s_2, s_3 , according to their values. However, since the four pions are identical, we can always number them in such a way that $s_1 \geq s_2 \geq s_3$. Table 1 makes this statement obvious. Thus, all events can be “squeezed” into one sextant of the C -plane. Table 1 also makes it evident that the remaining four-fold arbitrariness of pion labelling can be utilized to squeeze all events into one quadrant of the r -space, e.g. the basic one with $r_1 \geq 0, r_2 \geq 0$.

Of course, the r - and s -space are not completely disjoint. For one thing, they are linked by the inequalities

$$s_i > |r_i|, \quad \text{i.e.} \quad s_i^2 > t_i, \quad (33)$$

which follow from the definitions (6), (29), and (30), and Eq. (3). Geometrically, to each point (s_1, s_2, s_3) in the s -space [remember that it has to lie in the C -plane (32)] there corresponds a parallelepiped in the r -space which must contain the corresponding part of the mass shell. With our choice of the quadrant in the r -space, the eight corners of this parallelepiped are

$$\begin{aligned} &(s_1, s_2, s_3), (s_1, s_2, -s_3), (s_1, 0, s_3), (s_1, 0, -s_3), \\ &(0, s_2, s_3), (0, s_2, -s_3), (0, 0, s_3), (0, 0, -s_3). \end{aligned} \quad (34)$$

The intertwining of the r -space and the s -space is especially intimate at the boundary $\Delta_4 = 0$. In the new variables, Δ_4 takes the form

$$\begin{aligned} \Delta_4 = & \frac{1}{8} \left[(s_1^2 - t_1)(s_2^2 - t_2) + (s_2^2 - t_2)(s_3^2 - t_3) + (s_3^2 - t_3)(s_1^2 - t_1) \right] \\ & - \frac{1}{16} \left[(s_1^2 - t_1)^2 + (s_2^2 - t_2)^2 + (s_3^2 - t_3)^2 \right] - m^2(s_1 s_2 s_3 + r_1 r_2 r_3) \\ & + \frac{1}{2} m^4 (s_1^2 + s_2^2 + s_3^2 + t_1 + t_2 + t_3) - m^8. \end{aligned} \quad (35)$$

The analogous expressions for Δ_3 and Δ_2 may easily be obtained using Eqs. (11) and (12).

In the preceding Section we have defined the mass shell as that part of the C -plane and of the r -space that lies inside the boundary $\Delta_4 = 0$. The center of the mass shell is now characterized by

$$\begin{aligned} s_1 &= s_2 = s_3 = C/3, \\ r_1 &= r_2 = r_3 = 0. \end{aligned} \quad (36)$$

The restricted mass shell is defined as a suitable $(1/24)^{\text{th}}$ part of the entire mass shell, e.g. by the restrictions as chosen above:

$$s_1 \geq s_2 \geq s_3, \quad r_1 \geq 0, \quad r_2 \geq 0. \quad (37)$$

Of course, in the restricted mass shell, the center finds itself “at the edge”. While all events may be squeezed into the restricted mass shell by proper ordering of the pions, it may prove aesthetically more pleasing to copy them again into the other 23 congruent sections of the mass shell before presenting any plots. This procedure is commonly used when presenting triangular Dalitz plots of three-pion decays.

In the new variables the volume element $d\tau$ (referring to the full mass shell) has the form

$$d\tau = \frac{2\pi^2 \left(\prod_{i=1}^3 ds_i \right) \left(\prod_{i=1}^3 dr_i \right) \delta \left(\sum_{i=1}^3 s_i - C \right)}{M^2 \sqrt{\Delta_4}}. \quad (38)$$

Formally, Eq. (38) is no more complicated than Eq. (28). However, somewhat disappointingly, Δ_4 expressed in the form (35) seems, if anything, even more unwieldy than the form (8). The substantial advantages of the new scalar variables as compared to the old ones will only become apparent in the subsequent Sections.

4 Invariants and anti-invariants

The probability dW of decay per unit of proper time into an infinitesimal volume element $d\tau$ is proportional to the latter and defined as

$$dW = |A|^2 d\tau, \quad (39)$$

where A is the decay amplitude. In general, A is a complex quantity which depends in a complicated way on the dynamics of the decay but is a function of rather few quantum numbers characterizing the initial and final state. In our case, in addition to zero spins, the parity of the initial particle, its 4-momentum, and the 4-momenta of the final particles are the only relevant observables.

We shall also need the “reduced” decay probability w defined as

$$w = \frac{dW}{d\tau} = |A|^2. \quad (40)$$

We prefer to call the quantity w reduced rather than differential decay probability. The latter term is often used for the probability of decay into any part of the mass shell that is infinitesimal in all its five dimensions, but is not necessarily equal to $d\tau$ as defined by Eqs. (28) or (38). In the above expressions we have left undefined constants which become relevant only when a dynamical theory tries to predict the amplitude. For the purposes of the present article they may be considered included in the amplitude A .

Invariance under proper Lorentz transformations demands that the amplitude A be a function of the basic 4-scalars x_j (or, equivalently, s_i and r_i) and the basic 4-pseudoscalar ψ ; it is linear in the latter. The total intrinsic parity of the four identical final bosons is positive. If parity is conserved in the decay, the amplitude $A = A^+$ for the decay from a $J^P = 0^+$ initial state is a 4-scalar and the amplitude $A = A^-$ for the decay from a $J^P = 0^-$ initial state is a 4-pseudoscalar. We may write

$$A^+ = E, \quad (41)$$

$$A^- = \psi O. \quad (42)$$

Here, E and O are functions of the basic 4-scalars. The 4-scalar E has to be symmetric under permutations of the four identical bosons in the final state but exact symmetries do not impose any further restrictions on its form. Thus, E may be considered a “form factor” in the sense of Section 1, which could complete our discussion of the decay of an initial particle with $J^P = 0^+$: any symmetric 4-scalar function E of the four 4-momenta p_i (including a constant) is *a priori* allowed in Eq. (41).

The case of an initial particle with $J^P = 0^-$ is more complicated. The basic 4-pseudoscalar ψ is antisymmetric with respect to permutations of the four 4-momenta p_i ;

therefore their 4-scalar function O must be antisymmetric as well. The question of how to best express any possible function O has triggered the investigations reported in this article. Incidentally, if parity were not conserved in the decay, the amplitude A could be a linear superposition of A^+ and A^- .

From the point of view of group theory, the quantities appearing in Eqs. (41) and (42), i.e. A^+ , A^- , ψ , E , and O , are absolute invariants of the group

$$L^+ \otimes A_4, \quad (43)$$

which is the outer product of the proper (orthochronous) Lorentz group L^+ and the alternating group A_4 , both operating on the four 4-vectors p_i . It is a trivial exercise to find out the behaviour (sign change or no sign change) of the above quantities under the action of the larger groups: the full orthochronous Lorentz group L on one hand, and the symmetric group S_4 on the other. However, as explained in the preceding Section, it is advantageous to adopt a different perspective. Equations (41) and (42) guarantee the proper behaviour of the amplitudes A^+ and A^- under Lorentz transformations if we require that the quantities E and O be functions of the basic 4-scalars s_i and r_i . Thereafter, we may abandon the 4-vector variables p_i and forget about the Lorentz group. The proper behaviour of the amplitudes under permutations of the four pions is guaranteed by requiring that

- i) the quantities E and O must be absolute invariants of the group U [see Eq.(31)] of sign changes of the variables r_i ;
- ii) E must be an absolute invariant and O an anti-invariant of the symmetric group S_3 acting on the indices i of the variables s_i and r_i .

Concisely – albeit with less precision – we may say that E is an absolute invariant and O an anti-invariant of the group

$$G = U \rtimes S_3, \quad (44)$$

where U is defined by Eq. (31) and where the semidirect product sign $\rtimes$ reminds us that U is – and its factor group S_3 is not – a normal subgroup of G . The group G acts on the vectors

$$\vec{s} = (s_1, s_2, s_3), \quad (45)$$

$$\vec{r} = (r_1, r_2, r_3), \quad (46)$$

in the way specified in Table 1. In other words, we are only interested in the concrete representation of this group by matrices in the 6-dimensional $\vec{s}, \vec{r}$ -space. Absolute invariants are polynomials in the variables $s_1, s_2, s_3, r_1, r_2, r_3$ which do not change if the 6-dimensional vector $(s_1, s_2, s_3, r_1, r_2, r_3)$ is operated upon by any of these matrices; anti-invariants change sign if the matrix corresponds to an odd permutation of the indices 1, 2, 3 and remain unchanged otherwise. In Table 2 we illustrate the representation by one matrix for each of the five classes of S_4 -permutations. For connoisseurs we remark that while G is isomorphic to the symmetric group S_4 , the representation we are working in contains no pseudoreflections and the determinants of all matrices are equal to 1. This fact is intimately connected with the simultaneous action of S_3 on two vectors $\vec{s}$ and $\vec{t}$, the latter being defined as

$$\vec{t} = (t_1, t_2, t_3). \quad (47)$$

For future use we also define the group

$$H = U \rtimes A_3. \quad (48)$$

The exact definition of H is completely analogous to that of G , with S_3 replaced by A_3 . The quantities E and O are both absolute invariants of the group H .

From now on, when we talk about the group G we mean the group of all twenty-four 6×6 matrices illustrated in Table 2. Analogously, H has to be understood as the group of twelve matrices representing the first three classes of Table 2, i.e. the even permutations of four objects.

The term "anti-invariant" sounds familiar to physicists and it is nearly superfluous to explain it: an anti-invariant of S_3 (or G) changes sign if the indices $i = 1, 2, 3$ undergo an odd permutation, and does not change at all if the permutation is even; thus it is an absolute invariant of the alternating group A_3 . Mathematicians mostly refer to anti-invariants as "relative invariants (or semi-invariants) of alternating character". We shall most often use the terminology "invariant" for absolute invariant and "(anti)invariant" when we want to refer to both absolute and relative invariants. Instead of thinking in terms of invariants and anti-invariants of the group G , it is often simpler to consider them together as absolute invariants of the group H . Yet another remark is in place: a (complex) number is a trivial invariant and 1 is a trivial basic invariant but it often saves words to omit them from a list of invariants. Most of the time it is clear from the context whether the term "invariant" is meant to include unity, i.e. also all complex numbers.

As we shall see, the (anti)invariants of greatest interest for our purpose are polynomial (or "integral") (anti)invariants. An example of a polynomial invariant of the group G is the quantity Δ_4 of Eq. (35). Fortunately, polynomial absolute and relative invariants are the main subjects of interest in the theory of invariants, which is a well-developed and reasonably well-documented field of algebra. We shall repeatedly refer to the classical books of Burnside [11], Schur and Grunsky [12], and Weyl [8]. A short but easily readable introduction to the parts of invariant theory most relevant to our work is contained in an article by Sloane [13]; more advanced and very informative is the survey article by Stanley [14]. More recently, several books [15, 16, 17] have appeared, treating subjects of interest to us and containing references to previous work; the most extensive list of references is contained in a collection of articles on different topics in invariant theory [18].

Except when explicitly stated, we shall confine our further discussion to polynomial (anti)invariants but omit the adjective "polynomial" for simplicity. We shall assume, without proof, the validity of some theorems of invariant theory; nevertheless, the present article should be understandable to readers lacking any prior knowledge of this theory. Hilbert's 1st finiteness theorem guarantees that there exists a finite basis of invariants and anti-invariants of the group G of Eq. (44). Thus, any invariant polynomial can be expressed as a polynomial in the basic invariants, the coefficients being complex numbers. Clearly, the square of any anti-invariant is an invariant. Therefore, any anti-invariant polynomial is linear and homogeneous (i.e. a linear form) in the basic anti-invariants, the coefficients of the latter being invariants. Mathematicians would say that anti-invariants constitute a module over the ring of invariants; invariants may also be considered as a module over their own ring.

Emmy Noether [6], [8, Chapter VIII, §15], [12, Chapter III, §2], [13, p. 97] found a beautifully simple way of constructing a basis which, at the same time, represents a proof of Hilbert's 1st theorem for finite groups. We shall adapt her method to our needs. Consider any monomial

$$r_1^k r_2^{k'} r_3^{k''} s_1^m s_2^{m'} s_3^{m''}, \quad (49)$$

k, k', k'', m, m', m'' being positive integers or zero, and construct the sum of all such mono-

mials derived from the monomial (49) by applying all substitutions of Table 1. The sum is an invariant of the group G , unless it vanishes identically. If, before summing, we multiply every individual monomial (49) that corresponds to an odd permutation of 1, 2, 3 by -1 , the sum either vanishes identically or represents an anti-invariant. Clearly, (anti)invariants derived in this way from the monomial (49) are forms, i.e. homogeneous polynomials.

Among the three exponents k, k', k'' , at least one, say k'' , is not larger than the other two. Then we may write

$$r_1^k r_2^{k'} r_3^{k''} = R_3^{k''} r_1^{(k-k'')} r_2^{(k'-k'')}, \quad (50)$$

where all three exponents are positive integers or zero. Here,

$$R_3 = r_1 r_2 r_3 \quad (51)$$

is an invariant, a fact which is obvious from our discussion of Table 1. The summation of the four monomials (50) that correspond to a given ordering of 1, 2, 3 but to different sign possibilities (31) yields zero unless both exponents $k - k''$ and $k' - k''$ are even. Therefore, apart from the invariant R_3 of Eq. (51), we henceforth only have to consider invariants and anti-invariants formed from the "germs"

$$\begin{pmatrix} m & m' & m'' \\ n & n' & n'' \end{pmatrix} = s_1^m s_2^{m'} s_3^{m''} t_1^n t_2^{n'} t_3^{n''}, \quad (52)$$

n, n', n'' being positive integers or zero. In other words, except for R_3 , all other basic (anti)invariants are "simultaneous" (anti)invariants of the group S_3 in the vectors $\vec{s}$ and $\vec{t}$. It is obvious that this conclusion does not depend on our choice in Eq. (49) of k'' as not being larger than either k or k' . We shall refer to m, m', m'' as the m -exponents and to n, n', n'' as the n -exponents.

Noether's construction leads to a finite basis (called primary basis since, in the next Section, we are going to reduce it to a smaller one) because no m - or n -exponent larger than 2 needs to be considered. This follows from the fact that, for $i = 1, 2, 3$,

$$s_i^3 = S_1 s_i^2 - S_2 s_i + S_3, \quad (53)$$

with

$$S_1 = s_1 + s_2 + s_3, \quad (54)$$

$$S_2 = s_1 s_2 + s_2 s_3 + s_3 s_1, \quad (55)$$

$$S_3 = s_1 s_2 s_3, \quad (56)$$

being the final basic invariants (called elementary symmetric functions) of the group S_3 (the final basis is to be defined in the next Section). Equation (53) may be trivially verified for any of the three possible values of the index i . In the 3-dimensional s -space, any integral S_3 -invariant can be expressed as a polynomial in S_1, S_2, S_3 . Those belonging to the primary basis can be found among the invariants quoted in Appendix B.

We label the (anti)invariants with an index K defined by the dimension m^{2K} of the (anti)invariant in question. This labelling is conformable to the labelling of the quantities Δ_K ($K = 1, 2, 3, 4$) of Sections 2 and 3, and has already been applied in the definitions (51) and (54)–(56). We shall refer to K as the degree of the (anti)invariant.

There exists a single basic anti-invariant of the group S_3 , namely

$$H_3 = (s_1 - s_2)(s_2 - s_3)(s_3 - s_1) = s_1(s_2^2 - s_3^2) + s_2(s_3^2 - s_1^2) + s_3(s_1^2 - s_2^2). \quad (57)$$

Its square is an invariant; using Eq. (213) of Appendix A, we obtain

$$\Xi_6(H_3^2) = 4S_2^3 + 27S_3^2 + 4S_1^3S_3 - S_1^2S_2^2 - 18S_1S_2S_3 + H_3^2 = 0. \quad (58)$$

Equation (58) is an example of a syzygy belonging to the first link of the syzygy chain (also called syzygy of the 1st kind): a polynomial Ξ_K in (anti)invariants, homogeneous of degree K and fulfilling the following conditions:

- i) it vanishes identically in the vectors $\vec{s}$ and $\vec{r}$;
- ii) it does not vanish identically in the (anti)invariants, as does, for example, $(S_1S_2)^2 - S_1^2S_2^2$, which vanishes identically in S_1, S_2 , even if the value of $\vec{s}$ in S_1 is different from its value in S_2 ;
- iii) it does not contain a single (anti)invariant of degree K so as not to permit expressing that (anti)invariant as a polynomial in the others.

We use the term syzygy indiscriminately for the polynomial Ξ_K and the relation $\Xi_K = 0$. We further characterize the syzygy by a product of two (anti)invariants stated in parentheses after the symbol Ξ_K for the syzygy. This product appears in the syzygy with a constant coefficient which may be chosen at will by multiplying the syzygy with a suitable constant factor. In the syzygy (58) the coefficient has been chosen to be 1. Later (see Section 7), we shall be obliged to relax the last of the above conditions.

The word syzygy has been introduced into the classical invariant theory by one of its early practitioners J. J. Sylvester [19, 364F]. In ancient Greek $\sigma\upsilon\zeta\upsilon\gamma\acute{\iota}\alpha$ signified a pair of oxen yoked together. In modern Greek it means, not inappropriately, matrimony. Incidentally, without noticing it, we have already met a (mathematical) syzygy in Eq. (7) which represents the unique syzygy linking invariants of the group L^+ for four particles [8, Chapter II, §17].

The foregoing is all that has to be said about the basis of (anti)invariants of S_3 in the s -space. For (anti)invariants in the t -space we have to introduce a different notation, keeping in mind our labelling convention. We have

$$T_2 = t_1 + t_2 + t_3, \quad (59)$$

$$T_4 = t_1t_2 + t_2t_3 + t_3t_1, \quad (60)$$

$$T_6 = t_1t_2t_3, \quad (61)$$

$$H_6 = (t_1 - t_2)(t_2 - t_3)(t_3 - t_1) = t_1(t_2^2 - t_3^2) + t_2(t_3^2 - t_1^2) + t_3(t_1^2 - t_2^2), \quad (62)$$

as the basis of S_3 -(anti)invariants in the vector $\vec{t}$, i.e. in the t -space. However, R_3 of Eq. (51) rather than $T_6 = R_3^2$ belongs to the final basis of G -invariants. The syzygy linking the four polynomials (59)–(62) can be directly transcribed from Eq. (58) and so can the fundamental reduction formula (53), yielding

$$t_i^3 = T_2t_i^2 - T_4t_i + T_6. \quad (63)$$

Having constructed the part of our primary basis that corresponds to either all m -exponents or all n -exponents in Eq. (52) being zero, we have to face the task of constructing the remaining part of the basis, starting with “mixed germs” of the form (52), i.e. those with at least one of the m -exponents and one of the n -exponents different from zero. (Anti)invariants originating from such germs will be called “mixed”, omitting the specification “basic” as self-evident. In order to simplify this task, we observe that

- i) if all three m -exponents are different from zero, an appropriate power of S_3 can be factored out of the germ (52),
- ii) if all three n -exponents are different from zero, an appropriate power of T_6 can be factored out of the germ (52),
- iii) clearly, we do not lose generality if we assume $m'' = 0$ and $m \geq m'$,
- iv) because of relations (53) and (63) no exponent larger than 2 needs to be considered.

The five relevant configurations of m -exponents – called m -configurations for short and designated by symbols of the type $(mm'm'')$ – appear in the left column of Table 3.

The list of relevant n -configurations (designated by the same kind of symbols as the m -configurations) results from arbitrary permutation of the three exponents in each of the five m -configurations. Let us call all n -configurations resulting from any permutation of the three exponents in the symbol $(nn'n'')$ “ n -configurations of the type (n,n',n'') ”. There are six different permutations of (210) and three different ones otherwise. Thus, there are altogether 18 different n -configurations. Since each of them can be combined with any of the five m -configurations, 90 “mixed” configurations are possible. However, some of them are equal: whenever two of the m -exponents are equal (i.e. the m -configuration is one of the four listed in the upper and the middle box of Table 3), a transposition of the corresponding n -exponents does not change the invariant. In the case where the n -configuration is of the type $(1,0,0)$, $(2,0,0)$, $(1,1,0)$, or $(2,2,0)$, this equality reduces the number of different invariants by one. There are $4 \times 4 = 16$ such cases. In the case where the n -configuration is of the type $(2,1,0)$, the above equality reduces the number of different invariants by three, i.e. altogether by 12. As a result there exist only $90 - 16 - 12 = 62$ different mixed invariants.

Table 3 lists all possible combinations of m - and n -configurations leading to non-equal mixed invariants. The reason for grouping them in two categories will become obvious shortly. The n -configurations are listed in different rows according to the value of

$$n_{\text{tot}} = n + n' + n'' . \quad (64)$$

Analogously, we define m_{tot} by

$$m_{\text{tot}} = m + m' + m'' \quad (65)$$

and have

$$K = m_{\text{tot}} + 2n_{\text{tot}} . \quad (66)$$

Corresponding to configurations of Table 3 there exist 62 different mixed invariants of the form

$$\begin{aligned} & \begin{bmatrix} m & m' & 0 \\ n & n' & n'' \end{bmatrix} = \\ & = s_1^m s_2^{m'} t_1^n t_2^{n'} t_3^{n''} + s_2^m s_3^{m'} t_2^n t_3^{n'} t_1^{n''} + s_3^m s_1^{m'} t_3^n t_1^{n'} t_2^{n''} \\ & + s_2^m s_1^{m'} t_2^n t_1^{n'} t_3^{n''} + s_3^m s_2^{m'} t_3^n t_2^{n'} t_1^{n''} + s_1^m s_3^{m'} t_1^n t_3^{n'} t_2^{n''} \end{aligned} \quad (67)$$

with none of the exponents larger than 2, $m \geq m'$, and at least one of the n -exponents being zero. Together with the “pure” invariants S_1, S_2, S_3, T_2, R_3 , and T_4 they form a basis of 68 invariants for our problem (not counting the constant).

In the construction of anti-invariants by the above method a further simplification occurs. Starting from the general germ (52) with $m'' = 0$ we obtain the general mixed

anti-invariant

$$\left\{ \begin{array}{ccc} m & m' & 0 \\ n & n' & n'' \end{array} \right\} = \quad (68)$$

$$= s_1^m s_2^{m'} t_1^n t_2^{n'} t_3^{n''} + s_2^m s_3^{m'} t_2^n t_3^{n'} t_1^{n''} + s_3^m s_1^{m'} t_3^n t_1^{n'} t_2^{n''} \\ - \left[s_2^m s_1^{m'} t_2^n t_1^{n'} t_3^{n''} + s_3^m s_2^{m'} t_3^n t_2^{n'} t_1^{n''} + s_1^m s_3^{m'} t_1^n t_3^{n'} t_2^{n''} \right].$$

This anti-invariant vanishes identically if $m' = 0$ and $n' = n''$. It also vanishes identically if $m = m'$ and $n = n'$. Therefore, the n -configurations listed as the second category in Table 3 do not lead to anti-invariants. The total number of different anti-invariants of the type (68) is thus 46. Together with the anti-invariants H_3 and H_6 of Eqs. (57) and (62) they form a basis of 48 anti-invariants.

In the following, it will prove convenient to characterize any invariant belonging to the primary basis (except R_3) by the symbol

$$\left[\begin{array}{ccc} m & m' & m'' \\ n & n' & n'' \end{array} \right]$$

indicating its m - and n -configuration. Similarly, we shall characterize any anti-invariant belonging to the primary basis by the "antisymbol"

$$\left\{ \begin{array}{ccc} m & m' & m'' \\ n & n' & n'' \end{array} \right\}.$$

The rules for calculating with these (anti)symbols are stated in Appendix A.

We stress the fundamental simplicity of the primary basis. Among all possible basic (anti)invariants of the group G , only R_3 is specific to that group. All others are (anti)invariants of the group S_3 , simultaneous in the two 3-dimensional vectors $\vec{s}$ and $\vec{t}$.

5 Reduction of the primary to the final basis

We refer to the basis of 68 invariants and 48 anti-invariants constructed in the previous Section as the primary basis. It turns out that most of these (anti)invariants may be expressed as polynomials in elements of a subset of the primary basis. We shall call such a subset a final basis if no subset with a smaller number of elements forms a basis. (The term "minimal" basis would appear more appropriate; however, in invariant theory it is sometimes used in other contexts which could lead to misunderstandings.) In general, there exist syzygies among the members of a final basis. They permit a further reduction of the final basis if some of its members are allowed to be expressed rationally or algebraically (and not only as polynomials) in the others.

There exists an elegant method called "complete polarization" [8, Ch. II, §3] to obtain a final basis of invariants of our problem from the final basis (54)–(56) of pure invariants. It is equivalent to systematically decreasing in the symbols

$$\left[\begin{array}{ccc} m & m' & m'' \\ 0 & 0 & 0 \end{array} \right]$$

the values of m -exponents at the expense of increasing the values of n -exponents, followed by a proof that all symbols obtainable in this way, together with the original pure invariants (54)–(56), form a complete basis. This proof is based on complete induction with respect to the dimensionality of the vectors $\vec{s}$, $\vec{t}$ and can be extended to any number of vectors of common dimension n (but it does not include a demonstration that the basis obtained in this way is final). Unfortunately, the proof is not directly translatable into an analogous proof for anti-invariants.

However, in a special case like ours, the final basis can be found by a concise and elegant procedure based on general methods of invariant theory that we shall learn about in this and the following two Sections. This procedure, suggested by H. Kraft [20] is presented in Appendix C. It should be followed by a modern systematic derivation of the bases of all syzygy links. Nowadays, at least this second task would normally be assigned to a computer using powerful algebraic algorithms [16, 17], applicable to a large class of matrix groups.

However, as we shall see in this Section, it is also possible to use a primitive but somewhat tedious constructive method to discover the final basis of (anti)invariants and the corresponding first-link syzygies. To a physicist with no previous experience in invariant theory, this path is more easily accessible than its sophisticated alternative.

The explicit expression of a symbol

$$\begin{bmatrix} m & m' & 0 \\ n & n' & n'' \end{bmatrix}$$

of degree K as a polynomial in other symbols cannot, obviously, contain any symbols of higher degree. At any K , we may form all products of all possible invariants of lower degree such that the degree of the product is K . Using the rules of Appendix A, we may reduce any such product to a polynomial containing invariants of degree K , if at all, linearly with constant coefficients. By putting all terms obtained in the reduction of the product on the same side of the equation as the product, we obtain a polynomial in the invariants which vanishes identically in the vectors $\vec{s}, \vec{t}$. If it contains only invariants of degree lower than K , it either vanishes identically in these invariants or it represents a syzygy. Otherwise, it represents one member of a set of linear equations (with constant coefficients) in the mixed primary invariants of degree K . We start with $K = 3$ (no relations of interest occur for lower values of K) and proceed to $K = 12$. In the process it turns out that, for each K , the corresponding set of linear equations enables us to express all mixed primary invariants of degree K as polynomials in the basic pure invariants S_1, S_2, S_3, T_2, R_3 (actually, only $T_6 = R_3^2$ occurs in the process), T_4 , and three basic mixed invariants U_3, U_4 , and U_5 . Thus, these nine basic invariants constitute the final basis of invariants. (From now on we call basic only those (anti)invariants that belong to the final basis). We have chosen as the basic mixed invariants:

$$U_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = s_1(t_2 + t_3) + s_2(t_3 + t_1) + s_3(t_1 + t_2), \quad (69)$$

$$U_4 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = s_1 s_2(t_1 + t_2) + s_2 s_3(t_2 + t_3) + s_3 s_1(t_3 + t_1), \quad (70)$$

$$U_5 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} = t_1 t_2(s_1 + s_2) + t_2 t_3(s_2 + s_3) + t_3 t_1(s_3 + s_1). \quad (71)$$

As an example of an invariant expressed in the final basis, we quote the quantity Δ_4 of Eq. (35):

$$\begin{aligned} \Delta_4 = & \frac{1}{16} \left[4S_1^2 S_2 - S_1^4 - 8S_1 S_3 + 2(S_1^2 + 2S_2) T_2 - T_2^2 + 4T_4 - 4S_1 U_3 - 4U_4 \right] \\ & - m^2 (S_3 + R_3) + \frac{1}{2} m^4 (S_1^2 - 2S_2 + T_2) - m^8. \end{aligned} \quad (72)$$

In a completely analogous way, the primary basis of anti-invariants can be reduced to a final basis. In this case, for obvious reasons, all polynomials are linear forms in the anti-invariants. In addition to H_3 and H_6 , the final basis contains three mixed anti-invariants

which may be chosen as

$$J_3 = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{Bmatrix} = s_1(t_2 - t_3) + s_2(t_3 - t_1) + s_3(t_1 - t_2), \quad (73)$$

$$J_4 = \begin{Bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{Bmatrix} = s_1 s_2(t_1 - t_2) + s_2 s_3(t_2 - t_3) + s_3 s_1(t_3 - t_1), \quad (74)$$

$$J_5 = \begin{Bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{Bmatrix} = t_1 t_2(s_1 - s_2) + t_2 t_3(s_2 - s_3) + t_3 t_1(s_3 - s_1). \quad (75)$$

In Appendix B, all other primary (anti)invariants are expressed in terms of the basic ones. Notice that besides T_6 two additional auxiliary invariants,

$$\begin{aligned} U_6 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} &= s_1 s_2(t_3 t_1 + t_3 t_2) + s_2 s_3(t_1 t_2 + t_1 t_3) + s_3 s_1(t_2 t_3 + t_2 t_1) \\ &= \frac{1}{3} (4S_2 T_4 + S_1 T_2 U_3 - T_2 U_4 - S_1 U_5 - U_3^2), \end{aligned} \quad (76)$$

$$\begin{aligned} U_9 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} &= s_1 s_2 t_1 t_2 (s_1 t_2 + s_2 t_1) + s_2 s_3 t_2 t_3 (s_2 t_3 + s_3 t_2) + s_3 s_1 t_3 t_1 (s_3 t_1 + s_1 t_3) \\ &= -S_1 S_2 T_2 T_4 - 3S_3 T_6 + S_2 T_4 U_3 + U_4 U_5, \end{aligned} \quad (77)$$

and the additional auxiliary anti-invariant

$$\begin{aligned} J_6 = \begin{Bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{Bmatrix} &= s_1 s_2(t_3 t_1 - t_3 t_2) + s_2 s_3(t_1 t_2 - t_1 t_3) + s_3 s_1(t_2 t_3 - t_2 t_1) \\ &= \frac{1}{3} [(S_1 T_2 - U_3) J_3 + T_2 J_4 - S_1 J_5], \end{aligned} \quad (78)$$

have been introduced. The auxiliary (anti)invariants simplify the expressions in Appendix B but the deeper reason for their introduction will become apparent only as we go on. In the next Section we shall introduce the name of “good” basis for the basic and auxiliary (anti)invariants (51), (54)–(57), (59)–(60), (62), (69)–(71), and (73)–(78), i.e. all basic and auxiliary (anti)invariants with the exception of T_6 .

In the above reduction process, the symmetry of the (anti)symbols

$$\begin{bmatrix} m & m' & m'' \\ n & n' & n'' \end{bmatrix}$$

and

$$\begin{Bmatrix} m & m' & m'' \\ n & n' & n'' \end{Bmatrix}$$

with respect to the interchange of the two rows of exponents is useful as a check. We call this symmetry the s, t -symmetry. Under the s, t -symmetry operation, the basic and auxiliary (anti)invariants transform as stated in Table 4. Clearly, either two different (anti)invariants are (mutually) s, t -conjugate (e.g. S_1 and T_2) or a single invariant is “self-conjugate” (e.g. U_3) or a single anti-invariant is “self-anticonjugate” (e.g. J_3) under the s, t -symmetry operation.

The reduction process, if carried out to the letter, is very-time consuming owing to the fast increase of the number of products with their degree K . However, in carrying it

out, one soon develops an intuition for the products which lead to polynomials vanishing identically in the (anti)invariants, thus producing redundant results. The total work involved can be measured in terms of weeks rather than months. The result guarantees that no product of interest for obtaining the final basis has been missed. That the latter basis is indeed final can be proved by inspecting the results listed in Appendix B. We see that all possible products leading to mixed (anti)invariants with $K \leq 5$ occur, except $S_1^2 U_3$, $S_1^3 T_2$, and $S_1^2 J_3$. By explicitly evaluating these products we easily find that they lead to redundant identities. Thus, it is indispensable to introduce, in addition to pure (anti)invariants, three new basic invariants and three new basic anti-invariants, which means that the reduced basis is indeed final.

Of course, for any K , one of the derived invariants that depend on U_K could be chosen as the basic one instead of U_K ; the same statement applies to the anti-invariants J_K . For instance, the aforementioned complete polarization method leads to the mixed invariants

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

instead of U_4 and U_5 , respectively. In other words, the final basis is not unique. However, this ambiguity is of a trivial nature.

Appendix B reveals a remarkable simplicity. The mixed invariants U_3 , U_4 , U_5 , U_6 , and U_9 appear at most linearly in the primary basis. The anti-invariants H_3 , J_3 , J_4 , J_5 , J_6 , and H_6 are never multiplied by one of the mixed invariants U_K . Of course, these simple rules are only valid if the auxiliary invariants U_6 , U_9 , and J_6 are added to the final basis. They contain respectively U_3^2 , $U_4 U_5$, and $U_3 J_3$. On the other hand, no simple rule seems to limit the occurrences of pure basic invariants S_1 , S_2 , S_3 , T_2 , R_3 , and T_4 , except for the well-understood rule (see Section 4) that R_3 appears in any other primary (anti)invariant only as $T_6 = R_3^2$.

6 Good bases and their Molien functions

The simplicity pointed out at the end of the preceding Section is a general phenomenon characterizing the ring of invariants of any finite group [14, Theorem 3.2]. Our basic pure invariants S_1 , S_2 , T_2 , S_3 , R_3 , T_4 are sometimes called “free” [13]. (They also go by the awkward collective name of a “homogeneous system of parameters” with the abbreviation h. s. o. p.). Our basic and auxiliary mixed invariants U_3 , U_4 , U_5 , U_6 , U_9 are sometimes called the “transient” invariants [13] of the group G . Together with the basic and auxiliary anti-invariants H_3 , J_3 , J_4 , J_5 , J_6 , H_6 they form the set of transient invariants of the group H or, if one prefers, of transient (anti)invariants of the group G . The transient invariants and the free invariants constitute a so-called “good” basis of the group G . The transient (anti)invariants and the free invariants constitute a good basis of the group H . In later Sections we shall find it convenient to replace the above basic (anti)invariants by other equivalent sets of basic (anti)invariants. If any confusion could arise, we shall specify the set used in this Section as the “original” good basis.

To any good basis corresponds a Molien function of the form

$$\Phi(\lambda) = \frac{1 + \sum_m c_m \lambda^m}{\prod_n (1 - \lambda^n)^{d_n}}. \quad (79)$$

Here, d_n is the number of free invariants of a given degree n and c_m is the number of transient invariants of a given degree m in the good basis. From the above description of the good basis it follows that the Molien functions $\Phi_G(\lambda)$ and $\Phi_H(\lambda)$, for the groups (of matrices) G and H , respectively, are

$$\Phi_G(\lambda) = (1 + \lambda^3 + \lambda^4 + \lambda^5 + \lambda^6 + \lambda^9)/D, \quad (80)$$

$$\Phi_H(\lambda) = (1 + 3\lambda^3 + 2\lambda^4 + 2\lambda^5 + 3\lambda^6 + \lambda^9)/D, \quad (81)$$

with

$$D = (1 - \lambda)(1 - \lambda^2)^2(1 - \lambda^3)^2(1 - \lambda^4). \quad (82)$$

By an explicit calculation we can easily convince ourselves that these Molien functions are indeed equal – as they should be according to Molien's theorem [21] – to

$$\Phi_g(\lambda) = \frac{1}{g} \sum_{\mathcal{A}} \frac{1}{\det(\mathcal{I} - \lambda\mathcal{A})}, \quad (83)$$

with $g = G$ or H . The sum $\sum_{\mathcal{A}}$ extends over the g matrices $\mathcal{A}$ of the group g (of order g); $\mathcal{I}$ is the corresponding unit matrix. The symbol “det” signifies the determinant of the matrix given in the parentheses. In other words, $\Phi_g(\lambda)$ is the average over the group g of the inverse determinants of $(\mathcal{I} - \lambda\mathcal{A})$. When $\Phi_g(\lambda)$ is expanded into powers of λ , the result is known as the formal Molien (or Poincaré) series (sometimes inappropriately called polynomial [8, Ch.VII, §11])

$$\Phi_g(\lambda) = 1 + \sum_i b_i \lambda^i. \quad (84)$$

The coefficients b_i indicate the number of *linearly* independent invariants of degree i .

Here, at the latest, we understand why it is preferable not to talk about 1 as being one of the basic invariants. It should be included both among the free and among the transient invariants (albeit in a somewhat different capacity). As a transient invariant, it is represented by the 1 in the numerator of the Molien function (79) but it has to be omitted as a free invariant in the denominator or else the latter would vanish.

Actually, in what follows, we shall often include 1 among the transient invariants and occasionally assign it the symbol U_0 , always understanding that

$$U_0 = 1. \quad (85)$$

The set $U_0, U_3, U_4, U_5, U_6, U_9$ forms the basis of the “free” module of invariants over the subring of invariants that are expressible as polynomials in free invariants. We shall call such polynomials “pure coefficients” (of the transient invariants). Similarly, anti-invariants constitute a free module over the ring of pure coefficients with the basis H_3, J_3, J_4, J_5, J_6 , and H_6 . In modern mathematical literature, the term “basis” is often used only in the sense that we have just defined. All our previous “bases” (which include the free invariants) are then called “systems of generators”.

The degrees of our basic (anti)invariants respect some general theorems valid for matrix groups with $\det \mathcal{A} = \infty$ (for any $\mathcal{A}$). For example, the transient (anti)invariants may be grouped in pairs such that the sum of degrees of the two members of the pair is always equal (in our case, to nine) [14, p.503]. Such pairs are (U_0, U_9) , (U_3, U_6) , (U_4, U_5) , (H_3, H_6) , (J_3, J_6) , and (J_4, J_5) . The sum of degrees of the free invariants minus their number (six) is also nine [14, Corollary 3.9].

The anti-invariants by themselves have their own Molien function [14, Theorem 2.1] defined as

$$\Phi_{G,a}(\lambda) = \frac{1}{24} \sum_{\mathcal{A}} \frac{\pm 1}{\det(\mathcal{I} - \lambda \mathcal{A})}, \quad (86)$$

where the positive (negative) sign of the numerator applies to the even (odd) permutations of S_4 (see Tables 1 and 2). Explicitly, we obtain

$$\Phi_{G,a}(\lambda) = (2\lambda^3 + \lambda^4 + \lambda^5 + 2\lambda^6)/D, \quad (87)$$

as expected, since we must have

$$\Phi_H(\lambda) = \Phi_G(\lambda) + \Phi_{G,a}(\lambda). \quad (88)$$

7 Syzygies and the extended final basis

At the outset, I must admit that my understanding of the chain of syzygies leaves much to be desired. Fortunately, it seems to suffice for the questions of interest for physics (see Sections 9 and 10).

The term syzygy of the 1st kind (or, as we prefer to call it, of the first link) and our notation conventions have been explained in Section 4 [cf. Eq. (58) ff.]; by an additional convention we designate anti-invariant syzygies by the letter Λ and syzygies containing only invariants by the letter Σ . It is clear that any linear form in Σ -syzygies with invariant coefficients is again a Σ -syzygy. Mathematicians express this fact by saying that Σ -syzygies form an ideal in the ring of invariants. The Λ -, Σ -, and Ξ -syzygies together form an ideal in the ring of (anti)invariants. Such ideals have finite bases, i.e. any syzygy belonging to the ideal may be expressed as a linear form in a finite number of basic syzygies, with invariant or (anti)invariant coefficients, respectively.

To find a basis of an ideal of syzygies is a difficult undertaking, in general, nowadays best tackled by algebraic techniques using computers [17]. Our pedestrian approach described in Section 5 – while perfectly satisfactory for the reduction of the primary to the final basis of (anti)invariants – is deficient when it comes to finding the bases of the syzygy ideals. For one thing, the intuitive short cuts taken on the way to the final basis of (anti)invariants may have caused us to miss some syzygy of degree $K \leq 12$. Moreover, there may exist basic syzygies of degree $K > 12$.

However, the simplicity of the primary basis of (anti)invariants pointed out at the end of Section 5 creates the possibility of finding a basis of the first-link syzygy ideal. This simplicity may be considered a property of all (anti)invariants, not just of the primary basis. The reason is that any (anti)invariant can be reduced to the primary basis by applying relations of the type of Eq. (214) of Appendix A which introduce additional pure invariants only. Since an arbitrary (anti)invariant may contain arbitrary powers of transient (anti)invariants, we conclude that there must exist first-link syzygies permitting the “linearization” of any product of transient (anti)invariants. Thus, the ideal of first-link Σ -, Λ -, and Ξ -syzygies must contain all binary products of transient (anti)invariants. There are five transient invariants and six anti-invariants so that we expect $(11 \times 12)/2 = 66$ such first-link syzygies: 15 Σ -, 30 Λ -, and 21 Ξ -syzygies. However, for this counting to be correct, we have to introduce exceptions to the last condition imposed on an (anti)invariant if it is to qualify as a first-link syzygy [cf. Eq. (58) ff.]: Eqs. (76)–(78) have to be considered as syzygies of the form

$$\Sigma_6(U_3^2) = -4y_6 - w_3 U_3 + a_2 U_4 + a_1 U_5 + 3U_6 + U_3^2, \quad (89)$$

$$\Sigma_9(U_4 U_5) = -\chi_9 - 3z_9 + y_6 U_3 - U_9 + U_4 U_5, \quad (90)$$

$$\Lambda_6(U_3 J_3) = -w_3 J_3 - a_2 J_4 - a_1 J_5 + 3J_6 + U_3 J_3, \quad (91)$$

where the coefficients denoted by the auxiliary symbols of Table 5 in Appendix B are monomials in free invariants. From now on we shall call coefficients that are polynomials in free invariants “pure coefficients”. Using the equations of Appendices A and B it is easy (though somewhat tedious) to derive the remaining 63 first-link syzygies corresponding to the good basis of (anti)invariants.

A fundamental property of a good basis is the non-existence of syzygies linear in transient (anti)invariants (with pure coefficients) [14, Proposition 3.1]: the vanishing of a polynomial linear in transient (anti)invariants entails the vanishing of all its pure coefficients, identically in the free invariants. It follows that the 66 syzygies just enumerated constitute a complete basis of the ideal of (anti)invariant first-link syzygies. The 15 Σ -syzygies by themselves constitute a complete basis of the ideal of invariant first-link syzygies. Clearly, both bases are also minimal, i.e. they cannot be reduced to smaller bases as long as we stick with the good basis of (anti)invariants (cf. [13, p.100]). By multiplying with a suitable complex number, all basic syzygies may be brought into the form

$$P_0 + \sum_i P_i V_i + V_j V_k, \quad (92)$$

with P_i being pure coefficients and V_i transient (anti)invariants. It is easy to figure out the conditions on the coefficients P_i and on the character (invariant or anti-invariant) of the different V 's in expression (92) for the latter to constitute a Σ -, a Λ -, or a Ξ -syzygy, respectively.

For simplicity, we shall limit ourselves until further notice to invariants and Σ -syzygies; the generalization to (anti)invariants is straightforward. In order to understand how to arrive at syzygies of the second link, we imagine three transient invariants V_j, V_k, V_l , not all of them equal, say $V_j \neq V_k$. In contradistinction to our general labelling practice, the indices $j, k, l, \dots$ are here only enumerating subscripts and do not indicate the degree of the invariant. We have

$$V_j \Sigma_K(V_k V_l) - V_k \Sigma_{K'}(V_j V_l) = P_0 + \sum_i P_i V_i + \sum_m P_m V_j V_m - \sum_n P_n V_k V_n. \quad (93)$$

The first two terms on the right-hand side contain contributions from both terms on the left-hand side of the equation. The degrees $K, K', \dots$ of the first-link syzygies in this and the following equation follow from the (not indicated) degrees of the two invariants listed in the parentheses after the symbol Σ . We are always assuming that the product of these two invariants appears in the corresponding syzygy with the coefficient 1. Therefore, the product $V_j V_k V_l$ does not appear on the right-hand side of Eq. (93). From this equation we conclude that the expression

$$\Sigma_N^{(2)}(V_j V_k V_l) = V_j \Sigma_K(V_k V_l) - V_k \Sigma_{K'}(V_j V_l) - \sum_m P_m \Sigma_{K''}(V_j V_m) + \sum_n P_n \Sigma_{K'''}(V_k V_n) \quad (94)$$

is linear in the transient invariants (with pure coefficients). However, it clearly belongs to the ideal of first-link syzygies. The apparent contradiction is resolved if the expression vanishes *identically in the invariants* (and not only identically in the vectors $\vec{s}, \vec{t}$). The quantity $\Sigma_N^{(2)}$ is a syzygy “of the 2nd kind” or of the second link: a linear form in first-link syzygies with invariant coefficients which vanishes identically in the invariants. Its degree

is indicated by the subscript N , as usual; it is the sum of the degrees of the invariants V_j , V_k , and V_l .

There may exist several syzygies of the second link, forming an ideal with a finite basis in the ring of invariants. In general, some linear forms in second-link syzygies with invariant coefficients vanish identically in first-link syzygies. They form the ideal of syzygies of the third link, and so the chain continues. Hilbert's 2nd finiteness theorem asserts that the chain of syzygies has a finite number of links (if his 1st theorem is valid which it is in our case).

General theorems [14, Sections 6-8], particularly on the Molien function, help considerably in the study of higher links of the syzygy chain. For instance, in our case the syzygy chain can be shown to consist of five links (as many as the number of transient invariants) [14, Theorem 6.2]. The number of basic syzygies in the i^{th} link is designated by β_i and called the Betti number of the i^{th} link [14, Section 6]. For good reasons, mathematicians set by definition $\beta_0 = 1$. The fact that all our matrices $\mathcal{A}$ of Table 2 have $\det \mathcal{A} = 1$ ensures that $\beta_{(5-i)} = \beta_i$ [14, Section 8]. This relation and the construction (94) of second-link syzygies fixes our Betti numbers: $\beta_0 = 1$, $\beta_1 = 15$, $\beta_2 = 40$, $\beta_3 = 40$, $\beta_4 = 15$, $\beta_5 = 1$. The construction (94) also determines the degrees of the basic second-link syzygies, while the study of the Molien function (80) reveals that the unique syzygy of the fifth link has degree $K = 36$. (Incidentally, the Betti number β_5 is totally unrelated to the auxiliary symbol β_5 in Table 5 of Appendix B.)

To explicitly calculate by hand the bases of all these syzygy ideals seems hardly profitable. However, some more light can be thrown on our ring of invariants and its syzygy chain with a lesser investment in tedious algebra if we introduce yet a different but equally "good" basis of invariants. We first quote a few first-link Σ -syzygies in which the auxiliary mixed invariants U_6 and U_9 have been eliminated using Eqs. (76)–(77):

$$\Sigma_7(U_3U_4) = A_7 + B_4U_3 + C_1U_3^2 + D_3U_4 + E_2U_5 + U_3U_4, \quad (95)$$

$$\Sigma_8(U_3U_5) = A_8 + B_5U_3 + C_2U_3^2 + E_4U_4 + D_3U_5 + U_3U_5, \quad (96)$$

$$\Sigma_8(U_4^2) = F_8 + G_5U_3 + H_2U_3^2 + I_4U_4 + K_3U_5 + U_4^2, \quad (97)$$

$$\Sigma_{10}(U_5^2) = F_{10} + G_7U_3 + H_4U_3^2 + K_6U_4 + I_5U_5 + U_5^2, \quad (98)$$

$$\Sigma_9(U_3U_6) = L_9 + M_6U_3 + N_3U_3^2 - \frac{1}{3}U_3^3 + O_5U_4 + O_4U_5 + U_4U_5. \quad (99)$$

The syzygies of this list have been labelled by the products of the invariants that have led to their discovery. [The inconsistency of labelling a syzygy by a product of invariants, one of which has been eliminated from the basis, is due to the desire to distinguish Eqs. (90) and (99) without introducing a cumbersome new symbol.] Similarly as before, the notation for the quantities $A_K, B_K, \dots, O_K$ has been chosen in such a way that two quantities designated by the same letter are s, t -conjugate. These quantities are polynomials of degree K in free invariants. They are listed in Table 6 of Appendix B in terms of the auxiliary symbols of Table 5. We remark that $\Sigma_7(U_3U_4) \leftrightarrow \Sigma_8(U_3U_5)$ on one hand and $\Sigma_8(U_4^2) \leftrightarrow \Sigma_{10}(U_5^2)$ on the other are s, t -conjugate, while $\Sigma_9(U_3U_6)$ is self-conjugate.

All syzygies of the set (95)–(99) are clearly basic: none of them can be expressed as a linear form with invariant coefficients in the others. Equation (76) shows that we could choose U_3^2 as one of the transient invariants instead of U_6 . Similarly, Eq. (77) together with the syzygy $\Sigma_9(U_3U_6)$ permits the replacement of U_9 by U_3^3 . We shall refer to the good basis with the transient invariants

$$\hat{U}_3 = U_3, \quad \hat{U}_4 = U_4, \quad \hat{U}_5 = U_5, \quad \hat{U}_6 = U_3^2, \quad \hat{U}_9 = U_3^3 \quad (100)$$

(the free invariants remaining unchanged) as the “extended” final basis of invariants. Clearly the extended final basis is characterized by the same Molien function as the original good basis. However, it has the special property that two of its transient invariants ($\hat{U}_6$ and $\hat{U}_9$) are powers of a third one ($\hat{U}_3$). This property facilitates calculations but we should not forget that we cannot reduce the extended final basis back to the final one without losing the nice properties of a good basis. For instance, the analogue of the syzygy (89) in the extended final basis is

$$\Sigma_6(\hat{U}_3^2) = -\hat{U}_6 + \hat{U}_3^2, \quad (101)$$

which should be considered a first-link syzygy though, in the final basis, it clearly vanishes identically in U_3 and would thus not even qualify as a second-link syzygy, since it is not a linear form in first-link syzygies.

We shall not need an extended final basis of anti-invariants but, if desired, the basic anti-invariant J_6 could be replaced by $U_3 J_3$ in analogy to the above replacement of U_6 by U_3^2 .

We observe that the syzygies $U_5 \Sigma_7(U_3 U_4)$, $U_4 \Sigma_8(U_3 U_5)$, and $U_3 \Sigma_9(U_3 U_6)$ all contain the product $U_3 U_4 U_5$ and should lead to interesting relations. First, starting from these three syzygies, we form three new syzygies that contain the product $U_3 U_4 U_5$ but are otherwise linear in U_4 and U_5 with pure coefficients. As it may be explicitly checked, they are:

$$P_{12} = (U_5 + C_1 E_4) \Sigma_7(U_3 U_4) - [C_1(U_3 - D_3) + B_4] \Sigma_8(U_3 U_5) - E_2 \Sigma_{10}(U_5^2) - D_3 \Sigma_9(U_3 U_6), \quad (102)$$

$$\bar{P}_{12} = (U_4 + C_2 E_2) \Sigma_8(U_3 U_5) - [C_2(U_3 - D_3) + B_5] \Sigma_7(U_3 U_4) - E_4 \Sigma_8(U_4^2) - D_3 \Sigma_9(U_3 U_6), \quad (103)$$

$$Q_{12} = U_3 \Sigma_9(U_3 U_6) - O_5 \Sigma_7(U_3 U_4) - O_4 \Sigma_8(U_3 U_5). \quad (104)$$

Since these new quantities belong to the ideal of first-link syzygies, they all vanish, i.e. they are equal to each other, identically in the vectors $\vec{s}$ and $\vec{t}$:

$$P_{12} = Q_{12}, \quad (105)$$

$$\bar{P}_{12} = Q_{12}, \quad (106)$$

$$P_{12} = \bar{P}_{12}. \quad (107)$$

Clearly, the term $U_3 U_4 U_5$ occurs on both sides of any of these three equations, i.e. it cancels out. The equations are thus linear in U_4 and U_5 , with pure coefficients. The last equation is also linear in the other three transient invariants U_3 , U_3^2 , and U_3^3 . As before, we conclude that Eq. (107) must be valid identically in the invariants, i.e.

$$\Sigma_{12}^{(2)} = P_{12} - \bar{P}_{12} = [U_5 + C_1 E_4 + C_2(U_3 - D_3) + B_5] \Sigma_7(U_3 U_4) - [U_4 + C_2 E_2 + C_1(U_3 - D_3) + B_4] \Sigma_8(U_3 U_5) + E_4 \Sigma_8(U_4^2) - E_2 \Sigma_{10}(U_5^2) \quad (108)$$

constitutes a syzygy of the second link.

Owing to this second link-syzygy, Eqs. (105) and (106) are identical, i.e. they are just one equation. It permits the expression of U_3^4 in the extended final basis, i.e. linearly in its transient invariants U_3 , U_3^2 , U_3^3 , U_4 , U_5 , with pure coefficients. This expression, together with the syzygies $\Sigma_7(U_3 U_4)$ and $\Sigma_8(U_3 U_5)$, enables us to express U_3^5 in the extended final

basis, etc. All these expressions may be written as linear forms with invariant coefficients in just the five syzygies (95)–(99). We see how, in principle, any power of U_3 can be directly reduced to a polynomial (with pure coefficients) linear in the transient invariants of the extended final basis, without having to use Appendices A and B. Combined with the syzygies (95)–(99) this provides the possibility of directly expressing any monomial in U_3, U_4, U_5 in the extended final basis.

The syzygies $\Sigma_7(U_3U_4) = 0$ and $\Sigma_8(U_3U_5) = 0$ can be considered as two linear equations in the two unknowns U_4 and U_5 . The resolving determinant clearly does not vanish identically in the invariants (in fact, it is quadratic in U_3) and cannot vanish identically in $\vec{s}, \vec{t}$, or else it would constitute a first-link syzygy linear in the transient invariants of the extended final basis. Therefore, the two syzygies enable us to express U_4 and U_5 rationally in terms of U_3 and the free invariants. This possibility of expressing rationally all invariants in an n -dimensional space in terms of only $n + 1$ basic invariants is in agreement with a general theorem of the theory of invariants [11, Ch.XVII, Section 263]. The $n + 1$ basic invariants are said to constitute the “rationality basis” of invariants.

Expressing one of the remaining first-link syzygies (97)–(99) in the rationality basis and multiplying the result by the square of the resolving determinant yields a polynomial of sixth degree in U_3 with pure coefficients and with total degree $K > 18$. Clearly, this polynomial cannot vanish identically in the invariants but it must vanish identically in $\vec{s}, \vec{t}$; thus it belongs to the ideal of first-link syzygies and enables us to express U_3 algebraically in terms of the free invariants $S_1, S_2, T_2, R_3, S_3, T_4$, which constitute the “functional basis” of invariants. Since our representation space is 6-dimensional, no further reduction in the number of basic invariants is possible [11, Ch.XVII, Section 262], [13, p.96].

This completes our discussion of the extended final basis of invariants and of the chain of invariant syzygies. As mentioned before, the extension to (anti)invariants is straightforward and we shall try to be brief. Expressed in the final basis, seven anti-invariant syzygies are

$$\Lambda_6(U_3H_3) = D_3H_3 + K_3J_3 - 2E_2J_4 + U_3H_3, \quad (109)$$

$$\Lambda_7(U_3J_4) = -E_4H_3 + (B_4 - I_4)J_3 + D_3J_4 + E_2J_5 + C_1U_3J_3 + U_3J_4, \quad (110)$$

$$\Lambda_7(U_4H_3) = (I_4/2)H_3 + R_4J_3 - K_3J_4 + U_4H_3, \quad (111)$$

$$\Lambda_7(U_4J_3) = E_4H_3 + (B_4 - I_4/2)J_3 + E_2J_5 + C_1U_3J_3 + U_4J_3, \quad (112)$$

$$\Lambda_8(U_4J_4) = W_5J_3 + (I_4/2)J_4 + K_3J_5 + H_2U_3J_3 + U_4J_4, \quad (113)$$

$$\Lambda_8(U_5H_3) = (I_5/2)H_3 + X_5J_3 + (I_4 - 2B_4/3)J_4 - (3K_3 + Y_3)J_5 - (2E_2/3)U_3J_3 + U_5H_3, \quad (114)$$

$$\Lambda_9(U_5J_4) = (-K_6/2)H_3 + Z_6J_3 + (I_5 - B_5/3)J_4 + (B_4/3 - I_4/2)J_5 - (K_3/2)H_6 + (D_3/6)U_3J_3 + U_5J_4. \quad (115)$$

The pure coefficients of the transient anti-invariants are again denoted by the symbols defined in Table 6 of Appendix B. From the seven anti-invariant syzygies (109)–(115), seven additional ones can be obtained by s, t -symmetry. Their respective degrees are equal to or higher than the degrees of their listed s, t -partners. The trivial syzygy (91), the listed syzygies (109)–(115), and their s, t -partners represent a first set of 15 out of the 30 basic Λ -syzygies. The remaining set of 15 can be directly deduced from the first set by repeated use of the first set combined with the set of invariant syzygies (95)–(99).

It can be noticed that, without the trivial syzygy, the first set of basic Λ -syzygies represents a highly overdetermined system of 14 homogeneous linear equations for the

anti-invariants H_3, J_3, J_4, J_5, H_6 . Its consistency is ensured by a large number of vanishing determinants which must all belong to the ideal of first-link invariant syzygies. This system of equations permits the rational expression of all basic anti-invariants in terms of basic invariants and a single basic anti-invariant, e.g. H_3 . (Without actually calculating all relevant determinants of the system we could not be sure of these statements, were it not for the general theorems of invariant theory cited before in this Section.)

One example of the 21 Ξ -syzygies is the syzygy (58). Two other examples are

$$\Xi_6(H_3J_3) = \Gamma_6 - 3K_3U_3 + 6E_2U_4 + H_3J_3, \quad (116)$$

$$\Xi_7(H_3J_4) = \Phi_7 + \Psi_4U_3 + \Omega_3U_4 + H_3J_4, \quad (117)$$

where the additional pure coefficients designated by capital Greek letters are also listed in Table 6 of Appendix B. The above two syzygies can be considered as being a system of two linear equations for the two unknowns U_3 and U_4 . The latter can thus be rationally expressed in terms of basic anti-invariants and free invariants. Any of the two s, t -partners of the syzygies (116), (117) may then serve to rationally express U_5 in terms of basic anti-invariants and free invariants. We see that all transient invariants can be rationally expressed in terms of basic anti-invariants and free invariants, i.e. in terms of a single basic anti-invariant and free invariants. In conclusion, a single basic anti-invariant, say H_3 , and the free invariants form the rationality basis of (anti)invariants. From Eq. (58) it is then clear that the free invariants constitute the functional basis of (anti)invariants.

The syzygy chain of (anti)invariants has 11 links with even larger bases, in general, than is the case with invariants. The relation $\beta_{(11-i)} = \beta_i$ is valid but it seems much harder to study this chain than the chain of invariant syzygies (e.g. to explicitly calculate its Betti numbers). However, if we limit ourselves to anti-invariant syzygies, their chain resembles the chain of invariant syzygies [14, p.499].

8 Kinematical zeros and the infinitesimal basis

As explained in Section 4, symmetry requirements do *not* force the scalar amplitude A^+ to vanish anywhere on the mass shell: A^+ has no kinematical zeros. On the other hand, the pseudoscalar amplitude A^- clearly vanishes if $\psi = 0$, i.e. for coplanar events, which means at the 4-dimensional boundary of the 5-dimensional mass shell.

The pseudoscalar amplitude also vanishes if the scalar O of Eq. (42) is zero. For this to be necessarily the case, the five basic anti-invariants H_3, J_3, J_4, J_5, H_6 have to vanish simultaneously. The anti-invariant H_3 is zero (irrespective of the value of the vector $\vec{t}$) if either $s_1 = s_2$, or $s_2 = s_3$, or $s_3 = s_1$. Similarly, the anti-invariant H_6 is zero (irrespective of the value of the vector $\vec{s}$) if either $t_1 = t_2$, or $t_2 = t_3$, or $t_3 = t_1$. These conditions are a consequence of the defining equations (57) and (62). Checking what happens with the mixed anti-invariants when any of the above conditions is fulfilled, we immediately discover that the necessary and sufficient conditions for a kinematical zero of the scalar O are

– either

$$s_1 = s_2 \quad \text{and} \quad t_1 = t_2 \quad (\text{i.e. } r_1 = \pm r_2), \quad (118)$$

– or

$$s_2 = s_3 \quad \text{and} \quad t_2 = t_3 \quad (\text{i.e. } r_2 = \pm r_3), \quad (119)$$

– or

$$s_3 = s_1 \quad \text{and} \quad t_3 = t_1 \quad (\text{i.e. } r_3 = \pm r_1). \quad (120)$$

Each of these six conditions represents a 3-dimensional hyperplane in the 5-dimensional mass shell. We shall call these hyperplanes *O*-zero planes.

Kinematical zeros should be deducible directly from the symmetries of the decay. The only danger is that we may overlook some symmetry and thus miss some zeros. However, this may also happen if we rely exclusively on algebraic arguments. Clearly, the agreement between both methods provides an optimal guarantee that no zeros have been missed. Indeed, symmetry considerations [22] yield all kinematical zeros of the decay into three pions that have been algebraically derived in Ref. [1] (and a couple more).

In our case, the kinematical zeros of the pseudoscalar amplitude A^- follow from Aage Bohr's selection rule [23] which, in turn, follows from the mirror symmetry of the decay with respect to some plane. This selection rule states

$$\Pi_i(-1)^{\sum M_i} = \Pi_f(-1)^{\sum M_f}. \quad (121)$$

Here, Π_i and Π_f are the products of intrinsic parities of the initial and final particles, respectively. Similarly, $\sum M_i$ and $\sum M_f$ are the sums of *spin* magnetic quantum numbers with respect to the normal to the mirror symmetry plane, of the initial and final particles, respectively. No spins occur in our problem and $\Pi_f = +1$, while for a pseudoscalar initial particle $\Pi_i = -1$. Therefore, the amplitude A^- must vanish on any plane of mirror symmetry for the decay.

Clearly, the common plane of four coplanar particles in the final state is such a mirror symmetry plane. So is any plane defined by the two 3-momenta of any two final particles, say $\vec{p}_1$ and $\vec{p}_2$, if the other two 3-momenta $\vec{p}_3$ and $\vec{p}_4$ exhibit mirror symmetry with respect to this plane. The latter condition implies $p_1 \cdot p_3 = p_1 \cdot p_4$ and $p_2 \cdot p_3 = p_2 \cdot p_4$, i.e. $x_2 = x_6$ and $x_1 = x_5$, i.e. $s_1 = s_2$ and $r_1 = -r_2$. The six possible groupings of the four final particles into two dissimilar pairs (one pair defining the plane with respect to which the other pair exhibits mirror symmetry) lead to the complete set of hyperplanes of Eqs. (118)–(120).

The six *O*-zero planes cross at the center of the mass shell. It is of some interest to find out the behaviour of the basic (anti)invariants in the vicinity of the center. For this purpose it is useful to replace the variables s_i by

$$\sigma_i = s_i - S_1/3 \quad (i = 1, 2, 3), \quad (122)$$

entailing

$$\sum_{i=1}^3 \sigma_i = 0. \quad (123)$$

In order not to lose the s, t -symmetry, we also replace the variables t_i by

$$\tau_i = t_i - T_2/3 \quad (i = 1, 2, 3), \quad (124)$$

entailing

$$\sum_{i=1}^3 \tau_i = 0. \quad (125)$$

If the vectors $\vec{s}$ and $\vec{t}$ of an (anti)invariant are replaced by the corresponding vectors $\vec{\sigma}$ and $\vec{\tau}$, we designate the resulting (anti)invariant by a tilde ($\sim$) over its symbol. From the defining equations (55)–(57), (60)–(62), (69)–(71), and (73)–(78) we obtain

$$\tilde{S}_2 = S_2 - \frac{1}{3}S_1^2, \quad (126)$$

$$\tilde{S}_3 = S_3 - \frac{1}{3}S_1S_2 + \frac{2}{27}S_1^3, \quad (127)$$

$$\tilde{T}_4 = T_4 - \frac{1}{3}T_2^2, \quad (128)$$

$$\tilde{T}_6 = T_6 - \frac{1}{3}T_2T_4 + \frac{2}{27}T_2^3, \quad (129)$$

$$\tilde{U}_3 = U_3 - \frac{2}{3}S_1T_2, \quad (130)$$

$$\tilde{U}_4 = U_4 + \frac{1}{3}S_1U_3 - \frac{2}{3}S_2T_2 - \frac{2}{9}S_1^2T_2, \quad (131)$$

$$\tilde{U}_5 = U_5 + \frac{1}{3}T_2U_3 - \frac{2}{3}S_1T_4 - \frac{2}{9}S_1T_2^2, \quad (132)$$

$$\tilde{U}_6 = U_6 + \frac{1}{3}S_1U_5 + \frac{1}{3}T_2U_4 + \frac{1}{9}S_1T_2U_3 - \frac{4}{9}S_1^2T_4 - \frac{4}{9}S_2T_2^2, \quad (133)$$

$$\begin{aligned} \tilde{U}_9 = & U_9 + \frac{1}{3}S_1T_2U_6 - \frac{1}{3}S_2T_2U_5 - \frac{1}{3}S_1T_4U_4 + \frac{2}{27}(S_1^2T_2^2 - 3S_2T_2^2 - 3S_1^2T_4)U_3 \\ & + \frac{1}{9}(2S_1S_2T_2T_4 + S_1^3T_6 + S_3T_2^3 + S_1^3T_2T_4 + S_1S_2T_2^3 - 10S_1^3T_2^3/27), \end{aligned} \quad (134)$$

$$\tilde{H}_3 = H_3, \quad (135)$$

$$\tilde{J}_3 = J_3, \quad (136)$$

$$\tilde{J}_4 = J_4 + \frac{1}{3}S_1J_3, \quad (137)$$

$$\tilde{J}_5 = J_5 - \frac{1}{3}T_2J_3, \quad (138)$$

$$\tilde{J}_6 = J_6 + \frac{1}{3}S_1J_5 - \frac{1}{3}T_2J_4 - \frac{1}{3}S_1T_2J_3, \quad (139)$$

$$\tilde{H}_6 = H_6. \quad (140)$$

As before, $\tilde{T}_6$ is only an auxiliary invariant. The new free invariants are thus S_1 , T_2 , R_3 , $\tilde{S}_2$, $\tilde{S}_3$, and $\tilde{T}_4$ and the new transient (anti)invariants are $\tilde{U}_3$, $\tilde{U}_4$, $\tilde{U}_5$, $\tilde{U}_6$, $\tilde{U}_9$, $\tilde{H}_3$, $\tilde{J}_3$, $\tilde{J}_4$, $\tilde{J}_5$, $\tilde{J}_6$, and $\tilde{H}_6$ as given above. This new set of basic (anti)invariants constitutes yet another good basis which we shall refer to as the “infinitesimal” basis. Clearly, this basis is *not* “infinitesimal” in the sense of infinitesimal generators of a continuous group but it has advantageous infinitesimal properties in the vicinity of the center where all its members except S_1 vanish. Sufficiently close to the center, the orders of magnitude of the infinitesimal (anti)invariants can be easily estimated. If we designate the order of magnitude of the vector $\vec{\sigma}$ simply by σ and the order of magnitude of the vector $\vec{r}$ by r , the orders of magnitude of the above (anti)invariants are:

$$S_1 \sim 1, \quad \tilde{S}_2 \sim \sigma^2, \quad \tilde{S}_3 \sim \sigma^3, \quad T_2 \sim r^2, \quad R_3 \sim r^3, \quad \tilde{T}_4 \sim r^4, \quad \tilde{T}_6 \sim r^6, \quad (141)$$

$$\tilde{U}_3 \sim \sigma r^2, \quad \tilde{U}_4 \sim \sigma^2 r^2, \quad \tilde{U}_5 \sim \sigma r^4, \quad \tilde{U}_6 \sim \sigma^2 r^4, \quad \tilde{U}_9 \sim \sigma^3 r^6, \quad (142)$$

$$\tilde{H}_3 \sim \sigma^3, \quad \tilde{J}_3 \sim \sigma r^2, \quad \tilde{J}_4 \sim \sigma^2 r^2, \quad \tilde{J}_5 \sim \sigma r^4, \quad \tilde{J}_6 \sim \sigma^2 r^4, \quad \tilde{H}_6 \sim r^6. \quad (143)$$

We see that in the vicinity of the center the orders of magnitude of these basic (anti)invariants (contrary to some of the original ones) are given by their degrees K ; the obvious exception is S_1 .

Owing to Eqs. (123) and (125), the reduction equations (53) and (63) translate into

$$\sigma_i^3 = -\tilde{S}_2\sigma_i + \tilde{S}_3, \quad (144)$$

$$\tau_i^3 = -\tilde{T}_4\tau_i + \tilde{T}_6. \quad (145)$$

The algebra of 6-exponent (anti)symbols of Appendix A can be directly transcribed into an algebra of “ σ, τ -(anti)symbols”, simply by putting a tilde on any letter designating an (anti)invariant. This includes S_1 and T_2 which become $\tilde{S}_1 = \tilde{T}_2 = 0$, identically in $\vec{\sigma}, \vec{\tau}$. The σ, τ -(anti)symbols with the addition of S_1, T_2 , and R_3 constitute a primary basis of (anti)invariants. Its reduction rests on the algebra of σ, τ -(anti)symbols and on Eqs. (144), (145), and it requires a considerably smaller calculational effort than the reduction of the original primary basis. This can be appreciated by putting tildes on the auxiliary symbols of Table 5: out of 63 different symbols, only 15 do not vanish identically in the infinitesimal basis. However, of course, the long list of primary (anti)invariants in Appendix B does not become any shorter.

Someone may ask the question as to why we have not introduced the vectors $\vec{\sigma}$ and $\vec{\tau}$ from the beginning in order to simplify the calculations. The honest answer is, firstly, that I have not thought of it soon enough. However, secondly, our treatment as it stands should be more transparent than the alternative, since it is not always easy to keep apart $\tilde{S}_1 = \tilde{T}_2 = 0$ on one hand, and the free invariants S_1 and T_2 as members of the infinitesimal basis on the other hand. Nevertheless, by now we may feel sufficiently advanced to take advantage of the $\vec{\sigma}, \vec{\tau}$ -variables and put them to good use in Sections 10 and 11.

On a more fundamental level, the replacement of the variables s_1, s_2, s_3 by the variables $S_1, (\sigma_1 - \sigma_2)/\sqrt{2}, \sigma_3$ reduces the natural representation of the symmetric group S_3 into the 1-dimensional S_1 -space and a 2-dimensional irreducible space. The latter space is isomorphic to the familiar coordinate space of triangular Dalitz plots as used in Ref. [1]. Of course, the same statements apply to the τ -space but *not* to its underlying r -space. Only the fortunate circumstance that R_3 is the sole basic invariant explicitly depending on the components of $\vec{\tau}$, and that $T_6 = R_3^2$, makes it possible to treat the σ - and τ -spaces on an equal footing.

9 On intermediate resonances

Since the four π^0 from the decay of protonium are strongly interacting, narrow two- and three-pion resonances [24] (also called dipions and tripions, respectively) may appear as intermediate states of the decay. With π^0 's as the only particles in the final state, the possible dipions have isospin 0 or 2 (i.e. positive C-parity) and J^P -values $0^+, 2^+, 4^+, \dots$. On the other hand, any intermediate meson with positive C-parity, isospin 1, and any J^P -value except $J^P = 0^+$ may decay into three π^0 . (A tripion with isospin 3, even if it existed, could not be produced in protonium decay by an isospin-conserving interaction.) Of the well-known dipions, the narrowest are the $f_0(975)$ and the $f_2(1270)$ (with widths $\Gamma \approx 50$ MeV and 200 MeV, respectively). Apart from the anomalous case of the η , which is only about 1 keV wide and has a branching ratio of 32% for the isospin-forbidden decay into $3\pi^0$, the narrowest well-established tripion is $\pi_2(1670)$ with $\Gamma \approx 250$ MeV, while the even narrower $a_2(1320)$ probably has only a very small branching ratio for decay into $3\pi^0$.

Unfortunately, in general, the decay of protonium may feed (by a poorly understood mechanism) several other broad intermediate dipion and tripion states so that only a fit of experimental data to a sophisticated isobar model with many free parameters has any chance of determining the decay amplitude. In practice, this difficulty is exacerbated by the impossibility of completely disentangling the contributions of several possible initial “particles”, i.e. decaying states of protonium. Similar problems are likely to occur with any other realistic example of the decay of a boson into four identical spinless bosons.

In view of these practical difficulties it is not very interesting to dwell at length on

the kinematics (in the narrow sense of the word) of dipion and tripion intermediate states (see Ref. [7] for more details). It is amusing, though, that a narrow tripion with mass ν would lead to enhanced frequency of events along four planes in the r -space containing four non-adjacent faces of an octahedron. The six corners of this octahedron are situated on the coordinate axes of the r -space at the distance $\pm d$ from the origin with

$$d = M^2/2 + m^2 - \nu^2. \quad (146)$$

As ν^2 increases from its minimal value of $9m^2$, d decreases until it reaches zero at $\nu^2 = M^2/2 + m^2$, at which point the four planes characterizing the tripion intersect at the origin of the r -space. As ν^2 further increases, the octahedron expands again but now the four planes contain its complementary four faces.

Vice versa, any event may be partly characterized by the four squared masses of the four "tripions" it contains, instead of the value of $\vec{r}$. These four squared masses are linked by the condition that their sum is equal to $2M^2 + 4m^2$.

The decay into a pair of narrow dipions with masses μ_1 and μ_2 , respectively, would manifest itself by an enhancement along six planes in r -space that contain the faces of a cube and along three straight lines in the C -plane. The planes are perpendicular to the three coordinate axes of the r -space, respectively, and cut these axes at a distance $\pm b$ with

$$b = (\mu_1^2 - \mu_2^2)/2. \quad (147)$$

The three straight lines in the C -plane are defined respectively by $s_1 = a$, $s_2 = a$, and $s_3 = a$, with

$$a = (\mu_1^2 + \mu_2^2)/2 - 2m^2. \quad (148)$$

Vice versa, of course, the position of an event in the r -space and in the C -plane specifies the masses of the six dipions grouped in three pairs, the sum of their squared masses being equal to $M^2 + 8m^2$. The event is completely specified by the masses of these six dipions.

The appearance of octahedrons and cubes as kinematical loci of potential intermediate resonances is hardly accidental in view of the fact that the group S_4 is the symmetry group of both the octahedron and the cube [25, Ch. 18.4].

The two possible parities of the initial particle determine the possible J^P -values of intermediate tripions in a complementary way: the decay of a scalar boson may feed only the "natural" J^P -sequence of tripions $(0^-, 1^+, 2^-, 3^+, \dots)$, while a pseudoscalar boson may decay only via "unnatural" tripions $(1^-, 2^+, 3^-, \dots)$.

Any possible $2\pi^0$ resonance could appear as an intermediate state in the decay of a $J^P = 0^+$ boson. However, the J^P -values of intermediate $2\pi^0$ resonances in the decay of a $J^P = 0^-$ boson are severely restricted. If any one of the two dipions has $J^P = 0^+$, the decay is forbidden; the allowed dipion-dipion decay with the lowest possible J -values is the decay into two $J^P = 2^+$ dipions.

Again, since the analysis of experimental data is carried out by computer programs taking into account several overlapping intermediate resonances, it is not very interesting in itself to work out the algebraic restrictions imposed by a single isolated intermediate state on the decay amplitudes A^+ and A^- . However, a few illustrative cases may serve as examples for statements about intermediate states in the next two Sections. In the case of the amplitude $A^+ = E$ [see Eq. (41)] we treat the two simplest cases:

- i) an intermediate tripion with $J^P = 0^-$;
- ii) a pair of intermediate dipions, both with $J^P = 0^+$.

If the decay proceeds via an isolated intermediate tripion with $J^P = 0^-$, the decay amplitude A^+ is characterized by

$$E = f(x_1, x_2, x_3) + f(x_6, x_5, x_3) + f(x_1, x_5, x_4) + f(x_6, x_2, x_4). \quad (149)$$

Here, x_j ($j = 1, 2, \dots, 6$) are the basic 4-scalars defined by Eq. (6). The functions $f(x_i, x_j, x_k)$ are form factors containing the Breit-Wigner function of the intermediate tripion, the amplitude for its production, and the form factor for its decay into three π^0 . The form factor $f(x_i, x_j, x_k)$ is symmetric in x_i, x_j, x_k . This means that it depends only on the elementary symmetric functions of the latter three variables:

$$Z_1 = x_i + x_j + x_k, \quad (150)$$

$$Z_2 = x_i x_j + x_j x_k + x_k x_i, \quad (151)$$

$$Z_3 = x_i x_j x_k. \quad (152)$$

Therefore, $f(x_i, x_j, x_k)$ may be expanded into the formal series

$$f(x_i, x_j, x_k) = a_{(0)} + a_{(1)}Z_1 + a_{(2)}Z_1^2 + b_{(2)}Z_2 + a_{(3)}Z_1^3 + b_{(3)}Z_1Z_2 + c_{(3)}Z_3 + \dots. \quad (153)$$

Here, $a_{(K)}$ ($K = 0, 1, 2, \dots$), $b_{(K)}$ ($K = 2, 3, \dots$), $c_{(K)}$ ($K = 3, \dots$), $\dots$ are complex coefficients; their indices have been put in parentheses in order to prevent their values from being considered as contributing to the degree K of the terms that contain them.

The expansion (153) leads to the expression of E in the form

$$E = e_{(0)} + e_{(3)}U_3 + e_{(4)}U_4 + e_{(5)}U_5 + e_{(6)}U_6 + e_{(9)}U_9 \quad (154)$$

with

$$e_{(0)} = 4a_{(0)} + 2a_{(1)}S_1 + a_{(2)}(S_1^2 + T_2) + b_{(2)}S_2 + a_{(3)}(S_1^3 + 3S_1T_2)/2 \\ + b_{(3)}S_1S_2/2 + c_{(3)}S_3/2 + (6a_{(3)} + 3b_{(3)} + c_{(3)})R_3/2 + \dots, \quad (155)$$

$$e_{(3)} = b_{(3)}/2 + \dots, \quad (156)$$

and so forth.

As we have learned in the preceding three Sections, the coefficients $e_{(K)}$ can be made pure (if necessary with the help of first-link syzygies), so as to depend only on the free invariants $S_1, S_2, T_2, S_3, R_3, T_4$. Since the expansion (153) has been carried out only to 3rd order, T_4 does not explicitly occur in Eqs. (155)–(156) but there is no reason for it not to appear in higher orders. For the same reason, only the leading term of $e_{(3)}$ has been found and no information on the coefficients $e_{(K)}$ for $K > 3$ has been obtained. However, even to this order, the amplitude E depends only on 7 complex parameters $a_{(K)}, b_{(K)}, c_{(K)}$, while there exist 11 such parameters for an arbitrary amplitude A^+ expanded to 3rd order. This is the simplification introduced by the assumption of an isolated tripion intermediate state. In higher orders, the difference in the number of parameters increases dramatically. To arbitrary order, this difference can be found by comparing the Molien series (84) with $\mathbf{g} = \mathbf{G}$ on one hand and $\mathbf{g} = \mathbf{S}_3$ on the other.

If the decay proceeds via an isolated pair of intermediate dipions, both with $J^P = 0^+$, we have

$$E = g(s_1, t_1) + g(s_2, t_2) + g(s_3, t_3). \quad (157)$$

Here, the form factor $g(s_1, t_1)$ contains the two Breit-Wigner functions of the dipions formed from pions 2,3 and 1,4, respectively, as well as the form factors for their production

and decay. The form factor $g(s_1, t_1)$ does not explicitly depend on the variable r_1 but only on $t_1 = r_1^2$. This is so because this form factor g cannot change if the pair of pions 2,3 is exchanged with the pair of pions 1,4, i.e. g is a symmetric function of x_1 and x_6 . Thus, g may be expressed in terms of the elementary symmetric functions of x_1 and x_6 which are $s_1 = x_1 + x_6$ and $x_1 x_6$; clearly, the latter function may be replaced by $t_1 = (x_1 - x_6)^2$. As a consequence, E cannot contain odd powers of R_3 , which is the only basic invariant explicitly depending on the variables r_i .

Expanding $g(s_i, t_i)$ to 3rd order we obtain

$$g(s_i, t_i) = d_{(0)} + d_{(1)}s_i + d_{(2)}s_i^2 + h_{(2)}t_i + d_{(3)}s_i^3 + h_{(3)}s_i t_i + \dots, \quad (158)$$

leading to E of the form (154) with

$$e_{(0)} = 3d_{(0)} + d_{(1)}S_1 + d_{(2)}(S_1^2 - 2S_2) + h_{(2)}T_2 \\ + d_{(3)}(S_1^3 - 3S_1S_2 + 3S_3) + h_{(3)}S_1T_2 + \dots, \quad (159)$$

$$e_{(3)} = -h_{(3)} + \dots. \quad (160)$$

Again, up to 3rd order, E is characterized by the 6 complex parameters $d_{(K)}$ ($K = 0, 1, 2, 3$), $h_{(K)}$ ($K = 2, 3$) instead of the maximum of 11 parameters. To any order, the number of parameters could be obtained by a judicious use of the Molien series (84) for $\mathbf{g} = \mathbf{S}_2$, i.e. for large values of K this number is vastly inferior to the corresponding number in the case of an intermediate tripion (except if the latter, in its turn, decays via an isolated dipion).

Turning to the decay of a pseudoscalar boson with the decay amplitude $A^- = \psi O$ of Eq. (42) we again treat the two simplest cases:

- i) an intermediate tripion with $J^{PC} = 1^{-+}$. This is a meson with “exotic” quantum numbers and may not exist as a narrow resonance or, if it exists, it may not like to decay into $3\pi^0$.
- ii) a pair of intermediate dipions, both with $J^P = 2^+$. The simplest possibility with the lowest total mass is a pair of $f_2(1270)$. Protonium with the mass of 1877 MeV may perhaps decay into the low-energy tails of this pair of resonances; their relative angular momentum L can be 1 or 3.

We use the relativistic tensor formalism [26, and references therein] to treat these two cases. For more details and a treatment of the intermediate tripion with $J^P = 2^+$, see Ref. [7].

If the decay proceeds via an isolated intermediate tripion with $J^{PC} = 1^{-+}$, the decay amplitude turns out to be characterized by

$$O = \sigma(1, 2, 3)f(x_1, x_2, x_3) + \sigma(6, 5, 3)f(x_6, x_5, x_3) + \sigma(1, 5, 4)f(x_1, x_5, x_4) \\ + \sigma(6, 2, 4)f(x_6, x_2, x_4), \quad (161)$$

with

$$\sigma(i, j, k) = (x_i - x_j)(x_j - x_k)(x_k - x_i). \quad (162)$$

The form factors $f(x_i, x_j, x_k)$ have the same properties as those occurring in Eq. (149) and may be expanded according to Eq. (153). A general form of O is

$$O = \bar{o}_{(3)}H_3 + o_{(3)}J_3 + o_{(4)}J_4 + o_{(5)}J_5 + o_{(6)}J_6 + \bar{o}_{(6)}H_6, \quad (163)$$

$\bar{o}_{(3)}$, $o_{(3)}$, $o_{(4)}$, $o_{(5)}$, $o_{(6)}$, and $\bar{o}_{(6)}$ being pure coefficients. To 3rd order, of course, only $\bar{o}_{(3)}$ and $o_{(3)}$ are finite. It turns out that

$$\bar{o}_{(3)} = o_{(3)} = a_{(0)}/2. \quad (164)$$

In higher orders, the other coefficients appear as well and contain all free invariants including R_3 .

A pair of intermediate dipions, both with $J^P = 2^+$, are characterized by

$$\begin{aligned} O = & [(s_1 - s_2) + B(t_1 - t_2)] g(s_3, t_3) + [(s_2 - s_3) + B(t_2 - t_3)] g(s_1, t_1) \\ & + [(s_3 - s_1) + B(t_3 - t_1)] g(s_2, t_2). \end{aligned} \quad (165)$$

Here, B is a complex parameter measuring the relative strength of a higher order amplitude containing two more 4-momentum factors than the lower order amplitude. In the non-relativistic limit, the latter corresponds to $L = 1$ and the higher order amplitude to $L = 3$. The form factors $g(s_i, t_i)$ have the same properties as those contained in Eq. (157) and may be expanded according to Eq. (158). In this way we again obtain an expansion of O in the form (163) with, to 3rd order

$$\bar{o}_{(3)} = -d_{(2)}, \quad (166)$$

$$o_{(3)} = -h_{(2)} + Bd_{(1)}. \quad (167)$$

As discussed in the next Section, the most interesting result of this Section from the point of view of applications to data analysis may turn out to be the absence of odd powers of the free invariant R_3 from the pure coefficients in Eqs. (154) and (163). However, fundamentally, the method used in this Section is not satisfactory. Given a specific decay amplitude, we have not been able to find the pure coefficients representing it in a good basis without expanding them into formal power series in the variables s_i , r_i . The next Section explains how we might do better.

10 Purity space and data filters

In Section 4 we have stated that we could end our discussion of the scalar amplitude $A^+ = E$ with the requirement that E be a totally symmetric 4-scalar function of the four 4-momenta p_i ($i = 1, 2, 3, 4$). However, in subsequent Sections we have learnt that it is more informative to expand a polynomial (also called integral) function E in a good basis. As announced in Section 8, we have chosen the infinitesimal basis as the preferred basis for our present purpose, obtaining

$$E = \tilde{e}_{(0)} + \tilde{e}_{(3)}\tilde{U}_3 + \tilde{e}_{(4)}\tilde{U}_4 + \tilde{e}_{(5)}\tilde{U}_5 + \tilde{e}_{(6)}\tilde{U}_6 + \tilde{e}_{(9)}\tilde{U}_9, \quad (168)$$

where $\tilde{U}_K$ ($K = 0, 3, 4, 5, 6, 9$) with $\tilde{U}_0 = 1$ are the transient invariants of the infinitesimal basis of $\mathbf{G}$ -invariants and $\tilde{e}_{(K)}$ ($K = 0, 3, 4, 5, 6, 9$) are pure coefficients, i.e. polynomials in the free invariants S_1 , $\tilde{S}_2$, $\tilde{S}_3$, T_2 , R_3 , and $\tilde{T}_4$. The possibility of arriving at expression (168) depends crucially on the existence of first-link syzygies which permit the elimination of all monomials that are not linear in transient invariants. Conversely, the addition to expression (168) of any first-link syzygy [which would alter the form of (168) but not the value of E] should be considered as forbidden, since it would destroy the ‘‘purity’’ of the coefficients $\tilde{e}_{(K)}$. This purity has an important consequence. Contrary to the mixed transient (anti)invariants, the pure free invariants (and the pure transient anti-invariants $\tilde{H}_3$

and $\tilde{H}_6$) do not change if the components of the vector $\vec{r}$ are permuted among themselves at fixed vector $\vec{s}$ – and vice versa. Any combination of such independent permutations of the components of only one of the two vectors $\vec{s}$ and $\vec{r}$ may be called a purity test (though, as a special case, the purity tests include the simultaneous permutations of the components of both $\vec{s}$ and $\vec{r}$ according to Table 1). Pure integral functions of $\vec{s}$ and $\vec{r}$ are invariant against all purity tests. Obviously, we are confronted with a new kind of symmetry that may be called purity. It should be clearly understood, though, that purity is a mathematical consequence of Lorentz invariance and Bose-Einstein symmetrization and *not* a new, as yet unknown kind of physical symmetry.

Purity enables us to define an orthogonal basis in a vector space for amplitudes and decay probabilities that we shall call the “purity space”. (More appropriately but awkwardly, this space might be called the “impurity space”, since the integer value of the label K is a measure of the “impurity grade” of any of the basic vectors.) We allow the pure coefficients $\tilde{e}_{(K)}$ to be rational (not only integral) functions of the six free invariants and, in the standard way, we introduce a vector space over the field of such pure rational functions by considering the amplitude E to be a vector $|E\rangle$ which may, in general, be expressed as:

$$|E\rangle = \tilde{e}_{(0)}|\tilde{U}_0\rangle + \tilde{e}_{(3)}|\tilde{U}_3\rangle + \tilde{e}_{(4)}|\tilde{U}_4\rangle + \tilde{e}_{(5)}|\tilde{U}_5\rangle + \tilde{e}_{(6)}|\tilde{U}_6\rangle + \tilde{e}_{(9)}|\tilde{U}_9\rangle . \quad (169)$$

Here, we have really done nothing new except starting to think in terms of a vector space by introducing a new notation with the following “table of correspondence”:

$$A^+ = E \rightarrow |A^+\rangle = |E\rangle , \quad (170)$$

$$\tilde{U}_K \rightarrow |\tilde{U}_K\rangle \quad \text{for } K = 0, 3, 4, 5, 6, 9 . \quad (171)$$

We also introduce dual vectors by the following correspondence table:

$$(A^+)^* = E^* \rightarrow \langle A^+| = \langle E| , \quad (172)$$

$$\tilde{U}_K \rightarrow \langle \tilde{U}_K| \quad \text{for } K = 0, 3, 4, 5, 6, 9 , \quad (173)$$

i.e. with the understanding that complex numbers turn into their complex conjugates when vectors are replaced by their dual partners.

The decisive definition (made possible by purity) is that of the inner product

$$\langle \tilde{U}_L | \tilde{U}_K \rangle = \frac{1}{36} \sum_{36} \tilde{U}_L \tilde{U}_K \quad (L, K = 0, 3, 4, 5, 6, 9) \quad (174)$$

where the summation extends over the $6 \times 6 = 36$ purity tests applied to the product $\tilde{U}_L \tilde{U}_K$. Actually, since any product $\tilde{U}_L \tilde{U}_K$ is symmetric with respect to simultaneous permutations of the components of the vectors $\vec{s}$ and $\vec{r}$, we can calculate the sum in Eq. (174) by summing over the six different permutations of the components of $\vec{r}$, at fixed $\vec{s}$ – or vice versa – and multiplying the result by 6. Clearly, the scalar products $\langle \tilde{U}_L | \tilde{U}_K \rangle$ are pure polynomials.

By the standard Gram-Schmidt procedure, new basic vectors $|0\rangle, |3\rangle, |4\rangle, |5\rangle, |6\rangle, |9\rangle$, and their dual partners $\langle 0|, \langle 3|, \langle 4|, \langle 5|, \langle 6|, \langle 9|$ may be defined as linear superpositions – with pure rational coefficients – of the original vectors $|\tilde{U}_K\rangle$ and $\langle \tilde{U}_K|$, respectively, in such a way that the new vectors are mutually orthogonal (but *not* normalized!):

$$\langle L | K \rangle = 0 \quad \text{for } L \neq K . \quad (175)$$

Explicitly, the orthogonalization is carried out in Appendix D. The result is:

$$|0\rangle = |\tilde{U}_0\rangle, \quad (176)$$

$$|3\rangle = |\tilde{U}_3\rangle, \quad (177)$$

$$|4\rangle = |\tilde{U}_4\rangle + \frac{3\tilde{S}_3}{2\tilde{S}_2}|\tilde{U}_3\rangle, \quad (178)$$

$$|5\rangle = |\tilde{U}_5\rangle + \frac{3\tilde{T}_6}{2\tilde{T}_4}|\tilde{U}_3\rangle, \quad (179)$$

$$|6\rangle = |\tilde{U}_6\rangle + \frac{3\tilde{S}_3}{2\tilde{S}_2}|\tilde{U}_5\rangle + \frac{3\tilde{T}_6}{2\tilde{T}_4}|\tilde{U}_4\rangle + \frac{9\tilde{S}_3\tilde{T}_6}{4\tilde{S}_2\tilde{T}_4}|\tilde{U}_3\rangle - \frac{2}{3}\tilde{S}_2\tilde{T}_4|\tilde{U}_0\rangle, \quad (180)$$

$$|9\rangle = |\tilde{U}_9\rangle - \frac{2}{3}\tilde{S}_2\tilde{T}_4|\tilde{U}_3\rangle - \frac{2}{3}\tilde{S}_3\tilde{T}_6|\tilde{U}_0\rangle, \quad (181)$$

with

$$\langle 0|0\rangle = 1, \quad (182)$$

$$\langle 3|3\rangle = 2\tilde{S}_2\tilde{T}_4, \quad (183)$$

$$\langle 4|4\rangle = -\frac{\tilde{T}_4}{6\tilde{S}_2}\tilde{B}_6, \quad (184)$$

$$\langle 5|5\rangle = -\frac{\tilde{S}_2}{6\tilde{T}_4}\tilde{D}_{12}, \quad (185)$$

$$\langle 6|6\rangle = \frac{1}{72\tilde{S}_2\tilde{T}_4}\tilde{B}_6\tilde{D}_{12}, \quad (186)$$

$$\langle 9|9\rangle = \frac{1}{36}\tilde{B}_6\tilde{D}_{12}, \quad (187)$$

and

$$\tilde{B}_6 = 4\tilde{S}_2^3 + 27\tilde{S}_3^2, \quad (188)$$

$$\tilde{D}_{12} = 4\tilde{T}_4^3 + 27\tilde{T}_6^2. \quad (189)$$

Clearly, the signs and magnitudes of the quantities appearing in the array (182)–(187) are of decisive importance. We have

$$\tilde{S}_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = -(\sigma_1^2 + \sigma_2^2 + \sigma_3^2), \quad (190)$$

i.e. $\tilde{S}_2$ is negative except at the origin of the σ -space where it vanishes. Analogously, $\tilde{T}_4$ is negative except at the origin of the τ -space where it vanishes. The quantity $\tilde{B}_6$ vanishes whenever two components of the $\vec{\sigma}$ -vector are equal (of course, this condition is also satisfied at the origin of the σ -space) but is negative otherwise. This follows most simply from Eq. (58) written in the infinitesimal basis, i.e. with $\tilde{S}_1 = 0$:

$$\tilde{B}_6 = -\tilde{H}_6^2 = -(\sigma_1 - \sigma_2)^2(\sigma_2 - \sigma_3)^2(\sigma_3 - \sigma_1)^2. \quad (191)$$

Clearly, at the origin of the σ -space, also $\tilde{B}_6/\tilde{S}_2$ vanishes owing to the higher power of any component σ_i in the numerator as compared to the denominator. The analogous statements are valid for $\tilde{D}_{12}$ and the τ -space. Therefore, all squared norms $\langle K|K\rangle$ are positive or zero, as we could have proved right after the definition (174) of the inner product. A

posteriori, this vindicates our definition of the inner product and the introduction of the purity space. Whenever any square norm $\langle K|K \rangle$ vanishes, this simply means that the purity space has one dimension less and that the corresponding vector $|K\rangle$ is not needed in the expansion of $|E\rangle$. Clearly, at the origin of the σ, τ -space, i.e. for $\vec{\sigma} = 0, \vec{\tau} = 0$, we have

$$|E\rangle = \tilde{e}_{(0)}|0\rangle, \quad (192)$$

i.e.

$$\tilde{e}_{(0)} = E. \quad (193)$$

Elsewhere, if both vectors $\vec{\sigma}$ and $\vec{\tau}$ have each two equal components, the purity space is spanned only by $|0\rangle$ and $|3\rangle$. If only $\vec{\tau}$ has two equal components we need the vector $|4\rangle$, and if only $\vec{\sigma}$ has two equal components we need the vector $|5\rangle$, in addition to the vectors $|0\rangle$ and $|3\rangle$. Finally, if all components of $\vec{\sigma}$ are different from each other and the same is true about $\vec{\tau}$, the purity space is 6-dimensional and its orthogonal basis consists of all six vectors $|K\rangle$. In the most complicated case of a 6-dimensional purity space, the vector $|E\rangle$ may be written as

$$|E\rangle = \hat{e}_{(0)}|0\rangle + \hat{e}_{(3)}|3\rangle + \hat{e}_{(4)}|4\rangle + \hat{e}_{(5)}|5\rangle + \hat{e}_{(6)}|6\rangle + \hat{e}_{(9)}|9\rangle, \quad (194)$$

with

$$\hat{e}_{(K)} = \frac{\langle K|E\rangle}{\langle K|K\rangle} \quad (K = 0, 3, 4, 5, 6, 9). \quad (195)$$

Inserting into the latter equation the vector $\langle K|$ as obtained from the conjugate of the array (176)–(181) and considering the definition (174) of the inner product, we easily obtain any of the quantities $\hat{e}_{(K)}$ for any specific theoretically predicted amplitude E . The analogous statements apply to the special cases when the purity space has less than six dimensions.

Now, at last, we must generalize the “form factors” $\hat{e}_{(K)}$ by allowing them to be analytic functions of the free invariants, fulfilling certain conditions imposed by physical principles other than the ones relied upon in this article, in particular by causality. However, we should mention a somewhat moot point. In Section 7 we have shown that any totally symmetric scalar may be expressed as an algebraic function of only the free (i.e. pure) invariants which means that, in this sense, it is pure. We see that the notion of impurity is relevant only as long as we limit ourselves to integral and rational functions of the free invariants. It is probably not accidental that the algebraic equation enabling us to express the only mixed member of the rationality basis U_3 in terms of the free invariants is of sixth order in U_3 . In this way, the six grades of impurity might translate into something like the six Riemann sheets of an algebraic function.

As we shall see shortly, our main reason for introducing the purity space is rather mundane and we have not been worried about the singularities of the decay amplitude as a function of *complex* kinematical variables. If and when such properties of the amplitude are considered, the kinematical singularities as exposed here will have to be taken into account.

In our new notation, instead of Eq. (40), the reduced decay probability w is expressed as

$$w = \langle A^+|A^+ \rangle. \quad (196)$$

At this point, we make an interesting observation. The analogous reduced decay probability

$$w = \frac{1}{2J+1} \sum_M \frac{dW(M)}{d\tau} \quad (197)$$

for the decay of an *unpolarized* boson with *arbitrary* spin J and *arbitrary* parity into four π^0 's, averaged over the initial magnetic quantum numbers M , has the same symmetry properties as the scalar amplitude $A^+ = E$: it is a 4-scalar symmetric with respect to all permutations of the four π^0 's. Therefore, it can be considered as a vector $|w\rangle$ in the purity space and expanded in complete analogy to Eq. (194):

$$|w\rangle = \hat{w}_{(0)}|0\rangle + \hat{w}_{(3)}|3\rangle + \hat{w}_{(4)}|4\rangle + \hat{w}_{(5)}|5\rangle + \hat{w}_{(6)}|6\rangle + \hat{w}_{(9)}|9\rangle \quad (198)$$

with

$$\hat{w}_{(K)} = \frac{\langle K|w\rangle}{\langle K|K\rangle} \quad (K = 0, 3, 4, 5, 6, 9). \quad (199)$$

The quantities $\hat{w}_{(K)}$ may be called “partial reduced decay probabilities of impurity grade K ” or, for short, “ K -probabilities”. In an obvious way, the K -probabilities can be obtained from the reduced decay probability w and the results of this Section. The special cases discussed in connection with the expansion of the amplitude E apply here as well. (Incidentally, with the argument used to extend the application of the purity space to the decay of unpolarized particles of arbitrary spin and parity into four identical spinless bosons, we could further extend this application to the decay into four identical particles with spin, e.g. four photons, provided their polarization is not detected.)

Clearly, we are interested in using the above expressions for the analysis of experimental data. There, we encounter a difficulty. Some of the purity tests may convert an event on the mass shell to points outside the physical region. (Those points which stay inside the physical region remain also on the C -plane, i.e. on the mass shell, since S_1 is a pure invariant.) What to do in such a case is a matter of convention. After all, any comparison of data with a theoretical model will be done by first using the model to simulate the data. Applying the same convention to the simulation as it is used for data cannot lead to any error, provided the statistical uncertainties are handled correctly. One possible convention for dealing with points outside the physical region is simply to omit them.

Effectively, we propose to pass the data and their simulation through identical filters before comparing them. In principle, any filter may lead to interesting results. Simple models, i.e. those with few adjustable parameters, suggest by themselves the optimal filters. However, for protonium decays, simple models with any chance of fitting the data in a satisfactory way do not exist. The advantage of purity filters proposed here is their mathematical transparency. On one hand, the six K -probabilities with K ranging from 0 to 9 allow the complete reconstruction of the reduced decay probability w . On the other hand, K -probabilities with different values of K may have different sensitivities to particular aspects of the model. Perhaps it is justified to expect that higher impurities will contribute less to w than lower ones; this is necessarily the case in the vicinity of the center of the kinematical space, owing to our use of the infinitesimal basis. With luck, especially these higher K -probabilities may be more sensitive to some details of the model than the full reduced decay probability. However, only the greatest optimists could expect that the latter should contain contributions from only a few K -values: this would indicate that for some dynamical reasons K is a “good quantum number” for the decay.

Yet another interesting filter is suggested by the observation of Section 9 that amplitudes describing decays through intermediate dipion–dipion resonances do not contain odd powers of the basic invariant R_3 . Unfortunately, we have as yet no proof that this is true for unpolarized initial bosons with spins different from zero, but it seems probable. If it were indeed the case, the full reduced decay probability for protonium as it is experimentally obtained would not contain odd powers of R_3 , provided the decay proceeded entirely through dipion–dipion states. Subtracting data (and simulated data) in the basic octant of the r -space with $r_1 \geq 0, r_2 \geq 0, r_3 \geq 0$ from those in the second octant with $r_1 \geq 0, r_2 \geq 0, r_3 \leq 0$, we should obtain samples of subtracted data (and subtracted simulated data) which might be particularly sensitive to tripion intermediate states. Again, this subtraction cannot be carried out for those points in the second octant that have no counterpart in the basic octant. In view of Eq. (9), expression (72) for Δ_4 shows that this must be the case for part of the mass shell.

When applying filters, a practical difficulty arises with data of such low statistics that they cannot be binned. However, some methods for the analysis of unbinned data [27] might rather easily be adapted to the use of filters. Yet another problem may be caused by the special cases when the 6-dimensional purity space degenerates into a lower-dimensional space. Care must be exercised in order to ensure that the “unwanted” basic vectors do not get assigned large lengths in the vicinity of the origin and of the regions where two of the $\vec{\sigma}$ - or/and two of the $\vec{\tau}$ -components are equal. These are potential problems to be solved in the actual computer analysis of data and this is not the proper place to dwell on them.

Starting from the basic anti-invariants of the infinitesimal basis [see Eqs. (135)–(140)] we may define a set of basic vectors in yet another 6-dimensional purity space into which any antisymmetric scalar O may be expanded:

$$|O\rangle = \tilde{o}_{(3)}|\tilde{H}_3\rangle + \tilde{o}_{(3)}|\tilde{J}_3\rangle + \tilde{o}_{(4)}|\tilde{J}_4\rangle + \tilde{o}_{(5)}|\tilde{J}_5\rangle + \tilde{o}_{(6)}|\tilde{J}_6\rangle + \tilde{o}_{(6)}|\tilde{H}_6\rangle. \quad (200)$$

In order to distinguish the two 6-dimensional purity spaces, we shall refer to the original one as the purity space of invariants, and to the one we are discussing now as the purity space of anti-invariants. Of course, they are completely disjoint. As before, we introduce the dual purity space of anti-invariants and define the inner products in complete analogy to Eq. (174). Actually, we can likewise define an inner product of two vectors, one from the vector space of invariants and the other from the space of anti-invariants. It is easy to see that all such “disjoint” inner products vanish as they must for our formalism to be consistent.

When we try to introduce an orthogonal set of basic vectors in the purity space of anti-invariants, we run into a labelling problem, since there exist two different basic anti-invariants for both $K = 3$ and $K = 6$. We are forced to replace K by two labelling integers which may be chosen to be two different convenient linear combinations of n_{tot} and m_{tot} of Eqs. (64) and (65); in this way we are led to labels used in Ref. [5] and discussed in the next Section. The method employed to orthogonalize the basic vectors in the purity space of anti-invariants is that of Appendix D. Any vector $|O\rangle$ may then be expanded in these orthogonal vectors in complete analogy to Eqs. (194) and (195), and the pure coefficients of this expansion may be generalized to become form factors.

A comment is due concerning the potential use of a final instead of a good basis for the expansion of amplitudes and probabilities. As explained in Sections 5 and 7, our final basis is in a sense non-linear, since an arbitrary (anti)invariant cannot be expressed in this basis without, in general, the products U_3^2 , U_4U_5 , or U_3J_3 appearing in the expression.

Any other final basis (trivially related to ours) is also non-linear. Therefore, a final basis is not suitable for the introduction of a corresponding vector space, unless we extend it as in Section 7, in which case we end up dealing with a good basis again.

Finally, we should mention yet a different expansion of the totally antisymmetric scalar O which looks particularly simple:

$$O = (s_2 - s_3)f_1 + (s_3 - s_1)f_2 + (s_1 - s_2)f_3 + (t_2 - t_3)g_1 + (t_3 - t_1)g_2 + (t_1 - t_2)g_3. \quad (201)$$

Here, f_1 is the symbol for a scalar function of the pion 4-momenta $f(p_4, p_1; p_2, p_3)$ which is symmetric with respect to the interchanges of p_1 and p_4 , of p_2 and p_3 , as well as of the pair p_1, p_4 and the pair p_2, p_3 . We have

$$\begin{aligned} f_1 &= f(p_4, p_1; p_2, p_3), \\ f_2 &= f(p_4, p_2; p_3, p_1), \\ f_3 &= f(p_4, p_3; p_1, p_2). \end{aligned} \quad (202)$$

The functions g_1, g_2, g_3 are defined in complete analogy with f_1, f_2, f_3 except, of course, that the functions f and g may be different. Equation (201) is valid if O is replaced by any of the final basic anti-invariants H_3, J_3, J_4, J_5, H_6 , as we convince ourselves by taking a look at Eqs. (57), (62), and (73)–(75). However, it is also generally valid, since anti-invariants constitute a module over the ring of invariants, and multiplication by an invariant cannot change the symmetry properties of the functions f_i, g_i ($i = 1, 2, 3$). Notice, though, that Eq. (201) is not self-evident if O is replaced by the auxiliary anti-invariant J_6 of Eq. (78) which does not belong to the final basis.

Unfortunately, the expansion (201) is often non-unique. In general, a given O can be written in infinitely many different ways in the form (201). This can be seen, for example, by looking at Eq. (73) and noticing that the anti-invariant J_3 can be expressed partly by the first three and partly by the last three terms of Eq. (201). Clearly, this equation contains less information about the scalar O than expression (200), which vanishes identically only if all six form factors $\tilde{o}_{(3)}, \tilde{o}_{(3)}, \tilde{o}_{(4)}, \tilde{o}_{(5)}, \tilde{o}_{(6)}$, and $\tilde{o}_{(6)}$ do so. In contrast, expression (201) may vanish identically for some simple finite form factors, e.g. those resulting from $f = 1, g = 1$. The non-uniqueness of the form factors f_i, g_i has an unpleasant consequence. It is easy to write down expressions of the type (201) with two different sets of form factors f_i, g_i , representing the same scalar O . Without reducing the two seemingly different scalars to the form (200) it is sometimes difficult to see that they are actually equal. The problem is exacerbated when one is confronted with three or more seemingly different antisymmetric scalars which, in fact, depend only on a smaller number of basic anti-invariants of a good basis.

At the end, we should finally comment on the question that we have implicitly asked several times: Which is the “best” basis of (anti)invariants of the group G ? As we have seen, this question does not have a simple answer, and all we can say is that it depends on the application that we have in mind.

11 Discussion

Perhaps we should briefly return to our statement in the Introduction that the internal quantum numbers of the four identical bosons are irrelevant for our conclusions. As we have seen, our arguments rely only on Lorentz invariance and on the fundamental law of quantum mechanics concerning identical bosons: the amplitude for any process involving such particles must be invariant with respect to their permutations. We have

not found any need to introduce internal quantum numbers of the four π^0 's into the decay amplitude; the 4-momentum suffices to completely characterize a π^0 for our purpose. The situation would change if we considered decays into four pions of arbitrary charge which have to be described as identical particles with different coordinates in isotopic spin space. Then, of course, the decay amplitude depends on these coordinates as well, and invariance must be required with respect to *simultaneous* permutations of 4-momenta and isospin coordinates of the four pions.

Our results may be compared with those of Henley and Jacobsohn [5]. The scope of their work is much wider than ours. They treat both strong and electromagnetic decays of heavy bosons of zero strangeness, angular momentum $J \leq 2$, and isospin $I \leq 1$ (partly even $I \leq 3$), including n -pion final states for $n \leq 5$. On the other hand, they limit themselves to "leading" amplitudes, i.e. those with small K -values.

In order to discuss the correspondence between Ref. [5] and our results, we first have to reconvert our scalar kinematical variables $\vec{\sigma}$ and $\vec{r}$ [see Eqs. (29) and (122)] and the basic pseudoscalar ψ of Eq. (4) to expressions revealing their 4- and 3-momentum content. A convenient and easily checked form of these quantities is

$$\begin{aligned}\sigma_1 &= \frac{1}{3} [(p_1 - p_2) \cdot (p_4 - p_3) - (p_3 - p_1) \cdot (p_4 - p_2)] , \\ \sigma_2 &= \frac{1}{3} [(p_2 - p_3) \cdot (p_4 - p_1) - (p_1 - p_2) \cdot (p_4 - p_3)] , \\ \sigma_3 &= \frac{1}{3} [(p_3 - p_1) \cdot (p_4 - p_2) - (p_2 - p_3) \cdot (p_4 - p_1)] ,\end{aligned}\tag{203}$$

$$\begin{aligned}r_1 &= \frac{1}{2} [P \cdot (p_2 + p_3 - p_1 - p_4)] , \\ r_2 &= \frac{1}{2} [P \cdot (p_3 + p_1 - p_2 - p_4)] , \\ r_3 &= \frac{1}{2} [P \cdot (p_1 + p_2 - p_3 - p_4)] ,\end{aligned}\tag{204}$$

and

$$\psi = \epsilon_{\alpha\beta\gamma\delta} P^\alpha p_2^\beta p_3^\gamma p_4^\delta = \epsilon_{\alpha\beta\gamma\delta} p_1^\alpha P^\beta p_3^\gamma p_4^\delta = \epsilon_{\alpha\beta\gamma\delta} p_1^\alpha p_2^\beta P^\gamma p_4^\delta = \epsilon_{\alpha\beta\gamma\delta} p_1^\alpha p_2^\beta p_3^\gamma P^\delta .\tag{205}$$

In the rest frame of the decaying particle with mass M , ψ is given up to a sign by the product of M and the mixed product of any three of the four 3-momenta of the pions. In the same rest frame the scalars r_i are given by the product of M and the difference between the total energies of a pair of pions and the conjugate pair; clearly, we could have said kinetic instead of total energies. On the other hand, the mass M cannot be factored out of the scalars σ_i because they contain scalar products of two space-like vectors ($E_1 - E_2$, for instance, vanishes in the Lorentz frame where $\vec{p}_1 + \vec{p}_2 = 0$).

The four pions from the decay of a boson with mass M not much above the decay threshold of $4m$ (m being the pion mass) would necessarily be non-relativistic in the rest frame of the boson. Moreover, if this boson had a finite radius of the order of magnitude R , the *smallest* number N_{nr} of 3-momentum factors (each of the order of magnitude k) in any additive term of the decay amplitude would determine the penetration factor [28, Ch.VIII, Sec.5] of a "direct" decay into four pions to be of the order of $(kR)^{2N_{nr}}$. This is so because the wave function of each pion at the distance $R \ll k^{-1}$ is proportional to $(kR)^l$, l being the orbital angular momentum of the pion in the rest frame of the decaying boson,

and because the penetration factors of individual pions should be multiplied to yield the total penetration factor for the decay. The integer N_{nr} is sometimes called the “number of units of orbital angular momentum” in the decay. A look at Eqs. (203)–(205) convinces us that each factor σ_i or r_i contributes two units, and ψ contributes three units of orbital angular momentum to N_{nr} . Incidentally, for any particular decay, it is easy to find the value of N_{nr} by considering the quantum numbers of the simplest possible intermediate resonances.

However, as noted in Ref. [5], the number of 3-momentum factors in the decay amplitude should not be associated too closely with angular momentum. For one thing, any decay into more than two pions is inherently “dual”: a kinematical variable measuring a decay angle of a tripion (or dipion) in its rest frame is also determined by the invariant masses of all (other) possible dipions in the decay. A narrow intermediate dipion can strongly modulate the decay angular distribution of a different dipion, which does not necessarily mean that large angular momenta are involved. However, the Breit–Wigner form factor needed to properly describe the narrow intermediate tripion (or dipion) causes a slow convergence of series such as those given by Eqs. (153) and (158), i.e. it introduces important contributions of polynomial (anti)invariants with large values of K . Interpreting such values as indicating large orbital angular momenta in the decaying system would be misleading. Only the term in the amplitude with the smallest K -value determines N_{nr} and can be given an interpretation in terms of orbital angular momenta.

More generally, it could be said that the dynamics of the decay may strongly enhance powers of 3-momenta higher than the minimum allowed one, without necessarily introducing large orbital angular momenta. The “direct” decay should then be defined by its amplitude containing only one or at most a few monomials of lowest 3-momentum content.

Of course, it is questionable whether Nature ever chooses to let a strongly interacting boson decay “directly” into several pions. However, describing a potential direct decay by a few terms with small K -values in expansions of the scalars E and O into linearly independent monomials in basic (anti)invariants at least does not violate any exact symmetries or fundamental tenets of strong interaction physics. In contrast, direct decays are sometimes “described” by assuming a constant decay amplitude which would be correct for an initial scalar particle decaying into $4\pi^0$; however, it would violate Lorentz invariance and Bose–Einstein statistics in the case of a $4\pi^0$ -decay of a pseudoscalar particle. Alternatively, direct amplitudes are sometimes derived from amplitudes valid for narrow intermediate states by letting their total and partial widths tend to infinity. In general, this introduces an unphysical threshold behaviour and may cause the amplitude to become singular at the positions of Adler zeros, i.e. very near to the mass shell for a pionic final state.

In the other extreme of a very large initial mass M , all four pions in the final state are relativistic over most of the mass shell. At constant R , penetration factors should tend to constant values as the pion 3-momenta (i.e. total energies) become large. Still, at any given M , we can find a sufficiently small value of R such that the penetration factors should depend on 3-momentum magnitudes as outlined above in the non-relativistic case. In protonium decay, the effective R -value (called “annihilation radius”) is still a matter of contention. Quite possibly, the annihilation radius is sufficiently small so that even relativistic pions are subject to the influence of penetration factors linear in each 3-momentum factor contained in the leading monomial of the decay amplitude. However, as pointed out in Ref. [5, in particular Section III B], while ψ and $\vec{\sigma}$ contribute relativistically

and non-relativistically the same power of 3-momentum (3 and 2 for ψ and $\vec{\sigma}$, respectively), $\vec{r}$ contributes in the relativistic limit a factor of the order k and in the non-relativistic case a factor of the order k^2 . Therefore, relativistically, the lowest total power of 3-momenta is an integer $N_r \leq N_{nr}$. In our case of $4\pi^0$ -decay we have $N_{nr} = N_r = 0$ for a scalar initial boson, and $N_{nr} = 9$, $N_r = 7$ for a pseudoscalar one.

Henley and Jacobsohn call the integers N_{nr} and N_r “barrier factors” and symbolize them by L and Λ , respectively. However, they do not reserve the name and the symbols for the lowest total power of 3-momenta. Instead, they use these symbols in a classification of the final-state wave functions by the total power of 3-momenta. Therefore, L and Λ correspond to two linear combinations of our integers n_{tot} and m_{tot} of Eqs. (64) and (65). In the case of a scalar initial particle, the correspondence is $L = 2(m_{tot} + 2n_{tot}) = 2K$ and $\Lambda = 2(m_{tot} + n_{tot})$. For a pseudoscalar initial particle, the factor ψ in the amplitude A^- of Eq.(42) necessitates the addition of 3 on the right-hand side of the two equations in the preceding sentence. Still, Ref. [5] pays the greatest attention to small values of L and Λ . A few months after the discovery and the determination of the J^P -value of the ω meson, when Ref. [5] was published, direct decays must have appeared as the most important multipion decay channels of bosons yet to be found. Otherwise, as we have discussed above, the term “barrier” might have been misleading.

For the minimal values $L = N_{nr}$, $\Lambda = N_r$, Henley and Jacobsohn also list many decay matrix elements, corresponding to our decay amplitudes. Unfortunately, for the $4\pi^0$ -channel they do this only in the trivial case of a scalar initial particle where the lowest order amplitude is constant, and for the exotic initial boson with $J^{PC} = 1^{-+}$. Therefore, no detailed comparison with our work is possible. However, in conjunction with their Tables I and II, they give a prescription of how to find the number of independent amplitudes for a few small values of L and Λ . Besides the lowest order amplitudes, we find $L = 4$, $\Lambda = 2$ and $L = 4$, $\Lambda = 4$ in the scalar case, and $L = 9$, $\Lambda = 9$ in the pseudoscalar case. These barrier factors clearly correspond to our basic invariants $\tilde{T}_2$ and $\tilde{S}_2$, and our basic anti-invariant $\tilde{H}_3$, respectively. However, we also find that there should exist two independent amplitudes with $L = 9$, $\Lambda = 7$, while our treatment leaves no room for an additional amplitude with these barrier factors and independent of $\tilde{J}_3$. Unless this last finding is due to a simple misprint in Ref. [5] (it would take only a misplaced exponent of 2 in a Table full of numerical symbols), the disagreement is of fundamental significance for the understanding of our subject and it would be very desirable to resolve it.

The work presented in this article has been undertaken with the intention of applying it to the study of annihilation into $4\pi^0$ of antiprotons stopped in liquid hydrogen. Preliminary results of the Crystal Barrel Collaboration at CERN [27] on the behaviour of the differential annihilation probability in the vicinity of kinematical zeros indicate that the contribution of the initial S-state with $J^{PC} = 0^{-+}$ (often referred to as “S-wave annihilation”) is on the percent level, at most. Also the P-state with $J^{PC} = 1^{++}$ seems to contribute little, if anything, which is understandable since Ref. [5] quotes $N_{nr} = 6$, $N_r = 5$. Both P-states with $J^{PC} = 0^{++}$ and 2^{++} contribute to the $4\pi^0$ decay of protonium; according to Ref. [5] the latter of the P-states has $N_{nr} = N_r = 2$.

The absence of any substantial $4\pi^0$ yield from the initial S-state could have been expected considering its large N_{nr} , N_r -values, but it is deplorable. It would have been interesting to study such a highly forbidden decay with its great selectivity for definite intermediate states. The excitation of intermediate states in P-wave annihilation is under study. Hopefully, the techniques of treating experimental data as suggested in Section 10 will prove useful in this analysis.

12 Apologia (in place of a summary)

Unfortunately, it seems nearly impossible to concoct a short summary of an article such as this one, which has to use a substantial amount of mathematics in order to solve a somewhat academic problem in theoretical physics with the aim of applying the solution in the analysis of experimental data. Nowadays, even the language of the summary would have to be different for the three potential categories of readers. The casual reader who would like to inform himself by reading only the summary of this article will have to content himself with its Introduction. Incidentally, also the careful reader might find in the latter an acceptable surrogate for a proper summary, certainly better understandable as such than in its original role.

On the other hand, it should be appropriate for an author to step back and to ask himself at the end of his article about its significance. One may question his capability of producing even a remotely objective answer. At least, I shall try.

To a mathematician this article offers little else but a specific example for well-known mathematical theorems and of familiar mathematical techniques. (It is rather sobering to find the problem that we have solved in Sections 4 and 5 proposed as an exercise in a recent book on algorithms in invariant theory [17, p.28].) One mathematical point that has not been sufficiently elucidated in the article is the potential simplification of the syzygy chain due to the peculiarity of our transient (anti)invariants, some of which may be chosen as products of others. Probably, this obscurity is caused only by my ignorance of the relevant mathematical literature. Also, the special property of our (anti)invariants, namely the purity of the free invariants that has led to the introduction of the purity vector space, may have escaped the attention of mathematicians. To what extent is the purity of free invariants a general property of simultaneous invariants in several vectors, such as our $\vec{s}$ and $\vec{r}$ (not just $\vec{s}$ and $\vec{t}$, in which case it is evident)? Does the order of the algebraic equation leading to the functional basis of invariants always conform to our conjecture in Section 10?

To a theoretical physicist the article provides the “trivial” kinematical framework (in the wider sense of the word) for dynamical models needed to study the decay of a spinless particle into four identical spinless bosons. At present, only meson spectroscopists might be interested but to them the restriction to spinless and identical bosons is too severe. Fortunately, no fundamental reason seems to exist why the kinematical analysis of $4\pi^0$ decays as presented here could not be extended to initial bosons with spin and why the carrier space could not be enlarged to include isospin. These extensions would cover a sufficiently wide area of meson spectroscopy to be considered – together with the work of Pais [4] and of Byers and Yang [9] – as part of its kinematical foundations, especially if somebody succeeded in solving the problem of more than four identical bosons. Hopefully, the present article has demonstrated the relevance of invariant theory for such an endeavour.

Finally, an experimental meson spectroscopist is interested in the kinematical zeros of the decay amplitude. It is somewhat frustrating that the latter can be found by a symmetry argument taking about half a percent of the length of this article, while nearly all of it is needed to algebraically confirm the result. On the other hand, if some of the techniques for analyzing experimental data suggested in Section 10 prove successful, this would substantially increase the practical value of the article. At present, I cannot see how these techniques could have been imagined without the use of invariant theory. However, it is entirely possible that the techniques prove useless in practice.

13 Acknowledgments

This article would not have been conceived were I not a member of the Crystal Barrel Collaboration at CERN. I am indebted to all the other members of the Collaboration for the stimulating atmosphere in the CB Group and for the permission to mention our unpublished results. Specifically, I am grateful to P. Illinger for keeping me informed on the progress of his analysis of the $4\pi^0$ data.

The work reported here originates from a remark by C. Amsler. L. Montanet and J. M. Richard were instrumental in formulating the problem. I had a stimulating discussion concerning the first draft of this article with P. van Nieuwenhuizen. J. Mozrzymas – an enthusiastic advocate of applying group theory to all kinds of physical problems – was a source of encouragement; he also called my attention to the article by Sloane [13]. O. Forster spent many hours trying to convey to me some understanding of commutative algebra; I alone am to blame for his failures. R. Stora contributed important advice and critical remarks. At my peril, I disregarded his disapprobation of the introduction of purity space.

I owe much to an anonymous reviewer of *Annals of Physics*, not only for having the courage to recommend this article for publication on the basis of its original, still deficient, version. His questions and comments gave me ground for thought which – after an excruciatingly long delay – led to very substantial improvements of my work.

I am indebted to E. M. Henley for a clarifying correspondence on the contents of Ref. [5]. The final version of this article has been graciously commented upon by H. Kraft and R. P. Stanley. To H. Kraft I owe the central idea of Appendix C, several corrections, and an illuminating discussion, while R. P. Stanley called my attention to the newly published book by Sturmfels [17].

APPENDICES

A Algebra of 6-exponent symbols and antisymbols

In Section 4 we have defined the germ [see Eq. (52)]

$$\begin{pmatrix} m_a & m_b & m_c \\ n_a & n_b & n_c \end{pmatrix} = s_1^{m_a} s_2^{m_b} s_3^{m_c} t_1^{n_a} t_2^{n_b} t_3^{n_c}. \quad (206)$$

The absolute 6-exponent symbol

$$\begin{bmatrix} m_a & m_b & m_c \\ n_a & n_b & n_c \end{bmatrix}$$

(briefly called symbol) is defined as a sum of such germs

$$\begin{bmatrix} m_a & m_b & m_c \\ n_a & n_b & n_c \end{bmatrix} = \sum_{\text{all}} \begin{pmatrix} m_a & m_b & m_c \\ n_a & n_b & n_c \end{pmatrix} \quad (207)$$

where the summation indicated by $\sum_{\text{all}}$ extends over all six possible permutations of the three columns a, b, c . Alternatively, we may view the summation as extending over all six permutations of the indices $i = 1, 2, 3$ of the variables s_i, t_i with the exponents m_a, m_b, m_c staying fixed. The exponents $m_a, m_b, m_c, n_a, n_b, n_c$ are positive integers or zero.

The relative 6-exponent symbol

$$\left\{ \begin{matrix} m_a & m_b & m_c \\ n_a & n_b & n_c \end{matrix} \right\}$$

(briefly called antisymbol) is defined as

$$\left\{ \begin{matrix} m_a & m_b & m_c \\ n_a & n_b & n_c \end{matrix} \right\} = \sum_{\text{even}} \begin{pmatrix} m_a & m_b & m_c \\ n_a & n_b & n_c \end{pmatrix} - \sum_{\text{odd}} \begin{pmatrix} m_a & m_b & m_c \\ n_a & n_b & n_c \end{pmatrix} \quad (208)$$

where $\sum_{\text{even}}$ indicates summation over the three even and $\sum_{\text{odd}}$ summation over the three odd permutations of the three columns a, b, c .

Clearly, symbols are invariants and antisymbols are anti-invariants of the group G . Collectively, we call them (anti)symbols. Obviously, a symbol is not affected while an antisymbol changes its sign upon the transposition of any two columns, e.g.

$$\begin{bmatrix} m_a & m_b & m_c \\ n_a & n_b & n_c \end{bmatrix} = \begin{bmatrix} m_b & m_a & m_c \\ n_b & n_a & n_c \end{bmatrix}, \quad (209)$$

$$\left\{ \begin{matrix} m_a & m_b & m_c \\ n_a & n_b & n_c \end{matrix} \right\} = - \left\{ \begin{matrix} m_b & m_a & m_c \\ n_b & n_a & n_c \end{matrix} \right\}. \quad (210)$$

Therefore, (anti)symbols are not affected by cyclic permutations of the three columns. An antisymbol vanishes if two of its columns are equal, e.g. if $m_a = m_b$ and $n_a = n_b$.

From the definitions (207), (208) follow the rules for the multiplication of two (anti)symbols. We have

$$\begin{bmatrix} m_a & m_b & m_c \\ n_a & n_b & n_c \end{bmatrix} \begin{bmatrix} m'_a & m'_b & m'_c \\ n'_a & n'_b & n'_c \end{bmatrix} = \sum_{i \neq j \neq k} \begin{bmatrix} m_a + m'_i & m_b + m'_j & m_c + m'_k \\ n_a + n'_i & n_b + n'_j & n_c + n'_k \end{bmatrix}. \quad (211)$$

Here, the summation extends over the six possible permutations of the non-equal indices i, j, k . In each of the six terms of the sum, i, j, k are equal to a, b, c in some order. We also have

$$\left\{ \begin{matrix} m_a & m_b & m_c \\ n_a & n_b & n_c \end{matrix} \right\} \left\{ \begin{matrix} m'_a & m'_b & m'_c \\ n'_a & n'_b & n'_c \end{matrix} \right\} = \sum_{i \neq j \neq k} \left\{ \begin{matrix} m_a + m'_i & m_b + m'_j & m_c + m'_k \\ n_a + n'_i & n_b + n'_j & n_c + n'_k \end{matrix} \right\} \quad (212)$$

where the summation is defined as before. Notice, however, that the multiplication (212) is not symmetric between symbol and antisymbol. Finally, we have

$$\left\{ \begin{matrix} m_a & m_b & m_c \\ n_a & n_b & n_c \end{matrix} \right\} \left\{ \begin{matrix} m'_a & m'_b & m'_c \\ n'_a & n'_b & n'_c \end{matrix} \right\} = \sum_{i \neq j \neq k} \Pi(i, j, k) \left[\begin{matrix} m_a + m'_i & m_b + m'_j & m_c + m'_k \\ n_a + n'_i & n_b + n'_j & n_c + n'_k \end{matrix} \right]. \quad (213)$$

Here, $\Pi(i, j, k) = +1(-1)$ if i, j, k represents an even (odd) permutation of the original order a, b, c .

It is convenient to introduce special names for some simple (anti)symbols. They are listed in Appendix B together with the simplest symbol

$$\left[\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right] = 6.$$

Special names have been chosen for the (anti)symbols which form the final basis and for four other, auxiliary (anti)symbols T_6, U_6, U_9 , and J_6 . The nomenclature is such that the letter S indicates a pure s -symbol with all n -exponents equal to zero. The letter T indicates a pure t -symbol with all m -exponents equal to zero. The letter U indicates a "mixed" symbol, the letter J a "mixed" antisymbol. Finally, the letter H indicates a pure (s - or t -) antisymbol. The integer index K of any letter indicates the degree $K = m_a + m_b + m_c + 2(n_a + n_b + n_c)$ of the (anti)symbol.

Clearly, a factor S_3 can be factored out of any (anti)symbol with all three m -exponents different from zero and a factor T_6 can be factored out of any (anti)symbol with all three n -exponents different from zero. In this way, we are left with (anti)symbols containing at least one zero among its m -exponents and at least one zero among its n -exponents.

Any (anti)symbol containing an exponent larger than 2, say m_a , can be reduced by repeatedly using formulae such as

$$\left\{ \begin{matrix} m_a & m_b & m_c \\ n_a & n_b & n_c \end{matrix} \right\} = S_1 \left\{ \begin{matrix} m_a - 1 & m_b & m_c \\ n_a & n_b & n_c \end{matrix} \right\} - S_2 \left\{ \begin{matrix} m_a - 2 & m_b & m_c \\ n_a & n_b & n_c \end{matrix} \right\} \\ + S_3 \left\{ \begin{matrix} m_a - 3 & m_b & m_c \\ n_a & n_b & n_c \end{matrix} \right\} \quad (214)$$

for $m_a \geq 3$. This formula remains valid if curly brackets are replaced by square ones, i.e. antisymbols by symbols. Analogous formulae are valid for m_b or m_c , if they are larger than 2. They all follow from Eq. (53). Finally, completely analogous formulae are valid for the n -exponents, provided the invariants S_1, S_2, S_3 are replaced by the invariants T_2, T_4, T_6 , respectively.

B Primary basis of (anti)invariants expressed in terms of a good basis, and some auxiliary symbols

In this Appendix, the invariants (symbols) belonging to the primary basis are expressed as polynomials in the basic invariants of the original good basis. First, in order to make the presentation more transparent, auxiliary symbols of the type a_K or α_K for monomials in the pure invariants are defined and presented in Table 5. The notation has been chosen in such a way that two s, t -conjugate monomials (with different values of K , in general) are designated by the same letter. The arrow " $\leftrightarrow$ " calls attention to s, t -conjugation. Roman letters are reserved for monomials which appear both in the list of anti-invariants and in the list of invariants. Monomials that appear only in the latter are designated by Greek letters. In a few cases, two s, t -conjugate monomials have the same degree K ; one of them is marked by a bar, e.g. ϵ_6 and $\bar{\epsilon}_6$. Letters at the end of the alphabet ($w, y, z, \phi, \chi, \psi, \omega$) are reserved for self-conjugate monomials.

Next, in Table 6, pure coefficients occurring in the syzygies of Section 7 and designated by capital Roman and Greek letters are defined in terms of the auxiliary symbols of Table 5.

Table 7 (labelled "Invariants" on each page) presents the primary invariants in terms of the original good basis. The primary invariants are ordered according to their degree K ($1 \leq K \leq 12$). The basic invariants of the final basis and the three auxiliary invariants T_6, U_6 , and U_9 are indicated as such. The basic invariant R_3 cannot be represented by a 6-exponent s, t -symbol. The polynomial in basic and auxiliary invariants corresponding to any given 6-exponent s, t -symbol is coded in the form $F + a(1)U_3 + a(2)U_4 + a(3)U_5 + a(4)U_6 + a(5)U_9$. The quantities $F, a(i), (i = 1, \dots, 5)$ are polynomials in the pure (free) invariants and are listed in terms of the auxiliary symbols of Table 5 under the headings F, U_3, U_4, U_5, U_6 , and U_9 , respectively.

In total, 76 invariants are listed instead of the 68 mentioned in Section 4. This is accounted for by the $K = 0$ constant and by the pure symbols not belonging to the final basis, with one exponent equal to zero and none of the other two larger than two.

Finally, in Table 8 (labelled "Anti-invariants" on each page), anti-invariants (anti-symbols) belonging to the primary basis are expressed in terms of the basic anti-invariants of the original good basis. The presentation follows the pattern established for the invariants. Anti-invariants are coded in the form $b(1)H_3 + b(2)J_3 + b(3)J_4 + b(4)J_5 + b(5)J_6 + b(6)H_6$ and the coefficients $b(i), (i = 1, \dots, 6)$ are listed under the headings $H_3, J_3, J_4, J_5, J_6, H_6$, respectively.

The expressions in this Appendix can easily be checked for s, t -symmetry. The latter may require either internal symmetry of a single expression or mutual symmetry between two different expressions. Also, using the rules of Appendix A, it should be easy to verify any particular expression of interest.

C Simultaneous (anti)invariants in two vectors of the natural representation of S_3

In this Appendix, we present a concise derivation of a good basis of (anti)invariants for two copies of the natural representation of the symmetric group S_3 , i.e. a determination of the simultaneous basic S_3 -(anti)invariants in two (3-dimensional) vectors $\vec{x}$ and $\vec{y}$. Clearly, this is the main problem we are faced with in Section 5. This Appendix follows a suggestion by H. Kraft [20].

The crucial observation is that the (anti)invariants form a free module over the ring of free (and pure) invariants (cf. Section 6) which are *known at the outset*: they may be chosen to be the elementary symmetric functions X_1, X_2, X_3 and Y_1, Y_2, Y_3 in the two vectors $\vec{x}$ and $\vec{y}$, respectively, defined in complete analogy to Eqs. (54)–(56).

The matrices of the representation are illustrated by Table 2 with all negative signs converted to positive ones. Of course, the S_3 -classes corresponding to the S_4 -classes of Table 2 may be found by consulting Table 1. It is easy to compute the relevant Molien functions for invariants and anti-invariants (cf. Section 6). We obtain

$$\Phi(\lambda) = (1 + \lambda^2 + 2\lambda^3 + \lambda^4 + \lambda^6)/D, \quad (215)$$

$$\Phi_a(\lambda) = (\lambda^2 + 4\lambda^3 + \lambda^4)/D, \quad (216)$$

with

$$D = [(1 - \lambda)(1 - \lambda^2)(1 - \lambda^3)]^2. \quad (217)$$

Here it is decisive that the free invariants are known from the beginning. Otherwise, the denominators and the numerators of the Molien functions would not be uniquely defined [13, p.101], [14, p.481]. Since D correctly enumerates the free invariants, we may deduce from the Molien functions (215) and (216) the number of transient (anti)invariants of degree m . There must exist two transient invariants of degree 3 and one each of degrees 2, 4, and 6. We must have one transient anti-invariant of degree 2, one of degree 4 and four of degree 3. Of the latter, two are known at the outset: they are the pure anti-invariants $H_3^{(x)}$ and $H_3^{(y)}$, defined in complete analogy to Eq. (57).

Any (anti)invariant of the indicated degree may qualify as a transient (anti)invariant, provided that it is not linearly expressible in terms of other (previously defined) transient (anti)invariants, with pure coefficients. Clearly, the transient invariants and the transient anti-invariants, with the exception of $H_3^{(x)}$ and $H_3^{(y)}$, must be mixed. A general method to obtain mixed (anti)invariants starting from pure invariants is “polarization”, i.e. the application of the “polarizing” operators

$$D_x = y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2} + y_3 \frac{\partial}{\partial x_3}, \quad (218)$$

$$D_y = x_1 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_2} + x_3 \frac{\partial}{\partial y_3}, \quad (219)$$

$$E_x = y_1 \frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_3} + y_3 \frac{\partial}{\partial x_2} + y_3 \frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial x_3}, \quad (220)$$

$$E_y = x_1 \frac{\partial}{\partial y_2} + x_2 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_3} + x_3 \frac{\partial}{\partial y_2} + x_3 \frac{\partial}{\partial y_1} + x_1 \frac{\partial}{\partial y_3}, \quad (221)$$

and the “antipolarizing” operators

$$F_x = y_1 \frac{\partial}{\partial x_2} - y_2 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_3} - y_3 \frac{\partial}{\partial x_2} + y_3 \frac{\partial}{\partial x_1} - y_1 \frac{\partial}{\partial x_3}, \quad (222)$$

$$F_y = x_1 \frac{\partial}{\partial y_2} - x_2 \frac{\partial}{\partial y_1} + x_2 \frac{\partial}{\partial y_3} - x_3 \frac{\partial}{\partial y_2} + x_3 \frac{\partial}{\partial y_1} - x_1 \frac{\partial}{\partial y_3} . \quad (223)$$

Manifestly, a polarizing operator applied to an invariant yields another invariant or zero, and an antipolarizing operator applied to an invariant yields an anti-invariant or zero.

We may symbolize simultaneous (anti)invariants in the vectors $\vec{x}$, $\vec{y}$ by properly reinterpreted (anti)symbols of Appendix A. To prevent any possible misinterpretation we label these (anti)symbols by the lower indices xy . We obtain

$$V_2 \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{xy} = \frac{1}{2} D_x \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{xy} = D_x X_2 , \quad (224)$$

$$V_3 \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}_{xy} = \frac{1}{6} E_x \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{xy} = E_x X_3 , \quad (225)$$

$$V'_3 \stackrel{\text{def}}{=} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}_{xy} = \frac{1}{6} E_y \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}_{xy} = E_y Y_3 , \quad (226)$$

$$K_2 \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}_{xy} = \frac{1}{2} F_x (x_1^2 + x_2^2 + x_3^2) = \frac{1}{2} F_x (X_1^2 - 2X_2) , \quad (227)$$

$$K_3 \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right\}_{xy} = \frac{1}{6} F_x \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{xy} = F_x X_3 , \quad (228)$$

$$K'_3 \stackrel{\text{def}}{=} \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \right\}_{xy} = \frac{1}{6} F_y \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}_{xy} = F_y Y_3 . \quad (229)$$

Here, mixed invariants are symbolized by the letter V and mixed anti-invariants by the letter K. Their indices indicate their degree m . The (anti)invariants (224)–(229) qualify as the transient (anti)invariants of degree $m \leq 3$. Their number corresponds exactly to the number of transient (anti)invariants with $m \leq 3$ required by the Molien functions (215), (216) (recall the existence of the pure basic anti-invariants $H_3^{(x)}$ and $H_3^{(y)}$!).

Manifestly, the only potential linear relationships between these (anti)invariants would be of the form

$$aV_2 + bV_3 + cV'_3 + d = 0 , \quad (230)$$

$$eK_2 + fK_3 + gK'_3 = 0 , \quad (231)$$

a, b, c, d, e, f, g being pure coefficients. A fundamental property of free modules of (anti)-invariants [14, Sec. 3] (cf. also Section 7) guarantees that all these free coefficients vanish (identically in the free invariants).

We are still missing the transient invariants of degree 4 and 6 and the transient anti-invariant of degree 4. Nothing prevents us from choosing them as products of transient (anti)invariants that have been obtained till now. If we omit from this choice products of transient anti-invariants, we uniquely obtain

$$V_4 = V_2^2 \quad (232)$$

and

$$K_4 = V_2 K_2 , \quad (233)$$

but for V_6 we have still have some freedom left. Neither V_3^2 nor $V'_3{}^2$ represent acceptable alternatives for V_6 ; this can be seen by establishing the correspondence between the basic

(anti)invariants of this Appendix and those of the rest of our article. The correspondence is given by the correspondence of (anti)symbols containing the same exponents, with and without the lower indices xy , respectively. The choice $V_6 = V_3^2$ would lead to an s, t -invariant of degree $K = 8$, K being defined by Eq. (66), while the choice $V_6 = V_3'^2$ would yield $K = 10$. Neither is acceptable in view of the Molien function (80) of Section 6. On the other hand, either

$$V_6 = V_2^3 \tag{234}$$

or

$$V_6 = V_3 V_3' \tag{235}$$

is an acceptable alternative, since it leads to $K = 9$ (cf. also Section 7).

Upon inspection of the above results we realize that the free invariants $X_1, X_2, X_3, Y_1, Y_2, Y_3$ and the transient (anti)invariants $V_2, V_3, V_3', K_2, K_3, K_3', H_3^{(x)}, H_3^{(y)}$ constitute the final basis of (anti)invariants of the matrix group that we have been considering in this Appendix.

D Orthogonal basis in purity space

In this Appendix we explain how to set up an orthogonal basis of vectors $|K\rangle$ in the purity space of invariants by using the standard Gram-Schmidt procedure. We start from the vectors $|\tilde{U}_K\rangle$ which correspond to the transient invariants (including unity) of the infinitesimal basis [cf. the correspondence table (171) of Section 10].

We define the elements of a real symmetric Gram matrix G_{KL} by

$$G_{KL} = \langle \tilde{U}_K | \tilde{U}_L \rangle . \quad (236)$$

The elements G_{KL} are most simply calculated directly from the definition (174) and the expressions (69)–(71) and (76)–(77) for the transient invariants with the vectors $\vec{s}$ and $\vec{t}$ replaced by the vectors $\tilde{\sigma}$ and $\tilde{\tau}$, respectively. We obtain:

$$G_{00} = \langle \tilde{U}_0 | \tilde{U}_0 \rangle = 1 , \quad (237)$$

$$G_{03} = \langle \tilde{U}_0 | \tilde{U}_3 \rangle = 0 , \quad (238)$$

$$G_{04} = \langle \tilde{U}_0 | \tilde{U}_4 \rangle = 0 , \quad (239)$$

$$G_{05} = \langle \tilde{U}_0 | \tilde{U}_5 \rangle = 0 , \quad (240)$$

$$G_{06} = \langle \tilde{U}_0 | \tilde{U}_6 \rangle = \frac{2}{3} \tilde{S}_2 \tilde{T}_4 , \quad (241)$$

$$G_{09} = \langle \tilde{U}_0 | \tilde{U}_9 \rangle = \frac{3}{2} \tilde{S}_3 \tilde{T}_6 , \quad (242)$$

$$G_{33} = \langle \tilde{U}_3 | \tilde{U}_3 \rangle = 2 \tilde{S}_2 \tilde{T}_4 , \quad (243)$$

$$G_{34} = \langle \tilde{U}_3 | \tilde{U}_4 \rangle = -3 \tilde{S}_3 \tilde{T}_4 , \quad (244)$$

$$G_{35} = \langle \tilde{U}_3 | \tilde{U}_5 \rangle = -3 \tilde{S}_2 \tilde{T}_6 , \quad (245)$$

$$G_{36} = \langle \tilde{U}_3 | \tilde{U}_6 \rangle = \frac{9}{2} \tilde{S}_3 \tilde{T}_6 , \quad (246)$$

$$G_{39} = \langle \tilde{U}_3 | \tilde{U}_9 \rangle = \frac{4}{3} \tilde{S}_2^2 \tilde{T}_4^2 , \quad (247)$$

$$G_{44} = \langle \tilde{U}_4 | \tilde{U}_4 \rangle = -\frac{2}{3} \tilde{S}_2^2 \tilde{T}_4 , \quad (248)$$

$$G_{45} = \langle \tilde{U}_4 | \tilde{U}_5 \rangle = \frac{1}{2} \tilde{S}_3 \tilde{T}_6 , \quad (249)$$

$$G_{46} = \langle \tilde{U}_4 | \tilde{U}_6 \rangle = \tilde{S}_2^2 \tilde{T}_6 , \quad (250)$$

$$G_{49} = \langle \tilde{U}_4 | \tilde{U}_9 \rangle = -2 \tilde{S}_2 \tilde{S}_3 \tilde{T}_4^2 , \quad (251)$$

$$G_{55} = \langle \tilde{U}_5 | \tilde{U}_5 \rangle = -\frac{2}{3} \tilde{S}_2 \tilde{T}_4^2 , \quad (252)$$

$$G_{56} = \langle \tilde{U}_5 | \tilde{U}_6 \rangle = \tilde{S}_3 \tilde{T}_4^2 , \quad (253)$$

$$G_{59} = \langle \tilde{U}_5 | \tilde{U}_9 \rangle = -2 \tilde{S}_2^2 \tilde{T}_4 \tilde{T}_6 , \quad (254)$$

$$G_{66} = \langle \tilde{U}_6 | \tilde{U}_6 \rangle = 2 \tilde{S}_2^2 \tilde{T}_4^2 , \quad (255)$$

$$G_{69} = \langle \tilde{U}_6 | \tilde{U}_9 \rangle = 4 \tilde{S}_2 \tilde{S}_3 \tilde{T}_4 \tilde{T}_6 , \quad (256)$$

$$G_{99} = \langle \tilde{U}_9 | \tilde{U}_9 \rangle = \frac{4}{3} \tilde{S}_2^3 \tilde{T}_4^3 + 3 \tilde{S}_3^2 \tilde{T}_4^3 + 3 \tilde{S}_2^3 \tilde{T}_6^2 + \frac{45}{2} \tilde{S}_3^2 \tilde{T}_6^2 . \quad (257)$$

Whoever tries to casually check this list should not forget that we have $\tilde{S}_2 \leq 0$, $\tilde{T}_4 \leq 0$, while $\tilde{S}_3$ and $\tilde{T}_6$ can have either sign.

We are free to choose

$$|0\rangle = |\tilde{U}_0\rangle , \quad (258)$$

$$|3\rangle = |\tilde{U}_3\rangle . \quad (259)$$

In the next step, we set

$$|4\rangle = |\tilde{U}_4\rangle - a|3\rangle - b|0\rangle . \quad (260)$$

Requiring

$$\langle 3|4\rangle = \langle 0|4\rangle = 0 , \quad (261)$$

we obtain the values of the unknowns a and b and find

$$|4\rangle = |\tilde{U}_4\rangle + \frac{3\tilde{S}_3}{2\tilde{S}_2}|\tilde{U}_3\rangle . \quad (262)$$

From σ, τ -symmetry follows

$$|5\rangle = |\tilde{U}_5\rangle + \frac{3\tilde{T}_6}{2\tilde{T}_4}|\tilde{U}_3\rangle . \quad (263)$$

It turns out that $\langle 4|5\rangle = 0$ as we might have expected. In order to calculate $\langle 4|4\rangle$ we start from Eq. (260), obtaining

$$\langle 4|4\rangle = \langle 4|\tilde{U}_4\rangle . \quad (264)$$

Inserting into this equation the conjugate of Eq. (262), we arrive at

$$\langle 4|4\rangle = -\frac{\tilde{T}_4}{6\tilde{S}_2}\tilde{B}_6 , \quad (265)$$

and, by σ, τ -symmetry,

$$\langle 5|5\rangle = -\frac{\tilde{S}_2}{6\tilde{T}_4}\tilde{D}_{12} , \quad (266)$$

with $\tilde{B}_6$ and $\tilde{D}_{12}$ given by Eqs. (188) and (189).

Next, we set

$$|6\rangle = |\tilde{U}_6\rangle - c|5\rangle - d|4\rangle - e|3\rangle - f|0\rangle . \quad (267)$$

Requiring $\langle L|K\rangle = 0$ for $L \neq K \leq 6$, we obtain the values of the unknowns c, d, e, f in terms of the quantities $\langle L|\tilde{U}_6\rangle$ and $\langle L|L\rangle$ with $L < 6$. Using some of the previous expressions of this Appendix we may express the latter quantities in terms of the elements of the Gram matrix $\langle \tilde{U}_L|\tilde{U}_6\rangle$ and $\langle \tilde{U}_L|\tilde{U}_K\rangle$ with $L, K < 6$. We find

$$|6\rangle = |\tilde{U}_6\rangle + \frac{3\tilde{S}_3}{2\tilde{S}_2}|\tilde{U}_5\rangle + \frac{3\tilde{T}_6}{2\tilde{T}_4}|\tilde{U}_4\rangle + \frac{9\tilde{S}_3\tilde{T}_6}{4\tilde{S}_2\tilde{T}_4}|\tilde{U}_3\rangle - \frac{2}{3}\tilde{S}_2\tilde{T}_4|\tilde{U}_0\rangle \quad (268)$$

and, by the same method as before,

$$\langle 6|6\rangle = \frac{1}{72\tilde{S}_2\tilde{T}_4}\tilde{B}_6\tilde{D}_{12} . \quad (269)$$

One more step is needed to find $|9\rangle$ and $\langle \tilde{U}_9|\tilde{U}_9\rangle$; it reveals a welcome simplification in that $|\tilde{U}_9\rangle$ turns out to be orthogonal to $|4\rangle, |5\rangle$, and $|6\rangle$. The final results, i.e. the expressions of the orthogonal basic vectors $|K\rangle$ in terms of the original vectors $|\tilde{U}_L\rangle$ (with $L \leq K$) and the elements of the diagonalized Gram matrix are listed as arrays (176)–(181) and (182)–(187), respectively, in Section 10. As explicitly shown there, the Gram matrix is indeed positive semi-definite, i.e. positive definite except at a few places in the physical region where it is singular. However, its rank r is always positive ($r \geq 1$) which

is of decisive importance for our purposes. The value of the Gram determinant $|G|$ can be easily calculated from the array (182)–(187) to be

$$|G| = 6^{-6} \tilde{B}_6^3 \tilde{D}_{12}^3, \quad (270)$$

The “magic” integer 6 of our problem makes its most conspicuous appearance in the above value of $|G|$. We also note the simple relations between the elements of the diagonalized Gram matrix:

$$\langle 0|0\rangle\langle 9|9\rangle = \langle 3|3\rangle\langle 6|6\rangle = \langle 4|4\rangle\langle 5|5\rangle. \quad (271)$$

Table 1: Substitutions induced in the variables $r_i, s_i, t_i (i = 1, 2, 3)$ and $x_j (j = 1, \dots, 6)$ by the permutations of the four pions. The permutation designated by 1 is the identity.

Original permutation of p_1, p_2, p_3, p_4	Induced substitution of r_1, r_2, r_3	Induced permutation of the pairs $(s_1, t_1), (s_2, t_2), (s_3, t_3)$	Induced permutation of $x_1, x_2, x_3, x_4, x_5, x_6$
1	$+1 +2 +3$	1	1
(12)(34) (13)(24) (23)(14)	$-1 -2 +3$ $-1 +2 -3$ $+1 -2 -3$	1 1 1	(16)(25) (16)(34) (25)(34)
(123) (132) (124) (142) (134) (143) (234) (243)	$+2 +3 +1$ $+3 +1 +2$ $-3 +1 -2$ $+2 -3 -1$ $-2 -3 +1$ $+3 -1 -2$ $-3 -1 +2$ $-2 +3 -1$	(123) (132) (132) (123) (123) (132) (132) (123)	(123)(465) (132)(456) (142)(356) (124)(365) (153)(246) (135)(264) (145)(263) (154)(236)
(12) (13) (23) (14) (24) (34)	$+2 +1 +3$ $+3 +2 +1$ $+1 +3 +2$ $+1 -3 -2$ $-3 +2 -1$ $-2 -1 +3$	(12) (13) (23) (23) (13) (12)	(12)(56) (13)(46) (23)(45) (24)(35) (14)(36) (15)(26)
(1234) (1243) (1324) (1342) (1423) (1432)	$-3 -2 +1$ $-1 +3 -2$ $-2 +1 -3$ $-1 -3 +2$ $+2 -1 -3$ $+3 -2 -1$	(13) (23) (12) (23) (12) (13)	(1463)(25) (2354)(16) (1562)(34) (2453)(16) (1265)(34) (1364)(25)

Table 2: Illustration of our matrix representation of the group G . Zeros are indicated by dots for better visibility.

Permutation	Matrix
1	$\begin{pmatrix} +1 & . & . & . & . & . \\ . & +1 & . & . & . & . \\ . & . & +1 & . & . & . \\ . & . & . & +1 & . & . \\ . & . & . & . & +1 & . \\ . & . & . & . & . & +1 \end{pmatrix}$
(12)(34)	$\begin{pmatrix} +1 & . & . & . & . & . \\ . & +1 & . & . & . & . \\ . & . & +1 & . & . & . \\ . & . & . & -1 & . & . \\ . & . & . & . & -1 & . \\ . & . & . & . & . & +1 \end{pmatrix}$
(123)	$\begin{pmatrix} . & +1 & . & . & . & . \\ . & . & +1 & . & . & . \\ +1 & . & . & . & . & . \\ . & . & . & . & +1 & . \\ . & . & . & . & . & +1 \\ . & . & . & +1 & . & . \end{pmatrix}$
(12)	$\begin{pmatrix} . & +1 & . & . & . & . \\ +1 & . & . & . & . & . \\ . & . & +1 & . & . & . \\ . & . & . & . & +1 & . \\ . & . & . & +1 & . & . \\ . & . & . & . & . & +1 \end{pmatrix}$
(1234)	$\begin{pmatrix} . & . & +1 & . & . & . \\ . & +1 & . & . & . & . \\ +1 & . & . & . & . & . \\ . & . & . & . & -1 & . \\ . & . & . & -1 & . & . \\ . & . & . & +1 & . & . \end{pmatrix}$

Table 3: Combinations of m - and n -configurations leading to mixed (anti)invariants.

m -configurations	n -configurations
(100) or (200)	$1^{\text{st}} \text{ category } \left\{ \begin{array}{l} (010) \\ (020), (110) \\ (210), (102), (021) \\ (220) \end{array} \right.$ $2^{\text{nd}} \text{ category } \left\{ \begin{array}{l} (100) \\ (200), (011) \\ (022) \end{array} \right.$
(110) or (220)	$1^{\text{st}} \text{ category } \left\{ \begin{array}{l} (100) \\ (200), (011) \\ (210), (102), (021) \\ (022) \end{array} \right.$ $2^{\text{nd}} \text{ category } \left\{ \begin{array}{l} (001) \\ (002), (110) \\ (220) \end{array} \right.$
(210)	$1^{\text{st}} \text{ category } \left\{ \begin{array}{l} (100), (010), (001) \\ (200), (020), (002), (110), (101), (011) \\ (210), (102), (021), (201), (120), (012) \\ (022), (202), (022) \end{array} \right.$

Table 4: Tranformation of basic and auxiliary (anti)invariants under s,t-symmetry.

Pure invariants	Mixed invariants	Pure anti-invariant	Mixed anti-invariants
$S_1 \leftrightarrow T_2$	$U_3 \leftrightarrow U_3$	$H_3 \leftrightarrow H_6$	$J_3 \leftrightarrow -J_3$
$S_2 \leftrightarrow T_4$	$U_4 \leftrightarrow U_5$		$J_4 \leftrightarrow J_5$
$S_3 \leftrightarrow T_6$	$U_6 \leftrightarrow U_6$		$J_6 \leftrightarrow -J_6$
	$U_9 \leftrightarrow U_9$		

Table 5: Auxiliary symbols for monomials in pure invariants.

$a_1 = S_1 \quad \leftrightarrow \quad a_2 = T_2$ $c_2 = S_1^2 \quad \leftrightarrow \quad c_4 = T_2^2$ $e_3 = S_3 \quad \leftrightarrow \quad e_6 = T_6$ $g_4 = S_2^2 \quad \leftrightarrow \quad g_8 = T_4^2$ $i_5 = S_3 T_2 \quad \leftrightarrow \quad i_7 = S_1 T_6$	$b_2 = S_2 \quad \leftrightarrow \quad b_4 = T_4$ $d_3 = S_1 S_2 \quad \leftrightarrow \quad d_6 = T_2 T_4$ $f_4 = S_1 S_3 \quad \leftrightarrow \quad f_8 = T_2 T_6$ $h_4 = S_2 T_2 \quad \leftrightarrow \quad h_5 = S_1 T_4$ $j_7 = S_3 T_4 \quad \leftrightarrow \quad j_8 = S_2 T_6$
$w_3 = S_1 T_2 \quad y_6 = S_2 T_4 \quad z_9 = S_3 T_6$	
$\alpha_4 = S_1^2 T_2 \quad \leftrightarrow \quad \alpha_5 = S_1 T_2^2$ $\gamma_6 = S_1 S_3 T_2 \quad \leftrightarrow \quad \gamma_9 = S_1 T_2 T_6$ $\epsilon_6 = S_1^2 T_4 \quad \leftrightarrow \quad \bar{\epsilon}_6 = S_2 T_2^2$ $\eta_7 = S_3 T_2^2 \quad \leftrightarrow \quad \eta_8 = S_1^2 T_6$ $\kappa_8 = S_1 S_3 T_2^2 \quad \leftrightarrow \quad \kappa_{10} = S_1^2 T_2 T_6$ $\mu_8 = S_2^2 T_2^2 \quad \leftrightarrow \quad \mu_{10} = S_1^2 T_4^2$ $\xi_9 = S_1 S_2 T_6 \quad \leftrightarrow \quad \bar{\xi}_9 = S_3 T_2 T_4$ $\rho_{10} = S_2^2 T_2 T_4 \quad \leftrightarrow \quad \rho_{11} = S_1 S_2 T_4^2$ $\tau_{10} = S_2^2 T_6 \quad \leftrightarrow \quad \tau_{11} = S_3 T_4^2$	$\beta_5 = S_1 S_2 T_2 \quad \leftrightarrow \quad \beta_7 = S_1 T_2 T_4$ $\delta_6 = S_2^2 T_2 \quad \leftrightarrow \quad \delta_9 = S_1 T_4^2$ $\zeta_7 = S_1 S_2 T_2^2 \quad \leftrightarrow \quad \zeta_8 = S_1^2 T_2 T_4$ $\theta_7 = S_1 S_2 T_4 \quad \leftrightarrow \quad \theta_8 = S_2 T_2 T_4$ $\lambda_8 = S_1 S_3 T_4 \quad \leftrightarrow \quad \lambda_{10} = S_2 T_2 T_6$ $\nu_8 = S_2^2 T_4 \quad \leftrightarrow \quad \nu_{10} = S_2 T_4^2$ $\pi_{10} = S_1 S_3 T_2 T_4 \quad \leftrightarrow \quad \pi_{11} = S_1 S_2 T_2 T_6$ $\sigma_{10} = S_1 S_3 T_6 \quad \leftrightarrow \quad \sigma_{11} = S_3 T_2 T_6$ $v_{12} = S_1 S_3 T_4^2 \quad \leftrightarrow \quad \bar{v}_{12} = S_2^2 T_2 T_6$
$\phi_6 = S_1^2 T_2^2 \quad \chi_9 = S_1 S_2 T_2 T_4 \quad \psi_{12} = S_1 S_3 T_2 T_6 \quad \omega_{12} = S_2^2 T_4^2$	

Table 6: Pure coefficients occurring in the syzygies of Section 7.

$A_7 = 3j_7 + \zeta_7 - \eta_7 - \theta_7/3$	$\leftrightarrow$	$A_8 = 3j_8 + \zeta_8 - \eta_8 - \theta_8/3$
$B_4 = -(h_4 + \alpha_4/3)$	$\leftrightarrow$	$B_5 = -(h_5 + \alpha_5/3)$
$C_1 = a_1/3$	$\leftrightarrow$	$C_2 = a_2/3$
$D_3 = -2w_3/3$		
$E_2 = -(b_2 - c_2/3)$	$\leftrightarrow$	$E_4 = -(b_4 - c_4/3)$
$F_8 = -4(\lambda_8 - \nu_8/3)$	$\leftrightarrow$	$F_{10} = -4(\lambda_{10} - \nu_{10}/3)$
$G_5 = i_5 + \beta_5/3$	$\leftrightarrow$	$G_7 = i_7 + \beta_7/3$
$H_2 = -b_2/3$	$\leftrightarrow$	$H_4 = -b_4/3$
$I_4 = -4h_4/3$	$\leftrightarrow$	$I_5 = -4h_5/3$
$K_3 = 3e_3 - d_3/3$	$\leftrightarrow$	$K_6 = 3e_6 - d_6/3$
$L_9 = a_1(\zeta_8 - \eta_8)/3 + a_2(\zeta_7 - \eta_7)/3 + 2(\xi_9 + \bar{\xi}_9) - 11\chi_9/9 - 9z_9$		
$M_6 = (4y_6 - \epsilon_6 - \bar{\epsilon}_6 - 2\phi_6/3)/3$		$N_3 = 5w_3/9$
$O_4 = -(h_4/3 + \alpha_4/9)$	$\leftrightarrow$	$O_5 = -(h_5/3 + \alpha_5/9)$
$R_4 = 2g_4/3 - 2f_4$		$W_5 = \beta_5/3 - i_5$
$X_5 = 2a_1\alpha_4/9 - 2\beta_5/3$		$Y_3 = 2a_1c_2/9 - 6e_3$
$Z_6 = w_3^2/9 - y_6/6$		
$\Gamma_6 = 4\delta_6 + 6\gamma_6 - 2a_1\beta_5$		$\Phi_7 = -6(b_2 + c_2)i_5$
$\Psi_4 = 9f_4 - 2g_4$		$\Omega_3 = 9e_3 + 2d_3$

Table 7: Invariants

K	Symbol	F	U_3	U_4	U_5	U_6	U_9
0	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	6	0	0	0	0	0
1	basic $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$2a_1$	0	0	0	0	0
2	basic $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$2b_2$	0	0	0	0	0
	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$2c_2 - 4b_2$	0	0	0	0	0
	basic $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$2a_2$	0	0	0	0	0
3	basic $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$	$6e_3$	0	0	0	0	0
	$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$	$d_3 - 3e_3$	0	0	0	0	0
	basic $r_1 r_2 r_3 \stackrel{\text{def}}{=} R_3$	not applicable	n.a.	n.a.	n.a.	n.a.	n.a.
	basic $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	0	1	0	0	0	0
4	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$2w_3$	-2	0	0	0	0
	$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$2g_4 - 4f_4$	0	0	0	0	0
	basic $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	$2b_4$	0	0	0	0	0
	$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$	$2c_4 - 4b_4$	0	0	0	0	0

Invariants (cont.)

K	Symbol	F	U_3	U_4	U_5	U_6	U_9
4	basic $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	0	0	1	0	0	0
	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$2h_4$	0	-2	0	0	0
	$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$2\alpha_4$	$-2a_1$	-2	0	0	0
	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$	$-2h_4$	a_1	1	0	0	0
5	basic $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	0	0	0	1	0	0
	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$2h_5$	0	0	-2	0	0
	$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$	$2\alpha_5$	$-2a_2$	0	-2	0	0
	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$	$-2h_5$	a_2	0	1	0	0
	$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$	$\beta_5 - i_5$	0	$-a_1$	0	0	0
	$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$-\beta_5 - i_5$	b_2	a_1	0	0	0
	$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$	$\beta_5 - i_5$	$-b_2$	0	0	0	0
6	auxiliary $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$6e_6$	0	0	0	0	0
	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$	$d_6 - 3e_6$	0	0	0	0	0

Invariants (cont.)

K	Symbol	F	U_3	U_4	U_5	U_6	U_9
6	auxiliary $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	0	0	0	0	1	0
	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	$2y_6$	0	0	0	-2	0
	$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$	$-2y_6$	0	a_2	0	1	0
	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$2\bar{\epsilon}_6$	0	$-2a_2$	0	-2	0
	$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	$-2y_6$	0	0	a_1	1	0
	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	$2\epsilon_6$	0	0	$-2a_1$	-2	0
	$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$	$2\phi_6 + 4y_6$	$-2w_3$	$-2a_2$	$-2a_1$	-2	0
	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$	$-2\bar{\epsilon}_6 - 2\epsilon_6 + 2y_6$	w_3	a_2	a_1	1	0
	$\begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$-2\gamma_6$	e_3	b_2	0	0	0
	$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$2\delta_6$	$-2e_3$	$-2b_2$	0	0	0
7	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$	$\beta_7 - i_7$	0	0	$-a_2$	0	0
	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$	$-\beta_7 - i_7$	b_4	0	a_2	0	0
	$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$	$\beta_7 - i_7$	$-b_4$	0	0	0	0
	$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$	$\theta_7 - j_7$	0	0	$-b_2$	0	0

Invariants (cont.)

K	Symbol	F	U_3	U_4	U_5	U_6	U_9
7	$\begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	$\theta_7 - j_7$	0	0	0	$-a_1$	0
	$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	$-\theta_7 - j_7$	0	0	b_2	a_1	0
	$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \end{bmatrix}$	$\zeta_7 - \eta_7 + 2j_7$	0	$-w_3$	0	$-a_1$	0
	$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$	$-2\theta_7 - \zeta_7 - \eta_7 + 2j_7$	h_4	w_3	b_2	a_1	0
	$\begin{bmatrix} 2 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix}$	$\zeta_7 - \eta_7 + 2j_7$	$-h_4$	0	$-b_2$	0	0
8	$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix}$	$2g_8 - 4f_8$	0	0	0	0	0
	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$	$\theta_8 - j_8$	0	$-b_4$	0	0	0
	$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$	$\theta_8 - j_8$	0	0	0	$-a_2$	0
	$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$	$-\theta_8 - j_8$	0	b_4	0	a_2	0
	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix}$	$\zeta_8 - \eta_8 + 2j_8$	0	0	$-w_3$	$-a_2$	0
	$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$	$-2\theta_8 - \zeta_8 - \eta_8 + 2j_8$	h_5	b_4	w_3	a_2	0
	$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$	$\zeta_8 - \eta_8 + 2j_8$	$-h_5$	$-b_4$	0	0	0
	$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$	$-2\kappa_8 + 2\lambda_8 - 2\nu_8$	i_5	h_4	e_3	b_2	0
	$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	$4\lambda_8 + 2\mu_8$	$-2i_5$	$-2h_4$	$-2e_3$	$-2b_2$	0

Invariants (cont.)

K	Symbol	F	U_3	U_4	U_5	U_6	U_9
8	$\begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$	$2\nu_8$	0	0	$-2e_3$	$-2b_2$	0
	$\begin{bmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$-2\lambda_8$	0	0	e_3	b_2	0
9	$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix}$	$-2\gamma_9$	e_6	0	b_4	0	0
	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$	$2\delta_9$	$-2e_6$	0	$-2b_4$	0	0
	$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$	$\chi_9 - \bar{\xi}_9 - \xi_9 + 3z_9$	0	0	0	$-w_3$	-1
	$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$	$-\chi_9 - \bar{\xi}_9 - \xi_9 + 3z_9$	y_6	h_5	0	0	-1
	$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$	$-\chi_9 - \bar{\xi}_9 - \xi_9 + 3z_9$	y_6	0	h_4	0	-1
	auxiliary $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$	0	0	0	0	0	1
	$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$	$2\chi_9$	$-y_6$	$-h_5$	$-h_4$	0	1
	$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix}$	0	$-y_6$	0	0	w_3	1
10	$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$	$\rho_{10} + 2\sigma_{10} - \tau_{10}$	$-j_7$	$-y_6$	0	0	0
	$\begin{bmatrix} 2 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix}$	$\rho_{10} + 2\sigma_{10} - \tau_{10}$	0	0	$-i_5$	$-h_4$	0
	$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix}$	$-2\pi_{10} - \rho_{10} + 2\sigma_{10} - \tau_{10}$	j_7	y_6	i_5	h_4	0

Invariants (cont.)

K	Symbol	F	U_3	U_4	U_5	U_6	U_9
10	$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \end{bmatrix}$	$-2\kappa_{10} + 2\lambda_{10} - 2\nu_{10}$	i_7	e_6	h_5	b_4	0
	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix}$	$4\lambda_{10} + 2\mu_{10}$	$-2i_7$	$-2e_6$	$-2h_5$	$-2b_4$	0
	$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}$	$2\nu_{10}$	0	$-2e_6$	0	$-2b_4$	0
	$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix}$	$-2\lambda_{10}$	0	e_6	0	b_4	0
11	$\begin{bmatrix} 0 & 1 & 2 \\ 2 & 2 & 0 \end{bmatrix}$	$\rho_{11} + 2\sigma_{11} - \tau_{11}$	$-j_8$	0	$-y_6$	0	0
	$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix}$	$\rho_{11} + 2\sigma_{11} - \tau_{11}$	0	$-i_7$	0	$-h_5$	0
	$\begin{bmatrix} 2 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix}$	$-2\pi_{11} - \rho_{11} + 2\sigma_{11} - \tau_{11}$	j_8	i_7	y_6	h_5	0
12	$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix}$	$2\omega_{12} - 12\psi_{12}$	$-2z_9$	$-2j_8$	$-2j_7$	$-2y_6$	0
	$\begin{bmatrix} 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$	$10\psi_{12} - 2\bar{v}_{12} - 2v_{12}$	z_9	j_8	j_7	y_6	0

Table 8: Anti-invariants

K	Antisymbol	H_3	J_3	J_4	J_5	J_6	H_6
3	basic $\begin{Bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{Bmatrix}$	1	0	0	0	0	0
	basic $\begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{Bmatrix}$	0	1	0	0	0	0
4	basic $\begin{Bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{Bmatrix}$	0	0	1	0	0	0
	$\begin{Bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{Bmatrix}$	0	a_1	1	0	0	0
5	basic $\begin{Bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{Bmatrix}$	0	0	0	1	0	0
	$\begin{Bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{Bmatrix}$	0	a_2	0	-1	0	0
	$\begin{Bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \end{Bmatrix}$	$a_2/3$	$2b_2/3$	$a_1/3$	0	0	0
	$\begin{Bmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \end{Bmatrix}$	$a_2/3$	$-b_2/3$	$a_1/3$	0	0	0
	$\begin{Bmatrix} 2 & 0 & 1 \\ 1 & 0 & 0 \end{Bmatrix}$	$a_2/3$	$-b_2/3$	$-2a_1/3$	0	0	0
6	basic $\begin{Bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{Bmatrix}$	0	0	0	0	0	1
	auxiliary $\begin{Bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{Bmatrix}$	0	0	0	0	1	0
	$\begin{Bmatrix} 1 & 1 & 0 \\ 2 & 0 & 0 \end{Bmatrix}$	0	0	a_2	0	-1	0
	$\begin{Bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \end{Bmatrix}$	0	0	0	a_1	1	0

Anti-invariants (cont.)

K	Antisymbol	H_3	J_3	J_4	J_5	J_6	H_6
6	$\begin{Bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \end{Bmatrix}$	0	w_3	a_2	$-a_1$	-1	0
	$\begin{Bmatrix} 2 & 2 & 0 \\ 1 & 0 & 0 \end{Bmatrix}$	0	e_3	b_2	0	0	0
7	$\begin{Bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{Bmatrix}$	0	$-2b_4/3$	0	$a_2/3$	0	$a_1/3$
	$\begin{Bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \end{Bmatrix}$	0	$b_4/3$	0	$a_2/3$	0	$a_1/3$
	$\begin{Bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \end{Bmatrix}$	0	$b_4/3$	0	$-2a_2/3$	0	$a_1/3$
	$\begin{Bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \end{Bmatrix}$	$b_4/3$	0	0	$b_2/3$	$2a_1/3$	0
	$\begin{Bmatrix} 1 & 2 & 0 \\ 1 & 1 & 0 \end{Bmatrix}$	$b_4/3$	0	0	$-2b_2/3$	$-a_1/3$	0
	$\begin{Bmatrix} 2 & 0 & 1 \\ 1 & 1 & 0 \end{Bmatrix}$	$b_4/3$	0	0	$b_2/3$	$-a_1/3$	0
	$\begin{Bmatrix} 0 & 1 & 2 \\ 2 & 0 & 0 \end{Bmatrix}$	$(c_4 - 2b_4)/3$	$2h_4/3$	$w_3/3$	$-2b_2/3$	$-a_1/3$	0
	$\begin{Bmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \end{Bmatrix}$	$(c_4 - 2b_4)/3$	$-h_4/3$	$w_3/3$	$b_2/3$	$-a_1/3$	0
	$\begin{Bmatrix} 2 & 0 & 1 \\ 2 & 0 & 0 \end{Bmatrix}$	$(c_4 - 2b_4)/3$	$-h_4/3$	$-2w_3/3$	$b_2/3$	$2a_1/3$	0
8	$\begin{Bmatrix} 2 & 2 & 0 \\ 2 & 0 & 0 \end{Bmatrix}$	0	i_5	h_4	$-e_3$	$-b_2$	0
	$\begin{Bmatrix} 2 & 2 & 0 \\ 1 & 0 & 1 \end{Bmatrix}$	0	0	0	e_3	b_2	0

Anti-invariants (cont.)

K	Antisymbol	H_3	J_3	J_4	J_5	J_6	H_6
8	$\begin{Bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{Bmatrix}$	0	0	$b_4/3$	0	$-2a_2/3$	$b_2/3$
	$\begin{Bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \end{Bmatrix}$	0	0	$-2b_4/3$	0	$a_2/3$	$b_2/3$
	$\begin{Bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{Bmatrix}$	0	0	$b_4/3$	0	$a_2/3$	$b_2/3$
	$\begin{Bmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \end{Bmatrix}$	0	$-2h_5/3$	$-2b_4/3$	$w_3/3$	$a_2/3$	$(c_2 - 2b_2)/3$
	$\begin{Bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \end{Bmatrix}$	0	$h_5/3$	$b_4/3$	$w_3/3$	$a_2/3$	$(c_2 - 2b_2)/3$
	$\begin{Bmatrix} 2 & 0 & 0 \\ 2 & 0 & 1 \end{Bmatrix}$	0	$h_5/3$	$b_4/3$	$-2w_3/3$	$-2a_2/3$	$(c_2 - 2b_2)/3$
9	$\begin{Bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \end{Bmatrix}$	0	$-e_6$	0	b_4	0	0
	$\begin{Bmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{Bmatrix}$	$(d_6 - 3e_6)/6$	0	$-h_5/3$	$-h_4/3$	0	$(d_3 - 3e_3)/6$
	$\begin{Bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{Bmatrix}$	$(d_6 - 3e_6)/6$	$-y_6/3$	0	$h_4/3$	$w_3/3$	$(d_3 - 3e_3)/6$
	$\begin{Bmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{Bmatrix}$	$(d_6 - 3e_6)/6$	$y_6/3$	$h_5/3$	0	$-w_3/3$	$(d_3 - 3e_3)/6$
	$\begin{Bmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{Bmatrix}$	$(d_6 - 3e_6)/6$	0	$h_5/3$	$-h_4/3$	$-w_3/3$	$-(d_3 - 3e_3)/6$
	$\begin{Bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{Bmatrix}$	$(d_6 - 3e_6)/6$	$y_6/3$	0	0	$w_3/3$	$-(d_3 - 3e_3)/6$
	$\begin{Bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{Bmatrix}$	$(d_6 - 3e_6)/6$	$-y_6/3$	$-h_5/3$	$h_4/3$	0	$-(d_3 - 3e_3)/6$

Anti-invariants (cont.)

K	Antisymbol	H_3	J_3	J_4	J_5	J_6	H_6
10	$\begin{Bmatrix} 2 & 0 & 0 \\ 2 & 2 & 0 \end{Bmatrix}$	0	$-i_7$	$-e_6$	h_5	b_4	0
	$\begin{Bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \end{Bmatrix}$	0	0	e_6	0	$-b_4$	0
	$\begin{Bmatrix} 2 & 2 & 0 \\ 0 & 1 & 2 \end{Bmatrix}$	0	$j_7/3$	$y_6/3$	$-2i_5/3$	$-2h_4/3$	$(g_4 - 2f_4)/3$
	$\begin{Bmatrix} 2 & 2 & 0 \\ 1 & 2 & 0 \end{Bmatrix}$	0	$-2j_7/3$	$-2y_6/3$	$i_5/3$	$h_4/3$	$(g_4 - 2f_4)/3$
	$\begin{Bmatrix} 2 & 2 & 0 \\ 2 & 0 & 1 \end{Bmatrix}$	0	$j_7/3$	$y_6/3$	$i_5/3$	$h_4/3$	$(g_4 - 2f_4)/3$
11	$\begin{Bmatrix} 0 & 1 & 2 \\ 2 & 2 & 0 \end{Bmatrix}$	$(g_8 - 2f_8)/3$	$-j_8/3$	$-2i_7/3$	$y_6/3$	$2h_5/3$	0
	$\begin{Bmatrix} 1 & 2 & 0 \\ 2 & 2 & 0 \end{Bmatrix}$	$(g_8 - 2f_8)/3$	$2j_8/3$	$i_7/3$	$-2y_6/3$	$-h_5/3$	0
	$\begin{Bmatrix} 2 & 0 & 1 \\ 2 & 2 & 0 \end{Bmatrix}$	$(g_8 - 2f_8)/3$	$-j_8/3$	$i_7/3$	$y_6/3$	$-h_5/3$	0
12	$\begin{Bmatrix} 2 & 2 & 0 \\ 2 & 0 & 2 \end{Bmatrix}$	0	$-z_9$	$-j_8$	j_7	y_6	0

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