Time Measurement in Quantum Gravity

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Abstract

We discuss time measurement in quantum gravity. Using general
relativity for large distances and the uncertainty principle we find a
minimum time interval of the order of the Planck time, therefore the
uncertainty in time measurement is bounded from below.
In this letter we discuss a Gedanken experiment for the measurement of time and show the existence of a minimal observable time interval in quantum gravity. This result, in itself, is not completely new; the existence of a minimal observable length has already been proposed in the context of string theories in refs. [1, 2], the fact that the concept of horizon is not defined at scales smaller than Planck scale has been shown in [3]. Our new points are: 1) we do not consider strings, but use general properties of quantum gravity: the uncertainty principle and Schwarzschild solution at large distances. 2) we find that the minimum error in time measurement is not a constant of nature (Planck time) as might be expected, but a function of the distance between the observer and the observed event.

In order to measure time, one must have a clock located at a distance \( x \) from the observer. The observer obtains his information by looking at the clock, therefore the clock must emit at least one photon toward him.

There are three causes of error in this process of time measurement:

1. The clock’s accuracy—every clock has a minimum error \( \Delta t \).
2. The time it takes the photon to reach the observer has uncertainty due to the uncertainty of the metric caused by the clock’s energy uncertainty \( \Delta E \).
3. The size of the clock—the uncertainty in the distance that the photon had to travel in order to reach the observer is \( 2R \) (\( R \) is the clock’s radius), therefore this error contributes \( \frac{2R}{c} \) to the total error.

Classically there is no problem with this time measurement process, since we can eliminate those causes of error simultaneously, then the time it takes the photon to reach the observer is \( \frac{x}{c} \), and the clock’s time is exactly the difference between the time the photon reaches the observer and the time it takes the photon to reach him. The uncertainty principle is about to change the whole picture as we shall see.

In the absence of a theory of quantum gravity we do not know the law of gravitation at short distances but at large distances (relative to the Planck
length) general relativity would be a good approximation to quantum gravity.

Suppose \( R \leq x_c \) (\( x_c \) is the shortest distance for which we assume that general relativity is a good approximation to quantum gravity, thus \( x_c = \alpha \sqrt{\frac{G}{c^3}} \)) then the third cause of error will contribute \( F(R) \), in the absence of a theory of quantum gravity all we can say about \( F(R) \) is that \( F(x_c) = \frac{2}{c} x_c \) and \( F(R) > 0 \) for \( x_c > R > 0 \).

For distances larger than \( x_c \) we can use general relativity, thus for \( r > x_c \) Schwarzschild solution would be a fair assumption. For those distances we have

\[
ds^2 = -c^2 dt^2 (1 - \frac{2GE}{c^4 r}) + \frac{dr^2}{1 - \frac{2GE}{c^4 r}}
\]

where \( E \) is the energy of the clock, therefore the speed of light is

\[
v = \frac{dr}{dt} = c - \frac{2GE}{c^3 r}
\]

The time it takes the photon to reach the observer from \( x_c \) is

\[
T = \int_{x_c}^{x} \frac{dr}{v} = \frac{1}{c} (x - x_c) + \frac{2}{c^5} G E \log \frac{c^4 x - 2GE}{c^4 x_c - 2GE}
\]

Notice that \( \frac{2}{c^4} G E < x_c \), otherwise the photon will be locked at the clock’s black hole. We can use

\[
\log \frac{x-a}{y-a} > \log \frac{x}{y}, (x > y > a > 0)
\]

to obtain

\[
\Delta T > 2\frac{\Delta E}{c^5} G \log \frac{x}{x_c}
\]

Then we can use the uncertainty inequality \([4]\) \( \Delta t \Delta E \geq \hbar \) and \( F(R) > 0 \) to obtain

\[
\Delta T_{tot}(\Delta E) > \frac{\hbar}{\Delta E} + \frac{2\Delta E G \log \frac{x}{x_c}}{c^5}
\]

where \( \Delta T_{tot} \) is the error for the whole process.
Eq.(6) implies that there exists a minimum error

$$\Delta T_{\text{min}} = 2\sqrt{\frac{2}{c^3 G \hbar}} \sqrt{\log \frac{x}{x_c}}$$  \hspace{1cm} (7)

at

$$\Delta E = \left(\frac{\hbar c^5}{2G \log \frac{x}{x_c}}\right)^{\frac{1}{2}}$$  \hspace{1cm} (8)

Notice that as mentioned above $\Delta E < c^4 \frac{x}{2c}$, thus the relation in eq.(7) is satisfied only for $x > c^4 \frac{x}{2c}$.

If $x_c < x < c^4 \frac{x}{2c}$, then we obtain the minimum at

$$\Delta T_{\text{min}} = \frac{x_c}{c} \left(\frac{2}{\alpha^2} + \log \frac{x}{x_c}\right)$$  \hspace{1cm} (9)

at

$$\Delta E = \frac{x_c c^4}{2G}$$  \hspace{1cm} (10)

Suppose $R > x_c$, then we can use general relativity inside the clock, thus the third cause of error will contribute $\frac{2}{c} R$. The time it takes the photon to reach the observer from $R$ is

$$T = \int_{x_c}^{x} \frac{dr}{v} = \frac{1}{c} (x - R) + \frac{2}{c^3} G E \log \frac{c^4 x - 2GE}{c^4 R - 2GE}$$  \hspace{1cm} (11)

thus

$$\Delta T > \frac{2}{c^5} \Delta E \log \frac{x}{R}$$  \hspace{1cm} (12)

then

$$\Delta T_{\text{tot}}(\Delta E, R) = \Delta t + \Delta T + 2 \frac{R}{c} > \frac{\hbar}{\Delta E} + \frac{2}{c^5} \Delta E G \log \frac{x}{R} + 2 \frac{R}{c}$$  \hspace{1cm} (13)

$R > 2 \frac{\Delta E G}{c^3}$ (otherwise the photon will be locked at the clock’s black hole) therefore the function

$$f(R) = \frac{\Delta E G}{c^3} \log \frac{x}{R} + R$$  \hspace{1cm} (14)
is an increasing function, thus in order to measure time as well as possible we should use a clock with $R = x_e$ then

$$\Delta T_{tot}(\Delta E) > \frac{\hbar}{\Delta E} + \frac{2\Delta E G \log \frac{\Delta E}{\Delta t}}{c^5} + \frac{2}{c} x_e \Delta E + \frac{2\Delta E G \log \frac{\Delta E}{\Delta t}}{c^5}$$

(15)

and we are left with the same uncertainty as before ($R < x_e$).

Note that in the discussion above we use the most simple time measurement process, any particles that will be added must necessarily increase the uncertainty of the metric without decreasing $\Delta t$ thus the total error will get larger. Therefore it seems that the uncertainty mentioned above is a basic property of nature.

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References