$N = 2$ Superstring Theory Generates
Supersymmetric Chern-Simons Theories\(^1\)

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**Abstract**

We show that the action of self-dual supersymmetric Yang-Mills theory in four-dimensions, which describes the consistent massless background fields for $N = 2$ superstring, generates the actions for $N = 1$ and $N = 2$ supersymmetric non-Abelian Chern-Simons theories in three-dimensions after appropriate dimensional reductions. Since the latters play important roles for supersymmetric integrable models, this result indicates the fundamental significance of the $N = 2$ superstring theory controlling (possibly all) supersymmetric integrable models in lower-dimensions.

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1. Introduction

The mathematical conjecture [1] that all the integrable models in lower-dimensions would be generated by what is called self-dual Yang-Mills (SDYM) theory [2] in four-dimensions \( D = 4 \)
\(^3\) has attracted much attention in physics nowadays, due to the recent realization [3] that the consistent massless background fields for \( N = 2 \) open superstring theory are nothing else than the SDYM fields or self-dual supersymmetric Yang-Mills (SDSYM) fields. In particular, it has been discovered [4] that the consistent background fields for \( N = 2 \) open superstring must be \( N = 4 \) SDSYM, while it must be \( N = 8 \) self-dual supergravity (SDSG) for \( N = 2 \) closed superstring, if they are to be described by irreducible superfields.

Motivated by this development, we have recently studied various aspects of SDSYM and self-dual supergravity systems with different number of supersymmetries [5-8], which are obtained by some truncations of the maximally supersymmetric SDSYM or SDSG theory above.

We have also shown that some well-known supersymmetric integrable systems in lower-dimensions, such as supersymmetric KdV equations [9], supersymmetric KP equations [10], supersymmetric Wess-Zumino-Novikov-Witten models [11], topological models and \( W_{\infty} \)-gravity [12], as well as dilaton black-hole solution [13] are indeed generated by the \( N = 4 \) SDSYM theory after some dimensional reductions (DRs).

Most of them strongly indicate that there must be supersymmetric Chern-Simons (SCS) theory [14,15] with the vanishing field strength in \( D = 3 \), serving as an intermediate theory that connects the SDSYM theory in \( D = 4 \) and these lower-dimensional supersymmetric integrable models. As a matter of fact, already in the context of non-supersymmetric Chern-Simons (CS) theory there are lots of links known between the CS theory and soluble lattice models, Yang-Baxter equations, and monodromies of conformal field theories [16]. Therefore it is quite natural that the SCS theory has close relationship with the supersymmetric integrable models or topological models. Based on this indication, we try in this Letter to show that the actions of the \( N = 1 \) and \( N = 2 \) SCS theories are directly generated by the action of the SDSYM theory in \( D = 4 \) after an appropriate DR and truncation. We show that this link is not just at the field equation level, which has been already indicated in our previous paper [15], but is manifest at the action level after some peculiar DR.

\(^3\)We sometimes use also the notation \( D = (2, 2) \) in order to show the number of positive and negative signs in the metric, respectively. When such signature does not matter, we simply use \( D = 4 \), instead.
2. CS Theory out of SDYM Theory

Before we deal with the DR of the SDSYM theory, we try our DR for a purely bosonic non-supersymmetric SDYM theory [2,17].

The suitable starting action for the SDYM theory, which is analogous to the supersymmetric case later, is based on what is called Parkes-Siegel action [4,17]. This action has a propagating fundamental multiplier $\hat{G}^{\hat{a}\hat{b}} I$ with the usual Yang-Mills field strength $\hat{F}_{\hat{a}\hat{b}} I$:

$$I_{SDYM} = -\frac{1}{2} \int d^4 \hat{x} \hat{G}^{\hat{a}\hat{b}} \left( \hat{F}_{\hat{a}\hat{b}} I - \frac{1}{2} \hat{e}^{\hat{c}\hat{d}} \hat{F}_{\hat{c}\hat{d}} I \right).$$  \hfill (2.1)

We use the universal notation that any fields or indices with hats indicates those in $D = 4$, to be distinguished from the non-hatted fields in $D = 3$ treated later. Therefore $\hat{a}, \hat{b}, \cdots = 0,1,2,3$ are for the world coordinates in $D = 4$, and relevantly we use the signature $(\hat{\eta}_{\hat{a}\hat{b}}) = \text{diag.}(+, -, -, +)$.

The indices $i,j, \cdots$ are for the adjoint representations for the non-Abelian gauge group. As is easily seen, the field equation of $\hat{G}^{\hat{a}\hat{b}}$ gives the self-duality of $\hat{F}_{\hat{a}\hat{b}}$, while it has some gauge invariance under arbitrary shift in its self-dual part [4]. The lagrangian in (2.1) is by itself invariant under the Yang-Mills gauge transformation, not to mention the total action.

We now try to setup some DR rule to get a CS theory action out of (2.1). First of all, we have to notice that the DR is not very straightforward, because the lagrangian of (2.1) is gauge invariant, while the usual CS lagrangian [2] is non-invariant, yielding a total divergence instead. This indicates that the desirable DR we need should break gauge at each field level, which is somehow recovered at the action level in $D = 3$. In other words, the gauge invariance of total action is somehow restored under the specific DR prescription. We will clarify this point later.

Keeping this point in mind, we setup the following rule for the DR: First of all, we regard the third coordinate $\hat{x}^3 \equiv y$ to be periodic, e.g., $0 \leq y < 2\pi$. This requirement is not strict but advantageous, when we perform $y$-integrations later. We next setup the rule:

$$\hat{A}_a I(\hat{x}) = f(y) A_a I(x),$$
$$\hat{A}_3 I(\hat{x}) = 0,$$  \hfill (2.2a)
$$\hat{G}^{ab} I(\hat{x}) = \epsilon^{abc} g(y) A_c I(x),$$
$$\hat{G}^{ab} I(\hat{x}) = -g(y) A_a I(x).$$  \hfill (2.2b)

Under our universal rule, the indices $a, b, \cdots = 0,1,2$ are for the $D = (1,2)$ coordinates, and accordingly $(x^a) = (x^0, x^1, x^2)$ with $\epsilon^{012} = +1$. As is easily seen, (2.2b) is consistent with

\footnote{In this Letter we use the indices $0,1,2,3$ instead of $1,2,3,4$, unlike our previous paper [5]. The choice of the signature $\text{diag.}(+, -, -, +)$ is for convenience such that $y \equiv x^3$ can be the extra dimension. In this Letter, we do not use the underlined indices, in order to avoid messy expressions.}
the field equation of the original \( \hat{G}^{a\hat{b}} \) itself. Out of (2.2a) we can construct the components of \( \hat{F} \), as
\[
\hat{F}_{a}^{\hat{I}}(\hat{x}) = f(y) \left[ \partial_{a} A_{\hat{I}}(x) - \partial_{\hat{I}} A_{a}(x) \right] + f^{IJK} f(y)^{2} A_{a}^{J}(x) A_{\hat{I}}^{K}(x) ,
\]
\[
\hat{F}_{\hat{a}}^{I}(\hat{x}) = -f'(y) A_{a}^{I}(x) ,
\]
where \( f'(y) \equiv df(y)/dy \). Even though \( \hat{F}_{a}(\hat{x}) \) does not appear to be gauge covariant in \( D = 3 \), this will eventually pose no problem, as will be discussed shortly.

Using (2.2) and (2.3) it is straightforward to approach our “goal” action in \( D = 3 \) in the form:
\[
I_{SDYM} = -\int d^{3}x \int_{0}^{2\pi} dy \varepsilon^{abc} \left[ g(y)f(y)(2\partial_{a} A_{\hat{I}}) A_{c}^{\hat{I}} + g(y)f(y)^{2} f^{IJK} A_{a}^{I} A_{\hat{I}}^{J} A_{\hat{I}}^{K} - g(y)f'(y) A_{a}^{I} A_{\hat{I}}^{I} \right] .
\]
\[
(2.4)
\]
From now on, we sometimes omit the \( x \)-dependence in \( A_{a}^{I}(x) \), when it is clear from the environment. As mentioned earlier, the \( y \)-integral here is periodic, and we can specify the following integrals for the products of \( f(y) \) and \( g(y) \):
\[
\int_{0}^{2\pi} dy g(y)f(y) = c ,
\]
\[
\int_{0}^{2\pi} dy g(y)f(y)^{2} = \frac{2}{3} c ,
\]
\[
\int_{0}^{2\pi} dy g(y)f'(y) = 0 ,
\]
\[
(2.5)
\]
where \( c \) is some constant. There are non-unique choices for \( f(y) \) and \( g(y) \) satisfying these conditions, but a natural and simple example is
\[
f(y) = \cos (my) ,
\]
\[
g(y) = \frac{1}{3\pi} c \left[ 2 + 3 \cos (my) \right] , \quad (m = \pm 1, \pm 2, \cdots) ,
\]
where \( m \) is an arbitrary non-zero integer, and the peculiar cosine-functions are chosen such that the periodic boundary conditions for \( \hat{A} \) and \( \hat{G} \) are satisfied:
\[
\hat{A}_{\hat{a}}^{I}(x, y = 2\pi) = \hat{A}_{\hat{a}}^{I}(x, y = 0) ,
\]
\[
\hat{G}_{a\hat{b}}^{I}(x, y = 2\pi) = \hat{G}_{a\hat{b}}^{I}(x, y = 0) .
\]
\[
(2.7)
\]
The coefficients of terms in (2.5) are fixed in such a way that the final action \( I_{CS} \) has the right overall quantized coefficient \( n/(16\pi) \) \cite{2}, when we fix \( c \) as
\[
c = \frac{n}{16\pi} , \quad (n = \pm 1, \pm 2, \cdots) ,
\]
\[
(2.8)
\]
for any non-Abelian gauge group whose $\pi_3$-homotopy mapping is non-trivial.\textsuperscript{5} After the $y$-integration, (2.4) actually yields the $D = 3$ CS action, as desired:

$$I_{\text{SYM}}^{\text{DR}} \rightarrow I_{\text{CS}} = \frac{n}{16\pi} \int d^3 x \, e^{abc} \left[ A_a^l F_{bc}^l - \frac{1}{4} f^{1JK} A_a^l A_b^j A_c^K \right].$$

(2.9)

We may wonder about the loss of the original $D = 4$ gauge covariance in our DR rule (2.2). We can understand this as a kind of “hidden symmetry” of the original action, when the DR rule are restricted in a peculiar way. Intuitively, we can also understand that all the apparently gauge non-covariant terms in $\delta \hat{F}_{ab}^l$ do not contribute to the variation of the total action in $D = 3$ after the $y$-integration. We can see this in a more explicit computation, as follows: First, we rewrite $\hat{F}_{ab}^l$ as

$$\hat{F}_{ab}^l(x) = f(y) F_{ab}^l(x) + \left( f(y)^2 - f(y) \right) f^{1JK} A_a^J(x) A_b^K(x) ,$$

(2.10)

with manifest $D = 3$ quantities. Accordingly, $D = 3$ gauge transformations are

$$\delta \hat{F}_{ab}^l(x) = -f(y) f^{1JK} \epsilon^J(x) F_{ab}^K(x) + 2 \left( f(y)^2 - f(y) \right) f^{1JK} A_a^J(x) D_b \epsilon^K(x) ,$$

$$\delta \hat{F}_{ab}^l(x) = -f^J(y) D_a \epsilon^l(x) ,$$

$$\delta \hat{G}_{ab}^l(x) = g(y) \epsilon^{abc} D_c \epsilon^l(x) ,$$

$$\delta \hat{G}_{ab}^l(x) = -g(y) D^a \epsilon^l(x) .$$

(2.11)

The covariant derivative $D_a$ here is in terms of the usual $D = 3$ gauge field $A_a^l(x)$. Applying these rules to (2.4), we get

$$\delta I_{\text{SYM}} = -\int d^3 x \, \int_0^{2\pi} dy \, g(y) \epsilon^{abc} f^{1JK} \left[ 3 \left( f(y)^2 - f(y) \right) A_a^l A_b^J D_c \epsilon^K - f(y) F_{ab}^l A_c^J \epsilon^K \right]$$

$$= c \int d^3 x \, \partial_a \left[ \epsilon^{abc} f^{1JK} A_a^l A_b^J \epsilon^K \right].$$

(2.12)

To get the last side we have arranged all the terms into two categories: (i) Bilinear $A$-terms with one derivative, and (ii) Cubic $A$-terms. For the terms in (ii), we have used the Jacobi identity to show they all cancel each other. For terms in (i) we see their mutual cancellation after the use of the $y$-integrals (2.5) up to a surface term as in the last side of (2.12) under the $x$-integral, leaving the total action invariant.

The invariance of the total action after our “non-covariant” DR is, however, very natural, since the resultant $D = 3$ lagrangian is exactly the CS lagrangian which leaves the action (but not the lagrangian) invariant. In other words, the advantage of the action level DR

\textsuperscript{5}As some readers may have already noticed, the number $n$ is related to the “winding” number along the extra coordinate $y$ in the integrals (2.5), namely if we have e.g. $\int_0^{2\pi n} dy \, g(y) f(y) = cn$, then the factor $n$ in (2.8) is automatically generated for $c = 1/(16\pi)$. 

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is that once the CS action is obtained, it automatically guarantees the validity of the non-covariant terms in the DR we have adopted.

$$N = 1 \text{ SCS out of } N = 1 \text{ SSYM}$$

Once we have understood the DR rule for the non-supersymmetric case, it is straightforward to generalize it to the supersymmetric case of SSYM. We start with the action for the $$N = 4$$ SSYM theory, which describes the appropriate massless background fields of the $$N = 2$$ superstring theory [4]. As has been displayed in the papers [4,5], the maximally $$N = 4$$ supersymmetric SSYM theory [4] needs a peculiar multipler superfield for its lagrangian formulation based on what we call Parkes-Siegel formulation [4,17]. The field content of the $$N = 4$$ SSYM theory is $$(\hat{A}_*^l, \hat{G}_{ab}^l, \rho^l, \lambda^i, \hat{S}_i^l, \hat{T}_i^l)$$, where $$i, j, \ldots = 1, 2, 3$$ are the indices for what are called $$\alpha$$ and $$\beta$$-matrices acting on the $$N = 4$$ indices [5], and the barred fields\(^6\) are anti-chiral spinor fields. Its component field action is given by [4,5]

$$I^{N=4}_{\text{SYM}} = \int d^4x \left[ - \frac{1}{2} \hat{G}_{\hat{a}\hat{b}l}(\hat{F}_{\hat{a}\hat{b}l} - \frac{1}{2} \hat{G}_{\hat{a}\hat{b}l} \hat{F}_{\hat{a}\hat{b}l}) + \frac{1}{2}(\hat{D}_a \hat{S}_i^l)^2 - \frac{1}{2}(\hat{D}_a \hat{T}_i^l)^2 + 2(\hat{D} \hat{F}_{\hat{a}l}) \hat{D} \hat{F}_{\hat{a}l} + i f^{JK} \left\{ (\hat{X}^I \alpha_I) \hat{\hat{S}}_i^l K + (\hat{X}^I \beta_I) \hat{\hat{T}}_i^l K \right\} \right]. \tag{3.1}$$

where \(\hat{\hat{F}}\)’s are gamma matrices in \(D = 4\).

For our purpose of displaying a DR that gives a SCS theory, we can simplify (3.1) into an action with lower supersymmetry by some truncation. A simple choice is to go down to $$N = 1$$ SSYM, namely we have the (off-shell) prepotential superfield \(\hat{V}\) for $$(\hat{A}_*^l, \hat{X}_*^l; \hat{D}^l, \hat{\lambda}_*^l)$$ and the multipler (off-shell) superfield $$\hat{\Lambda}^l$$ for $$(\hat{G}_{ab}^l, \hat{\rho}_a^l; \hat{\varphi}_l, \hat{\psi}_{\hat{a}}^l)$$ with the action [5]

$$I^{N=1}_{\text{SYM}} = \int d^4x \int d^2\hat{\theta} \hat{\Lambda}_*^l \hat{\bar{W}}_{\hat{a}}^l \tag{3.2}$$

$$= \int d^4x \left[ - \frac{1}{2} \hat{G}_{\hat{a}\hat{b}}^l \left( \hat{F}_{\hat{a}\hat{b}l} - \frac{1}{2} \hat{G}_{\hat{a}\hat{b}l} \hat{F}_{\hat{a}\hat{b}l} \right) + i \hat{\rho}_{\hat{a}}^l (\hat{\hat{D}}_{\hat{a}} \hat{\Lambda}_*^l + \hat{\varphi}_l \hat{D}^l + \hat{\psi}_{\hat{a}}^l \hat{\lambda}_*^l) \right].$$

Due to its automatic and manifest closure of supersymmetry, we work on superfield as much as possible from now on. The multipler superfield $$\hat{\Lambda}_{\hat{a}}$$ is the superfield analog of the $$\hat{G}_{\hat{a}\hat{b}}$$-field in the previous non-supersymmetric case (2.1), while $$\hat{\lambda}_*^l$$, $$\hat{\varphi}_l$$ and $$\hat{\psi}_{\hat{a}}^l$$ are auxiliary fields. We can easily see in component fields, which fields are to be truncated in (3.1) to reach (3.2), but we skip the details here.

We are now ready to setup our DR rule for (3.2) now in terms of superfields, that will generate a SCS action in $$D = 3$$, which is a superspace analog of the non-supersymmetric

\(^6\)This convention is different from ref. [5] in order to avoid using tildes reserved for other purpose later.
case (2.2):

\[
\begin{align}
\hat{A}_\lambda(\hat{z}) &= \left\{ \begin{array}{ll}
\hat{A}_i(\hat{z}) = f(y)A_i(z) ,
\hat{A}_5(\hat{z}) = f(y)A_5(z) = 0 ,
\hat{A}_6(\hat{z}) = f(y)A_6(z) ,
\end{array} \right. \\
\end{align}
\tag{3.3a}
\]

\[
\hat{A}^\lambda(\hat{z}) = g(y)\Lambda^\lambda(z) .
\tag{3.3b}
\]

Here we are omitting the gauge indices $i, j, \ldots$ temporarily, and $(\hat{z}^A) = (\hat{x}^\hat{a}, \hat{\theta}^\hat{a}, \hat{\theta}^\hat{A})$ are the $D = 4, N = 1$ superspace coordinates, while $(z^A) = (x^a, \theta^\alpha)$ are the $D = 3, N = 1$ superspace coordinates. Notice that the numbers of the coordinates $\hat{\theta}^\hat{a}$ ($\hat{a} = 1, 2$) and of $\theta^\alpha$ ($\alpha = 1, 2$) are exactly the same, and we can even directly identify them, notwithstanding a subtlety about the difference in the inner products to be discussed later. Considering this, we can omit the hat-symbols on the $\theta$-coordinates from now on. Note that we can choose $0 \leq y \equiv x^3 < 2\pi$ as the previous section, and the functions $f$ and $g$ are exactly the same as before, satisfying (2.5).

We next express $\hat{W}^\hat{a}$ in terms of these superpotentials. To this end we use the notation in ref. [5] with necessary hats. After some arrangement, we get

\[
\hat{W}^\hat{a}(\hat{z}) = \frac{i}{2}(\hat{F}^\hat{a})^\hat{\delta}^\hat{\epsilon}_\hat{A} \left[ D^\hat{A}_\delta \hat{A}_\epsilon(\hat{z}) - \hat{\partial}_\epsilon \hat{A}^\hat{\delta}_\delta(\hat{z}) + [\hat{A}^\hat{\delta}_\delta(\hat{z}), \hat{A}_\epsilon(\hat{z})] \right]
\]

\[
= \frac{i}{2}(\hat{F}^\hat{a})^\hat{\delta}^\hat{\epsilon}_\hat{A} \left[ f(y)F^\hat{a}_\delta(\hat{z}) + (f(y)^2 - f(y))[\hat{A}^\hat{\delta}_\delta(\hat{z}), A_\epsilon(z)] \\
+ \delta_\epsilon^\lambda f'(y)A_\epsilon(z) - \delta_\epsilon^\lambda f'(y)\Lambda^\lambda(\hat{z}) \right] .
\tag{3.4}
\]

If we plug this into the starting action (3.2), and perform the $y$-integrals (2.5), we get

\[
I_{\text{SYM}}^{D=4} = c \int d^3x \int d^2\theta \left\{ A^\hat{a}I(z) \left\{ \frac{i}{2}(\hat{F}^\hat{a})^\hat{\delta}^\hat{\epsilon}_\hat{A} F^\hat{a}_\delta I(z) \right\} \\
+ \frac{i}{6}(\hat{F}^\hat{a})^\hat{\delta}^\hat{\epsilon}_\hat{A} f^{ijk} \hat{A}^\hat{a}l(\hat{z}) \hat{A}^\hat{a}(\hat{z}) A_\epsilon(z) A_\delta(z) \right\} .
\tag{3.5}
\]

Here $F^\hat{a}_\hat{b}$ is given purely by $D = 3$ fields:

\[
F^\hat{a}_\hat{b}(z) \equiv \hat{D}^\hat{a}_\delta A^\hat{b}_\delta(z) - \hat{D}^\hat{a}_\delta \hat{A}_\delta(z) + [\hat{A}_\delta(z), A_\delta(z)] ,
\tag{3.6}
\]

with the usage of the hatted $D = 4$ spinorial indices distinguished from the $D = 3$ ones, as a reminder for the subtlety about the inner products mentioned earlier.
We now have to deal with the DR of the fermionic indices. For this purpose, we first perform the DR for the gamma-matrices. Using the notation in ref. [5], we get
\[
\hat{\Gamma}^0 = \sigma_2 \otimes \tau_3 = \sigma_2 \otimes \gamma^0 ,
\]
\[
\hat{\Gamma}^1 = \sigma_2 \otimes (-\tau_1) = \sigma_2 \otimes \gamma^1 ,
\]
\[
\hat{\Gamma}^2 = \sigma_2 \otimes (-\tau_2) = \sigma_2 \otimes \gamma^2 ,
\]
\[
\hat{\Gamma}^3 = \sigma_1 \otimes I_2 .
\]
Here $\gamma^\alpha$ can be identified with the gamma matrices in the final $D = (1, 2)$ [15], as is understood as follows: According to the general analysis of fermions in diverse dimensions in ref. [18], a general Majorana spinor in $D = (1, 2)$ has the structure
\[
(\Psi_\alpha) = \begin{pmatrix} \psi \\ \psi^* \end{pmatrix} ,
\]
where the star-operation implies the complex conjugate. Compared with a two-component Majorana-Weyl spinor in $D = (2, 2)$ [5], eq. (3.8) has exactly the same structure. Therefore, we can directly identify the two components in the latter with the former, and similarly for the $\gamma$-matrix components in (3.7). This direct DR rule for spinors works smoothly everywhere, except for a difference in the inner products of both cases, due to different charge-conjugation matrices used as their “metrics”. To be more specific, in the original $D = (2, 2)$, the charge-conjugation matrix is defined by [5]
\[
(C_{\alpha\beta}) = \sigma_3 \otimes \tau_2 ,
\]
while in the final $D = (1, 2)$, it is to be [18]
\[
(C_{\alpha\beta}) = i \tau_2 .
\]
Due to the extra factor $(+i)$ in (3.10) compared with the chiral (upper two) components in (3.9), we get an extra factor, when performing the replacement for the inner product
\[
\psi^\alpha \chi_\alpha \rightarrow (+i) \psi^\alpha \chi_\alpha .
\]
However, this extra factor $(+i)$ is exactly cancelled by another factor $(-i)$ coming out of the $\sigma_2$-matrix in (3.7), when $\hat{\Gamma}$'s are replaced by $\gamma$'s. Additionally, we can identify the anti-chiral components $\overline{A}_\xi^\alpha$ in (3.3a) with $A_\alpha$, because the anti-chiral spinors in $D = (2, 2)$ are independent of each other, instead of being complex conjugate to each other [5].

After these considerations in (3.5), we see that
\[
I_{S\text{SYM}}^{N=1} \xrightarrow{\text{DR}} \frac{n}{16\pi} \int d^4 x \int d^2 \theta \left[ A^\alpha(z) W_\alpha(z) - \frac{i}{6} (\gamma^\alpha)^{\beta\gamma} A_\beta^I(z) A_\gamma^J(z) A_\delta^K(z) \right] \]
\[
= I_{S\text{CS}}^{N=1} ,
\]
which is nothing else than the action of the \( N = 1 \) SCS theory [15], because

\[
W_\alpha(z) = \frac{i}{2} (\gamma^c)_\alpha^\beta F^I_\beta(z) = \frac{i}{2} (\gamma^c)_\alpha^\beta (D_\beta A_\alpha(z) - \partial_\alpha A_\beta(z) + [A_\beta(z), A_\alpha(z)])
\]

(3.13)
is the field strength superfield in \( D = (1, 2), N = 1 \) SCS theory [14,15]! Thanks to the superfield notation, we do not have to worry about the compatibility of our DR with supersymmetry.

The superficial gauge non-covariance of our DR rule (3.3) can be again understood as the “hidden” symmetry of the original action under our specific DR rule, as in the non-supersymmetric case, whose detailed demonstration is skipped here. We have also seen that the analogy between the non-supersymmetric case and the supersymmetric one is pretty parallel, even sharing the same functions \( f(y) \) and \( g(y) \).

4. \( N = 2 \) SCS Theory out of \( N = 2 \) SDSYM Theory

Once we have understood the DR to get the \( N = 1 \) SCS theory, it is straightforward to think of higher supersymmetries, such as \( N = 2 \) SCS theory. The parallel structure about fermions between \( D = (2, 2), N = 2 \) and \( D = (1, 2), N = 2 \) also makes the DR easier, so that we give the main result here.

The starting \( N = 2 \) SDSYM action is in terms of two \( N = 2 \) real superfields \( \hat{\Lambda}^I \) and \( \hat{S}^I \) [5]:

\[
I_{\text{SDSYM}}^{N=2} = \int d^4 x \int d^2 \theta \ \hat{\Lambda}^I \hat{S}^I.
\]

(4.1)

This action is also easily obtained from the \( N = 4 \) SDSYM action (3.1) for the \( N = 2 \) superstring, by appropriate truncations [5] similarly to the previous \( N = 1 \) case.

To appeal to the intuition of the readers to figure out the right DR rule, we next give our “goal” action of the \( N = 2 \) SCS theory [15]:

\[
I_{\text{SCS}}^{N=2} = \frac{n}{16 \pi} \int d^3 x \int d^2 \theta \int_0^1 dt \left[ \hat{A}_I^I \left( \hat{S}^I + i k f^{IJK} \hat{A}^a_J \hat{A}^a_K \right) + \text{c.c.} \right].
\]

(4.2)

In this section, the \textit{barred} spinorial superfields in \( D = 3 \) are Dirac conjugate spinorial superfields [15]. The \( k \) is an appropriate real constant. The \( t \)-coordinate needed here is what we call “Vainberg coordinate” to make the lagrangian formulation possible in the \( N = 2 \) SCS theory [15]. Accordingly, all the \textit{tilted} superfields \( \bar{S}, \bar{A}_\alpha \) and \( \bar{A}_\alpha \) satisfy the
boundary conditions\(^7\)
\[
\tilde{S}^l(x, t = 1, \zeta) = S^l(x, \zeta), \quad \tilde{S}^l(x, t = 0, \zeta) = 0, \\
\tilde{A}_a(x, t = 1, \zeta) = A_a(x, \zeta), \quad \tilde{A}_a(x, t = 0, \zeta) = 0, \quad (idem. \ for \ \tilde{A}_a) \tag{4.3}
\]

where non-tilded superfields \(S^l(x, \zeta), A_a(x, \zeta)\) and \(\tilde{A}_a(x, \zeta)\) are purely \(D = 3\) superfields, and \(\zeta^a\) represents both \(\theta^a\) and \(\overline{\theta}^a\) for the \(D = 3, N = 2\) superspace. The potential superfields are defined by \([15]\)
\[
\tilde{D} + \tilde{A}_a(x, t, \zeta) \equiv e^{-\bar{V}(x, t, \zeta)/2} \tilde{D}_a e^{\bar{V}(x, t, \zeta)/2}, \quad \tilde{D} + \tilde{A}_a(x, t, \zeta) \equiv e^{-\bar{V}(x, t, \zeta)/2} \tilde{D}_a e^{\bar{V}(x, t, \zeta)/2}, \\
\tilde{D}_t + \tilde{A}_l(x, t, \zeta) \equiv e^{-\bar{V}(x, t, \zeta)/2} \tilde{D}_t e^{\bar{V}(x, t, \zeta)/2}, \quad \tilde{D}_t + \tilde{A}_l(x, t, \zeta) \equiv e^{-\bar{V}(x, t, \zeta)/2} \tilde{D}_t e^{\bar{V}(x, t, \zeta)/2}, \tag{4.4}
\]

Our appropriate DR rule is now obvious due to the simple structure of the final action (4.2). To generate the desired form of (4.2), we first identify the “Vainberg coordinate” \(t\) in (4.2) with \(y \equiv x^3\) in \(D = 4\), and then choose its range to be \(0 \leq t \equiv y \equiv x^3 < 1.\) The DR rule can be established also by following the previous \(N = 1\) case, in particular, the apparent similarity between the superfield contents in \(D = 3, N = 2\) SDSYM \([5-8]\) and \(D = 3, N = 2\) SCS \([15]\), namely we can identify the two sorts of \(\theta\)-coordinates in \(D = (2,2)\) and \(D = (1,2)\). Thus the DR rule is eventually easier than before:
\[
\hat{\Lambda}^l(\hat{\zeta}) = \frac{n}{16\pi} \left[ \tilde{A}_l(x, t, \zeta) + \tilde{A}_l(x, t, \zeta) \right], \\
\hat{\bar{S}}^l(\hat{\zeta}) = \left[ \tilde{S}^l(x, t, \zeta) + ik f^{lJK} A^o J(x, t, \zeta) \bar{A}_a K(x, t, \zeta) \right] + c.c. \tag{4.5}
\]

Applying this DR rule, we get the \(N = 2\) SCS theory \([15]\) at one stroke, because the integrand is simply
\[
\hat{\Lambda}^l \hat{\bar{S}}^l = \frac{n}{16\pi} \left[ (\tilde{A}_l + \tilde{A}_l) \tilde{S}^l + (\tilde{A}_l + \tilde{A}_l) \bar{A} \{ \tilde{A}_a, \tilde{A}_a \} \right], \tag{4.6}
\]
and using this in (4.1) yields
\[
I_{\text{SCS}}^{N=2} \xrightarrow{\text{DR}} I_{\text{SCS}}^{N=2}. \tag{4.7}
\]
Compared with the previous \(N = 1\) case, the \(N = 2\) DR is much easier, owing to the similarity already at the superfield level.

Interestingly enough, the “Vainberg coordinate” \(t\) in the \(N = 2\) SCS theory \([15]\) exactly coincides with the “extra coordinate” \(x^3\) in our DR from \(D = 4\). This result strongly

\(^7\)We are using the checks instead of the hats in ref. \([15]\), in order not to be confused with the hats for DRs.
suggests the validity and naturalness of our DR rule to get SCS theories out of the more fundamental $D = 4$ SDSYM theory. Relevantly, the $N = 2$ case needs no particular functions such as $f(y)$ or $g(y)$, compared with the previous $N = 1$ case, because the “Vainberg coordinate” is a built-in variable in the former.

5. Concluding Remarks

In this Letter, we have taken a very important step to show that the actions for $N = 1$ and $N = 2$ SCS theories in $D = (1, 2)$ are directly generated by the $N = 1$ and $N = 2$ SDSYM theories in $D = (2, 2)$, which are nothing else than truncated theories of the consistent $N = 4$ SDSYM background system for the $N = 2$ open superstring theory. As a by-product, we have also shown that an analogous DR rule for the non-supersymmetric case works, as well.

To our knowledge, there has been no explicit demonstration that the $D = 4$ SDYM action directly gives rise to the action of the CS theory, not to mention the corresponding supersymmetric cases. We believe that our result has provided the “missing link” between these two important theories, which in turn indicates the importance of the $N = 2$ (open) superstring theory.

In our prescription, we have used apparently gauge non-covariant DR rule, which at first sight seemed meaningless. However, we have seen that this can be always understood as the realization of some “hidden” symmetry in the original $D = 4$ action, when the DR rule is specified and the extra dimension is compactified, because all the gauge non-covariant contributions in the variation of the action result only in a total $x$-divergence within $D = 3$.

The success of our DR in the $N = 2$ case is more appealing, because it reveals the important aspect that the “Vainberg coordinate” exactly coincides with the “extra coordinate” in the DR. This strongly indicates the validity and naturalness of our DR prescription related the topological feature of the SCS theory.

Even though we did not demonstrate all the procedure in this Letter, it is more straightforward to generate what is called BF-theory [19] or supersymmetric BF theory from the original PS-formulation of SDYM theory or SDSYM theory. This is because the original PS-formulation lagrangian (2.1) or (3.1) has already a suggestive form of the lagrangian of the former theory. However, we stress the fact that not only BF-type theories, but also the more non-trivial SCS theories themselves are directly generated by the SDSYM theory in $D = 4$.

We have emphasized in this Letter the usage of actions instead of field equations to demonstrate the mechanism of our DR, because some fields can be truncated by hand in
some DR for field equations, which even though is straightforward at the classical level, becomes non-trivial at the quantum level. The action formulation is more explicit and manifest, when we have to consider also the topological effects and gauge invariance, as well as quantum effects in the system. As a matter of fact, our DR rule that appeared to be gauge non-covariant, turned out to be free of problems, when considered at the action level. Additionally, the quantization of the over-all coefficient has the strong topological significance, only when formulated in terms of action principle.

There was a similar attempt [15,20] in the past in the context of $N = 1$ heterotic superstring showing an interesting link between a CS theory and $D = 10$ background fields for the heterotic superstring. Our result in the present Letter indicates that $N = 2$ superstring has even a closer relationship with SCS theories than the $N = 1$ superstring.

We hope that our result has opened a new direction revealing the important link between the $N = 2$ superstring, SDSYM theories in $D = 4$ and SCS theories in $D = 3$, which in turn will play an important role for supersymmetric topological as well as integrable models in lower-dimensions.

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References


