The Zero Tension Limit of the Virasoro Algebra and the Central Extension

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Abstract

We argue that the Virasoro algebra for the closed bosonic string can be cast in a form which is suitable for the limit of vanishing string tension. In this form the limit of the Virasoro algebra gives the null string algebra. The anomalous central extension is seen to vanish as well when $T \to 0$.

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A string theory is characterized by a dimensionful parameter, the string tension, which plays the role played by the mass for point particle theories. In this respect the zero tension limit of a string theory is in some sense the analog of massless particles. The first to consider this analogy was Schild [1] (see also [2]), which considered strings whose geodesics are null surfaces. He called them ‘null strings’. An action for null string was given in [3], the generalization to spinning and superstrings was started in [4] and an Hamiltonian approach is due to Zheltukhin [5]. The issues of quantization where first discussed in [6] and then in [7, 8]. In [6] was found that the quantization of the null string does not give rise to critical dimensions, while in [7, 8] was pointed out that the issue depends crucially on the choice of ordering, and that ordering which gives the usual value of 26 for the number of critical dimensions are also consistent. When $T \rightarrow 0$ the Weyl invariance is substituted by conformal invariance (see [9] and references therein), and the attempt to quantize the theory keeping this invariance leads to restriction on the Hilbert space. A very interesting connection with topological theories is made in [10].

In this short note our aim is more modest, we will not dwell upon the full quantization, or the spectrum of the theory in the limit, more simply we discuss the fate of critical dimensions, when the zero tension limit of the usual Virasoro algebra is taken. Of course the string tension does not appear as a parameter in the Virasoro algebra, that can be taken to zero straightforwardly. What we will do is to show that the algebra of constraints of the bosonic string can be expressed in such a way that the tension does indeed appear as a parameter that can be sent to zero, and that once this limit is taken, the resulting algebra is the one of the null string. With this procedure the central extension can be seen to go to zero as well when the limit is taken.

It is well known (see for example [11]) that the reparametrization algebra of the closed\(^1\) bosonic string has two set of generators, called $L_n$ and $\hat{L}_n$ which satisfy the commutation algebra:

$$[L_m, L_n] = i(m - n)L_{m+n} + A_m \delta_{m+n,0} \quad (1)$$

with an identical algebra for the $\hat{L}$'s, which commute with the $L$'s. Here $A_m = \frac{d}{12}(m^3 - m)$ is the so called central extension, a consequence of the quantization, and which is the source of some of the most interesting issues discussed in the past ten years or so.

Let us briefly trace the origins of (1). We start from the string coordinates $X^\mu(\sigma, \tau)$, a string action, which after convenient gauge choices\(^1\), can be cast

\(^1\)In this paper we will deal with closed bosonic strings, the generalizations being straightforward.

\(^1\)For more details and notations see [11].
in the form:

\[ S = -\frac{T}{2} \int d\sigma d\tau \sqrt{h} \partial_\alpha \partial_\beta X \cdot \partial_\beta X \] (2)

Where \( h \) is the two dimensional Minkowski metric, and \( T \) is a parameter with the dimensions of inverse of the square of a length, the string tension indeed.

The equations of motion of this action are of the wave equation form

\[
\left( \frac{\partial^2}{\partial \sigma^2} - \frac{\partial^2}{\partial \tau^2} \right) X^\mu = 0 \ ,
\] (3)

this means that the \( X \), which are periodic in \( \sigma \), can be expanded as

\[ X^\mu(\sigma, \tau) = X^\mu_R(\tau - \sigma) + X^\mu_L(\tau + \sigma) \] (4)

with

\[
X^\mu_R = \frac{1}{2} e^\mu + \frac{\pi T}{2} p^\mu(\tau - \sigma) + \sqrt{\pi T} \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in(\tau - \sigma)}
\]

\[
X^\mu_L = \frac{1}{2} e^\mu + \frac{\pi T}{2} p^\mu(\tau + \sigma) + \sqrt{\pi T} \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in(\tau + \sigma)}
\] (5)

The residual gauge symmetry is represented by the constraints

\[ \phi^\perp = P^2 + T^2 (\partial_\sigma X)^2 = 0 \] (6)

\[ \phi^\parallel = P \cdot X = 0 \] (7)

Usually these constraints are split into left and right components and become

\[ \dot{X}_R^2 = \dot{X}_L^2 = 0 \ . \] (8)

The Fourier modes of this last equation are the usual generators of the Virasoro algebra:

\[ L_n = \frac{T}{2} \int_0^\tau e^{-2in\sigma} \dot{X}_R^2 d\sigma = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m} \alpha_m \] (9)

In the calculation of the algebra becomes crucial the issue of the ordering of the \( \alpha \)'s. The issue is quickly resolved noticing that

\[ \alpha_n^\dagger = \alpha_{-n} \] (10)

and noticing that the \( \alpha \) satisfy harmonic oscillator-like equations. This imposes the choice of normal ordering, with annihilator to the right of creators, and consequently the appearance of the anomalous central extension.

At the level of action obviously the \( T \to 0 \) limit cannot be taken in a simple, we will comment on the \( T \to 0 \) limit of the action later. The only
place where the limit can be safely made is at the level of the constraints (6) and (7).

In this limit the constraints become:

\[ \dot{\phi}^\perp = P^2 = 0 \]  
\[ \dot{\phi}^\parallel = P \cdot X = 0 \]  

These are the constraints of the null string [6] (here and in the following we will indicate the quantities pertaining to the null string with an hat). Let us very briefly see how one can find these constraints in the context of null string, which we now very briefly describe.

As we mentioned earlier it is impossible to take the straightforward limit of the action (2), and an alternative must be found. Various solutions have been proposed, starting from the original `action principle' described by Schild in his original article [1], other actions have been described in [2, 3, 12, 13]. For example the action of [3] is given in terms of the quantity \( V^\mu \nu = \partial_\sigma X^\nu \partial_\tau X^\mu - \partial_\sigma X^\mu \partial_\tau X^\nu \) which is some sort of generalization of the velocity, and the auxiliary antisymmetric tensor \( \Sigma_{\mu\nu} \) constrained to satisfy \( \Sigma_{\mu_1}^{\nu_1} \Sigma_{\mu_2}^{\nu_2} \cdots \Sigma_{\mu_d}^{\nu_d} = \epsilon_{\mu_1 \cdots \mu_d} \Sigma_{\nu_1}^{\mu_1} \cdots \Sigma_{\nu_d}^{\mu_d} = 0 \), with these fields the action reads:\n
\[ S_N = \int d\sigma d\tau V^\mu \nu \Sigma_{\mu\nu} \]  

After the elimination of \( \Sigma \) the equations of motions are:

\[ \partial_\tau^2 X = 0 \]  

They have to be contrasted with (3), where for the tensionful string we had harmonic oscillators, for null strings we have free particles. The constraints too are changed, and, as we mentioned above, are given by (11) and (12). The solution for the equations of motion is now of the free particle kind:

\[ X(\sigma, \tau) = \dot{x}(\sigma) + \dot{\phi}(\sigma)\tau \]  

and the Fourier expansions which substitutes (5) are:

\[ \dot{x}(\sigma) = a_n e^{i \sigma} \]  
\[ \dot{\phi}(\sigma) = b_n e^{i \sigma} \]  

The Fourier expansion for the constraints now gives:

\[ \dot{\phi}^\perp_n = \sum_m a_{-m+n} a_m \]  
\[ \dot{\phi}^\parallel_n = \sum_m a_{-m+n} b_m \]  

\(^4\)For more details we refer to [3].
Considering the Poisson bracket between the $a$'s and the $b$'s which is $\{a^\mu_n, b^\nu_m\} = \delta^\nu_m \delta^\mu_n$, one finds the following (classical) algebra for the fourier components of the constraints:

\[
\begin{align*}
\{ \hat{\phi}_n^\perp, \hat{\phi}_m^\perp \} &= 0 \\
\{ \hat{\phi}_n^\parallel, \hat{\phi}_m^\parallel \} &= (n - m) \hat{\phi}_n^\parallel + \hat{\phi}_m^\parallel \\
\{ \hat{\phi}_n^\parallel, \hat{\phi}_m^\perp \} &= (n - m) \hat{\phi}_n^\perp + \hat{\phi}_m^\perp
\end{align*}
\tag{18}
\]

This is a classical result, to find the quantum one it is crucial the issue of the normal ordering. It is in fact the commutation rules of the properly ordered generators of the Virasoro Algebra to give the central extension. In the null strings results are known to depend on the choice of ordering [6, 7, 8], therefore we first choose the ordering (in the tensionful case) to the usual normal ordering, which (remembering that $[\alpha_n, \alpha_m] = 1$, $\alpha_n^\dagger = \alpha_m^\dagger$) reads:

\[
: \alpha_n \alpha_m : = \alpha_m \alpha_n .
\tag{19}
\]

We now have to calculate the algebra for $\phi^\parallel$ and $\phi^\perp$ with this ordering. The calculation could be very long, but is simplified greatly from the observation that

\[
\begin{align*}
\phi^\perp &= T(L_m + \hat{L}_m) \\
\phi^\parallel &= 2(L_m - \hat{L}_m)
\end{align*}
\tag{20}
\]

Since we know the commutators of the $l$'s and $\hat{l}$'s, including the anomalous terms, we can just substitute them and calculate the commutators of $\phi^\parallel$ and $\phi^\perp$. A short calculation shows

\[
\begin{align*}
[\phi_m^\parallel, \phi_n^\parallel] &= m \phi_0^\parallel \\
[\phi_m^\perp, \phi_n^\perp] &= m \phi_0^\parallel T^2 \\
[\phi_m^\perp, \phi_n^\parallel] &= m \phi_0^\parallel + 2A_m T
\end{align*}
\tag{21}
\]

It is now possible to take the $T \to 0$ limit. The anomaly $A_m$ cancels in the first two cases, while it adds in the third. This algebra is anomaly free only for the usual 26 dimensions and intercept. For $T = 0$ this algebra reduces to the one of null string [6], with no anomalous term which is proportional to the tension.

It is worth noticing at this point that the absence of the central extension in the null string algebra depends on the choice of ordering. The hermitian (or Weyl) ordering [5, 6] for which $: x_n p_n :_{h.o.} = \frac{1}{2} (x_n p_n + p_n x_n)$ gives rise to no central extension. If one starts from the null string, without any reference to the tensionful string, then an alternative ordering [7, 8], with the positive modes of the $X$ and $P$ to the right of the negative ones, is possible. In this
case one finds the a central extension, together with the value of 26 for the central extension. The choice of ordering of course has consequences on the spectrum, which is continuous with the hermitian ordering, massless with the ordering of [7, 8]. In particular, we refer to [8, 9] for discussions of the issue of alternative orderings, and mass spectra.

The conclusion of this short note is that the $T \to 0$ limit of the string can be taken at the level of the Virasoro algebra consistently, and that the central extension (and of course with it the critical dimension) vanishes with $T$. A result coinciding with the one of the null string, with hermitian ordering of the operators.

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REFERENCES


