Hierarchical Models with Inverse-Square Interaction in Harmonic Confinement

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A hierarchy of quantum models with inverse-square pairwise interaction is studied for the system with confining harmonic potential. An approach based on renormalized harmonic-oscillators is used to construct the energy spectrum, which naturally results in the renormalized Hartree-Fock description of the interaction. In the limit of weak confinement, the model may serve as a canonical model for the multiple chiral Tomonaga-Luttinger liquids for the edge states of the fractional quantum Hall effect. For the $SU(\nu)$ symmetric case, we find that the excitation spectrum is classified in terms of the simple roots and the weights of $SU(\nu)$ Lie algebra.

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Quantum models with inverse-square ($1/r^2$) interaction have been studied extensively with renewed interest, and their fascinating aspects have been clarified [1-16]. Among others, a family of quantum systems confined by harmonic potential provides us with simple and fundamental models [1,2,17-22]: the excitation spectrum consists of *free* harmonic oscillators with the interaction effects being incorporated into the ground-state energy [2,17]. It has been shown that such a simple spectrum can be reproduced by the renormalized-harmonic oscillators (RHIO) which take into account the correlation effects via the repulsion of energy levels [22]. An analogous approach was proposed, which claims that the Hartree-Fock type treatment leads to the *exact* spectrum of the $1/r^2$ systems [2,3,13,18].

Another interesting point for the confined models is that they can constitute, in the limit of the weak confinement, microscopic models for the *chiral* Tomonaga-Luttinger liquid (TLL) which is described by the holomorphic piece of the $U(1)$ conformal field theory (CFT). We note that a typical example of the chiral TLL may be the edge states of the fractional quantum Hall effect (FQHE) [23]. This characteristic feature of confined models is contrasted to that for periodic models which are described by a pair of holomorphic and anti-holomorphic pieces of $U(1)$ CFT. Hence, the confined models with internal degrees of freedom may allow us to study the multiple chiral TLL related to the FQHE.

In this paper we study the energy spectrum of multicomponent electron models with the $1/r^2$ interaction confined by harmonic potential. In section II, we introduce a hierarchy of the models which has a close relationship to the edge states of a certain family of the FQHE. We then compute in III the excitation spectrum as well as the ground-state energy using the RHIO method. In the limit of the weak confinement potential, we will see that the model describes multiple chiral TLL for edge states of the FQHE. In IV, we further classify the spectrum in terms of the dressed charge matrix, from which one can observe the relationship to the underlying symmetry explicitly. In particular, we encounter the simple roots and the weights of the $SU(\nu)$ Lie algebra in the classification of the spectrum. We shall finally show

in V that the renormalized Hartree-Fock description is essentially the same as that of the RHO approach. A brief summary is given in VI.

II. HIERARCHY OF MODELS WITH CONFINEMENT

We introduce a hierarchy of the one-dimensional multicomponent electron models with $1/r^2$ interactions which are confined with harmonic potential,

$$H = -\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \sum_{\alpha, \beta} \sum_{j>i} \frac{g_{\alpha\beta}(g_{\alpha\beta} X_i^{\alpha\alpha} X_j^{\beta\beta} + X_i^{\alpha\beta} X_j^{\beta\alpha})}{(x_j - x_i)^2} + \frac{1}{2} \sum_{i=1}^N \omega_0^2 x_i^2, \quad (1)$$

where the operator $X_i^{\alpha\beta}$ changes the spin states of the i -th electron from β to α ($\alpha, \beta = 1, 2, \dots, \nu$). The $1/r^2$ interaction has a specific combination out of the direct and exchange terms, the meaning of which will become clear when the interaction effects are expressed in the renormalized Hartree-Fock description. The set of interaction parameters is given by the matrix form $\mathcal{G} = (g_{\alpha\beta})$ as,

$$\mathcal{G} = \begin{pmatrix} \lambda_1 & \cdots & \lambda_1 \\ \vdots & \ddots & \vdots \\ \lambda_1 & \cdots & \lambda_1 \end{pmatrix} + \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \lambda_2 & \cdots & \lambda_2 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \lambda_2 & \cdots & \lambda_2 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_\nu \end{pmatrix}. \quad (2)$$

At first glance, the introduction of the parameters seems somehow artificial. We note here that the above matrix corresponds to the topological-order matrix introduced in the Chern-Simons theory for the FQHE to construct a certain hierarchy of the Abelian FQHE with the filling [24],

$$f = \frac{1}{\lambda_1 + 1 - \frac{1}{\lambda_2 + 2 - \frac{1}{\lambda_3 + 3 - \frac{1}{\lambda_4 + 4 - \frac{1}{\lambda_5 + 5 - \frac{1}{\lambda_6 + 6 - \frac{1}{\lambda_7 + 7 - \frac{1}{\lambda_8 + 8 - \frac{1}{\lambda_9 + 9 - \frac{1}{\lambda_{10} + 10}}}}}}}}}}}, \quad (3)$$

for the case of even integers λ_α . We will show that this matrix also specifies the excitation spectrum for the above $1/r^2$ interaction, and thus plays a similar role as the topological-order matrix for the FQHE. In fact we will observe that the above model reproduces the

excitation spectrum expected for the edge states of the FQHE with the filling fraction f (for the model (1), the quantity f is defined by the compressibility χ_c as $f = \omega_0 \chi_c$). It is noted that analogous models have been previously discussed for periodic systems [26]. Such periodic models, however, do not have the direct correspondence to the edge states (chiral liquids), and some *ad hoc* procedure was introduced to impose the chiral constraint. We will see that the confined model introduced here may be naturally related to the chiral TLL for the edge states of the FQHE.

In order to construct the energy spectrum for the above Hamiltonian, let us first consider the $SU(\nu)$ symmetric model for which the ground-state energy [20,19] as well as the excitation energy [20] were obtained exactly. By analyzing the obtained results, we find a key feature of the $1/r^2$ interaction, i.e. the repulsion among energy levels which uniformly enlarges the spacing of levels with adjacent quantum numbers. Such repulsion effects can be incorporated via the renormalization of quantum numbers by introducing the two-body step-wise function $\lambda\theta(n_i - n_j)$ for two eigenstates labeled by n_i and n_j [22]. For example, in the case of spinless fermions the quantum number n_j is determined as

$$n_j = I_j + \lambda \sum_i \theta(n_j - n_i) \quad (4)$$

with the bare quantum number I_j ($= 0, 1, \dots$). Here the step function is defined as: $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x \leq 0$. We refer to this approach as the renormalized harmonic oscillator (RHIO) approach. It was already confirmed that the RHO indeed provides the exact spectrum for the $SU(\nu)$ symmetric model [22]. We will observe that this description is essentially the same as the renormalized Hartree-Fock treatment of the $1/r^2$ interaction [2,3,13,18].

It is straightforward to apply the idea of the RHIO to the hierarchical models (1) by making use of the nested Bethe-ansatz techniques developed for the periodic systems [27]. The energy in the RHIO is expressed in the form of *free* harmonic oscillators in terms of the renormalized quantum numbers $n_j^{(1)}$,

$$E = \omega_0 \sum_{j=1}^N (n_j^{(1)} + \frac{1}{2}), \quad (5)$$

where the interaction effects are taken into account via the renormalization of $n_j^{(1)}$. Following standard techniques [27], the quantum number $n_j^{(1)}$ may be determined by the following nested equations including the spin quantum numbers $n_j^{(\alpha)}$ ($\alpha = 2, \dots, \nu$),

$$n_j^{(1)} = I_j^{(1)} - \sum_m^{M_2} \theta(n_j^{(1)} - n_m^{(2)}) + \lambda_1 \sum_l^{M_1} \theta(n_j^{(1)} - n_l^{(1)}), \quad (6)$$

$$(\lambda_\alpha + 1) \sum_l^{M_\alpha} \theta(n_m^{(\alpha)} - n_l^{(\alpha)}) + I_m^{(\alpha)} = \sum_j^{M_{\alpha-1}} \theta(n_m^{(\alpha)} - n_j^{(\alpha-1)}) + \sum_s^{M_{\alpha+1}} \theta(n_m^{(\alpha)} - n_s^{(\alpha+1)}), \quad (7)$$

for $2 \leq \alpha \leq \nu$, where non-negative integers $I_j^{(\alpha)}$ ($= 0, 1, \dots$) are the bare quantum numbers which classify ν species of elementary excitations. Here we have introduced $M_\alpha = \sum_{\beta=\alpha}^\nu N_\beta$, where N_β is the number of electrons with β spin ($M_1 = N = \sum_{\beta=1}^\nu N_\beta$). It is noted that at each stage of the nested equations the repulsion effects due to $\lambda_\alpha \theta(n_i^{(\alpha)} - n_j^{(\alpha)})$ are included. Here we would like to mention that the integrability of the hierarchical models (1) has been proven, which may give the microscopic foundation of the RHO equations (6) and (7). The detail for the integrability will be reported elsewhere [28].

III. ENERGY SPECTRUM

Using the RHO equations let us express the energy spectrum in a closed form. First substitute the first line of the RHO equation (6) into the expression for the energy. We thus get

$$E/\omega_0 = \frac{N}{2} + \sum_{j=1}^N [I_j^{(1)} - \sum_m^{M_2} \theta(n_j^{(1)} - n_m^{(2)}) + \lambda_1 \sum_{l \neq j}^{M_1} \theta(n_j^{(1)} - n_l^{(1)})]. \quad (8)$$

We further make use of the second line of the RHO equations so as to eliminate $\sum_{j,m} \theta(n_j^{(1)} - n_m^{(2)})$. Iterating similar procedures step by step and making use of the relation,

$$\sum_{j \neq m} \theta(n_j^{(\alpha)} - n_m^{(\alpha)}) = \frac{1}{2} M_\alpha (M_\alpha - 1), \quad (9)$$

one finds the closed formula for the spectrum,

$$E/\omega_0 = \frac{N}{2} - \sum_{\alpha=1}^{\nu-1} (M_\alpha - 1) M_{\alpha+1} + \frac{1}{2} \sum_{\alpha=1}^{\nu} (\lambda_\alpha + 1 - \delta_{\alpha 1}) M_\alpha (M_\alpha - 1) + \sum_{\alpha=1}^{\nu} \sum_{j=1}^{M_\alpha} I_j^{(\alpha)}. \quad (10)$$

For the ground state, the quantum numbers $I_j^{(\alpha)}$ are given by non-negative successive integers

$$I_{j0}^{(\alpha)} = 0, 1, \dots, M_\alpha - 1, \quad \alpha = 1, 2, \dots, \nu. \quad (11)$$

The corresponding ground-state energy is computed as

$$E_0/\omega_0 = \frac{1}{2} \sum_{\alpha=1}^{\nu} [N_\alpha^2 + \lambda_\alpha M_\alpha (M_\alpha - 1)], \quad (12)$$

which indeed includes the interaction effects. The excitation energy is simply given by changing the quantum numbers from the ground-state distribution, resulting in

$$E/\omega_0 = E_0/\omega_0 + \sum_{j=1}^{M_\alpha} l^{(\alpha)} \quad (13)$$

with the particle number being fixed, where non-negative integers defined by

$$l^{(\alpha)} = \sum_{j=1}^{M_\alpha} [I_j^{(\alpha)} - I_{j0}^{(\alpha)}] \quad (14)$$

characterize particle-hole type excitations. It is seen from (13) that excitations with the fixed number of electrons do not include any effects of interactions, and hence the spacing of energy levels is equidistant as free oscillators [17]. This property is quite contrasted to the periodic models with the $1/r^2$ interaction for which particle-hole excitations are also affected by the interaction through the change of the velocity.

A. Electron basis and Hierarchical basis

We now rewrite the excitation spectrum in a transparent way in order to see the low-energy critical behavior more clearly. Let n_α be the change of the electron number with spin index α . The excitation spectrum then reduces to

$$\Delta E/\omega_0 = \sum_{\alpha=1}^{\nu} \mu_{\alpha} n_{\alpha} + \frac{1}{2} \vec{n}^t \mathcal{C}_e \vec{n} + \sum_{\alpha=1}^{\nu} l^{(\alpha)}. \quad (15)$$

Here the linear term in RHS corresponds to the chemical-potential term with

$$\mu_{\alpha} = N_{\alpha} + \sum_{\beta'=1}^{\alpha-1} \lambda_{\beta'} \sum_{\beta=\beta'}^{\nu} N_{\beta} - \frac{1}{2} \sum_{\beta=1}^{\alpha-1} \lambda_{\beta}, \quad (16)$$

which will be henceforth omitted since it is not important in the following discussions. The $\nu \times \nu$ matrix $\mathcal{C}_e$ which classifies the quadratic term in n_{α} is given by

$$\mathcal{C}_e = \mathcal{I} + \mathcal{G} \quad (17)$$

with the unit matrix $\mathcal{I}$ and the matrix $\mathcal{G}$ introduced for the interaction parameters (2). Here the quantum numbers are expressed by the vector $\vec{n}^t = (n_1, n_2, \dots, n_{\nu})$ and non-negative integers $l^{(\alpha)}$ introduced to label particle-hole excitations. It is instructive to note that the excitation is completely classified by the interaction matrix $\mathcal{G}$ introduced for the Hamiltonian.

Here we recall that the above classification of the spectrum is based on the quantum numbers related to electrons. We shall call the expression (15) as the *electron-basis* representation. Note that the spectrum can also be classified in terms of collective modes out of charge and spin excitations as in the ordinary one-dimensional electron models. We shall refer to this expression as the *hierarchical basis*, following the analogous terminology used in the FQHE [24,25]. Quantum numbers m_j ($j = 1, 3, \dots, \nu$) in the hierarchical basis are obtained from those for electron basis via the linear transformation of $\vec{m} = \mathbf{U}^{-1} \vec{n}$ with the $\nu \times \nu$ matrix $U_{\alpha\beta} = \delta_{\alpha\beta} - \delta_{\alpha(\beta-1)}$ [24,25,22]. After this transformation, m_1 labels the charge excitation, whereas m_{α} ($\alpha = 2, \dots, \nu$) expresses the spin excitations. The matrix $\mathcal{C}_e$ is now converted into the new matrix in the hierarchical basis [24,26],

$$\mathcal{C}_h = \begin{pmatrix} \lambda_1 + 1 & -1 & & & \\ -1 & \lambda_2 + 2 & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & & -1 & \lambda_{\nu} + 2 \end{pmatrix}, \quad (18)$$

which classifies the excitation energy as

$$\Delta E/\omega_0 = \frac{1}{2} \vec{m}^t \mathcal{C}_h \vec{m} + \sum_{\alpha=1}^{\nu} l^{(\alpha)}. \quad (19)$$

We wish to note here that the filling fraction defined by (3) is simply expressed by the above matrix as $f = (\mathcal{C}_h^{-1})_{11}$, and is directly related to the compressibility of the system χ_c : $f = \omega_0 \chi_c$.

It is instructive to observe here that the above matrix reduces to the $\text{OSp}(\nu, 1)$ Cartan matrix in the case of free $\text{SU}(\nu)$ electrons ($\lambda_{\alpha} = 0$ for $\alpha = 1, 2, \dots, \nu$). This is naturally understood in the following way. Note first that the spectrum of free electrons with $\text{SU}(\nu)$ spin degrees of freedom is classified by the $\nu \times \nu$ unit matrix $\mathcal{I}$ in the *electron* basis. The $\text{OSp}(\nu, 1)$ Cartan matrix is then obtained from the unit matrix by transforming the electron basis to the hierarchical basis. So, we can say that the $\text{OSp}(\nu, 1)$ Cartan matrix plays a fundamental role to specify the spectrum of free particles in the hierarchical basis. It is worthwhile to note that the integer QHE with the filling ν is also characterized by the $\text{OSp}(\nu, 1)$ Cartan matrix [24,25].

B. Multiple Tomonaga-Luttinger Liquids

Let us now consider the model in the limit of the weak confinement potential ($\omega_0 \rightarrow 0$). In such limiting cases, the system will become approximately uniform, resulting in the continuous excitation spectrum. It is expected in this case that the low-energy behavior can be described by collections of *gapless* harmonic fluids, namely TLL. A crucial point is that the present model confined by the harmonic potential naturally leads to the *chiral* TLL (holomorphic sector of $\text{U}(1)$ CFT). In fact one can find the one-to-one correspondence between the spectrum obtained above ($\omega_0 \rightarrow 0$) and the known spectrum for the chiral TLL. For example, if we regard the above spectrum as that for the uniform system of length L and with the velocity v , we find ω_0 to be replaced by $2\pi v/L$. This identification allows us to read the critical exponents for various correlation functions in the limit of $\omega_0 \rightarrow 0$, making

use of the finite-size scaling analysis [29]. For example the long-distance behavior of the field correlator for chiral electrons should have the power-law form

$$\langle \psi_a^\dagger(r) \psi_a(0) \rangle \simeq r^{-\eta_a} \quad (20)$$

with $\eta_a = 1 + \sum_{\beta=1}^{\alpha} \lambda_{\beta}$, which is obtained by setting $n_j = \delta_{j,a}$ for $j = 1, \dots, \nu$ in the expression (15). In contrast, the spin correlator and the charge correlator should not include the interaction parameters. One can easily check this by noting that these correlators conserve the electron number, so only the descendant fields of level $l^{(\alpha)}$ in (15) with $n_{\alpha} = 0$ ($\alpha = 1, \dots, \nu$) contribute to the asymptotic behavior, resulting in the canonical exponents. All these properties are in accordance with those predicted for the chiral TLL for the edge states using the effective CFT model [23]. Therefore the present model with confining potential may naturally reproduces the spectrum for the multiple (ν -component) *chiral* TLL in the limit of $\omega_0 \rightarrow 0$. It is instructive to observe that the spacing of the energy spectrum is precisely equidistant in the present model and that there are not any logarithmic corrections to the spectrum due to marginal operators. This demonstrates that the present model (with even integer λ_{α}) may serve as a canonical model which can describe the chiral TLL for the edge states of the FQHE with the filling fraction f (see (3)).

IV. DRESSED CHARGES, SIMPLE ROOTS AND WEIGHTS

We further convert the expression of the excitation spectrum into more fundamental formula by using what is called the dressed charge matrix $\mathcal{Z}$ [30]. We introduce the dressed charge matrix by rewriting the excitation spectrum (19) as,

$$\Delta E/\omega_0 = \frac{1}{2} \sum_{\alpha=1}^{\nu} (\mathcal{Z}_h \vec{m})_{\alpha}^2 + \sum_{\alpha=1}^{\nu} l^{(\alpha)}, \quad (21)$$

in the hierarchical basis. We thus find

$$\mathcal{Z}_h = \begin{pmatrix} \sqrt{q_1} & 0 & & 0 \\ -1/\sqrt{q_2} & \sqrt{q_2} & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & -1/\sqrt{q_{\nu}} & \sqrt{q_{\nu}} \end{pmatrix}, \quad (22)$$

where

$$q_{\alpha} = \frac{1}{p_{\alpha} - \frac{1}{p_{\alpha+1} - \dots - \frac{1}{p_{\nu}}}} \quad (23)$$

with $p_{\alpha} = \lambda_{\alpha} + 2 - \delta_{\alpha,1}$ and $q_{\nu} = p_{\nu}$. On the other hand, in the electron basis, the dressed charge matrix is rewritten as

$$\mathcal{Z}_e = \begin{pmatrix} \sqrt{q_1} & \sqrt{q_1} & \cdots & \sqrt{q_1} \\ -1/\sqrt{q_2} & -1/\sqrt{q_2} + \sqrt{q_2} & \cdots & -1/\sqrt{q_2} + \sqrt{q_2} \\ & \ddots & \ddots & \vdots \\ 0 & & -1/\sqrt{q_{\nu}} & -1/\sqrt{q_{\nu}} + \sqrt{q_{\nu}} \end{pmatrix}. \quad (24)$$

Let us now consider the simple case with $SU(\nu)$ symmetry, which is given by the interaction matrix in (2),

$$\mathcal{G} = \begin{pmatrix} \lambda_1 & \cdots & \lambda_1 \\ \vdots & \ddots & \vdots \\ \lambda_1 & \cdots & \lambda_1 \end{pmatrix}. \quad (25)$$

This model possesses the global $SU(\nu)$ symmetry for the spin sector. From (22) the $\nu \times \nu$ dressed charge matrix for the case reduces to

$$\mathcal{Z}_h = \begin{pmatrix} \sqrt{\lambda_1 + 1/\nu} & 0 & \cdots & 0 \\ -\sqrt{(\nu-1)/\nu} & & & \\ 0 & & \mathcal{Z}_h^{(\nu-1)} & \\ \vdots & & & \end{pmatrix}, \quad (26)$$

with the $(\nu-1) \times (\nu-1)$ matrix for the spin part

$$\mathcal{Z}_h^{(\nu-1)} = \begin{pmatrix} \sqrt{\nu/(\nu-1)} & 0 & \cdots & 0 \\ -\sqrt{(\nu-2)/(\nu-1)} & \sqrt{(\nu-1)/(\nu-2)} & \ddots & \\ 0 & \ddots & \ddots & 0 \\ \vdots & 0 & -1/\sqrt{2} & \sqrt{2} \end{pmatrix}. \quad (27)$$

We can express this matrix in terms of the $(\nu-1)$ -component vectors $\vec{h}_{\alpha}$ ($\alpha = 1, \dots, \nu-1$),

$$\mathcal{Z}_h^{(\nu-1)} = (\vec{h}_1, \vec{h}_2, \dots, \vec{h}_{\nu-1}), \quad (28)$$

where each component of $\vec{h}_\alpha$ is defined by

$$(\vec{h}_\alpha)_j = \sqrt{2}(\vec{a}_\alpha)_{\nu-j}, \quad j = 1, 2, \dots, \nu-1 \quad (29)$$

with the new set of vectors $\vec{a}_\alpha$,

$$\begin{aligned} \vec{a}_1 &= (1, 0, \dots, 0), \\ \vec{a}_2 &= \left(\frac{-1}{2}, \frac{\sqrt{3}}{2}, 0, \dots, 0\right), \\ &\dots \\ \vec{a}_m &= (0, \dots, 0, -\sqrt{\frac{m-1}{2m}}, \sqrt{\frac{m+1}{2m}}, 0, \dots, 0), \\ &\dots \\ \vec{a}_{\nu-1} &= (0, \dots, 0, -\sqrt{\frac{\nu-2}{2(\nu-1)}}, \sqrt{\frac{\nu}{2(\nu-1)}}). \end{aligned} \quad (30)$$

It is noteworthy that the above set of $(\nu-1)$ vectors $\vec{a}_\alpha$ constitute the system of the *simple roots* for the $SU(\nu)$ Lie algebra. Hence we observe that the simple roots can completely specify the spin excitation in the hierarchical basis.

Keeping this correspondence in mind we now turn to the excitation spectrum in the electron basis. From (24) the dressed charge matrix for the $SU(\nu)$ symmetric case reads

$$\mathcal{Z}_e = \begin{pmatrix} \sqrt{\lambda_1 + 1/\nu} & \sqrt{\lambda_1 + 1/\nu} & \dots & \sqrt{\lambda_1 + 1/\nu} \\ \vec{y}_\nu & \vec{y}_{\nu-1} & \dots & \vec{y}_1 \end{pmatrix}, \quad (31)$$

where $(\nu-1)$ -component vectors $\vec{y}_\alpha$ ($\alpha = 1, \dots, \nu$) is defined by

$$(\vec{y}_\alpha)_j = \sqrt{2}(\vec{w}_\alpha)_{\nu-j}, \quad j = 1, 2, \dots, \nu-1, \quad (32)$$

with the new vectors $\vec{w}_\alpha$,

$$\begin{aligned} \vec{w}_1 &= \left(\frac{1}{2}, \frac{1}{2\sqrt{3}}, \dots, \frac{1}{\sqrt{2j(j+1)}}, \dots, \frac{1}{\sqrt{2(\nu-1)\nu}}\right), \\ \vec{w}_2 &= \left(\frac{-1}{2}, \frac{1}{2\sqrt{3}}, \dots, \frac{1}{\sqrt{2j(j+1)}}, \dots, \frac{1}{\sqrt{2(\nu-1)\nu}}\right), \\ &\dots \end{aligned}$$

$$\begin{aligned} \vec{w}_m &= (0, \dots, 0, \frac{-(m-1)}{\sqrt{2m(m-1)}}, \frac{1}{\sqrt{2m(m+1)}}, \dots, \frac{1}{\sqrt{2(\nu-1)\nu}}), \\ &\dots \\ \vec{w}_\nu &= (0, \dots, 0, \frac{-(\nu-1)}{\sqrt{2\nu(\nu-1)}}). \end{aligned} \quad (33)$$

It is to be noted here again that the set of the vectors $\vec{w}_\alpha$ is nothing but the *weight vectors* for the defining representation of $SU(\nu)$ Lie algebra. Note that the roots and weights are related through

$$\vec{a}_\alpha = \vec{w}_\alpha - \vec{w}_{\alpha+1}, \quad \alpha = 1, 2, \dots, \nu-1, \quad (34)$$

which is the same as the transformation between the electron basis and the hierarchical basis. Therefore we find an interesting relationship between the excitation spectrum of the $1/r^2$ model and the underlying symmetry, i.e. two fundamental quantities, simple roots and weights in $SU(\nu)$ Lie algebra, classify the excitation spectrum for the $1/r^2$ models with $SU(\nu)$ spin symmetry in the hierarchical basis and in the electron basis, respectively, which are two fundamental bases to express the spectrum in one-dimensional electron systems.

V. RENORMALIZED HARTREE-FOCK DESCRIPTION

We now wish to make a brief comment on the renormalized Hartree-Fock description of the $1/r^2$ interaction, which has been developed to study the excitation, thermodynamics, etc [2,3,13,18]. Let us illustrate the idea by taking the spinless fermion model ($\nu = 1$) as an example. First recall that the free-oscillator spectrum in the confined system is expressed in terms of the occupation probability P_k ($=0,1$) of eigenstates specified by the level k : $E/\omega_0 = \frac{1}{2}N + \sum_{k=1}^N (k-1)P_k$. The interesting consequence found in [2,3,13,18] is that the effects due to the inverse-square interaction $\sum \lambda(\lambda+1)/(x_i - x_j)^2$ can be *exactly* evaluated via the Hartree-Fock type corrections,

$$E/\omega_0 = \frac{1}{2}N + \sum_{k=1}^N (k-1)P_k + \lambda \sum_{k>k'} P_k P_{k'}. \quad (35)$$

This may be referred to as the renormalized Hartree-Fock treatment of the interaction. The result implies that the two-body correlations are essential to effectively describe the $1/r^2$ interaction. In this sense this approach is expected to have the close relationship to the RHO method. In fact one can easily derive (35) from the RHO equations (6) and (7) by noting

$$\sum_{j \neq m} \theta(n_j - n_m) = \sum_{k > k'} P_k P_{k'}. \quad (36)$$

Therefore we can say that the *exact* Hartree-Fock description for $1/r^2$ interaction naturally follows from the uniform level-repulsion due to the momentum-independent phase-shift function of step-wise form. This characteristic property clearly distinguishes the $1/r^2$ models from the Bethe-ansatz solved models for which the phase-shift function depends strongly on the momentum, and hence the renormalized Hartree-Fock treatment cannot give the correct answer.

It is now straightforward to generalize the renormalized Hartree-Fock description to the hierarchical models. By substituting the RHO equations (6) and (7) into the energy (5), one readily arrives at the following expression for the energy,

$$E/\omega_0 = \frac{1}{2}N + \sum_{\alpha=1}^{\nu} \sum_{k=1}^{N_{\alpha}} (k-1)P_k^{(\alpha)} + \frac{1}{2} \sum_{\alpha, \beta} \sum_{k, k'} g_{\alpha\beta} P_k^{(\alpha)} P_{k'}^{(\beta)}, \quad (37)$$

where the sum is taken under the condition $(\alpha, k) \neq (\beta, k')$. It is clear from this expression that the set of interaction parameters $g_{\alpha\beta}$ introduced for the Hamiltonian (1) precisely correspond to the strength of the Hartree-Fock interactions among oscillators.

VI. SUMMARY

We studied the excitation spectrum for a hierarchy of confined electron models with the $1/r^2$ interaction. The excitation spectrum was classified in two alternative manners, i.e. in the electron basis and in the hierarchical basis, similarly to the FQHE. In the limit of the weak confining potential the present model reproduces the spectrum of the multiple chiral

TLL, which may serve as a canonical model for the edge states of the hierarchical FQHE. The classification of the spectrum in terms of simple roots and weights of Lie algebra seems instructive, making explicit the relationship of the excitation to the underlying symmetry. It was also shown that the renormalized Hartree-Fock approach, which enabled one to construct the full thermodynamics, is essentially the same as the RHO approach.

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